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A NOTE ON GUE MINORS, MAXIMAL BROWNIAN FUNCTIONALS AND LONGEST INCREASING SUBSEQUENCES

FLORENT BENAYCH-GEORGES AND CHRISTIAN HOUDRÉ

ABSTRACT. We present equalities in law between the spectra of the minors of a GUE matrix and some maximal functionals of independent Brownian motions. In turn, these results allow to recover the limiting shape (properly centered and scaled) of the RSK Young diagrams associated with a random word as a function of the spectra of these minors. Since the length of the top row of the diagrams is the length of the longest increasing subsequence of the random word, the corresponding limiting law also follows.

1. INTRODUCTION

It is by now well known that there exist strong and interesting connections between directed percolation and random matrices. The precise results we have in mind have their origins in the identity in law, due to Baryshnikov [2] and Gravner, Tracy and Widom [10], between the maximal eigenvalue of an $M \times M$ element of the GUE and a certain maximal functional of standard M -dimensional Brownian motion originating in queuing theory, with Glynn and Whitt [9]. This first result has seen many extensions and complements. For example, O’Connell and Yor [24] as well as Bougerol and Jeulin [5] obtained identities in law between (different) multivariate Brownian functionals and the spectrum of the GUE whose equivalence is shown in Biane, Bougerol and O’Connell [3]. Various related representations have also been put forward and studied for instance in Doumerc [7], Johansson [17, 19], O’Connell [23] to name but a few authors and pieces of work.

Our interest in such representations comes from the identification by Its, Tracy and Widom [25, 15, 16] of the limiting length (properly centered and scaled) of the longest increasing subsequence of a random word as the maximal eigenvalue of a certain random

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FBG: MAP5, Université Paris Descartes, 45, rue des Saints-Pères 75270 Paris Cedex 06, France. florent.benaych-georges@parisdescartes.fr. This work was done in part during this author’s stay at Georgia Institute of Technology, Atlanta.

CH: School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia, 30332-0160, USA. houdre@math.gatech.edu. This work was partially supported by the grant #246283 from the Simons Foundation and by a Simons Foundation Fellowship, grant #267336. Many thanks to the Laboratory MAS of École Centrale, Paris, and to the LPMA of the Université Pierre et Marie Curie for their hospitality while part of this research was carried out.

matrix. For example, in the case of a word with i.i.d. uniformly distributed letters in an alphabet of size M , the limiting law is the maximal eigenvalue of the $M \times M$ traceless GUE. Moreover, positively answering a conjecture of Tracy and Widom, Johansson [18] showed that the whole normalized limiting shape of the RSK Young diagrams associated with the random word is the spectrum of the $M \times M$ traceless GUE. Since the length of the top row of the diagrams is the length of the longest increasing subsequence of the random word the maximal eigenvalue result of [25] followed.

Limiting laws expressed in terms of maximal Brownian functionals are also obtained in [11]. These last representations involve *dependent* Brownian motions and do not clearly recover the results of [25] or [15, 16], which themselves are mainly derived by analytical techniques. To resolve this issue, we provide below an extension of Baryshnikov's result [2] on the identification of the multivariate law of the maximal eigenvalues of the principal minors of a GUE matrix with some maximal functionals of a standard multidimensional Brownian motion. This allows us to circumvent the analytical approach and provides a mixed combinatorial/probabilistic methodology to the solutions of these finite alphabet longest increasing subsequence problems. Our hope is that Theorem 1, below, will also be helpful to fully identify eigenvalues of random matrices as the limiting laws in the corresponding Markov random word problems (see Kuperberg's Conjecture 7 in [21]). In the Markovian setting, the analytical methodology is lacking, in contrast to the probabilistic one, and to date the limiting laws are mainly only expressed as Brownian functionals. Indeed, the multivariate functional appearing in Theorem 1 is exactly the one giving the limit law of the shape of the RSK image of a Markov random word in [12], the only difference being that the Brownian motions in [12] are correlated. This correlation issue in maximal functionals often amounts to adding a condition on the trace of the random matrix (as in [25, 18, 15, 16]). However, for general Markov random words the full identification of these functionals via random matrices remains open. For Markov random words with cyclic and symmetric transition matrix, the longest increasing subsequence will be asymptotically identified to the eigenvalues of some random matrices once we will have a more general version of Theorem 1 below where the Brownian motions are correlated. Our intuition is that to get such a generalization of Theorem 1, one needs to consider the minors of more general random matrices, namely Gaussian Hermitian matrices with general Gaussian vector as diagonal, independent of the off-diagonal entries, who are i.i.d. Besides providing the final touch to an essentially probabilistic proof of the random word asymptotics problem, our results also allow us to shed new lights on the queuing framework by providing, for example, joint limiting laws involving departing times and service times of individual customers.

2. STATEMENTS AND PROOFS OF THE RESULTS

Throughout, fix a positive integer M and consider:

- an $M \times M$ GUE matrix $H = [h_{ij}]$, *i.e.*, a standard Gaussian variable on the space of $M \times M$ Hermitian matrices endowed with the Euclidean scalar product given for any two such matrices X and Y by $X \cdot Y = \text{Tr } XY$,
- an M -dimensional standard Brownian motion $B = (B_k(t))_{t \in [0,1], k=1, \dots, M}$.

For each $k = 1, \dots, M$, denote by

$$(1) \quad \mu_1^k \geq \dots \geq \mu_k^k,$$

the eigenvalues of the principal $k \times k$ minor of H . Next, introduce the set

$$\mathcal{P} := \{\pi : [0, 1] \rightarrow \{1, \dots, M\} \text{ càdlàg, non-decreasing}\},$$

and for $\pi \in \mathcal{P}$, let

$$\Delta_\pi(B) := \int_0^1 dB_{\pi(t)}(t) = \sum_{j=1}^M (B_j(t_j) - B_j(t_{j-1})),$$

where $0 = t_0 \leq t_1 \leq \dots \leq t_M = 1$ are such that

$$\pi(\cdot) = \sum_{j=1}^{M-1} j \times 1_{[t_{j-1}, t_j)}(\cdot) + M \times 1_{[t_{M-1}, t_M]}(\cdot).$$

To complete our notations, for $\pi_1, \pi_2 \in \mathcal{P}$, we write $\pi_1 < \pi_2$ whenever $\pi_1(t) < \pi_2(t)$, for all $t \in [0, 1]$. Let us now state our first result which, in particular, when $\ell = 1$ below, identifies the joint law of the maximal eigenvalue of the principal minors of H and is therefore, in that case, already present in [2].

Theorem 1. *The following equality in law holds true:*

$$\left(\sum_{i=1}^{\ell} \mu_i^k \right)_{1 \leq \ell \leq k \leq M} \stackrel{\text{law}}{=} \left(\sup \left\{ \sum_{i=1}^{\ell} \Delta_{\pi_i}(B); \pi_1, \dots, \pi_\ell \in \mathcal{P}, \pi_1 < \dots < \pi_\ell \leq k \right\} \right)_{1 \leq \ell \leq k \leq M}$$

This theorem, which is certainly of interest on its own right and also has a process version (where the matrix H is replaced by a Dyson Brownian motion, see [1], and B is taken up to time t and not to time 1), is proved at the end of the paper. Let us present and prove at first some of its corollaries which motivated, in part, the present study. When combined with [11], the first corollary provides an alternative approach to [25] or [15, 16]. The second corollary makes full use of Theorem 1 and, when combined with [14] or [12], provides an alternative approach to [18], [15], [16]. In both results, and throughout, \implies indicates convergence in distribution.

Let us briefly recall the framework of the works just cited. Let $(X_i)_{i \geq 1}$ be a sequence of i.i.d. random variables on a totally ordered finite alphabet \mathcal{A} of cardinality k . Denote

the elements of \mathcal{A} by $\alpha_1, \dots, \alpha_k$ listed in such a way that if $p_i := \mathbb{P}(X_1 = \alpha_i)$, $i = 1, \dots, k$, then $p_1 \geq \dots \geq p_k$ (therefore this indexing of the letters in \mathcal{A} has nothing to do with the order used on \mathcal{A}). Next, decompose the alphabet \mathcal{A} into subsets $\mathcal{A}_1, \dots, \mathcal{A}_n$ in such a way that α_i and α_j belong to the same \mathcal{A}_m , $m = 1, \dots, n \leq k$, if and only if $p_i = p_j$. Finally, let LI_N be the length of the longest increasing subsequence of the random word

$$X_1 \cdots \cdots X_N.$$

Corollary 2. *Let $p_{\max} := p_1$, $k_1 := \#\mathcal{A}_1$ and let $H = [h_{ij}]$ be a $k_1 \times k_1$ GUE matrix with largest eigenvalue μ_{\max} . Then, as N tends to infinity,*

$$(2) \quad \frac{\text{LI}_N - Np_{\max}}{\sqrt{Np_{\max}}} \Longrightarrow \frac{\sqrt{1 - k_1 p_{\max}} - 1}{k_1} \sum_{j=1}^{k_1} h_{jj} + \mu_{\max}.$$

Proof. From Theorem 1, with the notation introduced above, and if B is now a k_1 -dimensional standard Brownian motion,

$$(3) \quad (\mu_1^j)_{1 \leq j \leq k_1} \cup (h_{jj})_{1 \leq j \leq k_1} \stackrel{\text{law}}{=} \left(\max_{0=t_0 \leq \dots \leq t_j=1} \sum_{i=1}^j (B_i(t_i) - B_i(t_{i-1})) \right)_{1 \leq j \leq k_1} \cup (B_j(1))_{1 \leq j \leq k_1},$$

since $h_{11} = \mu_1^1$ and for all $j = 2, \dots, k_1$, $h_{jj} = \sum_{i=1}^j \mu_i^j - \sum_{i=1}^{j-1} \mu_i^{j-1}$. Next, Corollary 3.3 in [11] asserts that

$$(4) \quad \frac{\text{LI}_N - Np_{\max}}{\sqrt{Np_{\max}}} \Longrightarrow \frac{\sqrt{1 - k_1 p_{\max}} - 1}{k_1} \sum_{j=1}^{k_1} B_j(1) + \max_{0=t_0 \leq \dots \leq t_{k_1}=1} \sum_{i=1}^{k_1} (B_i(t_i) - B_i(t_{i-1})),$$

combining (3) and (4) gives (2). \square

Denote by $\lambda_1 \geq \dots \geq \lambda_k$ the shape of the Young diagrams obtained by applying the RSK algorithm to the random word

$$X_1 \cdots \cdots X_N,$$

and let

$$\xi_i = \frac{\lambda_i - Np_i}{\sqrt{Np_i}},$$

$1 \leq i \leq k$, be the corresponding rescaled variables. Introduce next some independent GUE matrices H_1, \dots, H_n , where each H_j has size $k_j := \#\mathcal{A}_j$, and let

$$H := \begin{pmatrix} H_1 & & \\ & \ddots & \\ & & H_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{H}_1 & & \\ & \ddots & \\ & & \tilde{H}_n \end{pmatrix} := H - \text{Tr}(HJ)J,$$

where

$$J = \text{diag}(\sqrt{p_1}, \dots, \sqrt{p_k}).$$

Remark 3. Note that J is a unit vector of the space $\mathcal{H}_{k_1} \times \cdots \times \mathcal{H}_{k_n}$ endowed with the Euclidean product structure, so $H - \text{Tr}(HJ)J$ is the orthogonal projection onto J^\perp , so that its law is the law of H conditioned to belong to J^\perp .

Finally, define the random vector (μ_1, \dots, μ_k) by

$$(\mu_1, \dots, \mu_k) := (\text{ordered spectrum of } \tilde{H}_1, \dots, \text{ordered spectrum of } \tilde{H}_n).$$

Corollary 4. As $N \rightarrow \infty$,

$$(\xi_1, \dots, \xi_k) \implies (\mu_1, \dots, \mu_k).$$

Remark 5. The limiting law of LI_N , rescaled, is simply the law of μ_1 and is given by

$$(5) \quad \lambda_{\max}(H_1) - p_1 \text{Tr}(H_1) - \sqrt{p_1(1 - k_1 p_1)}Z,$$

where Z is a standard Gaussian random variable, independent of H_1 . Note also that this law only depends on p_1 and k_1 .

Proof. First,

$$\text{Tr}(HJ) = \sum_{j=1}^n \sqrt{p_{(j)}} \text{Tr} H_j,$$

where for all j , $p_{(j)} := p_\ell$ for $\ell \in \mathcal{A}_j$. So for each i , we have

$$\tilde{H}_i = H_i - \left(\sqrt{p_i} \sum_{j=1}^n \sqrt{p_{(j)}} \text{Tr} H_j \right) I,$$

where I is the corresponding identity matrix. Then, Theorem 3.1 and Remark 3.2 (iv) in [14] together with Theorem 1 allow to conclude. \square

In case the i.i.d. random variables generating the random word are replaced by an (irreducible, aperiodic) homogeneous Markov chain, with state space \mathcal{A} of cardinality k , the corresponding limiting laws are also given in terms of maximal Brownian functionals similar to those in Theorem 1 (see [12]). However, an important difference is that now the standard Brownian motion B is replaced by a correlated one \tilde{B} with, say, covariance matrix Σ instead of I . The possible identification of (the law of) these functionals as (the law of) maximal eigenvalues (or spectra) of random matrices has not been fully accomplished yet. In particular, for cyclic transition matrices P , in which case the stationary distribution is the uniform one, there is a curious dichotomy between alphabets of size at most three and size four or more. Indeed for $k \leq 3$, the cyclic hypothesis forces Σ to have a permutation-symmetric structure seen in the i.i.d. uniform case. For example, for $k = 3$, Σ is, a rescaled version of,

$$(6) \quad \Sigma_u := \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix},$$

and so (up to a multiplicative constant) and with $k = k_1 = 3$, $p_{\max} = p_1 = 1/3$, (2) continues to hold for cyclic Markov chains. For $k \geq 4$, the cyclicity constraint on P forces Σ to be cyclic but does no longer force the permutation-symmetric structure, and, say, for $k = 4$, Σ might differ from, a rescaled version of,

$$(7) \quad \Sigma_u := \begin{pmatrix} 1 & -1/3 & -1/3 & -1/3 \\ -1/3 & 1 & -1/3 & -1/3 \\ -1/3 & -1/3 & 1 & -1/3 \\ -1/3 & -1/3 & -1/3 & 1 \end{pmatrix}.$$

In fact, if

$$(8) \quad P = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_4 & p_1 & p_2 & p_3 \\ p_3 & p_4 & p_1 & p_2 \\ p_2 & p_3 & p_4 & p_1 \end{pmatrix},$$

then Σ is a rescaled version of Σ_u if and only if $p_3^2 = p_2 p_4$. Nevertheless, see [12], for $k \geq 2$,

$$(9) \quad \frac{\text{LI}_N - N/k}{\sigma\sqrt{N}} \implies \sup \left\{ \Delta_\pi(\tilde{B}); \pi \in \mathcal{P}, \pi \leq k \right\} = \max_{0=t_0 \leq \dots \leq t_k=1} \sum_{j=1}^k \left(\tilde{B}_j(t_j) - \tilde{B}_j(t_{j-1}) \right).$$

Assuming that in addition to be cyclic, P is also symmetric (for $k = 2$ the cyclic and symmetric assumptions are the same, and Chistyakov and Götze [6], see also [13], showed that the corresponding limiting law is a maximal eigenvalue) a diagonalization argument, combined with (9), leads to the following result.

Proposition 6. *Let $P := (p_{i,j})_{1 \leq i,j \leq k}$ be cyclic and symmetric, i.e., $P = (p(j-i))_{1 \leq i,j \leq k}$, where p is a k -periodic function defined on \mathbb{Z} such that $p(r) = p(-r)$, for all $r \in \mathbb{Z}$. Let*

$$(10) \quad \lambda_\ell := \sum_{r=1}^k p(r) \cos(2\pi(\ell-1)r/k) \quad (1 \leq \ell \leq k),$$

and let $(B_j)_{j=2,\dots,k}$ be a $(k-1)$ -dimensional standard Brownian motion on $[0, 1]$. Then,

$$(11) \quad \frac{\text{LI}_N - N/k}{\sigma\sqrt{N}} \implies \max_{0=t_0 \leq \dots \leq t_k=1} \left\{ \sqrt{\frac{2}{k}} \sum_{j=1}^k \sum_{r=1}^{\lfloor \frac{k-1}{2} \rfloor} \sqrt{\frac{1+\lambda_{r+1}}{1-\lambda_{r+1}}} \left(\cos\left(\frac{2\pi jr}{k}\right) (B_{2r}(t_j) - B_{2r}(t_{j-1})) + \sin\left(\frac{2\pi jr}{k}\right) (B_{2r+1}(t_j) - B_{2r+1}(t_{j-1})) \right) - \frac{1}{\sqrt{k}} \sqrt{\frac{1+\lambda_{\frac{k}{2}+1}}{1-\lambda_{\frac{k}{2}+1}}} \left(2 \sum_{j=1}^k (B_k(t_j) - B_k(t_{j-1})) - B_k(1) \right) \right\},$$

where the last term above is only present for k even.

Proof. Since P is symmetric, it can be diagonalized as $P = S\Lambda S^\top$, where Λ is the diagonal matrix formed with its eigenvalues $(\lambda_\ell)_{1 \leq \ell \leq k}$ (we will see below that these are the quantities defined at (10)) and where S is a matrix formed by the orthonormal column eigenvectors $(u_\ell)_{1 \leq \ell \leq k}$ where $u_1^\top = (1/\sqrt{k}, \dots, 1/\sqrt{k})$. Next, by Theorem 4.3 in [12], Σ , the covariance matrix of the k -dimensional correlated Brownian motion \tilde{B} , is given by $\Sigma = S\Lambda_\Sigma S^\top$, where Λ_Σ is the diagonal matrix with diagonal entries $0, (1 + \lambda_2)/(1 - \lambda_2), \dots, (1 + \lambda_k)/(1 - \lambda_k)$. Therefore, $\tilde{B} = S\sqrt{\Lambda_\Sigma}B$, where now B is a standard k -dimensional Brownian motion. Now the symmetric and cyclic structures imply that the eigenvalues of P are in fact the λ_ℓ 's defined at (10): $\lambda_\ell = \sum_{r=1}^k p(r) \cos(2\pi(\ell - 1)r/k)$, $1 \leq \ell \leq k$, (clearly they are not all simple since $\lambda_\ell = \lambda_{k-\ell+2}$, $\ell = 2, \dots, k$). The corresponding orthonormal column eigenvectors are

$$v_\ell := (v_{j,\ell})_{1 \leq j \leq k} = (\sqrt{2} \cos(2\pi(\ell - 1)j/k)/\sqrt{k})_{1 \leq j \leq k}, \quad \ell = 1, 2, \dots, \lfloor k/2 \rfloor + 1,$$

and

$$w_\ell := (w_{j,\ell})_{1 \leq j \leq k} = (\sqrt{2} \sin(2\pi(\ell - 1)j/k)/\sqrt{k})_{1 \leq j \leq k}, \quad \ell = 2, 3, \dots, \lfloor (k - 1)/2 \rfloor + 1.$$

Clearly, $v_1 = u_1$ is an eigenvector corresponding to the simple eigenvalue 1, while if k is even, $v_{(k/2)+1} = (1/\sqrt{k}, -1/\sqrt{k}, \dots, 1/\sqrt{k}, -1/\sqrt{k})$ is an eigenvector corresponding to the simple eigenvalue $\sum_{r=1}^k p(r) \cos(2\pi(k/2 + 1 - 1)r/k) = \sum_{r=1}^k (-1)^r p(r)$. Moreover, for $\ell = 2, 3, \dots, \lfloor (k - 1)/2 \rfloor + 1$, v_ℓ and w_ℓ share the same eigenvalue λ_ℓ . Therefore,

$$S = \left(v_1, v_2, w_2, \dots, v_{\lfloor \frac{k-1}{2} \rfloor + 1}, w_{\lfloor \frac{k-1}{2} \rfloor + 1}, v_{\frac{k}{2} + 1} \right),$$

where, above, the last column is only present if k is even. Next, from the transformation $\tilde{B} = S\sqrt{\Lambda_\Sigma}B$, and since $(\sqrt{\Lambda_\Sigma}B)_\ell = \sqrt{(1 + \lambda_{\lfloor \ell/2 \rfloor + 1})/(1 - \lambda_{\lfloor \ell/2 \rfloor + 1})}B_\ell$, $\ell = 2, \dots, k$, and $(\sqrt{\Lambda_\Sigma}B)_1 = 0$, then for $j = 1, \dots, k$,

$$(12) \quad \tilde{B}_j = \sum_{\ell=2}^k u_{j,\ell} \sqrt{\frac{1 + \lambda_{\lfloor \frac{\ell}{2} \rfloor + 1}}{1 - \lambda_{\lfloor \frac{\ell}{2} \rfloor + 1}}} B_\ell,$$

where $u_{j,\ell} = v_{j, \lfloor \ell/2 \rfloor + 1}$ or $u_{j,\ell} = w_{j, \lfloor \ell/2 \rfloor + 1}$, for ℓ even or odd. Therefore, for $j = 1, \dots, k$,

$$(13) \quad \tilde{B}_j = \sqrt{\frac{2}{k}} \sum_{r=1}^{\lfloor \frac{k-1}{2} \rfloor} \sqrt{\frac{1 + \lambda_{r+1}}{1 - \lambda_{r+1}}} \left(\cos\left(\frac{2\pi r j}{k}\right) B_{2r} + \sin\left(\frac{2\pi r j}{k}\right) B_{2r+1} \right) \\ + \frac{(-1)^{j+1}}{\sqrt{k}} \sqrt{\frac{1 + \lambda_{\frac{k}{2}+1}}{1 - \lambda_{\frac{k}{2}+1}}} B_k,$$

where the last term on the right of (13) is only present for k even. With (13), the sum on the right hand side of (9) becomes:

$$(14) \quad \sqrt{\frac{2}{k}} \sum_{j=1}^k \sum_{r=1}^{\lfloor \frac{k-1}{2} \rfloor} \sqrt{\frac{1+\lambda_{r+1}}{1-\lambda_{r+1}}} \left(\cos\left(\frac{2\pi jr}{k}\right) B_{2r}(t_j) + \sin\left(\frac{2\pi jr}{k}\right) B_{2r+1}(t_j) \right. \\ \left. - \cos\left(\frac{2\pi jr}{k}\right) B_{2r}(t_{j-1}) - \sin\left(\frac{2\pi jr}{k}\right) B_{2r+1}(t_{j-1}) \right) \\ + \frac{1}{\sqrt{k}} \sqrt{\frac{1+\lambda_{\frac{k}{2}+1}}{1-\lambda_{\frac{k}{2}+1}}} \sum_{j=1}^k (-1)^{j+1} (B_k(t_j) - B_k(t_{j-1})),$$

an expression only involving standard Brownian motions and where, again, the last term

$$(15) \quad \frac{1}{\sqrt{k}} \sqrt{\frac{1+\lambda_{\frac{k}{2}+1}}{1-\lambda_{\frac{k}{2}+1}}} \sum_{j=1}^k (-1)^{j+1} (B_k(t_j) - B_k(t_{j-1})) \\ = \frac{-1}{\sqrt{k}} \sqrt{\frac{1+\lambda_{\frac{k}{2}+1}}{1-\lambda_{\frac{k}{2}+1}}} \left(B_k(1) + \sum_{j=1}^{k-1} 2(-1)^j B_k(t_j) \right) \\ (16) \quad = \frac{-1}{\sqrt{k}} \sqrt{\frac{1+\lambda_{\frac{k}{2}+1}}{1-\lambda_{\frac{k}{2}+1}}} \left(2 \sum_{j=1}^k (B_k(t_j) - B_k(t_{j-1})) - B_k(1) \right),$$

is only present if k is even. □

Remark 7. Let us try to specialize the previous results in instances where further simplifications and identifications occur.

(i) For $k = 3$, and up to the factor $\sqrt{2(1+\lambda_2)/(k(1-\lambda_2))} = \sqrt{2(1+3p_1)/(3(3-3p_1))}$, the right-hand side of (11) becomes

$$\max_{0=t_0 \leq t_1 \leq t_2 \leq t_3=1} \sum_{j=1}^3 \left(\cos\left(\frac{2\pi j}{3}\right) (B_2(t_j) - B_2(t_{j-1})) + \sin\left(\frac{2\pi j}{3}\right) (B_3(t_j) - B_3(t_{j-1})) \right) \\ = \max_{0 \leq t_1 \leq t_2 \leq 1} \left(B_2(1) + \sqrt{3} B_3(t_1) - \frac{\sqrt{3}}{2} B_3(t_2) - \frac{3}{2} B_2(t_2) \right) \\ \stackrel{\text{law}}{=} \sqrt{\frac{2}{3}} \max_{0 \leq t_1 \leq t_2 \leq 1} \sum_{j=1}^2 \left(-\sqrt{\frac{j}{j+1}} B_j(t_{j+1}) + \sqrt{\frac{j}{j+1}} B_j(t_j) \right), \\ \stackrel{\text{law}}{=} \sqrt{\frac{2}{3}} \left(\max_{0=t_0 \leq t_1 \leq t_2 \leq t_3=1} \sum_{j=1}^3 (B_j(t_j) - B_j(t_{j-1})) - \frac{1}{3} \sum_{j=1}^3 B_j(1) \right),$$

where the last equality, in law, follows either by using, in (9), the simple linear transformation

$$(17) \quad \tilde{B}_j = \sqrt{\frac{2(1+\lambda_2)}{3(1-\lambda_2)}} \left(\sqrt{\frac{2}{3}} B_j - \sqrt{\frac{1}{6}} \sum_{i=1, i \neq j}^3 B_i \right), \quad j = 1, 2, 3,$$

which, by comparing covariances, is easily verified from (13); or, still by comparing covariances, by arguments such as those in the proof of Theorem 3.2 in [11]. Therefore, with the help of Theorem 1, and up to a scaling factor, the limiting law of LI_N is that of the maximal eigenvalue of the 3×3 traceless GUE.

(ii) For $k = 4$, Σ , the covariance matrix of $\tilde{B} = (\tilde{B}_j)_{j=1, \dots, 4}$ is given, up to a scaling constant, by:

$$(18) \quad \Sigma := \begin{pmatrix} 2\eta_2 + \eta_3 & -\eta_3 & -2\eta_2 + \eta_3 & -\eta_3 \\ -\eta_3 & 2\eta_2 + \eta_3 & -\eta_3 & -2\eta_2 + \eta_3 \\ -2\eta_2 + \eta_3 & -\eta_3 & 2\eta_2 + \eta_3 & -\eta_3 \\ -\eta_3 & -2\eta_2 + \eta_3 & -\eta_3 & 2\eta_2 + \eta_3 \end{pmatrix}.$$

where $\eta_2 = (1 + \lambda_2)/(1 - \lambda_2)$, $\lambda_2 = p_1 - p_3$, and $\eta_3 = (1 + \lambda_3)/(1 - \lambda_3)$, $\lambda_3 = p_1 - 2p_2 + p_3$. Clearly, Σ can differ from Σ_u , e.g., let $2\eta_2 = \eta_3$, i.e., let

$$P = \begin{pmatrix} p_1 & p_2 & \frac{p_2(1-2p_2)}{1+2p_2} & p_2 \\ p_2 & p_1 & p_2 & \frac{p_2(1-2p_2)}{1+2p_2} \\ \frac{p_2(1-2p_2)}{1+2p_2} & p_2 & p_1 & p_2 \\ p_2 & \frac{p_2(1-2p_2)}{1+2p_2} & p_2 & p_1 \end{pmatrix}.$$

Then, and up to the multiplicative constant $4\eta_2$, Σ becomes:

$$\Sigma := \begin{pmatrix} 1 & -1/2 & 0 & -1/2 \\ -1/2 & 1 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ -1/2 & 0 & -1/2 & 1 \end{pmatrix},$$

which is clearly different from, a rescaled version of, (7). In fact, if $\Sigma = \Sigma_u$, then clearly

$$(19) \quad \tilde{B}_j = \frac{\sqrt{3}}{2} B_j - \frac{1}{2\sqrt{3}} \sum_{i=1, i \neq j}^4 B_i, \quad j = 1, 2, 3, 4.$$

Conversely, and as easily seen, for a linear transformation such as

$$\tilde{B}_j = \alpha_j B_j - \sum_{i=1, i \neq j}^4 \beta_i B_i, \quad j = 1, 2, 3, 4,$$

to lead to Σ , one needs to Σ to be permutation-symmetric and, up to a multiplicative constant, the right-hand side of (11) becomes equal in law to

$$\max_{0=t_0 \leq t_1 \leq t_2 \leq t_3 \leq t_4=1} \sum_{j=1}^4 (B_j(t_j) - B_j(t_{j-1})) - \frac{1}{4} \sum_{j=1}^4 B_j(1),$$

and corresponds to the matrix P in (8) with $p_2 = p_3 = p_4$.

(iii) Finally, it is easy to see that the properties just described continue to hold for arbitrary dimension $k \geq 4$. In arbitrary dimension, if $\Sigma = \Sigma_u$ (the k -dimensional version of the matrix defined at (6) and (7)), then the linear transformation corresponding to (19) is given by

$$\tilde{B}_j = \sqrt{\frac{k-1}{k}} B_j - \sqrt{\frac{1}{k(k-1)}} \sum_{i=1, i \neq j}^k B_i, \quad j = 1, \dots, k,$$

Conversely, for a linear transformation such as

$$\tilde{B}_j = \alpha_j B_j - \sum_{i=1, i \neq j}^k \beta_i B_i, \quad j = 1, 2, \dots, k,$$

to lead to Σ , one needs to Σ to be permutation-symmetric. In either instance, and up to a multiplicative constant, the right-hand side of (11) has the same law as

$$\max_{0=t_0 \leq \dots \leq t_k=1} \sum_{j=1}^k (B_j(t_j) - B_j(t_{j-1})) - \frac{1}{k} \sum_{j=1}^k B_j(1),$$

which, in turn, via Theorem 1, is equal in law to the maximal eigenvalue of an element of the $k \times k$ traceless GUE.

In order to present our last corollary, we recall elements of the queuing framework where some of these maximal Brownian functionals originated (see [9]). There, one considers N single servers in series each with unlimited waiting space and FIFO discipline. At first the queuing system is empty then k customers are placed in the first queue. The service times $V_{i,j}$, $1 \leq i \leq k$, $1 \leq j \leq N$, of all the customers at all the queues are i.i.d. with mean 1 and variance σ^2 . Finally, let $D_{i,j}$ be the departure time of customer i , $1 \leq i \leq k$, from queue j , $1 \leq j \leq N$.

Introduce also the rescaled final departure time of customer i , $1 \leq i \leq k$,

$$\zeta_i := \frac{D_{i,N} - N}{\sqrt{N\sigma^2}},$$

and the rescaled total service time of the customer i , $1 \leq i \leq k$,

$$\phi_i := \frac{\sum_{j=1}^N V_{i,j} - N}{\sqrt{N\sigma^2}}.$$

The next corollary extends a result of Glynn and Whitt [9], who showed the weak convergence of ζ_k towards the μ_1^k .

Corollary 8. *Let $H = [h_{ij}]$ be a $k \times k$ GUE matrix, and for each $\ell = 1, \dots, k$, let μ_1^ℓ be the maximal eigenvalue of the $\ell \times \ell$ principal minor of H . Then, as N tends to infinity,*

$$(\zeta_1, \dots, \zeta_k, \phi_1, \dots, \phi_k) \implies (\mu_1^1, \mu_1^2, \dots, \mu_1^k, h_{11}, \dots, h_{\ell\ell}).$$

Proof. The techniques of [9] and [12], combined with Theorem 1, give the result noticing again that $h_{11} = \mu_1^1$, $h_{\ell\ell} = \sum_{j=1}^{\ell} \mu_j^\ell - \sum_{j=1}^{\ell-1} \mu_j^{\ell-1}$, $1 \leq \ell \leq k$. \square

3. PROOF OF THEOREM 1

Let $w_{N,M} := [w_{ij}]$ ($i = 1, \dots, N$, $j = 1, \dots, M$) be an array of i.i.d. geometric random variables with parameter $q \in (0, 1)$, i.e., with law $\sum_{k \geq 0} q^k (1 - q) \delta_k$. Such variables have mean $e := q/(1 - q)$ and variance $v := q/(1 - q)^2$.

Applying the RSK correspondence to $w_{N,M}$ (see e.g., [22] for an introduction to the RSK correspondence applied to arrays of integers) gives a pair (P, Q) of Young diagrams with the same shape. Let us denote the shape of these Young diagrams by

$$\lambda_1^M \geq \dots \geq \lambda_M^M.$$

The exponent M is here to emphasize on the dependence on the dimension M of the GUE matrix H (the dependence on N is implicit). In the same way, one can of course define, for each $k = 1, \dots, M$, the shape

$$\lambda_1^k \geq \dots \geq \lambda_k^k,$$

of the Young diagrams obtained by applying the RSK correspondence to the array $w_{N,k}$, which is the array $w_{N,M}$ where all but the first k columns have been removed. Note that

$$\left(\begin{array}{cccccccc} & & & & \lambda_1^1 & & & \\ & & & & & \lambda_2^2 & & \\ & & & & & & \lambda_3^3 & \\ & & \lambda_1^3 & & \lambda_2^3 & & & \\ & & \dots & & \dots & & \dots & \\ \lambda_1^M & & & & & & & \lambda_M^M \end{array} \right),$$

is a Gelfand-Tsetlin pattern, i.e., satisfies the interlacing inequalities $\lambda_i^k \geq \lambda_i^{k-1} \geq \lambda_{i+1}^k$ ($1 \leq i < k \leq M$).

Let us now define the random variables

$$(20) \quad \xi_i^k := \frac{\lambda_i^k - eN}{\sqrt{vN}}.$$

Then it is not hard to deduce the following lemma from [2] (this lemma is also stated as Proposition 2.12 in [20], with slightly different notation).

Lemma 9. *As N tends to infinity,*

$$\left(\begin{array}{ccccccc} & & & \xi_1^1 & & & \\ & & \xi_1^2 & & \xi_2^2 & & \\ & \xi_1^3 & & \xi_2^3 & & \xi_3^3 & \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ \xi_1^M & & & & & & \xi_M^M \end{array} \right) \implies \left(\begin{array}{ccccccc} & & & \mu_1^1 & & & \\ & & \mu_1^2 & & \mu_2^2 & & \\ & \mu_1^3 & & \mu_2^3 & & \mu_3^3 & \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ \mu_1^M & & & & & & \mu_M^M \end{array} \right),$$

where the μ_i^k s are the ones introduced in (1).

Next, for π, π' any two up-right paths in the set $\{1, \dots, N\} \times \{1, \dots, k\}$, we write $\pi < \pi'$, if for any $n \in \{1, \dots, N\}$, the intersection of the support of π with $\{n\} \times \{1, \dots, k\}$ is located strictly below the intersection of the support of π' with $\{n\} \times \{1, \dots, k\}$. We will also need the following lemma.

Lemma 10. *For each $1 \leq \ell \leq k \leq M$,*

$$(21) \quad \lambda_1^k + \dots + \lambda_\ell^k = \max_{\pi_1, \dots, \pi_\ell} \sum_{r=1}^{\ell} \sum_{(i,j) \in \pi_r} w_{ij},$$

where the max is over collections $\{\pi_1 < \dots < \pi_\ell\}$ of up-right paths in the set

$$\{1, \dots, N\} \times \{1, \dots, k\}$$

starting in the subset $\{1\} \times \{1, \dots, k\}$ and ending in the subset $\{N\} \times \{1, \dots, k\}$.

Proof. It is well known that (21) is true when the paths are only required to be pairwise disjoint, without any condition on the starting and ending points (see e.g., the third chapter of [8] and use the words/arrays equivalence described in [2] or [22]). Then, it is easy to see that any set of pairwise disjoint paths can be changed into a set of pairwise disjoint paths starting in $\{1\} \times \{1, \dots, k\}$ in such a way that the union of their supports does not decrease. Then, one can also change these paths in such a way to make them end in $\{N\} \times \{1, \dots, k\}$ but also such that the union of their supports does not decrease and that the starting points still have first coordinate 1. Note that a collection of such pairwise disjoint paths can always be re-indexed in such a way that $\pi_1 < \dots < \pi_\ell$. To finish the proof, it suffices then to notice that since the w_{ij} s are non-negative, enlarging the union of the supports of the paths never decreases the total weight. \square

To complete the proof of the theorem, note that any up-right path π_r as described in the previous lemma is a concatenation of at most k paths with fixed second coordinate and has length between N and $N + M$. Moreover, by Donsker theorem (see [4, 9]), the M -dimensional process

$$\left(\frac{1}{\sqrt{vN}} \sum_{i=1}^{\lfloor Nt \rfloor} (w_{ik} - e) \right)_{k=1, \dots, M}$$

converges in distribution (for the Skorohod topology) to the M -dimensional Brownian motion B . To finish the proof, apply both Lemma 9 and Lemma 10 . \square

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