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Large deviations and Gallavotti–Cohen principle for dissipative PDE's with rough noise

V. Jakšić* V. Nersesyan† C.-A. Pillet‡ A. Shirikyan§

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Abstract

We study a class of dissipative PDE's perturbed by an unbounded kick force. Under some natural assumptions, the restrictions of solutions to integer times form a homogeneous Markov process. Assuming that the noise is rough with respect to the space variables and has a non-degenerate law, we prove that the system in question satisfies a large deviation principle (LDP) in τ-topology. Under some additional hypotheses, we establish a Gallavotti–Cohen type symmetry for the rate function of an entropy production functional and the strict positivity and finiteness of the mean entropy production in the stationary regime. The latter result is applicable to PDE's with strong nonlinear dissipation.

AMS subject classifications: 35Q30, 35Q56, 37L55, 60B12, 60F10

Keywords: Dissipative PDE's, large deviation principle, occupation measures, entropy production, Gallavotti–Cohen fluctuation relation, Navier–Stokes system, Ginzburg–Landau equation, Burgers equation, reaction-diffusion system

Contents

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0 Introduction

Let H be a separable Hilbert space and let $S : H \to H$ be a continuous mapping. We consider a discrete-time Markov process defined by the equation

$$u_k = S(u_{k-1}) + \eta_k, \quad k \geq 1, \tag{0.1}$$

where $\{\eta_k\}$ is a sequence of i.i.d. random variables in H. This type of systems naturally arise when studying the large-time asymptotics of randomly forced PDE's, and we do not discuss here our motivation, referring the reader to Section 2.3 of the book [KS12]. Equation (0.1) generates a homogeneous family of Markov chains, and its ergodic theory is well understood in the case when S possesses a dissipativity property and the law of η_k is sufficiently non-degenerate. Namely, let us assume that

$$\|S(u)\| \leq q\|u\| + C \quad \text{for any } u \in H, \tag{0.2}$$

where $q < 1$ and C are some numbers not depending on u. If, in addition, the mapping S is compact in the sense that the image under S of any bounded set is relatively compact, then the existence of a stationary distribution can easily be proved with the help of the Bogolyubov–Krylov argument. The uniqueness of a stationary measure and its mixing properties are much more delicate questions, and in this paper we deal with a "rough" noise, in which case convergence to the unique stationary measure holds in the total variation distance. To describe the problems and results, let us assume that the law ℓ of the random variables η_k is a Gaussian measure. In this situation, the above-mentioned roughness condition takes the form:

(H) *The mapping S is continuous from H to the Cameron–Martin space of ℓ and is bounded on any ball.*

Under this hypothesis, the transition probabilities of the Markov family associated with (0.1) are all equivalent, and the uniqueness of a stationary measure and its stability in the total variation norm follows from the well-known Doob's theorem; e.g., see Chapter 4 in [DZ96]. We refer the reader to the pioneering articles [Yag47, Doo48] for first results of this type, to the book [MT93] for a general ergodic theory of Markov chains, and to the paper [BKL01] for a proof of the above-mentioned existence and stability result in the case of Navier–Stokes equations on the 2D torus.

The aim of this paper is twofold: firstly, to establish a large deviation principle (LDP) for occupation measures of (0.1) and some physically relevant functionals and, secondly, to derive a Gallavotti–Cohen type symmetry for the rate function corresponding to entropy production. Without going into technical details, we now describe our main results in the case of the 1D Burgers equation on the circle $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$. Namely, let us denote by H the space of square-integrable functions on $\mathbb{S}$ with zero mean value and consider the problem

$$\partial_t u - \nu\partial_x^2 u + u\partial_x u = h(x) + \eta(t,x), \tag{0.3}$$

$$u(0,x) = u_0(x). \tag{0.4}$$

Here $x \in \mathbb{S}$, $\nu > 0$ is a parameter, $h \in H$ is a fixed function, and $\eta(t,x)$ is a random process of the form

$$\eta(t,x) = \sum_{k=1}^{\infty} \eta_k(x)\delta(t-k), \tag{0.5}$$

where $\{\eta_k\}$ is a sequence of i.i.d. Gaussian random variables in H and $\delta(t)$ denotes the Dirac measure at zero. Normalising trajectories of (0.3) to be right-continuous and denoting $u_k = u(k,x)$, we see that the sequence $\{u_k\}$ satisfies Eq. (0.1), where $S : H \to H$ denotes the time-1 shift along trajectories of (0.3) with $\eta \equiv 0$. For any trajectory $\{u_k\}$, let $\boldsymbol{\zeta}_k(u_0)$ be the corresponding occupation measure:

$$\boldsymbol{\zeta}_k(u_0) = \frac{1}{k}\sum_{n=0}^{k-1} \delta_{\boldsymbol{u}_n}, \quad \boldsymbol{u}_n = (u_l, l \geq n),$$

where $\delta_{\boldsymbol{v}}$ denotes the Dirac mass concentrated at $\boldsymbol{v} = (v_l, l \geq 0)$ in the space of probability measures on $\boldsymbol{H} = H^{\mathbb{Z}_+}$. Thus, $\{\boldsymbol{\zeta}_k(u_0)\}$ is a sequence of random probability measures on $\boldsymbol{H}$, and we wish to investigate the problem of large deviations for it. Let us denote by V^s the space of functions in the Sobolev space of order s on $\mathbb{S}$ whose mean value is equal to zero.

Theorem A. *Let us assume that $h \in V^s$ for an integer $s \geq 0$ and the law ℓ of the i.i.d. random variables η_k is a centred Gaussian measure on H such that V^{s+1} is continuously embedded into its Cameron–Martin space. Then the*

discrete-time Markov process associated with (0.3) has a unique stationary measure μ, which is exponentially mixing in the sense that the law of any trajectory converges to μ in the total variation metric exponentially fast. Moreover, for any initial point $u_0 \in H$, the occupation measures $\boldsymbol{\zeta}_k(u_0)$ satisfy the LDP in the τ_p-topology with a good rate function not depending on u_0.

The reader is referred to Section 1 for the definition of the concepts used in this theorem. We now turn to the question of the Gallavotti–Cohen fluctuation principle. To this end, given a vector $a \in H$, denote by ℓ_a the image of ℓ under the translation in H by the vector a. The hypotheses of Theorem A imply that the shifted measure $\ell_{S(u)}$ is equivalent to ℓ. Thus, the transition kernel of the Markov chain defined by (0.1) is given by $P(u, \mathrm{d}v) = \ell_{S(u)}(\mathrm{d}v) = \rho(u, v)\ell(\mathrm{d}v)$, the density $\rho(u, v)$ being positive for any $u \in H$ and ℓ-almost every $v \in H$. This further implies that, for any $k > 0$, the law λ_k of the random variable u_k is equivalent to ℓ, irrespective of the law λ_0 of the initial condition u_0. In particular, the stationary measure μ is equivalent to ℓ. We denote by ρ its density. Thus, when discussing the long time behaviour of the system, we can assume that all the measures λ_k belong to the equivalence class of ℓ.

Adapting Gaspard's argument [Ga04] to our setup, we measure the entropy of the system at time k by the relative entropy of λ_k w.r.t. the stationary measure μ,

$$S(\lambda_k) = \mathrm{Ent}(\lambda_k \,|\, \mu) = -\int \log\left(\frac{\mathrm{d}\lambda_k}{\mathrm{d}\mu}\right) \mathrm{d}\lambda_k.$$

We note that the basic properties of relative entropy imply that $S(\lambda) \leq 0$, with equality if and only if $\lambda_k = \mu$. The change of entropy in one time step is given by

$$\delta S(\lambda) = S(\mathfrak{P}_1^* \lambda) - S(\lambda),$$

where $\mathfrak{P}_k^*$ denotes the Markov semigroup associated with the transition kernel P. Let $\boldsymbol{\lambda}$ be the law induced on $\boldsymbol{H}$ by the initial distribution λ. Define the following function on $\boldsymbol{H}$:

$$J(\boldsymbol{u}) = \log \frac{\rho(u_0)\rho(u_0, u_1)}{\rho(u_1)\rho(u_1, u_0)}. \tag{0.6}$$

In the third section of the Appendix, we shall show that, under the hypotheses of Theorem A,

$$\delta S(\lambda) = \mathrm{Ep}(\lambda) - \int_{\boldsymbol{H}} J(\boldsymbol{u})\boldsymbol{\lambda}(\mathrm{d}\boldsymbol{u}), \tag{0.7}$$

where the functional $\mathrm{Ep}(\,\cdot\,)$ is such that $\mathrm{Ep}(\lambda) \geq 0$ for all λ in the equivalence class of ℓ. Moreover, $\mathrm{Ep}(\lambda) = 0$ if and only if $\lambda = \mu$ and μ satisfies the detailed balance condition

$$\rho(u)\rho(u, v) = \rho(v)\rho(v, u), \tag{0.8}$$

$\ell \otimes \ell$-almost everywhere on $H \times H$. The validity of Eq. (0.8) is well known to be necessary and sufficient to ensure the time-reversal invariance of the Markov chain under the stationary law $\boldsymbol{\mu}$. The functional $\mathrm{Ep}(\,\cdot\,)$ is thus a measure of the breakdown of time-reversal invariance, a phenomenon usually connected

with the production of entropy. We shall therefore identify $\mathrm{Ep}(\lambda)$ with the entropy production rate of the system in the state λ. Reading Eq. (0.7) as an entropy balance relation, we may consequently interpret the observable J as the entropy dissipated into the environment, i.e., the integral of the outgoing entropy flux over the unit time interval. Note that the vanishing of the entropy flux observable J is equivalent to the detailed balance condition (0.8). We shall prove in Section 2.3 that the unique stationary measure μ does not satisfy the detailed balance relation, so that $\mathrm{Ep}(\lambda) > 0$ for all λ.

In terms of the random variables

$$\xi_k(\boldsymbol{u}) = \frac{1}{k}\sum_{n=0}^{k-1} \sigma(u_n, u_{n+1}), \tag{0.9}$$

where

$$\sigma(u,v) = \log\frac{\rho(u,v)}{\rho(v,u)},$$

we can write the entropy balance relation over k time steps as

$$\begin{aligned}\frac{1}{k}(S(\lambda_k) - S(\lambda_0)) = \frac{1}{k}\sum_{n=0}^{k-1}\mathrm{Ep}(\mathfrak{P}_n^*\lambda) - \int_{\boldsymbol{H}} \xi_k(\boldsymbol{u})\boldsymbol{\lambda}(\mathrm{d}\boldsymbol{u}) \\ - \frac{1}{k}\int \log\rho(u)(\lambda_k(\mathrm{d}u) - \lambda_0(\mathrm{d}u)).\end{aligned} \tag{0.10}$$

The last term on the right hand side of this relation (a so-called boundary term) becomes negligible in the large time limit. It vanishes in the stationary regime where the previous relation becomes

$$\mathrm{Ep}(\mu) = \int_{\boldsymbol{H}} \xi_k(\boldsymbol{u})\boldsymbol{\mu}(\mathrm{d}\boldsymbol{u}) = \int_{H\times H} \rho(u)\rho(u,v)\sigma(u,v)\ell(\mathrm{d}u)\ell(\mathrm{d}v). \tag{0.11}$$

In the third subsection of the Appendix, we shall briefly discuss the relation of the observable σ with time-reversal of the path measure $\boldsymbol{\mu}$ and its connection with dynamical (Kolmogorov–Sinai) entropy.

According to Eq. (0.11), the mean entropy flux is non-negative. By the law of large numbers, the sequence ξ_k converges $\boldsymbol{\mu}$-a.s. towards $\mathrm{Ep}(\mu)$. The Gallavotti–Cohen fluctuation relation is a statement about the large deviations of ξ_k from this limit. Roughly speaking, it says that

$$\frac{\boldsymbol{\mu}\left(\xi_k \simeq -r\right)}{\boldsymbol{\mu}\left(\xi_k \simeq +r\right)} \simeq \mathrm{e}^{-kr} \quad \text{for large } k.$$

The fact that the entropy production rate is non-negative and the definition of the entropy flux observable σ are part of the general theory of non-equilibrium statistical mechanics in the mathematical framework of deterministic and stochastic dynamical systems [ES94, GC95, Rue97, Kur98, Mae99, Ga04, RMM07, JPR11]. On the other hand, detailed dynamical questions like

strict positivity of the entropy production rate, LDP for the entropy flux, and validity of the Gallavotti–Cohen fluctuation relation can be answered only in the context of concrete models. In some cases, it is possible to relate the observable σ to the fluxes of some physical quantities, typically heat or some other forms of energy. In this respect, we refer the reader to [BM05] for the discussion of a closely related model. In this paper, we shall prove the following result.

Theorem B. *In addition to the hypotheses of Theorem A, assume that $h \in V^{2s+1}$ and the set of normalised eigenvectors of the covariance operator for ℓ coincides with the trigonometric basis in H. Then, for any initial condition $u_0 \in H$, the laws of the random variables (0.9) satisfy the LDP with a good rate function $I : \mathbb{R} \to [0, +\infty]$ not depending on u_0. Moreover, the entropy production rate is strictly positive,*

$$\mathrm{Ep}(\mu) = \int_{\boldsymbol{H}} \sigma(u_0, u_1) \boldsymbol{\mu}(\mathrm{d}\boldsymbol{u}) > 0, \tag{0.12}$$

and the Gallavotti–Cohen fluctuation relation[1] *holds for I:*

$$I(-r) = I(r) + r \quad \text{for } r \in \mathbb{R}. \tag{0.13}$$

There is an enormous literature on mathematical, physical, numerical, and experimental aspects of Gallavotti–Cohen fluctuation relation (some of the references can be found in [JPR11, RMM07]). The previous mathematically rigorous works closest to ours are [LS99, EPR99]. Lebowitz and Spohn [LS99], building on the previous work by Kurchan [Kur98], have developed general theory of Gallavotti–Cohen fluctuation relation for finite dimensional Markov processes with applications to various models, including diffusion and simple exclusion processes. In [EPR99], the authors consider a finite anharmonic chain coupled to two thermal reservoirs at its ends. Its analysis reduces to study of suitable finite dimensional Markov process with degenerate noise. In particular, the local Gallavotti–Cohen fluctuation relation for this model has been established in [RT02]. To the best of our knowledge, there were no previous mathematically rigorous studies of Gallavotti–Cohen fluctuation relation for nonlinear PDE's driven by a stochastic forcing (see, however, the papers [Gou07b, Gou07a] for a proof of LDP for the Navier–Stokes and Burgers equations perturbed by a rough white-noise force and [JNPS12] for the case of a smooth bounded kick force). On the physical level of rigour, Maes and coworkers [MRV01, MN03, Mae04] have examined in depth the fluctuation relation for stochastic dynamics. In a somewhat different spirit, inspired by the thermodynamic formalism of dynamical systems, we should also mention the works of Gaspard [Ga04] and Lecomte *et al.* [LAW06].

The LDP for the Burgers equation stated in Theorem A is true for other more complicated models, such as the Navier–Stokes system or complex Ginzburg–Landau equation, while the Gallavotti–Cohen fluctuation relation of Theorem B

[1] Relation (0.13) means, in particular, that $I(r) = +\infty$ if and only if $I(-r) = +\infty$.

remains valid for problems with strong nonlinear dissipation, such as the reaction–diffusion system with superlinear interaction. Moreover, the law of η_k does not need to be Gaussian, and the results we prove are true for a rather general class of decomposable measures; see Sections 1 and 2 for details.

The somewhat surprising fact that the global LDP for unbounded observables holds for Burgers and reaction-diffusion equation has its physical origin in the strong dissipation characterising these models. It is natural to expect that in more generic situations (like Navier–Stokes systems) only a local LDP and, hence, a local fluctuation relation hold (like in [RT02]). However, in the absence of a strong dissipative mechanism, our method of the proof of LDP for occupational measures is not suited for establishing local LDP for unbounded observables like the entropy flux. It is likely that more specific techniques that deal directly with LDP for the entropy flux are needed to analyse this question. We plan to address this problem in future publications.

The paper is organised as follows. In Section 1, we formulate our main abstract results on the large deviations and the Gallavotti–Cohen fluctuation theorem. Various applications of these results are discussed in Section 2. Sections 3 and 4 are devoted to proving the theorems announced in Section 1. The Appendix gathers some auxiliary results on decomposable measures and LDP for Markov chains and discusses the analogy between our models and heat conducting networks.

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Notation

Let X be a Polish space with a metric d. We always assume that it is endowed with its Borel σ-algebra $\mathcal{B}_X$. Given $R > 0$ and $a \in X$, we denote by $B_X(a, R)$ the closed ball in X of radius R centred at a. The following spaces are systematically used in the paper.

$\boldsymbol{X} = X^{\mathbb{Z}_+}$ denotes the direct product of countably many copies of X. The space $\boldsymbol{X}$ is endowed with the Tikhonov topology, and its elements are denoted by $\boldsymbol{u} = (u_n, n \geq 0)$. We write X^m for the direct product of m copies of X.

$C(X)$ is the space of continuous functions $f : X \to \mathbb{R}$. We denote by $C_b(X)$ the subspace of bounded functions in $C(X)$ and endow it with the natural norm $\|f\|_\infty = \sup_X |f|$.

$\mathcal{P}(X)$ denotes the space of probability measures on X. Given $\mu \in \mathcal{P}(X)$ and a

μ-integrable function $f : X \to \mathbb{R}$, we write

$$\langle f, \mu \rangle = \int_X f(u)\mu(\mathrm{d}u).$$

The total variation metric on $\mathcal{P}(X)$ is defined by

$$\|\mu_1 - \mu_2\|_{\mathrm{var}} = \frac{1}{2} \sup_{\|f\|_\infty \le 1} |\langle f, \mu_1 \rangle - \langle f, \mu_2 \rangle| = \sup_{\Gamma \in \mathcal{B}_X} |\mu_1(\Gamma) - \mu_2(\Gamma)|.$$

$C(J, H)$ denotes the space of continuous functions on an interval $J \subset \mathbb{R}$ with range in the Banach space H. We write $C_b(J, X)$ for the subspace of bounded functions and endow it with the natural norm

$$\|f\|_{L^\infty(J,H)} = \operatorname*{esssup}_{t \in J} \|f(t)\|_H.$$

$L^p(J, H)$ stands for the space of Borel-measurable functions $f : J \to H$ such that

$$\|f\|_{L^p(J,H)} = \left(\int_J \|f(t)\|_H^p \mathrm{d}t \right)^{1/p} < \infty.$$

In the case $p = \infty$, the above norm should be replaced by $\|f\|_{L^\infty(J,H)}$.

We denote by $C, C_1, C_2, \dots$ unessential positive numbers.

1 Main results

In this section, we introduce a class of discrete-time Markov processes and formulate a result on the existence, uniqueness, and exponential mixing of a stationary measure and the large deviation principle for the occupation measures and some unbounded functionals. We next discuss the Gallavottii–Cohen fluctuation theorem for an entropy production functional.

1.1 The model

Let H be a separable Hilbert space, let $S : H \to H$ be a continuous mapping, and let $\{\eta_k, k \ge 1\}$ be a sequence of i.i.d. random variables in H. We consider the stochastic system (0.1), supplemented with the initial condition

$$u_0 = u \in H. \tag{1.1}$$

Let us denote by $(u_k, \mathbb{P}_u)$ the Markov family generated by (0.1), (1.1), by $P_k(u, \Gamma)$ its transition function, and by $\mathfrak{P}_k : C_b(H) \to C_b(H)$ and $\mathfrak{P}_k^* : \mathcal{P}(H) \to \mathcal{P}(H)$ the corresponding Markov semigroups. Given a measure $\lambda \in \mathcal{P}(H)$, we write $\mathbb{P}_\lambda(\cdot) = \int_H \mathbb{P}_u(\cdot)\lambda(\mathrm{d}u)$. We shall always assume that S satisfies the two conditions below.

(A) Continuity and compactness. *There is a separable Banach space U compactly embedded into H such that S is continuous from H to U and is bounded on any ball.*

(B) Dissipativity. *There is a continuous function $\Phi : H \to \mathbb{R}_+$ bounded on any ball and such that $\Phi(u) \to +\infty$ as $\|u\| \to +\infty$ and*

$$\Phi(S(u) + v) \le q\,\Phi(u) + C(\Phi(v) + 1) \quad \text{for all } u, v \in H, \tag{1.2}$$

where $q < 1$ and $C \ge 1$ do not depend on u and v.

As for the random variables $\{\eta_k\}$, we assume that their law has a particular structure related to S. To formulate this condition, we shall use some concepts defined in Section 5.1. Given a vector $a \in H$ and a measure $\ell \in \mathcal{P}(H)$, we denote by $\theta_a : H \to H$ the shift operator in H taking u to $u + a$, by $\ell_a = \ell \circ \theta_a^{-1}$ the image of ℓ under θ_a, and by H_ℓ the set of all admissible shifts for ℓ.

(C) Structure of the noise. *The support of the measure $\ell := \mathcal{D}(\eta_1)$ coincides with H, and there is $\delta > 0$ such that*

$$\mathfrak{m}_\delta(\ell) := \int_H e^{\delta\Phi(u)}\,\ell(\mathrm{d}u) < \infty. \tag{1.3}$$

Moreover, the Banach space U defined in (A) *is contained in the subgroup of admissible shifts H_ℓ, and the mapping $\theta : U \to \mathcal{P}(H)$ that takes $a \in U$ to ℓ_a is continuous, provided that the space $\mathcal{P}(H)$ is endowed with the total variation norm.*

A sufficient condition for the validity of some of the above properties is given in Proposition 5.3. In the next two subsections, we formulate our main results on the exponential mixing, the LDP in the space of trajectories (or level-3 LDP), and the Gallavotti–Cohen fluctuation relation.

1.2 Exponential mixing and large deviations

For the reader convenience, we begin with some well-known definitions. Let X be a topological space, endowed with its Borel σ-algebra $\mathcal{B}_X$, and let $\mathcal{P}(X)$ be the set of probability measures on X, which is endowed with a regular[2] topology $\mathcal{T}$ and the corresponding Borel σ-algebra. Recall that a mapping $I : \mathcal{P}(X) \to [0, +\infty]$ is called a *rate function* if it is lower semicontinuous, and a rate function I is said to be *good* if its level sets are compact. For a Borel subset $\Gamma \subset \mathcal{P}(X)$, we write $I(\Gamma) = \inf_{\sigma\in\Gamma} I(\sigma)$.

Now let $\{\zeta_k\}$ be a sequence of random probability measures[3] on X defined on a measurable space $(\Omega, \mathcal{F})$, let Λ be an arbitrary set, and let $\mathbb{P}_\lambda$ be a family of probabilities on $(\Omega, \mathcal{F})$.

[2] Recall that a topological space $(Y, \mathcal{T})$ is said to be *regular* if any singleton is a closed subset, and for any closed set $F \subset X$ and any point $x \notin F$ there are disjoint open subsets G_1 and G_2 such that $F \subset G_1$ and $x \in G_2$.

[3] This means that ζ_k is a measurable mapping from $(\Omega, \mathcal{F})$ with range in the space $\mathcal{P}(X)$.

Definition 1.1. We shall say that $\{\zeta_k\}$ satisfies the *uniform LDP* with $\lambda \in \Lambda$ and a rate function I if the following two properties hold.

Upper bound. For any closed subset $F \subset \mathcal{P}(X)$, we have

$$\limsup_{k\to\infty} \frac{1}{k} \log \sup_{\lambda\in\Lambda} \mathbb{P}_\lambda\{\zeta_k \in F\} \le -I(F).$$

Lower bound. For any open subset $G \subset \mathcal{P}(X)$, we have

$$\liminf_{k\to\infty} \frac{1}{k} \log \inf_{\lambda\in\Lambda} \mathbb{P}_\lambda\{\zeta_k \in G\} \ge -I(G).$$

We now consider a particular case in which X is the product space $\boldsymbol{H} = H^{\mathbb{Z}_+}$, endowed with the Tikhonov topology. For any integer $k \ge 1$, consider the space $\mathcal{P}(H^k)$ endowed with the τ-topology, which is defined as the weakest topology with respect to which all the functionals $\mu \mapsto (f, \mu)$ with $f \in L^\infty(H^k)$ are continuous. We shall write $\mathcal{P}_\tau(H^k)$ to emphasise the τ-topology on $\mathcal{P}(H^k)$. The space of probability measures $\mathcal{P}(\boldsymbol{H})$ is endowed with the projective limit topology τ_p of the system $\{\mathcal{P}_\tau(H^k), k \ge 1\}$. In other words, τ_p is the weakest topology on $\mathcal{P}(\boldsymbol{H})$ with respect to which all the functionals $\mu \mapsto (f, \mu)$ with $f \in L^\infty(H^k)$ and any $k \ge 1$ are continuous.

Let us go back to system (0.1). Recall that a measure $\mu \in \mathcal{P}(H)$ is said to be *stationary* for a Markov family $(u_k, \mathbb{P}_u)$ if $\mathfrak{P}_1^*\mu = \mu$. We denote by $\boldsymbol{\zeta}_k$ the occupation measure in the trajectory space for a solution of (0.1); that is,

$$\boldsymbol{\zeta}_k = \frac{1}{k} \sum_{n=0}^{k-1} \delta_{\boldsymbol{u}_n}, \tag{1.4}$$

where $\boldsymbol{u}_n = (u_l, l \ge n)$, and $\{u_l\}$ is a trajectory of (0.1). The following theorem establishes uniqueness and mixing of a stationary measure for the Markov family associated with (0.1) and a uniform LDP for $\boldsymbol{\zeta}_k$ in the τ_p-topology. Its proof is given in Section 3.

Theorem 1.2. *Let Hypotheses* (A), (B), *and* (C) *be fulfilled and let* $(u_k, \mathbb{P}_u)$ *be the Markov family associated with* (0.1). *Then* $(u_k, \mathbb{P}_u)$ *has a unique stationary measure* μ, *and there are positive numbers* γ *and* C_1 *such that*

$$\|\mathfrak{P}_k^*\lambda - \mu\|_{\mathrm{var}} \le C_1 e^{-\gamma k}\biggl(1 + \int_H \Phi(u)\,\lambda(\mathrm{d}u)\biggr) \quad \text{for any } \lambda \in \mathcal{P}(H),\ k \ge 0. \tag{1.5}$$

Moreover, for any $c > 0$ *and any subset* $\Lambda \subset \mathcal{P}(H)$ *satisfying the condition*

$$\sup_{\lambda\in\Lambda} \int_H e^{c\,\Phi(u)} \lambda(\mathrm{d}u) < \infty, \tag{1.6}$$

the uniform LDP with $\lambda \in \Lambda$ *and a good rate function* $\boldsymbol{I} : \mathcal{P}(\boldsymbol{H}) \to [0, +\infty]$ *holds for the sequence of* $\mathbb{P}_\lambda$*-occupation measures* $\{\boldsymbol{\zeta}_k, k \ge 1\}$.

Theorem 1.2 combined with an approximation argument enables one to establish the LDP for various functionals of trajectories of (0.1) with moderate growth at infinity. To formulate the corresponding result, we shall need the concept of a stabilisable functional.

Let $\mathfrak{p} : H \to [0, +\infty]$ be a lower semi-continuous function. We shall say that $\mathfrak{p}$ is *uniformly stabilisable* for the Markov family $(u_k, \mathbb{P}_u)$ if there is an increasing continuous function $Q : \mathbb{R}_+ \to \mathbb{R}_+$ and a positive number γ such that

$$\mathbb{E}_u \exp\bigl(\mathfrak{p}(u_1) + \cdots + \mathfrak{p}(u_k)\bigr) \le Q(\|u\|)e^{\gamma k} \quad \text{for } k \ge 1,\ u \in H. \tag{1.7}$$

Theorem 1.3. *Under the hypotheses of Theorem 1.2, let $\mathfrak{p}$ be a uniformly stabilisable functional, let $m \ge 0$ be an integer, and let $f : H^{m+1} \to \mathbb{R}$ be a measurable function that is bounded on any ball and satisfies the condition*

$$\frac{|f(v_0, \dots, v_m)|}{\mathfrak{p}(v_0) + \cdots + \mathfrak{p}(v_m)} \to 0 \quad \textit{as } \|v_0\| + \cdots + \|v_m\| \to +\infty. \tag{1.8}$$

Then, for any measure $\lambda \in \mathcal{P}(H)$ satisfying the condition

$$\int_H \bigl(\exp\bigl(c\Phi(u)\bigr) + e^{\mathfrak{p}(u)}Q(\|u\|)\bigr)\lambda(\mathrm{d}u) < \infty, \tag{1.9}$$

with some $c > 0$, the $\mathbb{P}_\lambda$-laws of the real-valued random variables

$$\xi_k = \frac{1}{k}\sum_{n=0}^{k-1} f(u_n, \dots, u_{n+m}), \quad k \ge 1,$$

satisfy the LDP with a good rate function $I_f : \mathbb{R} \to [0, +\infty]$ not depending on λ.

Theorems 1.2 and 1.3 are applied in Section 2 to prove the LDP for various dissipative PDE's with random perturbations. In the next subsection, we discuss a symmetry property of the rate function for a particular choice of the observable f.

1.3 Gallavotti–Cohen fluctuation relation

The entropy flux observable for a general Markov family in H is defined by (0.6), provided that the transition function $P_1(u, \mathrm{d}v)$ possesses a density with respect to a reference measure $\ell \in \mathcal{P}(H)$,

$$P_1(u, \mathrm{d}v) = \rho(u, v)\ell(\mathrm{d}v), \tag{1.10}$$

and that $\rho(u, v) > 0$ for $\ell \otimes \ell$-almost every (u, v). If $(u_k, \mathbb{P}_u)$ is the Markov family associated with (0.1), then the existence of a density with respect to the law of η_k follows from Conditions (A) and (C), while a sufficient condition for its positivity on a set of full measure is given by Proposition 5.3. By the parameter version of the Radon–Nikodym theorem (see [Nov05]), if $(u_k, \mathbb{P}_u)$ possesses the Feller property, then one can choose ρ to be a measurable function in (u, v).

Given a stationary distribution μ of $(u_k, \mathbb{P}_u)$, we denote by $\boldsymbol{\mu}$ the corresponding path measure and note that μ is absolutely continuous with respect to ℓ, with the corresponding density given by

$$\rho(v) = \int_H \rho(z, v)\mu(\mathrm{d}z). \tag{1.11}$$

It is straightforward to check that $\rho(v) > 0$ for ℓ-almost every $v \in H$. We have the following simple result.

Lemma 1.4. *Let $(u_k, \mathbb{P}_u)$ be a Feller family of discrete-time Markov processes in H such that (1.10) holds for a reference measure $\ell \in \mathcal{P}(H)$ and a measurable function $\rho(u, v)$ that is positive $\ell \otimes \ell$-almost everywhere. Let μ be a stationary measure of $(u_k, \mathbb{P}_u)$ such that*

$$\int_H |\log \rho(v)|\, \mu(\mathrm{d}v) < \infty. \tag{1.12}$$

Then the negative part of σ is $\boldsymbol{\mu}$-integrable, and the mean value of σ with respect to $\boldsymbol{\mu}$ is non-negative.

Proof. We only need to prove the $\boldsymbol{\mu}$-integrability of the negative part of σ (which implies in particular that $\langle\sigma\rangle_\mu$, the mean value of σ with respect to $\boldsymbol{\mu}$, is well defined), because the fact that $\langle\sigma\rangle_\mu$ is non-negative follows immediately from (0.11) and the fact that $\mathrm{Ep}(\mu) \geq 0$. To this end, setting $\rho_{01} = \rho(v_0, v_1)$ and $\rho_{10} = \rho(v_1, v_0)$ and defining ρ to be the density of μ against ℓ, we write

$$\begin{aligned}
\int_{\boldsymbol{H}} \sigma^- \,\mathrm{d}\boldsymbol{\mu} &= \int_{H^2} I_{\{\rho_{01} \leq \rho_{10}\}} \Big|\log \frac{\rho_{01}}{\rho_{10}}\Big| \, P(\mathrm{d}v_0, \mathrm{d}v_1) \\
&= \int_{H^2} I_{\{\rho_{01} \leq \rho_{10}\}} \log \frac{\rho_{10}}{\rho_{01}} \, P(\mathrm{d}v_0, \mathrm{d}v_1) \\
&\leq \int_{H^2} I_{\{\rho_{01} \leq \rho_{10}\}} \Big(\log \frac{\rho_1 \rho_{10}}{\rho_0 \rho_{01}} - \log \frac{\rho_1}{\rho_0}\Big) P(\mathrm{d}v_0, \mathrm{d}v_1),
\end{aligned}$$

where $\rho_i = \rho(v_i)$ and $P(\mathrm{d}v_0, \mathrm{d}v_1) = \rho_0 \rho_{01} \ell(\mathrm{d}v_0)\ell(\mathrm{d}v_1)$. Using the inequality $\log x \leq x$ for $x > 0$, we see that the right-hand side of this inequality does not exceed

$$\int_{H^2} \rho_1 \rho_{10} \ell(\mathrm{d}v_0)\ell(\mathrm{d}v_1) + \int_{H^2} \big(|\log \rho_0| + |\log \rho_1|\big) P(\mathrm{d}v_0, \mathrm{d}v_1).$$

The first term of this expression is equal to 1, while the second is finite in view of (1.12). □

We now go back to the Markov family $(u_k, \mathbb{P}_u)$ associated with (0.1) and assume that Conditions (A)–(C) are fulfilled. Furthermore, we make the following hypothesis:

(D) Entropy production. *The densities $\rho(u,v)$ can be chosen so that the observable $\sigma(v_0,v_1)$ for $(u_k,\mathbb{P}_u)$ is well defined and bounded on any ball of $H\times H$. Moreover, there is a uniformly stabilisable functional $\mathfrak{p}:H\to[0,+\infty]$ such that*

$$\frac{|\sigma(v_0,v_1)|}{\mathfrak{p}(v_0)+\mathfrak{p}(v_1)}\to 0 \quad \text{as } \|v_0\|+\|v_1\|\to+\infty. \tag{1.13}$$

The following theorem establishes the LDP for the entropy production functional calculated on trajectories and the Gallavotti–Cohen fluctuation principle for the corresponding rate function.

Theorem 1.5. *Let us assume that Conditions* (A)–(D) *are fulfilled. Then, for any initial measure $\lambda\in\mathcal{P}(H)$ satisfying* (1.9)*, the LDP with a good rate function $I:\mathbb{R}\to[0,+\infty]$, independent of λ, holds for the $\mathbb{P}_\lambda$-laws of the real-valued random variables* (0.9)*. Moreover, if* (1.9) *is satisfied for $\lambda=\ell$, then the Gallavotti–Cohen fluctuation relation* (0.13) *holds for I.*

A proof of Theorem 1.5 is presented in Section 4, and its applications are discussed in Sections 2.3 and 2.4.

2 Applications

In this section, we discuss some applications of the results of the foregoing section to various dissipative PDE's perturbed by an unbounded kick force. We first prove that the hypotheses of Theorems 1.2 and 1.3 are satisfied for the 2D Navier–Stokes system and the complex Ginzburg–Landau equation. We next show that, in the case of equations with strong damping (such as the Burgers equation with periodic boundary conditions or a reaction-diffusion system with superlinear interaction), Theorem 1.5 is also applicable.

2.1 Two-dimensional Navier–Stokes system

We consider the Navier–Stokes system on the torus $\mathbb{T}^2\subset\mathbb{R}^2$. Let us denote by $\dot{L}^2$ the space of square-integrable vector fields on $\mathbb{T}^2$ with zero mean value, introduce the space

$$H=\bigl\{u\in\dot{L}^2:\operatorname{div}u=0 \text{ on } \mathbb{T}^2\bigr\}, \tag{2.1}$$

and write Π for the orthogonal projection in $\dot{L}^2$ onto H. Restricting ourselves to solutions and external forces with zero mean value with respect to the space variables and projecting the Navier–Stokes system onto H, we obtain the non-local evolution equation

$$\partial_t u+\nu Lu+B(u)=f(t). \tag{2.2}$$

Here $\nu>0$ is a parameter, $L=-\Delta$, $B(u)=\Pi(\langle u,\nabla\rangle u)$ is the nonlinear term, and f is an external force of the form

$$f(t)=h+\sum_{k=1}^{\infty}\eta_k\delta(t-k), \tag{2.3}$$

where $h \in H$ is a deterministic function, $\delta(t)$ is the Dirac mass at zero, and $\{\eta_k\}$ is a sequence of i.i.d. random variables in H. Normalising solutions of (2.2), (2.3) to be right-continuous and setting $u_k = u(k)$, we obtain relation (0.1), in which $S : H \to H$ stands for the time-one shift along trajectories of Eq. (2.2) with $f = h$. We recall that L is a positive self-adjoint operator in H with a compact inverse and denote by $\{e_j\}$ an orthonormal basis in H composed of the eigenfunctions of L, with the eigenvalues $\{\alpha_j\}$ indexed in a non-decreasing order. Let V^s be the domain of the operator $L^{s/2}$, so that $V^s = H^s \cap H$, where H^s is the Sobolev space of order s on $\mathbb{T}^2$.

The family of all trajectories for (0.1) form a discrete-time Markov process, which will be denoted by $(u_k, \mathbb{P}_u)$; see Section 2.3 in [KS12] for details. We now make the following hypothesis on the stochastic part of the external force (2.3).

Condition 2.1. *The i.i.d. random variables η_k have the form (cf. (5.2))*

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j, \tag{2.4}$$

where $\{b_j\}$ is a sequence of positive numbers such that

$$\sum_{j=1}^{\infty} b_j^2 < \infty, \tag{2.5}$$

and $\{\xi_{jk}\}$ is a sequence of independent scalar random variables whose laws possess densities $\tilde\rho_j \in C^1$ with respect to the Lebesgue measure, which are positive everywhere and satisfy (5.6) and (5.10).

Let us note that if the laws of ξ_{jk} are centred Gaussian measures with variances σ_j^2 belonging to a bounded interval separated from zero, then (5.6) and (5.10) are satisfied. The following result establishes the LDP for the occupation measures of $(u_k, \mathbb{P}_u)$.

Theorem 2.2. *Let $s \geq 2$ be an integer, let $h \in V^s$, and let η_k be random variables for which Condition 2.1 is fulfilled. Assume, in addition, that the law ℓ of η_k satisfies (1.3) with $\Phi(u) = \|u\|$ and some $\delta > 0$, and*

$$\sum_{j=1}^{\infty} b_j^{-2} \alpha_j^{-1-s} < \infty. \tag{2.6}$$

Then $(u_k, \mathbb{P}_u)$ has a unique stationary measure $\mu \in \mathcal{P}(H)$, which is exponentially mixing in the sense that (1.5) holds. Moreover, for any $c > 0$ and any subset $\Lambda \subset \mathcal{P}(H)$ satisfying (1.6), the uniform LDP with $\lambda \in \Lambda$ and a good rate function $\boldsymbol{I} : \mathcal{P}(\boldsymbol{H}) \to [0, +\infty]$ holds for the sequence of $\mathbb{P}_\lambda$-occupation measures (1.4).

Inequality (2.6) prevents the random kicks η_k to be very regular functions of x. Indeed, it is well known that $\alpha_j \sim j$ as $j \to \infty$; see [Mét78]. Hence, if $b_j = j^{-r}$ for $j \geq 1$, then the above theorem is applicable for any $r \in (1/2, s/2)$,

so that the regularity of η_k is at most $V^{\frac{s-1}{2}}$. Furthermore, by the Cauchy-Schwarz inequality, we have

$$+\infty = \sum_{j=1}^{\infty} \alpha_j^{-1} \le \Bigl(\sum_{j=1}^{\infty} b_j^2 \Bigr)^{1/2} \Bigl(\sum_{j=1}^{\infty} b_j^{-2} \alpha_j^{-2} \Bigr)^{1/2}.$$

If $s \le 1$, then (2.5) and (2.6) imply that the right-hand side of this inequality is finite. Since s is an integer, we see that it must satisfy the inequality $s \ge 2$. On the other hand, if $s \ge 2$, then the hypotheses of Theorem 2.2 are fulfilled for any Gaussian measure whose covariance operator is diagonal in the basis $\{e_j\}$ and has eigenvalues $\{b_j\}$ satisfying (2.5) and (2.6).

Proof of Theorem 2.2. We shall prove that the hypotheses of Theorem 1.3 hold for the Markov family in question. This will imply all the required results.

Step 1: Continuity and compactness. We claim that Condition (A) is satisfied for the pair (H, U), where H is defined by (2.1) and $U = V^{s+1}$. To see this, we apply a standard regularisation property for the 2D Navier–Stokes equations. Namely, as is proved in Chapter 17 of [Tay97], the time-1 shift $S : H \to H$ along trajectories of the deterministic Navier–Stokes system (2.2) (in which $f(t) \equiv h \in V^s$) maps H to V^{s+2}. Moreover, the image by S of any ball in H is a bounded subset in V^{s+2}. Since $S : H \to H$ is continuous and the embedding $V^{s+2} \subset V^{s+1}$ is compact, it follows that the mapping $S : H \to V^{s+1}$ is continuous and bounded on any ball.

Step 2: Dissipativity. We claim that inequality (1.2) holds with $\Phi(u) = \|u\|$, $q = e^{-\nu\alpha_1}$, and a sufficiently large C. Indeed, it is well known that (e.g., see inequality (2.25) in [Tem88])

$$\|S(u)\| \le q\|u\| + C, \quad u \in H,$$

where $C \ge 1$ does not depend on u. It follows that

$$\Phi(S(u) + v) \le q\|u\| + C + \|v\| \le q\,\Phi(u) + C\bigl(\Phi(v) + 1\bigr).$$

Step 3: Structure of the noise. The fact that $\operatorname{supp} \ell = H$ follows from (2.4) and the positivity of the coefficients b_j and densities $\tilde\rho_j$. The validity of (1.3) is required by the hypotheses of the theorem. It remains to prove that $V^{s+1} \subset H_\ell$ and the mapping $\theta : V^{s+1} \to \mathcal{P}(H)$ taking a vector a to the shifted measure ℓ_a is continuous. To this end, we shall show that inequality (5.11) holds, which implies that the hypotheses of Proposition 5.3 are satisfied. Denoting by C_s the sum of the series in (2.6) and using the Cauchy–Schwarz inequality, we derive

$$\sum_{j=1}^{\infty} b_j^{-1} |(v, e_j)| \le \Bigl(\sum_{j=1}^{\infty} b_j^{-2} \alpha_j^{-1-s} \Bigr)^{1/2} \Bigl(\sum_{j=1}^{\infty} |(v, e_j)|^2 \alpha_j^{s+1} \Bigr)^{1/2} = C_s^{1/2} \|v\|_{V^{s+1}}.$$

We have thus shown that Hypotheses (A)–(C) are satisfied with $q = e^{-\nu\alpha_1}$ and any sufficiently large $C > 0$. This completes the proof of the theorem. □

Corollary 2.3. *In addition to the hypotheses of Theorem 2.2, assume that the law of η_k and the initial measure $\lambda \in \mathcal{P}(H)$ satisfy the conditions*

$$\int_H \exp(\alpha\|u\|^2)\ell(\mathrm{d}u) < \infty, \quad \int_H \exp(\alpha\|u\|^2)\lambda(\mathrm{d}u) < \infty \tag{2.7}$$

for some $\alpha > 0$. Then, for any $\theta \in (0,2)$, the $\mathbb{P}_\lambda$-laws of the random variables

$$\xi_k = \frac{1}{k}\sum_{n=0}^{k-1} \|u_n\|^\theta$$

satisfy the LDP with a good rate function $I : \mathbb{R} \to [0,+\infty]$ not depending on λ.

Proof. As was shown above, the hypotheses of Theorem 1.2 are satisfied for the Markov family $(u_k, \mathbb{P}_u)$. Therefore, the required result will be established if we prove that the conditions of Theorem 1.3 hold for some uniformly stabilisable functional $\mathfrak{p}$.

Let us denote $\mathfrak{p}_\varepsilon(u) = \varepsilon\|u\|^2$ and recall the following estimate established in [KS12] (see there Step 2 of the proof of Proposition 2.3.8 and inequality (2.53)), provided that ℓ satisfies the first inequality in (2.7):

$$\mathbb{E}_u \exp\bigl(\mathfrak{p}_\varepsilon(u_1) + \dots + \mathfrak{p}_\varepsilon(u_k)\bigr) \le \exp(C\varepsilon\|u\|^2 + Ck), \quad k \ge 1, \tag{2.8}$$

where $C > 0$ is an absolute constant and $\varepsilon > 0$ is sufficiently small. Thus, the functional $\mathfrak{p}_\varepsilon$ is uniformly stabilisable and satisfies inequality (1.7) with $Q(r) = \exp(C\varepsilon r^2)$. It remains to note that convergence (1.8) holds for the continuous function $f(v) = \|v\|^\theta$, and condition (1.9) is fulfilled for $\varepsilon \ll 1$ and any measure $\lambda \in \mathcal{P}(H)$ satisfying the second inequality in (2.7) with some $\alpha > 0$. □

2.2 Complex Ginzburg–Landau equation

We consider the following equation on the torus $\mathbb{T}^d \subset \mathbb{R}^d$:

$$\partial_t u - (\nu + i)(\Delta - 1)u + ia|u|^2 u = f(t,x), \quad x \in \mathbb{T}^d. \tag{2.9}$$

Here $a > 0$ is a parameter, $u = u(t,x)$ is a complex-valued function, and f is a random process. We assume that f has the form (2.3), where $h \in L^2(\mathbb{T}^d)$ is a deterministic complex-valued function and $\{\eta_k\}$ is a sequence of i.i.d. random variables in the complex space $H^1(\mathbb{T}^d)$, where $H^s(\mathbb{T}^d) =: V^s$ is the Sobolev space of order s. If $d \le 4$, then the Cauchy problem for (2.9) is well posed in V^1 (e.g., see [KS04] or the book [Caz03] for the more complicated case of the Schrödinger equation). This means that, for any $u_0 \in V^1$, problem (2.9) has a unique solution satisfying the initial condition

$$u(0,x) = u_0(x). \tag{2.10}$$

Under the above hypotheses, the restrictions of solutions to (2.9) form a discrete-time Markov process $(u_k, \mathbb{P}_u)$ in the space V^1, which is regarded as a real Hilbert space with the scalar product

$$(u,v)_1 = (u,v) + \sum_{j=1}^{d} (\partial_j u, \partial_j v), \quad (u,v) = \operatorname{Re} \int_{\mathbb{T}^d} u\bar{v}\,\mathrm{d}x.$$

Let $\{e_j\}$ be the complete system of eigenfunctions of $-\Delta+1$, which are indexed in an increasing order of the corresponding eigenvalues α_j. We normalise e_j to be unit vectors in $V := V^1$. In what follows, we impose the following condition on η_k.

Condition 2.4. The i.i.d. random variables η have the form (2.4), where $\{b_j\}$ is a sequence of positive numbers satisfying (2.5), $\xi_{jk} = \xi^1_{jk} + i\xi^2_{jk}$, and ξ^l_{jk} are independent real-valued random variables. Moreover, the laws of ξ^l_{jk} possess densities $\tilde{\rho}^l_j \in C^1$ with respect to the Lebesgue measure, which are positive and satisfy (5.6) and (5.10).

Let us define the functional

$$\mathcal{H}(u) = \int_{\mathbb{T}^d} \Big(\frac{1}{2}|\nabla u(x)|^2 + \frac{1}{2}|u(x)|^2 + \frac{a}{4}|u(x)|^4 \Big)\,\mathrm{d}x.$$

The following result is an analogue of Theorem 2.2 in the case of the Ginzburg–Landau equation. Its proof is essentially the same, and we shall confine ourselves to outlining it.

Theorem 2.5. *Let $s \geq d$ be an integer, let $h \in V^{s-1}$, and let $\{\eta_k\}$ be a sequence random variables for which Condition 2.4 is fulfilled. Assume, in addition, that the law ℓ of η_k satisfies (1.3) with $\Phi(u) = (\mathcal{H}(u))^\theta$ for some positive numbers δ and θ, and inequality (2.6) holds. Then $(u_k, \mathbb{P}_u)$ has a unique stationary measure $\mu \in \mathcal{P}(V)$, which is exponentially mixing in the sense that (1.5) holds with $H = V$. Moreover, for any $c > 0$ and any subset $\Lambda \subset \mathcal{P}(V)$ satisfying condition (1.6) in which $H = V$, the uniform LDP with $\lambda \in \Lambda$ and a good rate function[4] $\boldsymbol{I} : \mathcal{P}(\boldsymbol{V}) \to [0, +\infty]$ holds for the sequence of $\mathbb{P}_\lambda$-occupation measures (1.4).*

Outline of the proof. We need to check Hypotheses (A)–(C), in which $\Phi(u)$ is defined in the statement of the theorem, and $S : V \to V$ stands for the time-1 shift along trajectories of problem (2.9) with $f(t) \equiv h$. The validity of (A) with $U = V^s$ is a standard fact of the regularity theory for parabolic systems (e.g., see Chapter 15 in the book [Tay97] and the references therein), and we shall not dwell on it. To check (B), let us note that if $u(t,x)$ is a solution of (2.9), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(u(t)) \leq -\nu\big(\|(\Delta-1)u\|^2 + a(|u|^2, |\nabla u|^2) + a\|u\|^4_{L^4}\big) + \big(-(\Delta-1)u + a|u|^2 u, f\big).$$

[4] We define $\boldsymbol{V} = V^{\mathbb{Z}_+}$.

It follows that if $f(t) \equiv h$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(u(t)) \le -\beta\mathcal{H}(u(t)) + M,$$

where $M = C(\|h\|_{L^4}^4 + 1)$ and $\beta > 0$. Applying the Gronwall inequality, we derive

$$\mathcal{H}(S(u)) \le e^{-\beta}\mathcal{H}(u) + \beta^{-1}M. \tag{2.11}$$

It is easy to see that $\mathcal{H}(z+v) \le (1+\alpha)\mathcal{H}(z) + C_\alpha\mathcal{H}(v)$ for any $u, v \in V$, where $\alpha > 0$ is arbitrary and $C_\alpha > 0$ depends only on α. Combining this inequality with (2.11), we obtain

$$\mathcal{H}(S(u)+v) \le (1+\alpha)e^{-\beta}\mathcal{H}(u) + C_\alpha\mathcal{H}(v) + (1+\alpha)\beta^{-1}M.$$

Choosing $\alpha > 0$ sufficiently small and raising the resulting inequality to power $\theta > 0$, we arrive at (1.2) with $\Phi(u) = (\mathcal{H}(u))^\theta$ and $H = V$. Finally, the verification of (C) is completely similar to that for the case of the Navier–Stokes system, and we omit it. □

As in the case of the Navier–Stokes system, we can derive from Theorem 2.5 some results on LDP for observables with moderate growth at infinity.

Corollary 2.6. *In addition to the hypotheses of Theorem 2.5, assume that the law ℓ of η_k and the initial measure $\lambda \in \mathcal{P}(V)$ satisfy the conditions*

$$\int_V \exp\bigl(\alpha\sqrt{\mathcal{H}(u)}\bigr)\ell(\mathrm{d}u) < \infty, \quad \int_V \exp\bigl(\alpha\sqrt{\mathcal{H}(u)}\bigr)\lambda(\mathrm{d}u) < \infty, \tag{2.12}$$

where $\alpha > 0$. Then, for any measurable function $f : V \to \mathbb{R}$ satisfying the condition $\frac{|f(u)|}{\sqrt{\mathcal{H}(u)}} \to 0$ as $\|u\|_V \to \infty$, the $\mathbb{P}_\lambda$-laws of the random variables

$$\xi_k = \frac{1}{k}\sum_{n=0}^{k-1} f(u_k)$$

satisfy the LDP with a good rate function not depending on λ.

Proof. As for the proof of Corollary 2.3, it suffices to show that $\mathfrak{p}_\varepsilon(u) = \varepsilon\sqrt{\mathcal{H}(u)}$ is a uniformly stabilisable functional. To this end, we use inequality (1.2) with $\Phi(u) = \sqrt{\mathcal{H}(u)}$. Setting $u = u_{n-1}$ and $v = \eta_n$ with $n = 1, \dots, k$, we derive

$$\Phi(u_n) \le q\,\Phi(u_{n-1}) + C(\Phi(\eta_n) + 1).$$

Summing up these inequalities, we obtain

$$\sum_{n=1}^{k}\Phi(u_n) \le C_1\Phi(u) + C_1\sum_{n=1}^{k}\Phi(\eta_n) + C_1 k.$$

The independence of η_k now implies that

$$\mathbb{E}_u \exp(\mathfrak{p}_\varepsilon(u_1) + \dots + \mathfrak{p}_\varepsilon(u_k)) \le e^{\varepsilon C_1(\Phi(u)+k)} \biggl(\int_V e^{\varepsilon C_1 \Phi(z)} \ell(\mathrm{d}z) \biggr)^k .$$

Taking into account the first condition in (2.12), we see that $\mathfrak{p}_\varepsilon$ is uniformly stabilisable for $\varepsilon \ll 1$. It remains to note that, in view of the second condition in (2.12), inequality (1.9) is also satisfied for $\varepsilon \ll 1$. □

2.3 Burgers equation

Let us consider problem (0.3)–(0.5). Our aim is to establish Theorems A and B stated in the Introduction. In view of Theorem 1.2, to prove Theorem A, it suffices to check the validity of Hypotheses (A)–(C), in which $U = V^{s+1}$. The fact that $S : H \to V^{s+1}$ is continuous and bounded on any ball is a standard regularity result, and we omit it. Inequality (1.2) with $\Phi(u) = \|u\|$ is also well known, and the validity of (1.3) with any $\delta > 0$ follows from the Fernique theorem; e.g., see Theorem 2.8.5 in [Bog98]. To check the remaining hypotheses in (C), recall that the subgroup of admissible shifts for a Gaussian measure coincides with its Cameron–Martin space; see Theorem 2.4.5 in [Bog98]. Hence, the continuous inclusion of $U = V^{s+1}$ into H_ℓ holds in view of the hypotheses of Theorem A. Finally, to prove the continuity of $\theta : V^{s+1} \to \mathcal{P}(H)$, we use the following estimate for the total variation norm between shifts of a Gaussian measure (see Lemma 2.4.4 in [Bog98]):

$$\|\ell_a - \ell_{a'}\|_{\mathrm{var}} \le 2\bigl(1 - \exp\bigl\{-\tfrac14 \|a - a'\|^2_{H_\ell}\bigr\}\bigr)^{1/2}. \tag{2.13}$$

Here $a, a' \in H_\ell$ are arbitrary vectors, and $\|\cdot\|_{H_\ell}$ denotes the norm in the Cameron–Martin space of ℓ:

$$\|a\|^2_{H_\ell} = \sum_{j=1}^\infty b_j^{-2} a_j^2, \quad a = (a_1, a_2, \dots),$$

where a is expanded in the eigenbasis of the covariance operator for ℓ. Since V^{s+1} is continuously embedded in H_ℓ, we see that the shift operator θ is continuous from V^{s+1} to $\mathcal{P}(H)$. This completes the proof of Theorem A.

We now turn to Theorem B. In view of Theorem 1.5, to prove the LDP and the Gallavotti–Cohen relation for the rate function, it suffices to find a uniformly stabilisable function $\mathfrak{p} : H \to \mathbb{R}_+$ such that (1.13) holds and to check (1.9) for $\lambda = \ell$. Exactly the same argument as for the 2D Navier–Stokes system or Ginzburg–Landau equation shows that $\mathfrak{p}_\varepsilon(u) = \varepsilon\|u\|^2$ with $\varepsilon > 0$ is a uniformly stabilisable functional, and the corresponding function Q entering (1.7) can be chosen to be $Q_\varepsilon(r) = \exp(C\varepsilon r^2)$, where $C > 0$ does not depend on ε. By Fernique theorem, it follows that condition (1.9) is satisfied for ℓ, provided that $\varepsilon > 0$ is sufficiently small.

We now prove the boundedness of $\sigma(v_0, v_1)$ on balls of $H \times H$ and convergence (1.13). By the hypotheses of Theorem B, the measure ℓ can be decomposed in the standard trigonometric basis in H and written in the form (5.1), where μ_j denotes the centred normal law on $\mathbb{R}$ with variance b_j^2. It follows from (5.4) that

$$\rho(u, v) = \exp\bigl(-\tfrac{1}{2}\|S(u)\|_b^2 + (S(u), v)_b\bigr), \tag{2.14}$$

where we set

$$(u, v)_b = \sum_{j=1}^{\infty} b_j^{-2} u_j v_j, \quad \|u\|_b = (u, u)_b^{1/2}.$$

Combining (2.14) and (0.6), we see that

$$\sigma(v_0, v_1) = \tfrac{1}{2}\|S(v_0)\|_b^2 - \tfrac{1}{2}\|S(v_1)\|_b^2 - (S(v_0), v_1)_b + (S(v_1), v_0)_b. \tag{2.15}$$

We now need the following lemma, which is a consequence of the Kruzhkov maximum principle [Kru69]; its proof in the more difficult stochastic case can be found in [Bor13, Section 3].

Lemma 2.7. *Let $h \in V^m$ for some integer $m \geq 2$. Then the image of S is contained in V^{m+1}, the mapping $S : H \to V^{m+1}$ is continuous, and there is $K_m > 0$ such that*

$$\|S(u)\|_{m+1} \leq K_m \quad \textit{for any } u \in H. \tag{2.16}$$

Now note that the continuity of the embedding $V^{s+1} \subset H_\ell$ implies the inequality

$$\|w\|_{b^2}^2 := \sum_{j=1}^{\infty} |w_j|^2 b_j^{-4} \leq C \sum_{j=1}^{\infty} |w_j|^2 (1 + |j|^2)^{2(s+1)} = C\|w\|_{2(s+1)}^2, \tag{2.17}$$

where $w \in V^{2(s+1)}$ and $w_j = (w, e_j)$. Combining this with inequality (2.16) and relation (2.15), we obtain

$$\begin{aligned}
|\sigma(v_0, v_1)| &\leq \frac{1}{2}\bigl(\|S(v_0)\|_b^2 + \|S(v_1)\|_b^2\bigr) + \|v_0\|\,\|S(v_1)\|_{b^2} + \|v_0\|\,\|S(v_1)\|_{b^2} \\
&\leq \frac{C}{2}\bigl(\|S(v_0)\|_{s+1}^2 + \|S(v_1)\|_{s+1}^2\bigr) + C\|v_0\|\,\|S(v_1)\|_{2(s+1)} + C\|v_0\|\,\|S(v_1)\|_{2(s+1)} \\
&\leq CK_s^2 + CK_{2s+1}\bigl(\|v_0\| + \|v_1\|\bigr).
\end{aligned} \tag{2.18}$$

We see that Condition (D) is fulfilled for the Burgers equation. Thus, the LDP and the Gallavotti–Cohen symmetry hold for the entropy production and the corresponding rate function. We note that a similar argument leads to the following rough estimate on the density of the measure $\mathfrak{P}^*\lambda$ which applies, in particular to the stationary measure $\mu = \mathfrak{P}^*\mu$,

$$\mathrm{e}^{-C(1+\|u\|)} \leq \frac{\mathrm{d}\mathfrak{P}^*\lambda}{\mathrm{d}\ell}(u) \leq \mathrm{e}^{C(1+\|u\|)},$$

where the constant C is independent of λ. It easily follows from this estimate that the boundary term in the entropy balance relation (0.10) is indeed $O(k^{-1})$ for large k provided the measure λ satisfies (1.9).

It remains to prove the positivity and finiteness of the mean entropy production $\langle\sigma\rangle_\mu$. As was explained in the introduction, we always have $\langle\sigma\rangle_\mu \ge 0$, and the equality holds if and only if the detailed balance condition (0.8) is satisfied $\ell\otimes\ell$ almost everywhere. Recalling (2.14), we can write this condition as

$$\exp\bigl(-\tfrac12\|S(v_1)\|_b^2 + (S(v_1),v_0)_b\bigr)\int_H \rho(z,v_1)\mu(\mathrm{d}z) \\ = \exp\bigl(-\tfrac12\|S(v_0)\|_b^2 + (S(v_0),v_1)_b\bigr)\int_H \rho(z,v_0)\mu(\mathrm{d}z). \quad (2.19)$$

It follows from inequality (2.17) and Lemma 2.7 that the left- and right-hand sides are continuous functions on $H\times H$, and since $\operatorname{supp}(\ell\otimes\ell)$ coincides with the whole space, we see that relation (2.19) must hold for all $(v_0,v_1)\in H\times H$. Taking the logarithm of both sides of (2.19), replacing v_1 by λv_1, and dividing by λ, we derive

$$(S(v_0),v_1)_b = \frac{1}{\lambda}\log\int_H e^{\lambda(v_1,S(z))_b}\exp\bigl(-\tfrac12\|S(z)\|_b^2\bigr)\mu(\mathrm{d}z) + \lambda^{-1}r(\lambda), \quad (2.20)$$

where we set

$$r(\lambda) = \tfrac12\bigl(\|S(v_0)\|_b^2 - \|S(\lambda v_1)\|_b^2\bigr) + (S(\lambda v_1),v_0) + \log\int_H \rho(z,v_0)\mu(\mathrm{d}z).$$

It follows from inequality (2.16) with $m=2s+1$ that r is a bounded function of $\lambda\in\mathbb{R}$, so that the second term on the right-hand side of (2.20) goes to zero as $\lambda\to+\infty$. Since the first term on the right-hand side does not depend on v_0, passing to the limit in (2.20) as $\lambda\to+\infty$, we conclude that

$$\bigl(S(v_0),v_1\bigr)_b = C(v_1) \quad \text{for all } v_0,v_1\in H,$$

where $C(v_1)$ depends only on v_1. It follows that $S(v)$ is a constant function on H. This contradicts the backward uniqueness of solutions for the Burgers equation.

To prove the finiteness of $\langle\sigma\rangle_\mu$, note that, in view of (2.18), we have

$$\langle\sigma\rangle_\mu \le \int_{H\times H}|\sigma(v_0,v_1)|\,\boldsymbol{\mu}(\mathrm{d}v_0,\mathrm{d}v_1) \le C\biggl(1+\int_H\|v\|\mu(\mathrm{d}v)\biggr). \quad (2.21)$$

The integral on the right-hand side of this inequality is equal to

$$\mathbb{E}_\mu\|u_1\| \le \mathbb{E}_\mu\|S(u_0)\| + \mathbb{E}\|\eta_1\| < \infty,$$

where we used inequality (2.16). The proof of Theorem B is complete.

2.4 Reaction-diffusion system

Let $D \subset \mathbb{R}^d$ be a bounded domain with C^∞-smooth boundary ∂D. We consider the problem

$$\dot{u} - a\Delta u + g(u) = f(t,x), \tag{2.22}$$

$$u\big|_{\partial D} = 0, \tag{2.23}$$

$$u(0,x) = u_0(x). \tag{2.24}$$

Here $u = (u_1, \dots, u_l)^t$ is an unknown vector function, a is an $l \times l$ matrix such that

$$a + a^t > 0, \tag{2.25}$$

$g \in C^\infty(\mathbb{R}^l, \mathbb{R}^l)$ is a given function, and f is a random process of the form (2.3). We assume that $g \in C^1$ satisfies the following growth and dissipativity conditions:

$$\langle g(u), u\rangle \geq -C + c|u|^{p+1}, \tag{2.26}$$

$$g'(u) + g'(u)^t \geq -CI, \tag{2.27}$$

$$|g'(u)| \leq C(1 + |u|)^{p-1}, \tag{2.28}$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^l$, $g'(u)$ is the Jacobi matrix for g, I is the identity matrix, c and C are positive constants, and $1 < p \leq \frac{d+2}{d-2}$. As in the case of the 2D Navier–Stokes system, problem (2.22), (2.23) generates a discrete-time Markov process denoted by $(u_k, \mathbb{P}_u)$, and our aim is to study the LDP for the occupation measures (1.4).

Let us denote by $\{e_j\}$ an orthonormal basis in $H = L^2(D, \mathbb{R}^l)$ composed of the eigenfunctions of the Dirichlet Laplacian $-\Delta$ and by V^s the domain of the operator $(-\Delta)^{s/2}$. The following theorem can be established by a literal repetition of the arguments used in the case of the Navier–Stokes system, and therefore we omit its proof.

Theorem 2.8. *In addition to the above hypotheses, assume that $s \geq d$ is an integer, $h \in V^s$, the function $g(u)$ belongs to C^s and vanishes at $u = 0$ together with its derivatives up to order s, and $\{\eta_k\}$ is an i.i.d. sequence of random variables satisfying Condition 2.1 such that (2.6) and (1.3) hold with $\Phi(u) = \|u\|$. Then $(u_k, \mathbb{P}_u)$ has a unique stationary measure $\mu \in \mathcal{P}(H)$, which is exponentially mixing. Moreover, for any $c > 0$ and any subset $\Lambda \subset \mathcal{P}(H)$ satisfying (1.6) the uniform LDP with $\lambda \in \Lambda$ and a good rate function $\boldsymbol{I} : \mathcal{P}(\boldsymbol{H}) \to [0, +\infty]$ holds for the sequence of $\mathbb{P}_\lambda$-occupation measures (1.4).*

We now turn to the Gallavotti–Cohen fluctuation principle. The following result is an analogue of Theorem B for the reaction–diffusion system.

Theorem 2.9. *In addition to the hypotheses of Theorem 2.8, let us assume that $h \in V^{2s+1}$, the function g belongs to C^{2s+1} and vanishes at $u = 0$ together with its derivatives up to order $2s+1$, the orthonormal basis entering decomposition (5.1) for the measure ℓ coincides with the eigenbasis $\{e_j\}$, the measure ℓ*

satisfies the first condition in (2.7)*, and the second inequality in* (5.10) *holds for any* $y \in \mathbb{R}$ *and* $A \in [0, A_0]$*, with an arbitrary* $A_0 > 0$ *and constant* $C_1 = C_1(A_0)$*. Then, for any initial point* $u_0 \in H$*, the laws of the random variables* (0.9)*, in which* $\sigma(v_0, v_1)$ *is the entropy production functional for* $(u_k, \mathbb{P}_u)$*, satisfy the LDP with a good rate function* $I : \mathbb{R} \to [0, +\infty]$ *not depending on* u_0*. Moreover, the Gallavotti–Cohen fluctuation relation* (0.13) *holds for* I*.*

It is easy to check that the conditions imposed on ℓ are satisfied for any Gaussian measure on H such that its Cameron–Martin space contains V^{s+1}, and the eigenvectors of its covariance operator coincide with the eigenbasis $\{e_j\}$ of the Dirichlet Laplacian in $L^2(D, \mathbb{R}^l)$.

Proof. We shall show that the hypotheses of Theorem 1.5 are fulfilled; this will imply all required results. As was mentioned above, the verification of Conditions (A) and (B), in which $U = V^{2(s+1)}$ and $\Phi(u) = \|u\|$, is similar to the case of the Navier–Stokes system, and therefore we do not dwell on it.

Let us check Condition (D). To prove the positivity of ρ and the continuity of the shift operator $\theta : V^{s+1} \to \mathcal{P}(H)$, in view of Proposition 5.3, it suffices to check inequality (5.12). To this end, we first note that (2.6) implies the inequality

$$\sum_{j=1}^{\infty} b_j^{-2} |w_j|^2 \le \sup_{j \ge 1} \bigl(|w_j|^2 \alpha_j^{s+1} \bigr) \sum_{j=1}^{\infty} b_j^{-2} \alpha_j^{-s-1} \le C_1 \|w\|_{V^{s+1}}^2, \tag{2.29}$$

where $w \in V^{s+1}$ and $w_j = (w, e_j)$. Setting $w_j = v_j \alpha_j^{\frac{s+1}{2}}$ in (2.29) and using again (2.6), we obtain

$$\sum_{j=1}^{\infty} b_j^{-2} |v_j| \le \biggl(\sum_{j=1}^{\infty} b_j^{-2} \alpha_j^{-s-1} \biggr)^{1/2} \biggl(\sum_{j=1}^{\infty} b_j^{-2} |v_j|^2 \alpha_j^{s+1} \biggr)^{1/2} \le C_2 \|v\|_{V^{2(s+1)}}.$$

We now prove that $\sigma(v_0, v_1)$ is bounded on balls of $H \times H$ and satisfies (1.13) for some uniformly stabilisable functional $\mathfrak{p}$. Exactly the same argument as for the Ginzburg–Landau equation shows that $\mathfrak{p}_\varepsilon(u) = \varepsilon \|u\|^2$ is uniformly stabilisable, and one can take $Q(r) = \exp(C\varepsilon r^2)$ in (1.7). For $v_0, v_1 \in H$, let us write

$$v_i = \sum_{j=1}^{\infty} v_{ij} e_j, \quad S(v_i) = \sum_{j=1}^{\infty} S_j(v_i) e_j, \quad i = 0, 1.$$

Let ρ_j be the density of the law of $b_j \xi_{jk}$, so that $\rho_j(r) = b_j^{-1} \tilde{\rho}_j(r/b_j)$. Combining this relation with (5.4) and (0.6), we obtain

$$\begin{aligned} \sigma(v_0, v_1) &= \sum_{j=1}^{\infty} \biggl(\log \frac{\rho_j\bigl(v_{0j} - S_j(v_1)\bigr)}{\rho_j(v_{0j})} - \log \frac{\rho_j\bigl(v_{1j} - S_j(v_0)\bigr)}{\rho_j(v_{1j})} \biggr) \\ &= \sum_{j=1}^{\infty} \bigl(\Xi_j(v_0, v_1) - \Xi_j(v_1, v_0) \bigr), \end{aligned} \tag{2.30}$$

where we set

$$\Xi_j(v_0,v_1)=\log\tilde\rho_j\bigl(\tfrac{v_{0j}-S_j(v_1)}{b_j}\bigr)-\log\tilde\rho_j\bigl(\tfrac{v_{0j}}{b_j}\bigr).$$

Let us define $A_{ij}=b_j^{-1}|S_j(v_i)|$ and $A=\sup_{i,j}A_{ij}$, where the supremum is taken over $i=0,1$ and $j\ge1$. The second inequality in (5.10) implies that

$$\begin{aligned}|\Xi_j(v_0,v_1)|&=\left|\int_0^1\frac{\mathrm d}{\mathrm d\theta}\log\tilde\rho_j\biggl(\frac{v_{0j}-\theta S_j(v_1)}{b_j}\biggr)\mathrm d\theta\right|\le\int_{-A_{1j}}^{A_{1j}}\frac{|\tilde\rho_j'(b_j^{-1}v_{0j}-r)|}{\tilde\rho_j(b_j^{-1}v_{0j}-r)}\mathrm dr\\&\le C_3(A)\bigl(b_j^{-1}|v_{0j}|+1\bigr)A_{1j}.\end{aligned}$$

A similar estimate holds for $\Xi(v_1,v_0)$. Substituting them into (2.30) and using the Cauchy–Schwarz inequality and condition (2.5), we obtain

$$\begin{aligned}|\sigma(v_0,v_1)|&\le C_4\sum_{j=1}^\infty b_j^{-1}\Bigl(\bigl(b_j^{-1}|v_{0j}|+1\bigr)|S_j(v_1)|+\bigl(b_j^{-1}|v_{1j}|+1\bigr)|S_j(v_0)|\Bigr)\\&\le C_5\bigl(\|v_0\|+1\bigr)\|S(v_1)\|_{b^2}+C_5\bigl(\|v_1\|+1\bigr)\|S(v_0)\|_{b^2},\end{aligned}\tag{2.31}$$

where the norm $\|\cdot\|_{b^2}$ is defined in (2.17). We now need the following lemma, established at the end of this section.

Lemma 2.10. *Let (2.25)–(2.28) be satisfied, let $g\in C^m$ be a function vanishing at $u=0$ together with its derivative up to order m, and let $h\in V^m$ for some integer $m\ge0$. Then the image of S is contained in V^{m+1}, and there is $K_m>0$ such that*

$$\|S(v)\|_{V^{m+1}}\le K_m\quad\textit{for any } v\in H.\tag{2.32}$$

It follows from (2.29) and (2.32) with $m=2s+1$ that

$$\|S(v)\|_{b^2}^2\le C_6\sum_{j=1}^\infty\alpha_j^{2(s+1)}|S_j(v)|^2=C_6\|S(v)\|_{2(s+1)}^2\le C_6K_{2s+1}^2,\quad v\in H.$$

Substituting this inequality into (2.31), we obtain

$$|\sigma(v_0,v_1)|\le C_7\bigl(\|v_0\|+\|v_1\|\bigr),\quad v_0,v_1\in H.\tag{2.33}$$

This implies the required properties of σ.

It remains to show that (1.9) is also satisfied. To this end, we note that the integrand in (1.9) does not exceed $C_8\exp(C_8\varepsilon\|u\|^2)$. In view of the first inequality in (2.7), this function is integrable with respect to ℓ, provided that $\varepsilon>0$ is sufficiently small. The proof of Theorem 2.9 is complete. □

Proof of Lemma 2.10. In view of the regularising property of the resolving operator for the reaction-diffusion system, it suffices to prove that, if $u(t,x)$ is the solution of (2.22)–(2.24) with $f(t,x)\equiv h(x)$ and $u_0\in H$, then

$$\|u(\tfrac12,\cdot)\|\le K\quad\text{for any initial function } u_0\in H,\tag{2.34}$$

where $K > 0$ does not depend on u_0.

Taking the scalar product in L^2 of Eq. (2.22) (in which $f \equiv h$) with u, using (2.26), and carrying out some standard transformations, we derive

$$\partial_t \|u\|^2 + \delta \|\nabla u\|^2 + 2c\|u\|_{L^{p+1}}^{p+1} \le C_1 + \|h\|^2,$$

where δ and c are positive numbers. Since $\|u\|_{L^{p+1}} \ge C_2\|u\|$, we see that the function $\varphi(t) = \|u(t)\|^2$ satisfies the differential inequality

$$\varphi' + 2c_1\varphi^{(p+1)/2} \le C_1(1 + \|h\|^2),$$

where $c_1 > 0$. It follows that, as long as $\varphi(t) \ge K_1 := c_1^{-1}\big(C_1(1+\|h\|^2)\big)^{2/(p+1)}$, we have

$$\varphi' + c_1\varphi^{(p+1)/2} \le 0.$$

Resolving this differential inequality, we obtain

$$\varphi(t) \le \big(\|u_0\|^{(1-p)/2} + C_2 t\big)^{-2/(p-1)} \le \big(K_1^{(1-p)/2} + C_2 t\big)^{-2/(p-1)}, \tag{2.35}$$

provided that $\varphi(s) \ge K_1$ for $0 \le s \le K_1$. Denoting by K_2 the value of the right-hand side of (2.35) with $t = 1/2$, we conclude that inequality (2.34) holds with $K = \max(K_1, K_2)$. □

Finally, we discuss briefly the question of strict positivity and finiteness of the entropy production rate. If the measure ℓ is Gaussian, then exactly the same argument as in the case of the Burgers equation shows that the entropy production rate is strictly positive and finite in the stationary regime. However, these two properties are not related to the Gaussian structure of the noise and remain valid under more general hypotheses. Indeed, the finiteness of $\langle\sigma\rangle_\mu$ follows from inequality (2.33) and the fact the first moment of the stationary measure μ is finite. On the other hand, the strict positivity of $\langle\sigma\rangle_\mu$ holds under some additional hypotheses. Since the corresponding argument is technically rather complicated, we confine ourselves here to outlining the proof in the model case when $H = \mathbb{R}$. Namely, let us consider the Markov family associated with Eq. (0.1) in which $S : \mathbb{R} \to \mathbb{R}$ is a non-constant continuous mapping with bounded image and $\{\eta_k\}$ is a sequence of i.i.d. random variables in $\mathbb{R}$ whose law ℓ has a continuous density θ against the Lebesgue measure that has the form

$$\theta(y) = \exp\big(-a|y|^\alpha + b(y)\big), \quad r \in \mathbb{R}, \tag{2.36}$$

where $\alpha \in (1, 2]$, $a > 0$, and b is a bounded continuous function. As was explained in Section 2.3, the entropy production rate is zero if and only if (cf. (2.19))

$$\begin{aligned}\exp\big(-a|v_1 - S(v_0)|^\alpha + b(v_1 - S(v_0))\big) \int_H \rho(z, v_0)\mu(\mathrm{d}z) \\ = \exp\big(-a|v_0 - S(v_1)|^\alpha + b(v_0 - S(v_1))\big) \int_H \rho(z, v_1)\mu(\mathrm{d}z).\end{aligned}$$

Taking the logarithm of both sides of this relation and carrying out some simple transformations, we derive

$$-a|v_1 - S(v_0)|^\alpha = \log \int_{\mathbb{R}} \exp\bigl(-a|v_1 - S(z)|^\alpha + b(v_1 - S(z))\bigr)\mu(\mathrm{d}z) + r_1(v_0, v_1), \tag{2.37}$$

where we denote by r_i bounded functions of their arguments. Now note that

$$|v_1 - S(v_0)|^\alpha = v_1^\alpha - \alpha S(v_0) v_1^{\alpha-1} + r_2(v_0, v_1) v_1^{\alpha-2} \quad \text{as } v_1 \to +\infty.$$

Substituting this expression into (2.37) and dividing by $a\alpha v^{\alpha-1}$, we obtain

$$S(v_0) = \frac{1}{a\alpha v_1^{\alpha-1}} \biggl(\log \int_{\mathbb{R}} \exp\bigl(a\alpha S(z) v_1^{\alpha-1} - a r_2 v_1^{\alpha-2} + b(v_1 - S(z))\bigr)\mu(\mathrm{d}z) + r_1 \biggr).$$

Letting $v_1 \to +\infty$, we obtain

$$S(v_0) = C \quad \text{for all } v_0 \in \mathbb{R}.$$

This contradicts the condition that S is non-constant and proves the strict positivity of the entropy production rate.

3 Exponential mixing and LDP

In this section, we prove Theorems 1.2 and 1.3. To this end, we show that the Markov family in question satisfies the four hypotheses of Proposition 5.4, so that the LDP holds in the space of trajectories. We next use an approximation argument to establish the LDP for functionals with moderate growth at infinity.

3.1 Proof of Theorem 1.2

Lyapunov function

Let us show that $\Phi(u)$ satisfies (5.17). Indeed, in view of (1.2), we have

$$\int_H \Phi(v) P_1(u, \mathrm{d}v) = \mathbb{E}\,\Phi(S(u) + \eta_1) \le q\,\Phi(u) + C\,\mathbb{E}\bigl(\Phi(\eta_1) + 1\bigr).$$

This inequality coincides with (5.17) in which $M = \mathbb{E}\,\Phi(\eta_1) + C$, and the finiteness of M follows from (1.3).

Uniform strong Feller

We first note that $P_1(u, \cdot) = \ell_{S(u)}$. By Condition (A), the mapping S is continuous from H to U, and by Condition (C), the mapping $\theta : U \to \mathcal{P}(H)$ is continuous from U to $\mathcal{P}(H)$. We see that the mapping $u \mapsto P_1(u, \cdot)$ is continuous as the composition of two continuous mappings.

Irreducibility

By condition (C), the support of ℓ coincides with H. Since the measure $P_1(u,\cdot)$ is a translation of ℓ, the same property holds for it, and we see that $P_1(u,G)>0$ for any non-empty open set $G\subset H$.

Super-exponential recurrence

Given $\varepsilon>0$, let us denote by $\mathcal{K}_\varepsilon$ a compact subset in H such that $\ell(\mathcal{K}_\varepsilon)>1-\varepsilon$. We claim that (5.18) and (5.19) hold for $\mathcal{C}=B_U(R)+\mathcal{K}_\varepsilon$ with $R\gg 1$ and $\varepsilon\ll 1$. The proof of this fact is divided into three steps.

Step 1. Let σ_ρ be the first hitting time of the set $\{u\in H:\Phi(u)\le\rho\}$, which is denoted by $\{\Phi\le\rho\}$ in what follows. Suppose that, for any $\beta>0$ and sufficiently large $\rho=\rho(\beta)>0$, we have proved the inequality

$$\mathbb{E}_u e^{\beta\sigma_\rho}\le C_1 e^{\alpha\Phi(u)-\alpha\rho+\beta}\quad\text{for any } u\in H, \tag{3.1}$$

where $\alpha=q\delta/C$, the numbers q, C, and δ are defined in Conditions (B) and (C), and $C_1>0$ is an absolute constant. In this case, the validity of (5.18) with the above choice of $\mathcal{C}$ can be derived by a standard argument (e.g., see Section 3.3.2 of [KS12]). Indeed, choosing R so large that $S(\{\Phi\le\rho\})\subset B_U(R)$, we see that

$$\inf_{u\in\{\Phi\le\rho\}} P_1(u,\mathcal{C})\ge 1-\varepsilon. \tag{3.2}$$

Let us introduce an increasing sequence of stopping times by the relations

$$\sigma_0'=\sigma_\rho,\quad \sigma_n'=\min\{k\ge\sigma_{n-1}'+1:\Phi(u_k)\le\rho\}.$$

Setting $\sigma_n=\sigma_n'+1$, we conclude from (3.2) and the strong Markov property that

$$P_u(m):=\mathbb{P}\Big(\bigcap_{n=1}^m\{u_{\sigma_n}\notin\mathcal{C}\}\Big)\le\varepsilon^m. \tag{3.3}$$

We shall show in Step 2 that

$$\mathbb{E}_u e^{\beta\sigma_m}\le Q_\beta^m e^{\alpha\Phi(u)-\alpha\rho}\quad\text{for } m\ge 0,\ u\in H, \tag{3.4}$$

where $Q_\beta>1$ depends only on β. Using (3.3), (3.4), and the Chebyshev inequality, for any positive integers m and M we write

$$\begin{aligned}\mathbb{P}_u\{\tau_{\mathcal{C}}\ge M\}&=\mathbb{P}_u\{\tau_{\mathcal{C}}\ge M,\sigma_m<M\}+\mathbb{P}_u\{\tau_{\mathcal{C}}\ge M,\sigma_m\ge M\}\\&\le\mathbb{P}_u\{\tau_{\mathcal{C}}>\sigma_m\}+\mathbb{P}_u\{\sigma_m\ge M\}\le P_u(m)+e^{-\beta M}\,\mathbb{E}_u e^{\beta\sigma_m}\\&\le\varepsilon^m+Q_\beta^m e^{-\beta M+\alpha\Phi(u)-\alpha\rho}.\end{aligned}$$

Choosing $m\sim\frac{M}{\log Q_\beta}$ and $\varepsilon\sim Q_\beta^{1-\beta}$, we derive

$$\mathbb{P}_u\{\tau_{\mathcal{C}}\ge M\}\le C_2\big(1+e^{\alpha\Phi(u)-\alpha\rho}\big)e^{-(\beta-1)M},$$

whence, for any $A < \beta - 1$, it follows that

$$\mathbb{E}_u e^{A\tau_C} \le C_3\big(1 + e^{\alpha\Phi(u)-\alpha\rho}\big). \tag{3.5}$$

Since $\beta > 0$ was arbitrary, we see that (5.18) holds with any $A > 0$ and a suitable compact set $\mathcal{C}(A) \subset H$. Moreover, taking if necessary a larger constant $C > 0$ in inequality (1.2), we can make α smaller than the number $c > 0$ entering (1.6). Then, integrating (3.5) with respect to $\lambda(\mathrm{d}u)$, we conclude that (5.19) is also satisfied.

Step 2. We now prove (3.4). To this end, we introduce the stopping time $\sigma'_\rho = \min\{k \ge 1 : \Phi(u_k) \le \rho\}$. In view of (3.1) and the Markov property, we have

$$\mathbb{E}_u e^{\beta\sigma'_\rho} \le C_4 e^{\alpha\Phi(u)-\alpha\rho+2\beta}.$$

Combining this inequality with the strong Markov property and the fact that $u_{\sigma'_n} \in \{\Phi \le \rho\}$, we derive

$$\begin{aligned}\mathbb{E}_u e^{\beta\sigma'_m} &= \mathbb{E}_u\big(\mathbb{E}_u\big\{e^{\beta\sigma'_m} \,|\, \mathcal{F}_{\sigma'_{m-1}}\big\}\big) = \mathbb{E}_u\big(e^{\beta\sigma'_{m-1}}\mathbb{E}_{u(\sigma'_{m-1})}e^{\beta\sigma'_\rho}\big)\\ &\le C_5(\beta)\mathbb{E}_u e^{\beta\sigma'_{m-1}},\end{aligned}$$

where $\mathcal{F}_\sigma$ denotes the σ-algebra associated with the stopping time σ, and we write $u(\sigma'_n)$ for $u_{\sigma'_n}$. Iterating the above inequality and using the definition of σ_m, we obtain the required estimate (3.4).

Step 3. It remains to prove (3.1). First note that, in view of the inequality $I_{\{\sigma_\rho>1\}} \le \exp(\delta'\Phi(u_1) - \delta'\rho)$, where $\delta' > 0$, and relations (1.2) and (1.3), we have

$$\begin{aligned}\mathbb{E}_u\big(e^{\alpha\Phi(u_1)}I_{\{\sigma_\rho>1\}}\big) &\le e^{-\delta'\rho}\,\mathbb{E}_u e^{(\alpha+\delta')\Phi(u_1)}\\ &\le e^{-\delta'\rho+(\alpha+\delta')(q\Phi(u)+C)}\mathfrak{m}_{C(\alpha+\delta')}(\ell).\end{aligned}$$

Choosing $\delta' = (1-q)\delta/C$ and recalling that $\alpha = q\delta/C$, we obtain

$$\mathbb{E}_u\big(e^{\alpha\Phi(u_1)}I_{\{\sigma_\rho>1\}}\big) \le C_2\mathfrak{m}_\delta(\ell)\, e^{\alpha\Phi(u)-\delta'\rho}. \tag{3.6}$$

We now introduce the quantities $p_k(u) = \mathbb{E}_u(e^{\alpha\Phi(u_k)}I_{\{\sigma_\rho>k\}})$. Combining (3.6) with the Markov property, we obtain

$$\begin{aligned}p_{k+1}(u) &= \mathbb{E}_u\big(e^{\alpha\Phi(u_{k+1})}I_{\{\sigma_\rho>k+1\}}\big) = \mathbb{E}_u\big\{I_{\{\sigma_\rho>k\}}\,\mathbb{E}_{u_k}\big(e^{\alpha\Phi(u_1)}I_{\{\sigma_\rho>1\}}\big)\big\}\\ &\le C_2\mathfrak{m}_\delta(\ell)\,\mathbb{E}_u\big(e^{\alpha\Phi(u_k)-\delta'\rho}I_{\{\sigma_\rho>k\}}\big) \le C_2\mathfrak{m}_\delta(\ell)e^{-\delta'\rho}p_k(u).\end{aligned}$$

Iterating this inequality and using (3.6), we get

$$\mathbb{E}_u\big(e^{\alpha\Phi(u_k)}I_{\{\sigma_\rho>k\}}\big) \le e^{\alpha\Phi(u)-(\delta'\rho-C_3)k} \le e^{\alpha\Phi(u)-(\beta+1)k},$$

where $C_3 = \log(C_2\mathfrak{m}_\delta(\ell))$ and $\rho > 0$ is so large that $\delta'\rho - C_3 \ge \beta + 1$. It follows that

$$\mathbb{P}_u\{\sigma_\rho > k\} \le e^{-\alpha\rho}\mathbb{E}_u\big(e^{\alpha\Phi(u_k)}I_{\{\sigma_\rho>k\}}\big) \le e^{\alpha\Phi(u)-\alpha\rho-(\beta+1)k}. \tag{3.7}$$

Inequality (3.1) with an arbitrary $\beta > 0$ and a sufficiently large $\rho = \rho(\beta) > 0$ is a simple consequence of (3.7).

3.2 Proof of Theorem 1.3

Step 1: Scheme of the proof of LDP. We shall derive the LDP for the laws of ξ_k as a consequence of Theorem 1.2. To this end, we essentially repeat the argument used by Gourcy [Gou07b, Gou07a] in the case of the Navier–Stokes and Burgers equations. It is based on Lemma 2.1.4 of [DS89], which implies that the LDP with the rate function[5]

$$I_f(r) := \inf\{\boldsymbol{I}(\boldsymbol{\nu}) : \langle f, \boldsymbol{\nu}\rangle = r\} \tag{3.8}$$

will be established for the $\mathbb{P}_\lambda$-laws of ξ_k if we prove the following two properties:

(a) Let $f_j = (f \wedge j) \vee (-j)$ and let $I_m : \mathcal{P}(H^{m+1}) \to [0, +\infty]$ be defined by

$$I_m(\nu) = \inf\{\boldsymbol{I}(\boldsymbol{\nu}) : \boldsymbol{\nu} \in \mathcal{P}(\boldsymbol{H}), \Pi_m \boldsymbol{\nu} = \nu\},$$

where $\Pi_m : \boldsymbol{H} \to H^{m+1}$ denotes the natural projection sending the vector $\boldsymbol{v} = (v_n, n \geq 0)$ to $(v_0, \dots, v_m)$ and $\boldsymbol{I}$ is the rate function constructed in Theorem 1.2. Then, for any $L > 0$, we have

$$\sup_\nu |\langle f_j - f, \nu\rangle| \to 0 \quad \text{as } j \to \infty, \tag{3.9}$$

where the supremum is taken over all $\nu \in \mathcal{P}(H^{m+1})$ such that $I_m(\nu) \leq L$.

(b) For any $\delta > 0$, we have

$$\limsup_{k\to+\infty} \frac{1}{k} \log \mathbb{P}_\lambda\big\{\big|\big\langle f_j - f, \zeta_k^{(m)}\big\rangle\big| > \delta\big\} \to -\infty \quad \text{as } j \to \infty, \tag{3.10}$$

where $\zeta_k^{(m)}$ denote the occupation measures

$$\zeta_k^{(m)} = \frac{1}{k} \sum_{n=0}^{k-1} \delta_{u_n(m)}, \quad u_n(m) = (u_n, \dots, u_{n+m}). \tag{3.11}$$

To prove (a), we shall need the following lemma, which gives a lower bound for I_m in terms of the stabilisable functional $\mathfrak{p}$. Its proof is given at the end of this section.

Lemma 3.1. *Let the hypotheses of Theorem 1.2 be fulfilled and let $\mathfrak{p}(u)$ be a stabilisable functional for $(u_k, \mathbb{P}_u)$. Then*

$$I_m(\nu) \geq \frac{1}{m+1} \int_{H^{m+1}} \sum_{n=0}^{m} \mathfrak{p}(v_n)\, \nu(\mathrm{d}v_0, \dots, \mathrm{d}v_m) - \gamma \quad \textit{for any } \nu \in \mathcal{P}(H^{m+1}). \tag{3.12}$$

[5] We may consider f as a measurable function on $\boldsymbol{H}$ depending only on the first $m+1$ components of the argument $\boldsymbol{v} = (v_n, n \geq 0)$, so that the integral $\langle f, \boldsymbol{\nu}\rangle$ makes sense.

Step 2: Proof of (a). We first note that, in view of (3.12), if $I_m(\nu) \le L$, then

$$\int_{H^{m+1}} \sum_{n=0}^{m} \mathfrak{p}(v_n)\, \nu(\mathrm{d}v_0, \dots, \mathrm{d}v_m) \le (m+1)(L+\gamma). \tag{3.13}$$

Furthermore, since f is bounded on the balls of H^{m+1}, we have

$$A_j := \big\{(v_0, \dots, v_m) \in H^{m+1} : |f(v_0, \dots, v_m)| \ge j\big\} \subset B_m(r_j)^c, \quad j \ge 1, \tag{3.14}$$

where $B_m(r)$ denotes the ball in H^{m+1} of radius r centred at zero and $\{r_j\} \subset \mathbb{R}_+$ is a sequence going to $+\infty$ with j. It follows from (1.8) that

$$\frac{|f(v_0, \dots, v_m)|}{\mathfrak{p}(v_0) + \dots + \mathfrak{p}(v_m)} \le \varepsilon_j \quad \text{for } (v_0, \dots, v_m) \in B_m(r_j)^c, \tag{3.15}$$

where $\varepsilon_j \to 0$ as $j \to \infty$. Combining (3.13)–(3.15), we write

$$\begin{aligned}
\int_{H^{m+1}} |f_j - f|\, \mathrm{d}\nu &\le \int_{A_j} |f|\, \mathrm{d}\nu \le \int_{B_m(r_j)^c} |f|\, \mathrm{d}\nu \\
&\le \varepsilon_j \int_{H^{m+1}} \sum_{n=0}^{m} \mathfrak{p}(v_n)\, \nu(\mathrm{d}v_0, \dots, \mathrm{d}v_m) \\
&\le \varepsilon_j (m+1)(L+\gamma).
\end{aligned}$$

This implies the required convergence (3.9).

Step 3: Proof of (b). Using (3.15), (1.7), and the Chebyshev inequality, we write

$$\begin{aligned}
\mathbb{P}_\lambda\big\{\big|\langle f_j - f, \zeta_k^{(m)}\rangle\big| > \delta\big\} &= \mathbb{P}_\lambda\bigg\{\bigg|\frac{1}{k}\sum_{n=0}^{k-1}(f - f_j)(u_n, \dots, u_{n+m})\bigg| > \delta\bigg\} \\
&\le \mathbb{P}_\lambda\bigg\{\sum_{n=0}^{k-1} |f(u_n, \dots, u_{n+m})|\, I_{A_j}(u_n, \dots, u_{n+m}) > \delta k\bigg\} \\
&\le \mathbb{P}_\lambda\bigg\{\sum_{n=0}^{k+m-1} \mathfrak{p}(u_n) > \frac{\delta}{(m+1)\varepsilon_j} k\bigg\} \\
&\le C_{m,\gamma} \exp\big\{-k\big(\tfrac{\delta}{(m+1)\varepsilon_j} - \gamma\big)\big\} \int_H e^{\mathfrak{p}(u)} Q(\|u\|)\, \lambda(\mathrm{d}u).
\end{aligned}$$

Since $\varepsilon_j \to 0$ as $j \to \infty$, this implies the required convergence (3.10). This completes the proof of Theorem 1.3.

Proof of Lemma 3.1. Let $\mathfrak{p}_j : H \to \mathbb{R}_+$ be an increasing sequence of bounded continuous functions such that $\mathfrak{p}_j(u) \to \mathfrak{p}(u)$ for any $u \in H$. For instance, we can take

$$\mathfrak{p}_j(u) = j \wedge \inf_{v \in H} (\mathfrak{p}(v) + j\, \|u - v\|).$$

By the Varadhan lemma (see [DZ00, Section 4.3]),

$$\lim_{k\to\infty}\frac{1}{k}\log\mathbb{E}_u\exp\bigl(\tfrac{k}{m+1}\langle\mathfrak{p}_j(v_0)+\dots+\mathfrak{p}_j(v_m),\zeta_k^{(m)}\rangle\bigr)$$
$$=\sup_{\nu\in\mathcal{P}(H^{m+1})}\biggl(\frac{1}{m+1}\int_{H^{m+1}}\sum_{n=0}^{m}\mathfrak{p}_j(v_n)\,\nu(\mathrm{d}v_0,\dots,\mathrm{d}v_m)-I_m(\nu)\biggr). \quad (3.16)$$

On the other hand, since $\mathfrak{p}_j\le\mathfrak{p}$ and $\mathfrak{p}$ is a stabilisable functional, in view of (1.7) we have

$$\mathbb{E}_u\exp\bigl(\tfrac{k}{m+1}\langle\mathfrak{p}_j(v_0)+\dots+\mathfrak{p}_j(v_m),\zeta_k^{(m)}\rangle\bigr)\le\mathbb{E}_u\exp\biggl(\sum_{n=0}^{k+m-1}\mathfrak{p}(u_n)\biggr)$$
$$\le Q(\|u\|)e^{\gamma(k+m-1)+\mathfrak{p}(u)}.$$

Substituting this inequality into (3.16), for any $\nu\in\mathcal{P}(H^{m+1})$ we obtain

$$\frac{1}{m+1}\int_{H^{m+1}}\sum_{n=0}^{m}\mathfrak{p}_j(v_n)\,\nu(\mathrm{d}v_0,\dots,\mathrm{d}v_m)-I_m(\nu)\le\gamma.$$

The required inequality (3.12) follows now from the Fatou lemma. □

4 Gallavotti–Cohen fluctuation theorem for the entropy production functional

In this section, we prove Theorem 1.5. To this end, we first note that, by Theorem 1.3, the $\mathbb{P}_\lambda$-laws of the random variables (0.9) satisfy the LDP with the good rate function (cf. (3.8))

$$I(r):=\inf\{\boldsymbol{I}(\boldsymbol{\nu}):\langle\sigma,\boldsymbol{\nu}\rangle=r\}. \quad (4.1)$$

We shall use the Varadhan lemma and a symmetry property of the Feynman–Kac semigroup to prove the Gallavotti–Cohen fluctuation principle.

For any $\alpha\in\mathbb{R}$ we define (formally) a family of linear mappings by the relation

$$(\mathfrak{P}_k^{\alpha\sigma}f)(u)=\mathbb{E}_u\biggl\{\exp\biggl(-\alpha\sum_{n=0}^{k-1}\sigma(u_n,u_{n+1})\biggr)f(u_k)\biggr\},\quad k\ge0,$$

where $f\in C(H)$. In view of Condition (D) and inequality (1.7), the function $\mathfrak{P}_k^{\alpha\sigma}f$ is continuous on H for any $f\in C_b(H)$. We claim that [6]

$$(\mathfrak{P}_k^{\alpha\sigma}f,g)_\ell=(f,\mathfrak{P}_k^{(1-\alpha)\sigma}g)_\ell\quad\text{for all } k\ge0, \quad (4.2)$$

[6] The right- and left-hand sides of (4.2) are well defined as integrals of positive functions. Relation (4.2) means, in particular, that if one of them is infinite, then so is the other.

where $f, g \in C(H)$ are arbitrary non-negative functions, and we set

$$(f,g)_\ell = \int_H f(u)g(u)\,\ell(\mathrm{d}u).$$

Indeed, the Markov property for $(u_k, \mathbb{P}_u)$ implies that, for any continuous function $f \geq 0$, we have

$$\mathfrak{P}^{\alpha\sigma}_{k+l} f = \mathfrak{P}^{\alpha\sigma}_k(\mathfrak{P}^{\alpha\sigma}_l f), \quad k,l \geq 0.$$

Therefore it suffices to prove (4.2) for $k = 1$. Using the definitions of σ and $\mathfrak{P}^{\alpha\sigma}_1$ and the Fubini theorem, we write

$$\begin{aligned}
(\mathfrak{P}^{\alpha\sigma}_1 f, g)_\ell &= \int_H \mathbb{E}_u\big\{e^{-\alpha\sigma(u_0,u_1)} f(u_1)\big\} g(u)\,\ell(\mathrm{d}u) \\
&= \int_H \left(\int_H e^{\alpha(\log\rho(v,u) - \log\rho(u,v))} f(v)\rho(u,v)\ell(\mathrm{d}v)\right) g(u)\ell(\mathrm{d}u) \\
&= \int_H f(v)\left(\int_H e^{(1-\alpha)(\log\rho(u,v) - \log\rho(v,u))} g(u)\rho(v,u)\ell(\mathrm{d}u)\right)\ell(\mathrm{d}v) \\
&= \int_H f(v)\mathbb{E}_v\big\{e^{-(1-\alpha)\sigma(u_0,u_1)} g(u_1)\big\}\ell(\mathrm{d}v) = (f, \mathfrak{P}^{(1-\alpha)\sigma}_1 g)_\ell.
\end{aligned}$$

We can now derive (0.13). Since the measure ℓ satisfies (1.9), the LDP with the good rate function (4.1) holds for the $\mathbb{P}_\ell$-laws of the sequence of random variables (0.9). Therefore, by the Varadhan lemma, we have

$$\lim_{k\to\infty} \frac{1}{k} \log \mathbb{E}_\ell \exp(-\alpha k \xi_k) = \sup_{r\in\mathbb{R}} \big(-\alpha r - I(r)\big) =: I^*(-\alpha), \tag{4.3}$$

where $\alpha \in \mathbb{R}$ is arbitrary, and I^* stands for the Legendre transform of I. Now note that

$$\mathbb{E}_\ell \exp(-\alpha k \xi_k) = \big(\mathfrak{P}^{\alpha\sigma}_k \mathbf{1}, \mathbf{1}\big)_\ell,$$

where $\mathbf{1} : H \to \mathbb{R}$ stands for the function identically equal to 1. Substituting this relation into (4.3) and using (4.2), for any $\alpha \in \mathbb{R}$ we derive

$$I^*(-\alpha) = \lim_{k\to\infty} \frac{1}{k} \log\big(\mathfrak{P}^{\alpha\sigma}_k \mathbf{1}, \mathbf{1}\big)_\ell = \lim_{k\to\infty} \frac{1}{k} \log\big(\mathfrak{P}^{(1-\alpha)\sigma}_k \mathbf{1}, \mathbf{1}\big)_\ell = I^*(\alpha - 1).$$

Combining this with the well-known relation $I(r) = \sup_{\alpha\in\mathbb{R}}(\alpha r - I^*(\alpha))$, we obtain (0.13). This completes the proof of Theorem 1.5.

5 Appendix

5.1 Admissible shifts of decomposable measures

Let H be a separable Hilbert space endowed with its Borel σ-algebra $\mathcal{B}_H$. Given $\mu \in \mathcal{P}(H)$ and $a \in H$, we denote by $\theta_a : H \to H$ the shift operator

by the vector a (that is, $\theta_a u = u + a$) and by $\mu_a = \mu \circ \theta_a^{-1}$ the image of μ under θ_a. Recall that $a \in H$ is called an *admissible shift* for μ if μ_a is absolutely continuous with respect to μ. We denote by H_μ the set of all admissible shifts for μ and by $\rho_\mu(a;u) = \frac{d\mu_a}{d\mu}$ the corresponding densities. It is straightforward to check that H_μ is an additive subgroup in H.

We shall say that μ is a *decomposable measure* if there is an orthonormal basis $\{e_j\}$ in H such that

$$\mu = \bigotimes_{j=1}^{\infty} \mu_j, \tag{5.1}$$

where $\mu_j = \mu \circ \mathsf{P}_j^{-1}$, and $\mathsf{P}_j : H \to H$ is the orthogonal projection to the straight line spanned by e_j. It is clear that if μ is a decomposable measure, then it is the law of a random variable of the form [7]

$$\eta = \sum_{j=1}^{\infty} \xi_j e_j, \tag{5.2}$$

where $\{\xi_j\}$ is a sequence of independent scalar random variables such that $\mathcal{D}(\xi_j) = \mu_j$. A proof of the following result can be found in [GS80] (see Theorem 5 in Section VII.2).

Proposition 5.1. *Let μ be a decomposable measure such that μ_j possesses a density ρ_j with respect to the Lebesgue measure on $\mathbb{R}$ for any $j \geq 1$. Then $a \in H_\mu$ if and only if the series*

$$\sum_{j=1}^{\infty} \bigl(\log \rho_j\bigl(\xi_j - (a, e_j)_H\bigr) - \log \rho_j(\xi_j)\bigr) \tag{5.3}$$

converges almost surely. In this case, the corresponding density is given by

$$\rho_\mu(a;u) = \exp\biggl(\sum_{j=1}^{\infty} \log \frac{\rho_j(u_j - a_j)}{\rho_j(u_j)}\biggr), \quad u \in H, \tag{5.4}$$

where we set $u_j = (u, e_j)_H$ and $a_j = (a, e_j)_H$.

Let us note that, in view of the Kolmogorov zero-one law, series (5.3) either converges a.s. or diverges a.s. In the latter case, the measures μ_a and μ are mutually singular. Furthermore, if $a \in H_\mu$, then $\mu \ll \mu_a$. What has been said implies that, under the hypotheses of the proposition, the subset $H_\mu \subset H$ is a group, and the measures μ_a and $\mu_{a'}$ with $a, a' \in H_\mu$ are absolutely continuous with respect to each other, with the corresponding density given by

$$\rho_\mu(a, a'; u) = \frac{d\mu_a}{d\mu_{a'}}(u) = \exp\biggl(\sum_{j=1}^{\infty} \log \frac{\rho_j(u_j - a_j)}{\rho_j(u_j - a'_j)}\biggr), \quad u \in H. \tag{5.5}$$

[7] For instance, one can take the random variables $\xi_j = (u, e_j)_H$ on the probability space $(H, \mathcal{B}_H, \mu)$.

We now wish to estimate the total variation distance between two admissible shifts of a decomposable measure. To this end, we assume that μ_j is the law of a random variable of the form $\xi_j = b_j\tilde{\xi}_j$, where $\{b_j\}$ is a sequence of positive numbers and $\tilde{\xi}_j$ is a random variable whose law is absolutely continuous with respect to the Lebesgue measure, and the corresponding density $\tilde{\rho}_j \in C^1$ is positive everywhere and satisfies the inequality

$$\mathrm{Var}(\tilde{\rho}_j) \le C \quad \text{for all } j \ge 1, \tag{5.6}$$

where $\mathrm{Var}(\cdot)$ denotes the total variation of a function and $C > 0$ does not depend on j.

Proposition 5.2. *Let μ be a decomposable measure satisfying the above hypotheses. Then for any $a, a' \in H_\mu$ we have*

$$\|\mu_a - \mu_{a'}\|_{\mathrm{var}} \le \frac{C}{2}\sum_{j=1}^{\infty}\frac{|a_j - a'_j|}{b_j}, \tag{5.7}$$

where C is the same constant as in (5.6).

Proof. Let us recall that

$$\|\mu_a - \mu_{a'}\|_{\mathrm{var}} = \frac{1}{2}\int_H |\rho_\mu(a;u) - \rho_\mu(a';u)|\,\mu(du). \tag{5.8}$$

In view of (5.4), we have

$$\rho_\mu(a;u) - \rho_\mu(a';u) = \sum_{k=1}^{\infty} D_k(a,a';u)\frac{\rho_k(u_k - a_k) - \rho_k(u_k - a'_k)}{\rho_k(u_k)},$$

where

$$D_k(a,a';u) = \exp\biggl(\sum_{j=1}^{k-1}\log\frac{\rho_j(u_j - a_j)}{\rho_j(u_j)} + \sum_{j=k+1}^{\infty}\log\frac{\rho_j(u_j - a'_j)}{\rho_j(u_j)}\biggr).$$

Substituting the above relation into (5.8), using decomposition (5.1), and recalling that $\mu_j(du_j) = \rho_j(u_j)\,du_j$, we obtain

$$\|\mu_a - \mu_{a'}\|_{\mathrm{var}} \le \frac{1}{2}\sum_{k=1}^{\infty}\int_{\mathbb{R}} |\rho_k(u_k - a_k) - \rho_k(u_k - a'_k)|du_k. \tag{5.9}$$

The mean value theorem implies that

$$\rho_k(u_k - a_k) - \rho_k(u_k - a'_k) = \int_0^1 \rho'_k\bigl(u_k - \theta a_k - (1-\theta)a'_k\bigr)d\theta\,(a_k - a'_k).$$

Combining this with (5.9) and the relation $\rho_k(r) = b_k^{-1}\tilde{\rho}_k(b_k^{-1}r)$, we obtain (5.7). □

Finally, we need some sufficient conditions ensuring the continuity of the shift operator and the positivity of density for shifted measures.

Proposition 5.3. *Let μ be the same as in Proposition 5.2 and let the densities $\tilde\rho_j$ satisfy the inequalities*

$$\int_{\mathbb{R}} |r|\tilde\rho_j(r)\,\mathrm{d}r \le C_1, \quad \int_{-A}^{A} \frac{|\tilde\rho_j'(y-r)|}{\tilde\rho_j(y-r)}\,\mathrm{d}r \le C_1(|y|+1)A \tag{5.10}$$

for $j \ge 1$, $y \in \mathbb{R}$, and $A \in [0,1]$. Let $U \subset H$ be a Banach space such that

$$\sum_{j=1}^{\infty} b_j^{-1}|(v,e_j)| \le C_2\|v\|_U \quad \text{for any } v \in U, \tag{5.11}$$

where $\{e_j\}$ is the orthonormal basis entering (5.1). Then the inclusion $U \subset H_\mu$ holds, and the density $\rho_\mu(a;u)$ is positive for $(a,u) \in U \times U$. Moreover, the function $\theta : U \to \mathcal{P}(H)$ taking $a \in U$ to μ_a is Lipschitz continuous, provided that $\mathcal{P}(H)$ is endowed with the total variation norm. Finally, if there is $C_3 > 0$ such that

$$\sum_{j=1}^{\infty} b_j^{-2}|(v,e_j)| \le C_3\|v\|_U \quad \text{for any } v \in U, \tag{5.12}$$

then the density $\rho_\mu(a;u)$ is positive on $U \times H$.

Proof. If we know that $a, a' \in H_\mu$, then combining (5.7) and (5.11), we obtain immediately the Lipschitz continuity of the mapping θ. Thus, we need to show the inclusion $U \subset H_\mu$ and the positivity of ρ_μ on $U \times U$ (and on $U \times H$ under the additional condition (5.12)).

In view of Proposition 5.1, the required inclusion will be established if we prove that

$$\sum_{j=1}^{\infty} \int_H \bigl|\log\rho_j(u_j - a_j) - \log\rho_j(u_j)\bigr|\,\mu(du) < \infty \quad \text{for any } a \in U. \tag{5.13}$$

To prove this, note that, in view of the second inequality in (5.10), we have

$$\begin{aligned}\bigl|\log\rho_j(u_j - a_j) - \log\rho_j(u_j)\bigr| &\le \int_{-\theta_j}^{\theta_j} \frac{|\tilde\rho_j'(b_j^{-1}u_j - r)|}{\tilde\rho_j(b_j^{-1}u_j - r)}\,\mathrm{d}r \\ &\le C_1(b_j^{-1}|u_j| + 1)\theta_j \quad \text{for } j \ge N,\end{aligned} \tag{5.14}$$

where $\theta_j = b_j^{-1}|a_j|$ and $N \ge 1$ is the least integer such that $\theta_j \le 1$. Using the Fubini theorem and decomposition (5.1), we obtain

$$\begin{aligned}\sum_{j=N}^{\infty} \int_H \bigl|\log\rho_j(u_j - a_j) - \log\rho_j(u_j)\bigr|\,\mu(du) &\le C_1 \sum_{j=N}^{\infty} \theta_j \int_{\mathbb{R}} (b_j^{-1}|u_j| + 1)\rho_j(u_j)\,\mathrm{d}u_j \\ &= C_1 \sum_{j=N}^{\infty} \theta_j \int_{\mathbb{R}} (|v_j| + 1)\tilde\rho_j(v_j)\,\mathrm{d}v_j.\end{aligned}$$

The first inequality in (5.10) and inequality (5.11) with $v = a$ now imply that (5.13) holds.

To establish the positivity of the density $\rho_\mu(a;u)$ on $U \times U$, recall that it is given by (5.4). Therefore it suffices to show that

$$\Delta_N(a;u) := \sum_{j=N}^{\infty} \bigl|\log \rho_j(u_j - a_j) - \log \rho_j(u_j)\bigr| < \infty \tag{5.15}$$

for $a, u \in U$. To this end, note that, by (5.14) and (5.11), we have

$$\Delta_N(a;u) \le C_1 \sum_{j=N}^{\infty} \bigl(b_j^{-2}|u_j|\,|a_j| + b_j^{-1}|a_j|\bigr) \le C_1 C_2 \|a\|_U \bigl(\|u\|_U + 1\bigr). \tag{5.16}$$

Finally, to establish the positivity of $\rho_\mu(a;u)$ on $U \times H$ under the additional condition (5.12), it suffices to prove that (5.15) holds for $a \in U$ and $u \in H$. This follows immediately from the first inequality in (5.16). □

5.2 Exponential mixing and LDP for Markov chains

Let X be a separable Banach space with a norm $\|\cdot\|$ and let $(u_k, \mathbb{P}_u)$ be a family of Markov chains in X. Given $\lambda \in \mathcal{P}(X)$, we define the probability measure $\mathbb{P}_\lambda(\cdot) = \int_X \mathbb{P}_u(\cdot)\lambda(\mathrm{d}u)$ and denote by $\mathbb{E}_\lambda$ the corresponding mean value. Recall that we denote by $P_k(u,\Gamma)$ the transition function for $(u_k, \mathbb{P}_u)$ and by $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ the corresponding Markov semigroups. Given a closed subset $K \subset X$, let τ_K be the first positive hitting time of K:

$$\tau_K = \min\{k \ge 1 : u_k \in K\}.$$

The following proposition is a consequence of general results on mixing and LDP established in [MT93, Chapters 15 and 16] and [Wu01] (see Theorem 2.1 and Proposition A.2).

Proposition 5.4. *Let a Markov process $(u_k, \mathbb{P}_u)$ and a subset $\Lambda \subset \mathcal{P}(X)$ be such that the following hypotheses hold.*

Lyapunov function. *There is a continuous function $\Phi : X \to \mathbb{R}_+$ which is bounded on any ball of X and goes to $+\infty$ as $\|u\| \to \infty$ such that*

$$\int_X \Phi(v) P_1(u, \mathrm{d}v) \le q\,\Phi(u) + M \quad \text{for all } u \in X, \tag{5.17}$$

where $q < 1$ and M are some positive constants.

Uniform strong Feller. *The mapping $u \mapsto P_1(u,\cdot)$ is continuous from X to the space $\mathcal{P}(X)$ endowed with the total variation norm.*

Irreducibility. *We have $P_1(u, G) > 0$ for any $u \in X$ and any non-empty open set $G \subset X$.*

Hyper-exponential recurrence. *For any $A > 0$ there is a compact subset $\mathcal{C} = \mathcal{C}(A) \subset X$ such that*

$$\sup_{u \in B} \mathbb{E}_u \exp(A\tau_{\mathcal{C}}) < \infty, \tag{5.18}$$

$$\sup_{\lambda \in \Lambda} \mathbb{E}_\lambda \exp\bigl(A\tau_{\mathcal{C}}\bigr) < \infty, \tag{5.19}$$

where $B \subset X$ is an arbitrary ball in X. Then $(u_k, \mathbb{P}_u)$ possesses a unique stationary measure $\mu \in \mathcal{P}(X)$, which is exponentially mixing in the sense that

$$\|P_k(u, \cdot) - \mu\|_{\mathrm{var}} \le Ce^{-\gamma k}\bigl(1 + \Phi(u)\bigr) \quad \textit{for all } u \in X \textit{ and } k \ge 0, \tag{5.20}$$

where C and γ are positive constants. Moreover, the LDP in the τ_p-topology holds for the $\mathbb{P}_\lambda$-occupation measures (1.4), uniformly with respect to $\lambda \in \Lambda$.

This result implies, in particular, that for any ball $B \subset X$ the LDP holds for the $\mathbb{P}_u$-occupation measures (1.4) uniformly with respect to $u \in B$.

5.3 The entropy balance equation

In this section, we consider the entropy balance equation stated in the introduction. We prove that the entropy production functional $\mathrm{Ep}(\,\cdot\,)$ is non-negative and vanishes if and only if the detailed balance condition is satisfied. We also show how the entropy flux observable σ relates to time-irreversibility.

Let us set $f = \mathrm{d}\lambda/\mathrm{d}\ell$ and define the density transfer operator by

$$(\mathcal{R}f)(v) = \int f(u)\rho(u, v)\ell(\mathrm{d}u).$$

A direct calculation shows that (0.7) holds with

$$\mathrm{Ep}(\lambda) = \int_{\boldsymbol{H}} -\log\left(\frac{(\mathcal{R}f)(u_1)\rho(u_1, u_0)}{f(u_0)\rho(u_0, u_1)}\right)\boldsymbol{\lambda}(\mathrm{d}\boldsymbol{u}). \tag{5.21}$$

Jensen's inequality yields

$$\begin{aligned}
\mathrm{Ep}(\lambda) &\ge -\log\int \frac{(\mathcal{R}f)(u_1)\rho(u_1, u_0)}{f(u_0)\rho(u_0, u_1)}\boldsymbol{\lambda}(\mathrm{d}\boldsymbol{u}) \\
&= -\log\int f(u)\rho(u, v)\frac{(\mathcal{R}f)(v)\rho(v, u)}{f(u)\rho(u, v)}\ell(\mathrm{d}u)\ell(\mathrm{d}v) \\
&= -\log\int (\mathcal{R}^2 f)(u)\ell(\mathrm{d}u) = 0.
\end{aligned}$$

Moreover, this inequality is saturated if and only if

$$(\mathcal{R}f)(v)\rho(v, u) = cf(u)\rho(u, v),$$

$\ell \otimes \ell$-almost everywhere for some constant c. Integrating this relation over $\ell(\mathrm{d}v)$ yields $\mathcal{R}^2 f = cf$ and $c = 1$ follows from one more integration over ℓ. We deduce

that $\mathcal{R}f - f$ is either 0 or an eigenvector of $\mathcal{R}$ to the eigenvalue -1. Since the second alternative contradicts the mixing property of the stationary measure μ, we conclude that $\mathcal{R}f = f$, so f is the density of a stationary measure. Inserting this relation into Eq. (5.21) yields

$$\begin{aligned} 0 = \mathrm{Ep}(\lambda) &= \int f(u)\rho(u,v) \log\left(\frac{f(u)\rho(u,v)}{f(v)\rho(v,u)}\right) \ell(\mathrm{d}u)\ell(\mathrm{d}v) \\ &= \frac{1}{2}\int (f(u)\rho(u,v) - f(v)\rho(v,u)) \log\left(\frac{f(u)\rho(u,v)}{f(v)\rho(v,u)}\right) \ell(\mathrm{d}u)\ell(\mathrm{d}v), \end{aligned}$$

which clearly implies the detailed balance relation.

To connect the observables J and σ with time reversal of the stationary Markov chain, we follow Maes and Netočný [MN03]. Denote by $\mathbb{P}_\mu^{(k)}$ the measure induced by μ on the finite segment $(u_0, \dots, u_k)$ of the Markov chain and by $\pi_k : (u_0, \dots, u_k) \mapsto (u_k, \dots, u_0)$ the time-reversal map on this segment. The relative entropy $\mathrm{Ent}(\mathbb{P}_\mu^{(k)} | \mathbb{P}_\mu^{(k)} \circ \pi_k^{-1})$ is given by

$$\begin{aligned} &- \int \log\left(\frac{\mathrm{d}\mathbb{P}_\mu^{(k)}}{\mathrm{d}\mathbb{P}_\mu^{(k)} \circ \pi_k^{-1}}\right) \mathrm{d}\mathbb{P}_\mu^{(k)} \\ &= - \int \log\left(\frac{\rho(u_0)\rho(u_0,u_1)\cdots\rho(u_{k-1},u_k)}{\rho(u_k)\rho(u_k,u_{k-1})\cdots\rho(u_1,u_0)}\right) \mathbb{P}_\mu^{(k)}(\mathrm{d}u_0, \dots, \mathrm{d}u_k) \\ &= - \int \sum_{n=0}^{k-1} J \circ \phi^n(\boldsymbol{u}) \boldsymbol{\mu}(\mathrm{d}\boldsymbol{u}), \end{aligned}$$

where ϕ denotes the left shift. Thus, we have

$$-\frac{1}{k}\,\mathrm{Ent}(\mathbb{P}_\mu^{(k)} | \mathbb{P}_\mu^{(k)} \circ \pi_k^{-1}) = \langle \sigma \rangle_\mu, \tag{5.22}$$

which provides an alternative proof of the inequality $\langle \sigma \rangle_\mu \geq 0$. As noticed by Gaspard [Ga04], the ergodicity of $\boldsymbol{\mu}$ implies that

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{k} \log\left(\rho(u_0)\rho(u_0,u_1)\cdots\rho(u_{k-1},u_k)\right) \\ &\qquad = \int \rho(u)\rho(u,v) \log \rho(u,v) \ell(\mathrm{d}u)\ell(\mathrm{d}v) = h_+, \\ &\lim_{k\to\infty} \frac{1}{k} \log\left(\rho(u_k)\rho(u_k,u_{k-1})\cdots\rho(u_1,u_0)\right) \\ &\qquad = \int \rho(u)\rho(u,v) \log \rho(v,u) \ell(\mathrm{d}u)\ell(\mathrm{d}v) = h_-, \end{aligned}$$

for $\boldsymbol{\mu}$-almost every $\boldsymbol{u} \in \boldsymbol{H}$, where h_+ is the entropy per unit time (or entropy rate, or Kolmogorov–Sinai entropy) of the stationary Markov chain and h_- is the time-reversed entropy per unit time. Thus, relation (5.22) can be strengthened to

$$-\lim_{k\to\infty} \frac{1}{k} \log\left(\frac{\mathrm{d}\mathbb{P}_\mu^{(k)}}{\mathrm{d}\mathbb{P}_\mu^{(k)} \circ \pi_k^{-1}}(\boldsymbol{u})\right) = h_- - h_+ = \langle \sigma \rangle_\mu,$$

and the strict positivity of the entropy production rate translates into $h_- > h_+$.

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