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# Riemann problem for a particle-fluid coupling

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## Abstract

We present a model of coupling between a point wise particle and a compressible inviscid fluid following the Euler equations. The interaction between the fluid and the particle is achieved through a drag force. It writes as the product of a discontinuous function and a Dirac measure. After defining the solution, we solve the Riemann problem with a fixed particle for arbitrary data. We exhibit a set of condition on the drag force under which there exists a unique self-similar solution.

**Key words and phrases:** Fluid-particle interaction; Euler equations; Riemann problem; nonconservative product.

**2010 Mathematics Subject Classification:** primary 35L65, 76N10; secondary 35L67, 35R06.

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## 1 Introduction

In this paper we introduce and study a simple one-dimensional model of fluid-structure interaction. We consider a compressible and inviscid fluid, which will be governed by the isothermal Euler equations. At point  $x$  and time  $t$ , it has velocity  $u(t, x)$  and density  $\rho(t, x)$ . The particle is assumed to be pointwise, of mass  $m$ , and we denote by  $h(t)$ ,  $h'(t)$  and  $h''(t)$  its position, velocity and acceleration at time  $t$ . The interaction between the fluid and the particle is achieved through a drag force  $D$ , reflecting the fact that both tend to share the same velocity. Our model writes

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, & (1a) \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho)) = -D(\rho, u - h'(t))\delta_{h(t)}(x), & (1b) \\ mh''(t) = D(\rho(t, h(t)), u(t, h(t)) - h'(t)), & (1c) \\ (\rho(0, x), u(0, x)) = (\rho_0(x), u_0(x)), \\ (h(0), h'(0)) = (0, v_0), \end{cases}$$

where  $p$  is the pressure law. We suppose that  $D$  has the same sign as  $u - h'$ , which is natural for a drag force. As a result, Equation (1c) shows formally that the particle accelerates (respectively decelerates) when the fluid's velocity  $u(t, h(t))$  at its position is larger (respectively smaller) than its own velocity  $h'(t)$ . We choose the isothermal pressure law  $p(\rho) = c^2 \rho$  to avoid vacuum related issues, but the results could probably be extended without major difficulty in the adiabatic case  $p(\rho) = \rho^\gamma, 1 < \gamma \leq 3$ . This model is a generalization of the one first introduced in [LST08], and later deeply investigate in [AS12], [ALST10] and [ALST13], in which the fluid followed a scalar Burgers equation.

In the past few years, the interaction between an incompressible and viscous fluid and rigid bodies has been widely studied. Many papers deal with the existence of weak or strong solutions ([SMST02], [GLS00], [Fei03b], [DE99] and [VZ03]). The matter of collision has also been extensively investigated ([VZ06], [Sta04], [Hil07], [HT09]). Some other works consider compressible fluid or elastic bodies ([BST12], [CDEG05], [DE00] and [Fei03a]). The model we study strongly differs from those works as the equation governing the fluid is inviscid. In this framework, the d'Alembert's paradox states that in an incompressible and inviscid fluid, no drag force exerts on a body moving at a constant speed. Birds should therefore not be able to fly. An answer to this paradox is that, even at very high Reynolds number, the effect of viscosity cannot be neglected in a thin layer around the body. In our model, this paradox is somehow ignored, as we directly impose a drag force  $D$  between the fluid and the particle. According to Newton's law, the particle follows the ordinary differential equation (1c)

$$mh''(t) = D.$$

The action-reaction principle is taken into account in equation (1b) on the momentum of the Euler equation: the particle applies the force  $-D$  on the fluid. We suppose that the interaction is local: it applies only at point  $h(t)$ . Equation (1a) ensures that the fluid mass is conserved. This approach proved to be successful in the toy model [LST08] with a Burgers' fluid. In particular, it allows collisions between two particles having different velocities, unlike in the viscous case [VZ06], and we are able to recover the drafting kissing tumbling phenomenon. This work is strongly related to [BCG14], in which a coupling between a particle and the Euler equations is presented. However, the modelization is quite different, the particle being taken into account through conservation of mass and energy, while we enforce in the present work a drag force. Moreover, the nature of the theoretical results are different and complementary. In [BCG14], a local in time existence to the Cauchy problem for the fully coupled system is proved for small subsonic data. In the present work, we consider the Riemann problem for a particle having constant velocity, without making any assumption on the data.

We emphasize the fact that in model (1), the particle and the fluid do not share the same velocity. We do not impose any no slip condition as in works presented above. It can be either justified by saying that this condition is no longer relevant in an inviscid fluid, or by considering that the particle is porous and allows some fluid to pass through. It constitutes the main difficulty of this model. Indeed, as shocks develop in finite time in hyperbolic systems like (1a)-(1b) even with  $D = 0$ , the velocity  $u$  and the density  $\rho$  of the fluid have no reason to be continuous along the particle trajectory  $h$ . A first consequence is that the source term in (1b) is not well defined. A second one is that the ODE (1c) the particle satisfies must be considered in a weak sense. This paper focuses on the Riemann problem for the uncoupled problem

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0 \\ \partial_t (\rho u) + \partial_x (\rho u^2 + p(\rho)) = -D(\rho, u - h'(t)) \delta_{vt} \\ \rho(0, x) = \rho_L \mathbf{1}_{x < 0} + \rho_R \mathbf{1}_{x > 0} \\ u(0, x) = u_L \mathbf{1}_{x < 0} + u_R \mathbf{1}_{x > 0} \end{cases} \quad (2)$$

where the particle has a constant speed  $v$ . The difficulties arising from the coupling between an ODE and a PDE disappear, but the key point of the nonconservative source term remains. Let us outline the organization of the paper and sketch the main results. The first section is entirely devoted to the definition of the solution of (1). We exhibit an entropy condition that takes into account the particle. Then we focus on the case of a "linear" drag force  $D(\rho, u - h') = \lambda \rho (u - h')$ , where  $\lambda$  is a non-negative coefficient, and give a rigorous definition of the nonconservative product

$$\lambda \rho (u - h'(t)) \delta_{h(t)}.$$

The particle is seen as an interface, where two conditions are imposed. They link the quantities  $u(t, h(t)^\pm)$ ,  $\rho(t, h(t)^\pm)$  and  $h'(t)$ . For a motionless particle, those conditions express that the fluid mass is conserved through the particle, and that the loss of charge  $\rho u^2 + c^2 \rho$  through the particle is proportional to the mass flow  $\rho u$ . The second section is devoted to the solution of the Riemann problem for a particle moving at constant speed  $v$ . We state in Theorems 3.9 and 3.11 that the system (2) has a unique self-similar entropy solution. We exhibit two different types of solutions and recover the natural asymptotics when  $\lambda$  tends to infinity (solid wall with no slip condition) and  $\lambda$  tends to 0 (lack of particle). The main difficulty is that the Dirac measure in the source term corresponds to a linearly degenerate field and our system is not strictly hyperbolic. This may lead to resonance phenomena when two families of waves interact. Near resonance, Riemann problems with such source terms have been investigated in a conservative framework in [IT92] and later extended in the nonconservative framework in [GL04]. Away from resonance, the particle trajectory can be treated as a noncharacteristic boundary (see [BCG10], [BCG12] and [BCG14]). Our contribution is that, unlike in those frameworks, we solve the Riemann problem for all choices of parameters  $(\rho_L, u_L)$ ,  $(\rho_R, u_R)$  and  $v$ , without making any assumptions on their smallness or their resonant character. In the third section we extend this result for other type of frictions. In Theorem 4.5, we exhibit a sufficient condition on the form of the drag force to keep uniqueness, namely that the  $|D|$  should be an decreasing function of  $\rho$  and an increasing function of  $\rho(u - h')$ . We also recover, in some particular cases, the existence of up to three solutions, as shown in a general framework in [IT92] and [GL04], and well known for fluid in a nozzle with discontinuous cross-section [LT07] and for the shallow water equation with discontinuous decreasing topography [LT03].

## 2 Definition of the solutions

This section is devoted to the definition of the solution of the coupled system (1). The isothermal Euler equations are inviscid, so  $\rho$  and  $u$  can be discontinuous along the particle's trajectory  $h$  and the product  $D(\rho, (h' - u))\delta_h(x)$  does not make sense. Following [LST08], we consider two different regularizations in Sections 2.1 and 2.2. The first one consists in adding a vanishing viscosity to the equation. Passing to the non-regularized limit we deduce an entropy inequality for (1). The second regularization is a thickening of the particle. We focus on the linear drag force  $D(\rho, u - h') = \lambda\rho(u - h')$ , where  $\lambda$  is a non-negative friction coefficient, and give a precise definition of the non-conservative product when  $\lambda\rho(u - h')\delta_{h(t)}$ . Discussion for other drag forces is trickier and postponed to Section 4.

### 2.1 Entropy inequality

Let us first focus on the following classical regularization of problem (1), where we add a vanishing viscosity to the Euler equation:

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x q^\varepsilon = \varepsilon \partial_{xx} \rho^\varepsilon, \\ \partial_t q + \partial_x \left( \frac{(q^\varepsilon)^2}{\rho^\varepsilon} + c^2 \rho^\varepsilon \right) = -D(\rho^\varepsilon, u^\varepsilon - h'(t))\delta_{h(t)} + \varepsilon \partial_{xx} q^\varepsilon, \\ mh''(t) = D(\rho^\varepsilon(t, h(t)), u^\varepsilon(t, h(t)) - h'(t)), \\ (\rho^\varepsilon(0, x), q^\varepsilon(0, x)) = (\rho_0^\varepsilon(x), q_0^\varepsilon(x)), \\ (h(0), h'(0)) = (h_0, v_0). \end{cases} \quad (3)$$

Here,  $q = \rho u$  denotes the momentum of the fluid, and we only assume that the drag force  $D$  has the same sign than  $u - h'$ . In [LST08] and [Dom02] it is proven that the system

$$\begin{cases} \partial_t u^\varepsilon + \partial_x \frac{(u^\varepsilon)^2}{2} = \varepsilon \partial_{xx} \rho^\varepsilon - \lambda(u^\varepsilon - h')\delta_{h(t)}(x) \\ mh''(t) = \lambda(u^\varepsilon - h'), \\ u^\varepsilon(0, x) = u^0(x), \\ (h(0), h'(0)) = (h_0, v_0). \end{cases}$$

admits regular solution when  $u^0$  is regular. Such a result is much more difficult to obtain in the system case (3), and we did not attempt to prove it. However, existence and regularity results for the parabolic regularization of the Euler equation ([Che86]) and for the Navier Stokes regularization ([Hof98] and [Lio98]) exist. In order to derive an entropy inequality for our fluid particle coupling (1), we assume that if the initial data  $(\rho^0, u^0)$  are smooth, the solutions of its regularization (3) are also

smooth. Let  $E(\rho, q)$  and  $G(\rho, q)$  be a flux-entropy flux pair, with  $E$  convex, such that  $\partial_q E$  is a function of  $u = \frac{q}{\rho}$  denoted by  $J$ .

**Example 2.1.** The usual entropy-entropy flux pair

$$E(\rho, q) = \frac{q^2}{2\rho} + c^2 \rho \log(\rho) \quad \text{and} \quad G(\rho, q) = \frac{q}{\rho}(E(\rho, q) + c^2 \rho)$$

fulfills this assumption: we have  $J(u) = u$ .

For the sake of simplicity we introduce

$$U_0^\varepsilon = (\rho_0^\varepsilon, q_0^\varepsilon), \quad U^\varepsilon = (\rho^\varepsilon, q^\varepsilon) \quad \text{and} \quad F(U^\varepsilon) = \left( q^\varepsilon, \frac{(q^\varepsilon)^2}{\rho^\varepsilon} + c^2 \rho^\varepsilon \right).$$

Let  $\Phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$  be a *non-negative* smooth function, and multiply the first equation of (3) by  $\Phi \partial_\rho E$  and the second equation by  $\Phi \partial_q E = \Phi J$ . Let us add the two equations and integrate over  $\mathbb{R}_+ \times \mathbb{R}$ . We obtain

$$\begin{aligned} & \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi \nabla_{(\rho, q)} E(U^\varepsilon) \cdot \partial_t U^\varepsilon \, dt \, dx + \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi \nabla_{(\rho, q)} E(U^\varepsilon) \cdot \partial_x F(U^\varepsilon) \, dt \, dx \\ & - \varepsilon \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi \nabla_{(\rho, q)} E(U^\varepsilon) \cdot \Delta U^\varepsilon \, dt \, dx = - \int_{\mathbb{R}_+} [D(\rho^\varepsilon, u^\varepsilon - h'(t)) \Phi J(u^\varepsilon)](t, h(t)) \, dt. \end{aligned} \quad (4)$$

We first treat the left hand side of (4) by integrating by parts. The first term gives

$$\begin{aligned} \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi \nabla_{(\rho, q)} E(U^\varepsilon) \cdot \partial_t U^\varepsilon \, dt \, dx &= \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi \partial_t E(U^\varepsilon) \, dt \, dx \\ &= - \iint_{\mathbb{R}_+ \times \mathbb{R}} E(U^\varepsilon) \partial_t \Phi \, dt \, dx - \int_{\mathbb{R}} \Phi(0, \cdot) E(U_0^\varepsilon) \, dx \end{aligned}$$

and the second term yields

$$\begin{aligned} \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi \nabla_{(\rho, q)} E(U^\varepsilon) \cdot \partial_x F(U^\varepsilon) \, dt \, dx &= \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi \nabla_{(\rho, q)} E(U^\varepsilon) \cdot [DF(U^\varepsilon) \partial_x U^\varepsilon] \, dt \, dx \\ &= \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi \nabla_{(\rho, q)} G(U^\varepsilon) \cdot \partial_x U^\varepsilon \, dt \, dx \\ &= \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi \partial_x G(U^\varepsilon) \, dt \, dx \\ &= - \iint_{\mathbb{R}_+ \times \mathbb{R}} G(U^\varepsilon) \partial_x \Phi \, dt \, dx. \end{aligned}$$

Let us now tackle the third term. We have

$$\begin{aligned} \nabla_{(\rho, q)} E(U^\varepsilon) \cdot \Delta U^\varepsilon &= \sum_{i=1}^2 \partial_i E(U^\varepsilon) \partial_{xx} U_i^\varepsilon \\ &= \sum_{i=1}^2 \left[ \partial_x (\partial_i E(U^\varepsilon) \partial_x U_i^\varepsilon) - \left( \sum_{j=1}^2 \partial_{ji} E(U^\varepsilon) \partial_x U_j^\varepsilon \right) \partial_x U_i^\varepsilon \right] \\ &= \partial_{xx} E(U^\varepsilon) - \sum_{i=1}^2 \sum_{j=1}^2 (\partial_x U_j^\varepsilon) (\partial_{ji} E(U^\varepsilon)) (\partial_x U_i^\varepsilon), \end{aligned}$$

thus we obtain

$$\begin{aligned} -\varepsilon \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi \nabla_{(\rho, q)} E(U^\varepsilon) \cdot \Delta U^\varepsilon \, dt \, dx &= -\varepsilon \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi E(U^\varepsilon) \partial_{xx} \Phi \, dt \, dx \\ &\quad + \varepsilon \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi \sum_{i=1}^2 \sum_{j=1}^2 (\partial_x U_j^\varepsilon) \partial_{ij} E(U^\varepsilon) (\partial_x U_i^\varepsilon) \, dt \, dx. \end{aligned}$$

Remark that as  $\Phi$  is non-negative and  $E$  is convex, the last term is non-negative. We now focus on the right hand side of (4). Let us multiply the ODE in the third equation of (3) by  $J(h'(t))\Phi(t, h(t))$ . We have

$$\int_{\mathbb{R}_+} -mh''(t)J(h'(t))\Phi(t, h(t)) + D(\rho^\varepsilon, u^\varepsilon - h'(t))(t, h(t))J(h'(t))\Phi(t, h(t))dt = 0,$$

which reads, with  $P$  an antiderivative of  $J$ ,

$$\int_{\mathbb{R}_+} -m[P(h'(t))]' \Phi(t, h(t)) + D(\rho^\varepsilon, u^\varepsilon - h'(t))(t, h(t))J(h'(t))\Phi(t, h(t))dt = 0.$$

We can therefore replace the right hand side of (4) by

$$\int_{\mathbb{R}_+} mP(h'(t))\partial_t(\Phi(t, h(t)))dt + \int_{\mathbb{R}_+} D(\rho^\varepsilon, u^\varepsilon - h'(t))(J(h'(t)) - J(u^\varepsilon))\Phi(h)dt + mP(v_0)\Phi(0, h_0).$$

Remark that  $J(u) = \partial_q E(1, u)$  is nondecreasing, as the restriction of  $E$  to the line  $\rho = 1$  is convex, and that  $D$  has the same sign as  $u^\varepsilon - h'$ . Thus the second term is non-positive. To conclude, we add the different terms and drop the two non-positive ones to obtain

$$\begin{aligned} & \iint_{\mathbb{R}_+ \times \mathbb{R}} E(U^\varepsilon)\partial_t \Phi dt dx + \iint_{\mathbb{R}_+ \times \mathbb{R}} G(U^\varepsilon)\partial_x \Phi dt dx + \int_{\mathbb{R}_+} mP(h'(t))\partial_t(\Phi(t, h(t)))dt \\ & + \int_{\mathbb{R}} \Phi(0, \cdot)E(U_0^\varepsilon) dx + mP(v_0)\Phi(0, h_0) \geq -\varepsilon \iint_{\mathbb{R}_+ \times \mathbb{R}} \Phi E(U^\varepsilon)\partial_{xx} \Phi dt dx. \end{aligned}$$

Last, we formally pass to the limit as  $\varepsilon \rightarrow 0$  and get the following entropy inequality for the coupled problem (1):

$$\begin{aligned} & \iint_{\mathbb{R}_+ \times \mathbb{R}} E(U)\partial_t \Phi dt dx + \iint_{\mathbb{R}_+ \times \mathbb{R}} G(U)\partial_x \Phi dt dx + \int_{\mathbb{R}_+} mP(h'(t))\partial_t(\Phi(t, h(t)))dt \\ & + \int_{\mathbb{R}} \Phi(0, \cdot)E(U_0) dx + mP(v_0)\Phi(0, h_0) \geq 0. \end{aligned} \quad (5)$$

*Remark 2.2.* When the test function  $\Phi$  is supported on  $\{(t, x), x > h(t)\}$  or on  $\{(t, x), x < h(t)\}$ , the inequality (5) reduces to the classical entropy inequality for the isothermal Euler equation without source term.

## 2.2 How to handle the nonconservative product

In this section we assume that the particle trajectory  $h$  is given, and more precisely that it moves at constant speed  $v$ :  $h(t) = vt$ . We consider the “linear” drag force  $D(\rho, u - h') = \lambda\rho(u - h')$ , where  $\lambda$  is a non-negative parameter. We denote by  $H$  the Heaviside function  $H(x) = \mathbf{1}_{x>0}$ . Introducing the momentum  $q = \rho u$  and the new unknown  $w(t, x) = H(x - vt)$  allows us to rewrite the system (1a)-(1b) in the framework of hyperbolic equation,

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{\rho} + c^2 \rho \right) + \lambda(q - \rho v)\partial_x w = 0, \\ \partial_t w + v\partial_x w = 0. \end{cases} \quad (6)$$

Its quasilinear form is

$$\partial_t \begin{pmatrix} \rho \\ q \\ w \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ c^2 - u^2 & 2u & \lambda\rho(u - v) \\ 0 & 0 & v \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ q \\ w \end{pmatrix} = 0. \quad (7)$$

The eigenvalues of the Jacobian matrix are  $u + c$ ,  $u - c$  and  $v$ . This system is strictly hyperbolic whenever  $u \neq v \pm c$ . In the resonant case  $u \pm c = v$ , the matrix cannot be put in a diagonal form. Such resonant systems have been studied in [GL04] and [IT92]. However, with this source term, our system does not fall neither in the framework of [IT92], because it is not conservative, neither in the framework of [GL04], because one of the hypothesis on the source term (namely 1.7) is not satisfied. Following [LST08], [GL04], [IT92], [SV03] and [CLS04], we use a thickening of the particle to define the nonconservative product. Let  $H^\varepsilon$  be an approximation of the Heaviside function such that:

- $H^\varepsilon \in \mathcal{C}^0(\mathbb{R}) \cap \mathcal{C}^1((-\varepsilon/2, \varepsilon/2))$ ;
- $H^\varepsilon$  is nondecreasing;
- $H^\varepsilon(x) = 0$  if  $x \leq -\varepsilon/2$  and  $H^\varepsilon(x) = 1$  if  $x \geq \varepsilon/2$ .

We replace the Dirac measure by its regularization  $(H^\varepsilon)'$  to obtain the regularized system

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x q^\varepsilon & = 0; \\ \partial_t q^\varepsilon + \partial_x \left( \frac{(q^\varepsilon)^2}{\rho^\varepsilon} + c^2 \rho^\varepsilon \right) & = -\lambda(q^\varepsilon - v\rho^\varepsilon)(H^\varepsilon)'(x - vt). \end{cases} \quad (8)$$

We are interesting in what is happening inside the particle. In the spirit of traveling waves, we look for solution only depending on  $x - vt$ . With such a regularized source term, the values of the solutions at the extremities of the particle depend neither on the size of the thickened particle  $\varepsilon$ , nor on the choice of the regularization  $H^\varepsilon$  (satisfying the three hypotheses above). This allows us to define the source term  $\rho(u - v)\delta_{h'(t)}$  as an interface condition.

**Lemma 2.3.** *Let  $(\rho^\varepsilon, q^\varepsilon)$  be a piecewise  $\mathcal{C}^1$  solution of (8) on  $[-\varepsilon/2, \varepsilon/2]$  that only depends on  $\xi = x - vt$ . Then, on every interval where the solution is smooth, the quantity  $q^\varepsilon - v\rho^\varepsilon$  remains constant, while*

$$\left( \frac{(q^\varepsilon - v\rho^\varepsilon)^2}{\rho^\varepsilon} + c^2 \rho^\varepsilon - \lambda(q^\varepsilon - v\rho^\varepsilon)H^\varepsilon \right)' = 0.$$

If the solution is discontinuous at a point  $\xi^0 \in (-\varepsilon/2, \varepsilon/2)$ , then

$$\begin{cases} q^\varepsilon(\xi_0^-) - v\rho(\xi_0^-) = q^\varepsilon(\xi_0^+) - v\rho(\xi_0^+) := \alpha^\varepsilon(\xi^0); \\ \left( \frac{(\alpha^\varepsilon(\xi^0))^2}{\rho(\xi_0^-)} + c^2 \rho(\xi_0^-) \right) - \left( \frac{(\alpha^\varepsilon(\xi^0))^2}{\rho(\xi_0^+)} + c^2 \rho(\xi_0^+) \right) = \lambda \alpha^\varepsilon(\xi^0). \end{cases} \quad (9)$$

*Proof.* Let  $\rho^\varepsilon(x - vt)$  and  $q^\varepsilon(x - vt)$  be a piecewise  $\mathcal{C}^1$  solution of (8), only depending on  $\xi = x - vt$ . On each of its intervals of smoothness, it satisfies the following equations:

$$\begin{cases} -v(\rho^\varepsilon)' + (q^\varepsilon)' & = 0, \\ -v(q^\varepsilon)' + \left( \frac{(q^\varepsilon)^2}{\rho^\varepsilon} + c^2 \rho^\varepsilon \right)' & = -\lambda(q^\varepsilon - v\rho^\varepsilon)(H^\varepsilon)'(x - vt). \end{cases} \quad (10)$$

The first equation of (10) directly gives that  $q^\varepsilon - v\rho^\varepsilon$  remains constant. We denote by  $\alpha^\varepsilon$  this conserved quantity (which a priori depends on the considered interval). Replacing  $q^\varepsilon$  by  $\alpha^\varepsilon + v\rho^\varepsilon$  in the second line of (10) yields

$$-v^2(\rho^\varepsilon)' + \left( \frac{(\alpha^\varepsilon)^2 + 2\alpha^\varepsilon v\rho^\varepsilon + v^2(\rho^\varepsilon)^2}{\rho^\varepsilon} + c^2 \rho^\varepsilon \right)' = -\lambda \alpha^\varepsilon (H^\varepsilon)'$$

As  $\alpha^\varepsilon$  and  $v$  are constant, we obtain that

$$\left( \frac{(\alpha^\varepsilon)^2}{\rho^\varepsilon} + c^2 \rho^\varepsilon - \lambda \alpha^\varepsilon H^\varepsilon \right)' = 0.$$

On the other hand, if  $(\rho^\varepsilon, q^\varepsilon)$  has a discontinuity at a point  $\xi_0 \in (-\varepsilon/2, \varepsilon/2)$ , it verifies the two relations:

$$\begin{cases} q(\xi_0^+) - q(\xi_0^-) = v(\rho(\xi_0^+) - \rho(\xi_0^-)), \\ \left( \frac{q^2(\xi_0^+)}{\rho(\xi_0^+)} + c^2 \rho(\xi_0^+) \right) - \left( \frac{q^2(\xi_0^-)}{\rho(\xi_0^-)} + c^2 \rho(\xi_0^-) \right) = v(q(\xi_0^+) - q(\xi_0^-)), \end{cases}$$

and we obtain the result by introducing the conserved quantity  $\alpha(\xi_0) = q^\varepsilon(\xi_0^-) - v\rho^\varepsilon(\xi_0^-) = q^\varepsilon(\xi_0^+) - v\rho^\varepsilon(\xi_0^+)$ .  $\square$

*Remark 2.4.* The relations (9) are nothing but the Rankine-Hugoniot relations for a shock having speed  $v$  in the isothermal Euler equations.

**Theorem 2.5.** *Let us denote by  $\mathcal{G}_\lambda(v)$  the set of pair  $((\rho_-, q_-), (\rho_+, q_+))$  of  $(\mathbb{R}_+ \times \mathbb{R})^2$  verifying the four relations*

- $q_- - v\rho_- = q_+ - v\rho_+$ . We denote by  $\alpha$  this quantity;

- $\left(\frac{\alpha^2}{\rho_-} + c^2\rho_-\right) - \left(\frac{\alpha^2}{\rho_+} + c^2\rho_+\right) = \lambda\alpha$ ;
- If  $\alpha > 0$  and  $\rho_- \geq \frac{|\alpha|}{c}$ , then  $\rho_+ \geq \frac{|\alpha|}{c}$ ;
- If  $\alpha < 0$  and  $\rho_+ \geq \frac{|\alpha|}{c}$ , then  $\rho_- \geq \frac{|\alpha|}{c}$ .

Then, if  $(\rho^\varepsilon, q^\varepsilon)$  is a piecewise  $C^1$  entropy solution of (8) that only depends on  $\xi = x - vt$ , then the pair of states

$$((\rho^\varepsilon(-\varepsilon/2), q^\varepsilon(-\varepsilon/2)), (\rho^\varepsilon(\varepsilon/2), q^\varepsilon(\varepsilon/2)))$$

belongs to  $\mathcal{G}_\lambda(v)$ . Conversely, if  $((\rho_-, q_-), (\rho_+, q_+))$  belongs to  $\mathcal{G}_\lambda(v)$ , there exists  $(\rho^\varepsilon, q^\varepsilon)$  a piecewise  $C^1$  entropy solution of (8) on  $[-\varepsilon/2, \varepsilon/2]$  that only depends on  $\xi = x - vt$ , such that

$$\rho^\varepsilon(-\varepsilon/2) = \rho_-, \quad \rho^\varepsilon(\varepsilon/2) = \rho_+, \quad q^\varepsilon(-\varepsilon/2) = q_- \quad \text{and} \quad q^\varepsilon(\varepsilon/2) = q_+.$$

*Proof.* A straightforward consequence of Lemma 2.3 is that the state  $(\rho_-, q_-) = (\rho^\varepsilon(-\varepsilon/2), q^\varepsilon(-\varepsilon/2))$  on the left of the particle and the state  $(\rho_+, q_+) = (\rho^\varepsilon(\varepsilon/2), q^\varepsilon(\varepsilon/2))$  on its right are linked by the relations

$$\begin{cases} q_- - v\rho_- = q_+ - v\rho_+, \\ \left(\frac{(q_- - v\rho_-)^2}{\rho_-} + c^2\rho_-\right) - \left(\frac{(q_+ - v\rho_+)^2}{\rho_+} + c^2\rho_+\right) = \lambda(q_- - v\rho_-), \end{cases} \quad (11)$$

for any choice of  $\varepsilon$  and any choice of  $H^\varepsilon$  satisfying the hypotheses of Lemma 2.3.

We now fix a state  $(\rho_-, q_-) \in \mathbb{R}_+^* \times \mathbb{R}$  on the left of the particle, and look for the  $(\rho_+, q_+) \in \mathbb{R}_+^* \times \mathbb{R}$  verifying the relations (11). We focus on how they are reached through the particle, paying particular attention to the discontinuities in order to obtain a Lax solution. We denote by  $f_\alpha$  the function  $\rho \mapsto \frac{\alpha^2}{\rho} + c^2\rho$ . The second line of Equations (11) rewrites

$$f_\alpha(\rho_-) - f_\alpha(\rho_+) = \lambda\alpha.$$

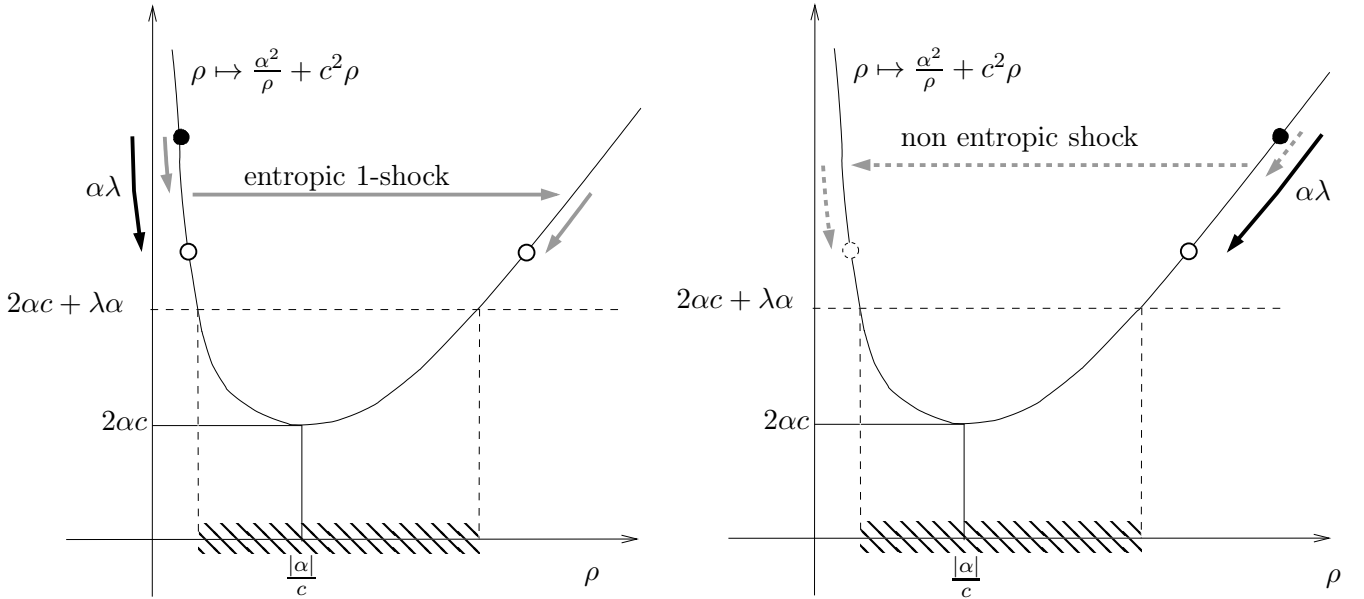


Figure 1: How to reach the density  $\rho_+$  (white circles) from the density  $\rho_-$  (black circle) when  $\alpha$  is positive. On the left is the supersonic case  $u_- - v > c$ ; on the right is the subsonic case  $u_- - v \leq c$ .

As depicted on Figure 1, the function  $f_\alpha$  decreases on  $(0, \frac{|\alpha|}{c})$  and increases on  $(\frac{|\alpha|}{c}, +\infty)$ . Thus there exists at most two  $(\rho_+, q_+)$  satisfying the relations (11), one on each side of  $\frac{|\alpha|}{c}$ . As  $H^\varepsilon$  is nondecreasing, the solution lying on the other side of  $|\alpha|/c$  than  $\rho_-$  cannot be reached continuously. The Rankine-Hugoniot relations (9) stipulates that through a discontinuity in (8), both  $\alpha$  and  $f_\alpha$  remain constant. Therefore this solution can be reached by decreasing, for any  $\theta$  in  $[0, 1]$ , of  $\theta\lambda\alpha$  on the same side of  $|\alpha|/c$  than  $\rho_-$ , then jumping horizontally on the other side of the curve, which



corresponds exactly to a discontinuity in the solution, and finally decreasing of  $(1 - \theta)\lambda\alpha$ . Remark that each choice of  $\theta$  yields to the same value of  $\rho_+$ . Let us now focus on whether such jumps are entropy satisfying or not. By (9), they are nothing but shocks at speed  $v$  in the Euler equations. Suppose that  $\alpha$  is positive, and denote by  $\xi_0$  the position of the jump. If  $\rho(\xi_0^-) > |\alpha|/c$  then  $\rho(\xi_0^-) > \rho(\xi_0^+)$ . If it was a Lax shock, it should be a 2-shock. Its speed will be

$$v = \frac{q(\xi_0^+)}{\rho(\xi_0^+)} + c\sqrt{\frac{\rho(\xi_0^-)}{\rho(\xi_0^+)}}$$

which rewrites

$$c\sqrt{\frac{\rho(\xi_0^-)}{\rho(\xi_0^+)}} = -\frac{\alpha}{\rho(\xi_0^+)},$$

and contradicts the fact that  $\alpha > 0$ . This entropy violating shock corresponds to the dotted gray line on Figure 1. On the other hand if  $\rho(\xi_0^-) < |\alpha|/c$ , we have that  $\rho(\xi_0^-) < \rho(\xi_0^+)$ , and the discontinuity should be a 1-shock. We obtain

$$c\sqrt{\frac{\rho(\xi_0^-)}{\rho(\xi_0^+)}} = \frac{\alpha}{\rho(\xi_0^+)},$$

which does not contradict the sign of  $\alpha$ . This Lax 1-shock is the bold gray line on Figure 1. Similarly, we obtain that if  $\alpha$  is negative, the jump is an entropy satisfying shock if and only if  $\rho(\xi_0^+) < |\alpha|/c$ . In particular, when  $(\rho_-, q_-)$  is fixed such that  $\alpha = q_- - v\rho_- > 0$  and  $\rho_- > \frac{\alpha}{c}$ , only the solution on the right of  $\frac{\alpha}{c}$  is entropy satisfying.  $\square$

*Remark 2.6.* If  $\alpha$  is positive and  $\rho_-$  belongs to the interval

$$\left[ \frac{\alpha}{c} + \frac{\lambda\alpha - \alpha\sqrt{4c\lambda + \lambda^2}}{2c^2}, \frac{\alpha}{c} + \frac{\lambda\alpha + \alpha\sqrt{4c\lambda + \lambda^2}}{2c^2} \right]$$

there is no solution.

*Remark 2.7.* Shocks can only link a supersonic state (in the sense that  $\rho > \frac{|\alpha|}{c}$ , which rewrites  $|u - v| > c$ ) to a subsonic state (in the sense  $|u - v| < c$ ) (from left to right if  $\alpha > 0$ , from right to left if  $\alpha < 0$ ). As a consequence, the Lax solutions of (8) contain at most one discontinuity.

### 2.3 Definition of the solution

Let us now reformulate the ODE (1c). The source term in (1b) is the exact opposite of the left hand side in (1c), so the total impulsion is formally conserved through time:

$$\frac{d}{dt} \left[ \int_{\mathbb{R}} \rho(t, x)u(t, x)dx + mh'(t) \right] = 0. \quad (12)$$

We can use this additional property of the model to give a precise definition of the ODE (1c).

**Proposition 2.8.** *Let  $(\rho, u)$  be a solution of (1a)-(1b) such that for almost every  $t > 0$ , the traces around the particle exist and are such that*

$$((\rho_-(t), q_-(t)), (\rho_+(t), q_+(t))) \in \mathcal{G}_\lambda(h'(t)).$$

*Then, it satisfies the conservation of total impulsion (12) if and only if for almost every  $t > 0$ ,*

$$mh''(t) = c^2(\rho_-(t) - \rho_+(t)) \left( 1 - \frac{(u_-(t) - h'(t))(u_+(t) - h'(t))}{c^2} \right), \quad (13)$$

*where the subscripts  $\pm$  indicates the left and right traces around the particle:  $\rho_\pm(t) = \rho(t, h(t)_\pm)$ .*

*Proof.* If  $((\rho_-(t), q_-(t)), (\rho_+(t), q_+(t)))$  belongs to  $\mathcal{G}_\lambda(h'(t))$ , then  $q_-(t) - h'(t)\rho_-(t)$  is equal to  $q_+(t) - h'(t)\rho_+(t)$ . We denote by  $\alpha(t)$  this quantity, and replace  $q = \rho u$  by  $\alpha + h'\rho$ . We have

$$\begin{aligned}
mh''(t) &= -\partial_t \int_{\mathbb{R}} \rho(t, x)u(t, x)dx \\
&= -\partial_t \int_{-\infty}^{h(t)} \rho(t, x)u(t, x)dx - \partial_t \int_{h(t)}^{+\infty} \rho(t, x)u(t, x)dx \\
&= h'(\rho_+u_+ - \rho_-u_-) + (\rho_-u_-^2 + c^2\rho_-) - (\rho_+u_+^2 + c^2\rho_+) \\
&= h'^2(\rho_+ - \rho_-) + \rho_- \left( \frac{\alpha^2}{\rho_-^2} + 2h' \frac{\alpha}{\rho_-} + h'^2 \right) - \rho_+ \left( \frac{\alpha^2}{\rho_+^2} + 2h' \frac{\alpha}{\rho_+} + h'^2 \right) + c^2(\rho_- - \rho_+) \\
&= h'^2(\rho_+ - \rho_-) + \alpha^2 \left( \frac{1}{\rho_-} - \frac{1}{\rho_+} \right) + h'^2(\rho_- - \rho_+) + c^2(\rho_- - \rho_+) \\
&= (\rho_- - \rho_+) \left( c^2 - \frac{\alpha^2}{\rho_- \rho_+} \right) \\
&= c^2(\rho_- - \rho_+) \left( 1 - \frac{(u_- - h')(u_+ - h')}{c^2} \right). \quad \square
\end{aligned}$$

*Remark 2.9.* In the case of the linear drag force considered here, the ODE (1c) makes actually sense, since the quantity  $\rho(u - h'(t))$  is conserved through the particle. However, the reformulation (13) holds for any choice of drag force, since the quantity  $q - h'\rho$  is conserved through the particle, whatever the drag force is. We will consider other types of friction in Section 4. Moreover, the ODE (13) does not seem to depend of the drag force: this dependence is hidden in the assumption

$$((\rho_-(t), q_-(t)), (\rho_+(t), q_+(t))) \in \mathcal{G}_\lambda(h'(t))$$

because the germ does depend on the drag force.

Thanks to the previous reformulation of the ODE and on the entropy inequality (5), we define the entropy solutions of the coupled problem (1):

**Definition 2.10.** Assume that  $(\rho_0, q_0) \in L^\infty(\mathbb{R})^2$  and  $v_0 \in \mathbb{R}$ . A triplet  $(\rho, q, h) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \times L^\infty(\mathbb{R}_+ \times \mathbb{R}) \times W_{loc}^{2, \infty}(\mathbb{R}_+)$  is called an entropy solution of the problem (1) if:

- $(\rho, q)$  is a weak solution of the isothermal Euler equations on the sets  $\{(t, x) \in \mathbb{R}_+^* \times \mathbb{R} : x > h(t)\}$  and  $\{(t, x) \in \mathbb{R}_+^* \times \mathbb{R} : x < h(t)\}$ .
- For any entropy-entropy flux pair  $(E, G)$  such that  $E$  is convex and  $\partial_q E(\rho, q) = J\left(\frac{q}{\rho}\right)$ , for any non-negative test function  $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R})$ , we have

$$\begin{aligned}
\iint_{\mathbb{R}_+ \times \mathbb{R}} E(U) \partial_t \Phi \, dt \, dx + \iint_{\mathbb{R}_+ \times \mathbb{R}} G(U) \partial_x \Phi \, dt \, dx + \int_{\mathbb{R}_+} mP(h'(t)) \partial_t (\Phi(t, h(t))) \, dt \\
+ \int_{\mathbb{R}} \Phi(0, \cdot) E(U_0) \, dx + mP(v_0) \Phi(0, v_0) \geq 0
\end{aligned}$$

where  $P$  is a given antiderivative of  $J$ ;

- For almost every  $t > 0$ , the traces around the particle exist and belong to the germ at speed  $h'(t)$ :

$$((\rho(t, h(t)^-), q(t, h(t)^-)), (\rho(t, h(t)^+), q(t, h(t)^+))) \in \mathcal{G}_{h'(t)};$$

- For almost every  $t > 0$ , the particle is driven by the ODE:

$$mh''(t) = c^2(\rho_-(t) - \rho_+(t)) \left( 1 - \frac{(u_-(t) - h'(t))(u_+(t) - h'(t))}{c^2} \right).$$

*Remark 2.11.* In the case of the scalar conservation law, it is not necessary to require the existence of the traces. Indeed, as the solution is a classic solution on  $\{x < h'\}$ , strong traces exist (see [Pan07] and [Vas01]). However, such a result is much harder to obtain in the system case.

### 3 Riemann problem for a particle with a constant fixed velocity

In this section we focus on the uncoupled problem where the particle has a constant speed equal to some given  $v$  in  $\mathbb{R}$ . Moreover, we consider a class of very specific initial data, which consist in piecewise constant functions for the density  $\rho$  and for the momentum  $q = \rho u$ , with a single discontinuity falling exactly on the initial position of the particle. The problem under study in this section is the Riemann problem:

$$\begin{cases} \partial_t \rho + \partial_x q & = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{\rho} + c^2 \rho \right) & = -\lambda(q - v\rho)\delta_{vt}, \\ \rho(0, x) & = \rho_L \mathbf{1}_{x < 0} + \rho_R \mathbf{1}_{x > 0}, \\ q(0, x) & = q_L \mathbf{1}_{x < 0} + q_R \mathbf{1}_{x > 0}, \end{cases} \quad (14)$$

where  $(\rho_L, q_L)$  and  $(\rho_R, q_R)$  belong to  $\mathbb{R}_+^* \times \mathbb{R}$ . As the particle's trajectory is a straight line, we look for self-similar solutions of (14), i.e. solutions that only depend on  $x/t$ . In this section, we prove that there exists a unique such self-similar solution of (14).

Let  $(\rho, q)$  be such a solution and denote by  $(\rho_-, q_-)$  (respectively  $(\rho_+, q_+)$ ) the traces of the density and the momentum on the left (respectively on the right) of the particle, i.e. on the line  $(t, (vt)_-)$  (respectively on the line  $(t, (vt)_+)$ ). Then

$$(\rho|_L, q|_L) = \begin{cases} (\rho, q) & \text{on } x < vt \\ (\rho_-, q_-) & \text{on } x \geq vt \end{cases} \quad \text{and} \quad (\rho|_R, q|_R) = \begin{cases} (\rho_+, q_+) & \text{on } x \leq vt \\ (\rho, q) & \text{on } x > vt \end{cases}$$

are self-similar solution of the *classical* isothermal Euler equations (without source term) for the initial data

$$(\rho(0, x), q(0, x)) = (\rho_L \mathbf{1}_{x < 0} + \rho_- \mathbf{1}_{x > 0}, q_L \mathbf{1}_{x < 0} + q_- \mathbf{1}_{x > 0})$$

and

$$(\rho(0, x), q(0, x)) = (\rho_+ \mathbf{1}_{x < 0} + \rho_R \mathbf{1}_{x > 0}, q_+ \mathbf{1}_{x < 0} + q_R \mathbf{1}_{x > 0}).$$

Conversely, if  $(\rho|_L, q|_L)$  and  $(\rho|_R, q|_R)$  are the restriction on the sets  $\{(t, x), x < vt\}$  and  $\{(t, x), x > vt\}$  of solutions of the *classical* isothermal Euler equations on the with initial data

$$(\rho_L \mathbf{1}_{x < 0} + \underline{\rho} \mathbf{1}_{x > 0}, q_L \mathbf{1}_{x < 0} + \underline{q} \mathbf{1}_{x > 0}) \quad \text{and} \quad (\bar{\rho} \mathbf{1}_{x < 0} + \rho_R \mathbf{1}_{x > 0}, \bar{q} \mathbf{1}_{x < 0} + q_R \mathbf{1}_{x > 0}),$$

such that the corresponding traces  $((\rho_-, q_-), (\rho_+, q_+))$  on the line  $x = vt_{\pm}$  belongs to the germ  $\mathcal{G}_v$ , then

$$(\rho, q) = \begin{cases} (\rho|_L, q|_L) & \text{on } x < vt \\ (\rho|_R, q|_R) & \text{on } x > vt \end{cases}$$

is a solution of (14). In Section 3.1 we study those restrictions of the solutions of the Euler equation. More precisely, we describe the set  $U_-(\rho_L, q_L, v)$  of all the values that a Riemann solution for the classical Euler equations between  $(\rho_L, q_L)$  and a state  $(\underline{\rho}, \underline{q})$  can take on the line  $x = vt$  and the set  $U_+(\rho_R, q_R, v)$  of all the values that a Riemann solution for the classical Euler equations between  $(\bar{\rho}, \bar{q})$  and  $(\rho_R, q_R)$  can take on the line  $x = vt$ . The traces  $(\rho_-, q_-)$  and  $(\rho_+, q_+)$  around the particle should be respectively chosen in those sets. Then in Sections 3.2 and 3.3 we prove that there is one and only one solution to the Riemann problem (14), or in other words, that there is a unique way to pick  $(\rho_-, q_-)$  in  $U_-(\rho_L, q_L, v)$  and  $(\rho_+, q_+)$  in  $U_+(\rho_R, q_R, v)$  such that  $((\rho_-, q_-), (\rho_+, q_+))$  belongs to the germ  $\mathcal{G}_v$ . We first show that if  $u_L - v \leq c$  and  $u_R - v \geq -c$ , any potential traces  $(\rho_{\pm}, q_{\pm})$  around the particle inherits the property  $|u_{\pm} - v| \leq c$  and conclude in that case. It will be referred as the subsonic case. In the other case, referred to as the supersonic case and studied in Section 3.3, we prove that either both traces around the particle verify  $|u_{\pm} - v| \leq c$  or they both verify  $|u_{\pm} - v| > c$ .

*Remark 3.1.* In all the sequel, the terms *subsonic* and *supersonic* are used *in the framework of the particle*. For example we say that a solution is subsonic if the difference between the velocity of the particle and the fluid's velocity on both side of the particle is smaller than the speed of sound  $c$ .

### 3.1 Accessible states around the particle

Let us start with some very classical results on the isothermal Euler equations without source term that will be useful to determine the solution of (14).

**Lemma 3.2.** *The isothermal Euler equations without source term*

$$\begin{cases} \partial_t \rho + \partial_x q & = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{\rho} + c^2 \rho \right) & = 0, \end{cases}$$

is a strictly hyperbolic system. The eigenvalues of its Jacobian matrix are

$$\lambda_i(\rho, q) = \frac{q}{\rho} + (-1)^i c$$

and the corresponding normalized eigenvectors are

$$r_i(\rho, q) = \begin{pmatrix} \frac{(-1)^i \rho}{c} \\ \frac{(-1)^i q}{c} + \rho \end{pmatrix}.$$

They define two genuinely nonlinear fields. The  $i$ -th rarefaction waves express

$$\rho(s) = \begin{cases} \rho_L & \text{if } s \leq s_L = \frac{q_L}{\rho_L} + (-1)^i c, \\ \rho_L e^{\frac{(-1)^i}{c}(s-s_L)} & \text{if } s_L \leq s \leq s_R, \\ \rho_R = \rho_L e^{\frac{(-1)^i}{c}(s_R-s_L)} & \text{if } s_R \leq s, \end{cases}$$

and

$$q(s) = \begin{cases} q_L & \text{if } s \leq s_L = \frac{q_L}{\rho_L} + (-1)^i c, \\ [q_L + \rho_L(s - s_L)] e^{\frac{(-1)^i}{c}(s-s_L)} & \text{if } s_L \leq s \leq s_R, \\ q_R = [q_L + \rho_L(s_R - s_L)] e^{\frac{(-1)^i}{c}(s_R-s_L)} & \text{if } s_R \leq s. \end{cases}$$

The speed of the  $i$ -shock is

$$\sigma_i = \frac{q_L}{\rho_L} + (-1)^i c \sqrt{\frac{\rho_R}{\rho_L}}$$

and shocks are entropy satisfying if and only if  $u_L \geq u_R$ .

We recall below a well-known result on the structure of the Riemann solution.

**Proposition 3.3.** *For all  $(\rho_a, q_a)$  and  $(\rho_b, q_b)$  in  $\mathbb{R}_+^* \times \mathbb{R}$ , there exists a unique self-similar entropy solution of the Riemann problem*

$$\begin{cases} \partial_t \rho + \partial_x q & = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{\rho} + c^2 \rho \right) & = 0, \\ \rho(0, x) & = \rho_a \mathbf{1}_{x < 0} + \rho_b \mathbf{1}_{x > 0}, \\ q(0, x) & = q_a \mathbf{1}_{x < 0} + q_b \mathbf{1}_{x > 0}. \end{cases}$$

It consists of the succession of a 1-wave (rarefaction or shock) linking  $(\rho_L, q_L)$  to an intermediate state  $(\rho_*, q_*)$  followed by a 2-wave (rarefaction or shock) linking  $(\rho_*, q_*)$  to  $(\rho_R, q_R)$ . We denote by

$$W(x/t; (\rho_a, q_a), (\rho_b, q_b))$$

this unique solution.

*Proof.* The proof of those two results are classical and can be found for example in [GR96].  $\square$

Those tools allow us to describe the set of the accessible states from  $(\rho_L, q_L)$  on the left of a particle moving at speed  $v$

$$U_-(\rho_L, q_L, v) = \{W(v^-; (\rho_L, q_L), (\underline{\rho}, \underline{q})), (\underline{\rho}, \underline{q}) \in \mathbb{R}_+^* \times \mathbb{R}\}.$$

and the set of the accessible states from  $(\rho_R, q_R)$  on the right of a particle moving at speed  $v$

$$U_+(\rho_R, q_R, v) = \{W(v^+; (\bar{\rho}, \bar{q}), (\rho_R, q_R)), (\bar{\rho}, \bar{q}) \in \mathbb{R}_+^* \times \mathbb{R}\}$$

in which the left and right traces around the particle must be chosen. According to Definition 2.5, the quantity  $q - v\rho$  must be conserved through the particle, so it is more convenient to reason with the variables  $(\rho, q - v\rho)$  rather than with the initial unknowns  $(\rho, q)$ . For this purpose we denote  $\alpha = q - v\rho$  and for any indices  $i$ ,  $\alpha_i = q_i - v\rho_i$ . We introduce the sets

$$V_-(\rho_L, \alpha_L, v) = \{(\rho_-, q_- - v\rho_-) : (\rho_-, q_-) \in U_-(\rho_L, \alpha_L + v\rho_L, v)\}$$

and

$$V_+(\rho_R, \alpha_R, v) = \{(\rho_+, q_+ - v\rho_+) : (\rho_+, q_+) \in U_+(\rho_R, \alpha_R + v\rho_R, v)\}.$$

Proving that there exists a unique solution to the Riemann problem (14) is equivalent to prove that

$$\mathcal{G}_v \cap (U_-(\rho_L, q_L, v) \times U_+(\rho_R, q_R, v))$$

consists in a unique pair of states  $((\rho_-, q_-), (\rho_+, q_+))$ . We now give a precise description of the sets  $V_-(\rho_L, \alpha_L, v)$  and  $V_+(\rho_R, \alpha_R, v)$ .

**Lemma 3.4.** *Let  $(\rho_L, \alpha_L) \in \mathbb{R}_+^* \times \mathbb{R}$  and  $v \in \mathbb{R}$ . Then*

$$V_-(\rho_L, \alpha_L, v) = \{(\rho_L, \alpha_L)\} \cup \Gamma_-^{sub} \cup \Omega_-^{sup},$$

where  $\Gamma_-^{sub}$  is the graph of a decreasing function  $f_-^{sub} : [\rho_{L,ex}, +\infty] \rightarrow \mathbb{R}$  for some  $\rho_{L,ex} > 0$ , included in  $\{(\rho, \alpha) : -c\rho \leq \alpha \leq c\rho\}$ , and  $\Omega_-^{sup}$  is the strict hypograph of a decreasing function  $f_-^{sup} : (0, +\infty) \rightarrow \mathbb{R}$ , included in  $\{(\rho, \alpha) : \alpha < -c\rho\}$ .

**Lemma 3.5.** *Let  $(\rho_R, \alpha_R) \in \mathbb{R}_+^* \times \mathbb{R}$  and  $v \in \mathbb{R}$ . Then*

$$V_+(\rho_R, \alpha_R, v) = \{(\rho_R, \alpha_R)\} \cup \Gamma_+^{sub} \cup \Omega_+^{sup}$$

where  $\Gamma_+^{sub}$  is the graph of a increasing function  $f_+^{sub} : [\rho_{R,ex}, +\infty] \rightarrow \mathbb{R}$  for some  $\rho_{R,ex} > 0$ , included in  $\{(\rho, \alpha) : -c\rho \leq \alpha \leq c\rho\}$ , and  $\Omega_+^{sup}$  is the strict epigraph of a increasing function  $f_+^{sup} : (0, +\infty) \rightarrow \mathbb{R}$ , included in  $\{(\rho, \alpha) : \alpha > c\rho\}$ .

Those sets are respectively depicted on the left and on the right of Figure 2. The subscript *ex* refers to the extremity of the graph that lies between  $\alpha = -c\rho$  and  $\alpha = c\rho$ .

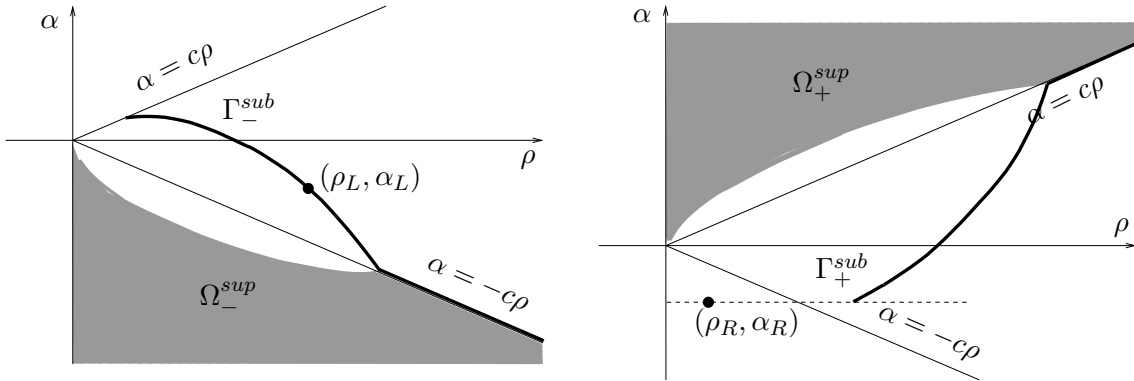


Figure 2: On the left, the set  $V_-(\rho_L, \alpha_L, v)$  when  $-c\rho_L < \alpha_L < c\rho_L$ . On the right, the set  $V_+(\rho_R, \alpha_R, v)$  when  $\alpha_R < c\rho_R$ .

*Remark 3.6.* By definition of  $\alpha$  we have

$$\{(\rho, \alpha) : -c\rho \leq \alpha \leq c\rho\} = \{(\rho, u) - c \leq u - v \leq c\},$$

so  $\Gamma_-^{sub}$  and  $\Gamma_+^{sub}$  contain only subsonic states (in the framework of the particle), while  $\Omega_-^{sup}$  and  $\Omega_+^{sup}$  contain only supersonic states (in the framework of the particle).

*Proof (Lemma 3.4).* We fix the particle velocity  $v$  and a left state  $(\rho_L, \alpha_L)$ . We recall that  $q_L = \alpha_L + v\rho_L$  and that  $u_L = \frac{q_L}{\rho_L}$ . Let us first exhibit all the states  $(\rho_*, q_*)$  that can be linked to  $(\rho_L, q_L)$

with a 1-wave traveling at speed smaller than  $v$ . According to Lemma (3.2), with a 1-rarefaction wave we can reach all the states  $(\rho, q)$  in the set

$$\left\{ \left( \rho_L e^{-\frac{(s-(u_L-c))}{c}}, [q_L + \rho_L(s - (u_L - c))] e^{-\frac{(s-(u_L-c))}{c}} \right), u_L - c \leq s \leq v \right\}.$$

This set is empty if  $u_L - c > v$ . Parametrized by  $\rho$  it rewrites

$$\left\{ \left( \rho, \left[ u_L - c \ln \left( \frac{\rho}{\rho_L} \right) \right] \rho \right), \rho_L e^{-\frac{v-(u_L-c)}{c}} \leq \rho \leq \rho_L \right\}.$$

Let us denote in this case  $\rho_{L,ex} = \rho_L e^{-\frac{v-(u_L-c)}{c}}$ . The states  $(\rho_*, q_*)$  accessible through a 1-shock traveling slower than  $v$  satisfy

$$\begin{cases} \frac{q_L}{\rho_L} - c\sqrt{\frac{\rho_*}{\rho_L}} < v & \text{and } \rho_* > \rho_L, \\ q_* = q_L + \left( \frac{q_L}{\rho_L} - c\sqrt{\frac{\rho_*}{\rho_L}} \right) (\rho_* - \rho_L). \end{cases}$$

We easily pass to the  $(\rho, \alpha)$  variable: with a 1-wave traveling slower than  $v$ , we can reach all the states  $(\rho, \alpha)$  with  $\alpha = f_-(\rho)$ , where

$$f_-(\rho) = \begin{cases} \left[ \frac{\alpha_L}{\rho_L} - c \ln \left( \frac{\rho}{\rho_L} \right) \right] \rho & \text{if } \rho_{L,ex} \leq \rho \leq \rho_L, \\ \alpha_L + \left( \frac{\alpha_L}{\rho_L} - c\sqrt{\frac{\rho}{\rho_L}} \right) (\rho - \rho_L) & \text{if } \max(\rho_{L,ex}, \rho_L) < \rho, \end{cases}$$

and

$$\rho_{L,ex} = \begin{cases} \rho_L e^{-\frac{v-s_L}{c}} & \text{if } s_L = u_L - c \leq v, \\ \left( \frac{u_L - v}{c} \right)^2 \rho_L & \text{if } u_L - c > v. \end{cases}$$

If  $u_L - v < c$ , this graph regroups all the 1-shocks and the 1-rarefaction waves leading to a density higher than  $\rho_{L,ex} = \rho_L e^{-\frac{v-(u_L-c)}{c}}$ , while if  $u_L - v \geq c$ , this graphs contains only the 1-shocks leading to a density higher than  $\rho_{L,ex} = \rho_L \left( \frac{u_L - v}{c} \right)^2 = \frac{\alpha_L}{c^2 \rho_L}$ . We check that  $f_-$  is concave and decreasing. Moreover, if  $u_L - c \leq v$ ,  $f_-(\rho_{L,ex}) = c\rho_{L,ex}$  and  $f_-^l(\rho_{L,ex}) = 0$ , while if  $u_L - c > v$ ,  $f_-(\rho_{L,ex}) = \alpha_L$ . In particular,  $f_-(\rho) < c\rho$  for any  $\rho > \rho_{L,ex}$  as shown on Figure (3).

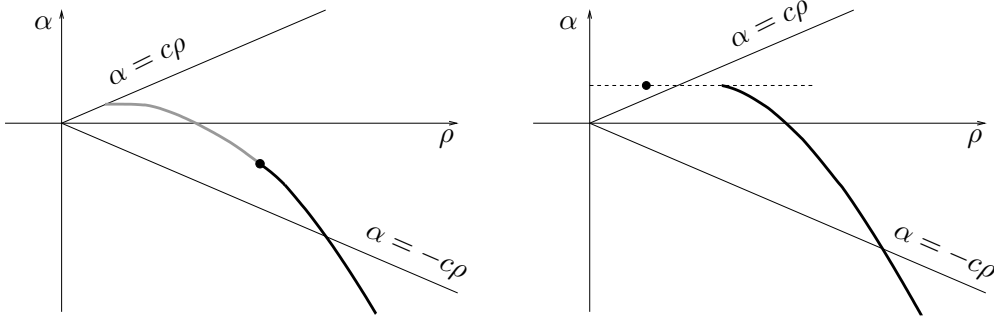


Figure 3: Accessible states via a 1-wave starting from  $(\rho_L, q_L)$ (black dot) in the  $(\rho, \alpha)$ -plane. Left, the case  $u_L - v \leq c$  and right, the case  $u_L - v > c$ . In gray are the 1-rarefaction waves and in black are the 1-shocks.

Let us now stop at any state  $(\rho_*, q_*)$  belonging to the graph of  $f_-$ , and continue with a 2-wave traveling at speed smaller than  $v$ . The set of all the states  $(\rho_-, q_-)$  that can be joined from  $(\rho_*, q_*)$  with a 2-wave traveling at speed smaller than  $v$  is

$$\left\{ \left( \rho_* e^{-\frac{(s-(u_*+c))}{c}}, [q_* + \rho_L(s - s_*)] e^{-\frac{(s-(u_*+c))}{c}} \right), u_* + c \leq s \leq v \right\}.$$

This set is empty if  $u_* + c > v$ , and can be parametrized by  $\rho$  by

$$\left\{ \left( \rho, \left[ u_* + c \ln \left( \frac{\rho}{\rho_*} \right) \right] \rho \right), \rho_* \leq \rho \leq \rho_* e^{-\frac{v-(u_*+c)}{c}} \right\}.$$

With a 2-shock slower than  $v$ , we can reach all the states  $(\rho_-, q_-)$  such that

$$\begin{cases} \frac{q_*}{\rho_*} + c\sqrt{\frac{\rho_-}{\rho_*}} < v & \text{and } \rho_- < \rho_*, \\ q_- = q_* + \left(\frac{q_*}{\rho_*} + c\sqrt{\frac{\rho_-}{\rho_*}}\right)(\rho_- - \rho_*). \end{cases}$$

Therefore the 2-waves traveling at speed smaller than  $v$ , starting from  $(\rho_*, q_*)$ , are:

- If  $u_* - v \leq -c$ , all the 2-shocks and the 2-rarefaction waves leading to a density smaller than  $\rho_* e^{\frac{v-(u_*+c)}{c}}$ ;
- If  $-c < u_* - v < 0$ , only the 2-shock leading to a density smaller than

$$\rho_{*,ex} = \rho_* \left(\frac{v - u_*}{c}\right)^2 = \frac{\alpha_*^2}{c^2 \rho_*};$$

- If  $u_* - v \geq 0$ , there are no such 2-waves.

The Figure (4) resumes the first two cases.

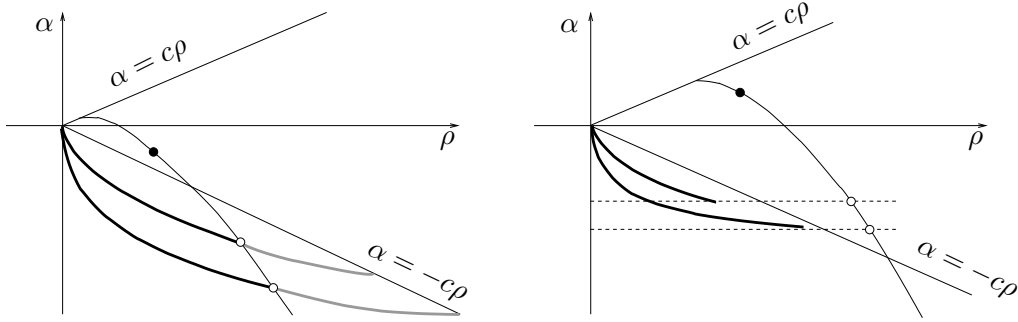


Figure 4: Accessible states from  $(\rho_L, q_L)$  (black dot) via a 1-wave stopped in  $(\rho_*, q_*)$  (white dots) followed by a 2-wave in the  $(\rho, \alpha)$ -plane. Left, the case where  $u_* - v \leq -c$  and right, the case  $-c < u_{L*} - v < 0$ . In gray are the 2-rarefaction waves and in black are the 2-shocks.

We now prove that, as emphasized in Figure 4, all the states reached from  $(\rho_*, q_*)$  through a 2-wave traveling at speed smaller than  $v$  are such that  $\alpha \leq -c\rho$ . This is easy to check for the 2-rarefaction waves and for the 2-shock when  $u_* - v \leq -c$ . When  $-c < u_* - v < 0$ , the 2-shocks require some extra attention. The reachable densities are smaller than  $\frac{\alpha_*^2}{c^2 \rho_*} < \rho_*$ . We introduce the auxiliary function, defined for  $\rho \leq \rho_*$ ,

$$\begin{aligned} A(\rho) &= q_* + \left(\frac{q_*}{\rho_*} + c\sqrt{\frac{\rho}{\rho_*}}\right)(\rho - \rho_*) - v\rho \\ &= \rho \left(\frac{\alpha_*}{\rho_*} + c\sqrt{\frac{\rho}{\rho_*}} - c\sqrt{\frac{\rho_*}{\rho}}\right). \end{aligned}$$

We have

$$A'(\rho) = \frac{\alpha_*}{\rho_*} + \frac{3}{2}c\sqrt{\frac{\rho}{\rho_*}} - \frac{1}{2}c\sqrt{\frac{\rho_*}{\rho}},$$

and  $\rho \mapsto \frac{3}{2}c\sqrt{\frac{\rho}{\rho_*}} - \frac{1}{2}c\sqrt{\frac{\rho_*}{\rho}}$  increases on  $[0, \rho_*]$ , so  $A$  is convex. On the other hand

$$\begin{aligned} \frac{\alpha_*}{\rho_*} + c\sqrt{\frac{\rho}{\rho_*}} - c\sqrt{\frac{\rho_*}{\rho}} &\leq \frac{\alpha_*}{\rho_*} + c\sqrt{\frac{\rho}{\rho_*}} + c^2 \frac{\rho_*}{\alpha_*} \\ &\leq -2c + c\sqrt{\frac{\rho}{\rho_*}} && \text{as } X \mapsto X + \frac{c^2}{X} \text{ decreases on } [-c, 0] \\ &\leq -c. \end{aligned}$$

Therefore  $A'(\rho_{*,ex}) \leq 0$  and  $A$  decreases and is smaller than  $-c\rho$ .

To conclude, the set of the states reached from  $(\rho_L, \alpha_L)$  through a 1-wave slower than the particle is the graph of  $f_-$ , and the set of the states reached from  $(\rho_L, \alpha_L)$  through a 1-wave followed by a 2-wave, both of them traveling at speed smaller than the particle, which is a family of curves, entirely included in  $\{(\rho, \alpha), \alpha \leq -c\rho\}$ . We can separate those states in two categories, as depicted on the left of Figure 2:

- the subsonic ones (i.e the ones such that  $-c\rho \leq \alpha \leq c\rho$ ), which constitute the graph of the function

$$f_-^{sub}(\rho) = \begin{cases} f_-(\rho) & \text{if } \rho_{L,ex}\rho \leq \rho_{-c} \\ -c\rho & \text{if } \rho_{-c} \leq \rho \end{cases}$$

where  $\rho_{-c}$  is the only density such that  $f_-(\rho_{-c}) = -c\rho$ . We regroup those states in  $\Gamma_-^{sub}$ ;

- the supersonic ones, and more precisely the states such that  $\alpha < -c\rho$ , which formed the hypograph of the function

$$f_-^{sup}(\rho) = \begin{cases} f_-\left(\frac{f_-(\rho)^2}{c^2\rho}\right) & \text{if } 0 < \rho < \rho_{-c} \\ -c\rho & \text{if } \rho_{-c} \leq \rho. \end{cases}$$

We regroup those states in  $\Omega_-^{sup}$ . □

*Proof (Lemma 3.5).* We do not prove this lemma which is exactly similar to Lemma 3.4. The curve of the states accessible by a 2-wave traveling faster than  $v$  and ending in  $(\rho_R, q_R)$ , can be parametrized by

$$f_+(\rho) = \begin{cases} \left[ \frac{\alpha_R}{\rho_R} + c \ln\left(\frac{\rho}{\rho_R}\right) \right] \rho & \text{if } \rho_{R,ex} \leq \rho \leq \rho_R, \\ \alpha_R + \left( \frac{\alpha_R}{\rho_R} + c\sqrt{\frac{\rho}{\rho_R}} \right) (\rho - \rho_R) & \text{if } \max(\rho_{R,ex}, \rho_R) < \rho, \end{cases}$$

where

$$\rho_{R,ex} = \begin{cases} \rho_R e^{\frac{v-(u_R+c)}{c}} & \text{if } u_R + c \geq v, \\ \left(\frac{v-u_R}{c}\right)^2 \rho_R & \text{if } u_R + c < v. \end{cases}$$

If  $u_R - v \geq -c$  it is possible to follow all the 2-shocks and some 2-rarefaction waves. In this case  $f_+(\rho_{R,ex})$  belongs to the line  $\alpha = -c\rho$  and  $f'_+(\rho_{R,ex}) = 0$ . On the contrary if  $u_R - v < -c$ , we can only follow 2-shocks, and the state  $(\rho_{R,ex}, \alpha_{R,ex})$  with the highest density we can reach verifies

$$\alpha_{R,ex} = \alpha_R \quad \text{and} \quad \rho_{R,ex} = \frac{\alpha_R^2}{c^2\rho_R}.$$

We easily check that that  $f_+$  is convex, increasing and crosses the line  $\alpha = c\rho$  for a unique density that we denote by  $\rho_c$ .

After similar computations for the 1-wave ending on a state  $(\rho_*, q_* = f_+(\rho_*))$  we obtain

$$f_+^{sub}(\rho) = \begin{cases} f_+(\rho) & \text{if } \rho_{R,ex} \leq \rho \leq \rho_c, \\ c\rho & \text{if } \rho_c \leq \rho, \end{cases}$$

and

$$f_+^{sup} = \begin{cases} f_+\left(\frac{f_+(\rho)^2}{c^2\rho}\right) & \text{if } 0 < \rho < \rho_c, \\ c\rho & \text{if } \rho_c \leq \rho. \end{cases}$$

□

### 3.2 Resolution of the Riemann problem in the subsonic case

We are now in position to solve the Riemann problem (14) for a particle moving at speed  $v$ . In the sequel, we denote by  $(\rho_+, q_+)$  the trace on the left of the particle, i.e. on the line  $x = (vt)_-$  and by  $(\rho_-, q_-)$  the trace on its right, i.e. on the line  $x = (vt)_+$ .



**Lemma 3.7.** *If  $(\rho_L, q_L)$  and  $(\rho_R, q_R)$  verify*

$$u_L - v \leq c \quad \text{and} \quad -c \leq u_R - v$$

*then  $(\rho_-, q_-)$  and  $(\rho_+, q_+)$  are necessary subsonic, i.e.,*

$$-c \leq u_- - v \leq c \quad \text{and} \quad -c \leq u_+ - v \leq c.$$

*Proof.* This is a straightforward consequence of Proposition 2.5 that describe the germ and of the form of  $V_-(\rho_L, \alpha_L, v)$  and  $V_+(\rho_R, \alpha_R, v)$  exhibited in Lemmas 3.4 and 3.5 and resumed on Figure 2. Proposition 2.5 implies that  $q_- - v\rho_- = q_+ - v\rho_+$ . We denote by  $\alpha$  this quantity. If  $\alpha = 0$  there is nothing to prove. Suppose  $\alpha > 0$ . Lemma 3.4 shows that  $V_-(\rho_L, \alpha_L, v)$  contains only states such that  $u - v \leq c$ . As a consequence, we necessary have that  $\rho_- \geq \frac{\alpha}{c}$ , and that  $\rho_+ \geq \frac{\alpha}{c}$  by the third point of Proposition 2.5. If  $\alpha < 0$ , Lemma 3.4 shows that  $V_+(\rho_R, \alpha_R, v)$  contains only states verifying  $u - v \geq -c$ . Using the last point of Proposition 2.5, we obtain that both  $\rho_+$  and  $\rho_-$  are bigger than  $\frac{|\alpha|}{c}$ .  $\square$

We are now looking for  $(\rho_-, \alpha_-)$  in  $\Gamma_-^{sub}$  and  $(\rho_+, \alpha_+)$  in  $\Gamma_+^{sub}$ . The quantity  $\alpha = q - v\rho$  being conserved through the particle, it is more convenient to parametrize the accessible sets  $\Gamma_-^{sub}$  and  $\Gamma_+^{sub}$  by  $\alpha$  rather than by  $\rho$ . For this purpose, we introduce  $g_-^{sub}$  and  $g_+^{sub}$ , the inverses of  $f_-^{sub}$  and  $f_+^{sub}$ . They exist because these two functions are strictly monotone. We now have

$$\Gamma_-^{sub} = \{(\rho, \alpha), \rho = g_-^{sub}(\alpha), \alpha \leq f_-^{sub}(\rho_{L,ex})\}$$

and

$$\Gamma_+^{sub} = \{(\rho, \alpha), \rho = g_+^{sub}(\alpha), \alpha \geq f_+^{sub}(\rho_{R,ex})\}.$$

**Lemma 3.8.** *The function*

$$\Delta(\alpha) = \left( \frac{\alpha^2}{g_-^{sub}(\alpha)} + c^2 g_-^{sub}(\alpha) \right) - \left( \frac{\alpha^2}{g_+^{sub}(\alpha)} + c^2 g_+^{sub}(\alpha) \right) - \lambda \alpha$$

- *is strictly decreasing on  $(0, f_-^{sub}(\rho_{L,ex}))$  where  $g_-^{sub} \geq g_+^{sub}$ ;*
- *is strictly decreasing on  $(f_+^{sub}(\rho_{R,ex}), 0)$  where  $g_-^{sub} \leq g_+^{sub}$ .*

*Proof.* We compute the derivative of  $\Delta$ .

$$\begin{aligned} \Delta'(\alpha) &= \left[ \left( \frac{\alpha^2}{g_-^{sub}} + c^2 g_-^{sub} \right) - \left( \frac{\alpha^2}{g_+^{sub}} + c^2 g_+^{sub} \right) - \lambda \alpha \right]' \\ &= \left( \frac{2\alpha}{g_-^{sub}} - \frac{\alpha^2 (g_-^{sub})'}{(g_-^{sub})^2} + c^2 (g_-^{sub})' \right) - \left( \frac{2\alpha}{g_+^{sub}} - \frac{\alpha^2 (g_+^{sub})'}{(g_+^{sub})^2} + c^2 (g_+^{sub})' \right) - \lambda \\ &= 2\alpha \left( \frac{1}{g_-^{sub}} - \frac{1}{g_+^{sub}} \right) + (g_-^{sub})' \left( c^2 - \frac{\alpha^2}{(g_-^{sub})^2} \right) - (g_+^{sub})' \left( c^2 - \frac{\alpha^2}{(g_+^{sub})^2} \right) - \lambda \end{aligned}$$

Thanks to the hypothesis on the signs of  $\alpha$  and  $g_-^{sub} - g_+^{sub}$ , the first term  $2\alpha \left( \frac{1}{g_-^{sub}} - \frac{1}{g_+^{sub}} \right)$  is non-positive. Moreover  $g_-^{sub}$  decreases,  $g_+^{sub}$  increases, and  $\Gamma_{\pm}^{sub}$  contains only subsonic states, so that  $c^2 - \frac{\alpha^2}{g_{\pm}^{sub}} \geq 0$ .  $\square$

We are now in position to state the result in the subsonic case.

**Theorem 3.9.** *Let  $v \in \mathbb{R}$ ,  $(\rho_L, \alpha_L) \in \mathbb{R}_+^* \times \mathbb{R}$  and  $(\rho_R, \alpha_R) \in \mathbb{R}_+^* \times \mathbb{R}$  such that*

$$-c \leq u_L - v \leq c \quad \text{and} \quad -c \leq u_R - v \leq c.$$

*Then the set*

$$\mathcal{G}_v \cap (V_-(\rho_L, \alpha_L, v) \times V_+(\rho_R, \alpha_R, v))$$

*is reduced to a unique element, that belongs to the subsonic triangle*

$$\{(\rho, \alpha), -c\rho \leq \alpha \leq c\rho\}^2 = \{(\rho, u), -c \leq u - v \leq c\}^2.$$

*In other words, the Riemann problem (14) admits a unique solution, which is entirely subsonic.*

*Proof. Step 1: Properties of the traces.*

Suppose

$$((\rho_-, \alpha_-), (\rho_+, \alpha_+)) \in \mathcal{G}_v \cap (V_-(\rho_L, \alpha_L, v) \times V_+(\rho_R, \alpha_R, v))$$

then  $\alpha_- = \alpha_+$  by Definition 2.5. We denote by  $\alpha$  this quantity. In Lemma 3.7 we proved that

$$-c \leq u_- - v \leq c \quad \text{and} \quad -c \leq u_+ - v \leq c.$$

Looking closer at the definition of the germ we see that:

- If  $\alpha = 0$  then  $\rho_- = \rho_+$  and  $q_- = q_+$ ;
- If  $\alpha > 0$ , then  $\rho_- > \rho_+$ ;
- If  $\alpha < 0$ , then  $\rho_- < \rho_+$ .

*Step 2: It exists a unique  $\alpha_0$  such that  $g_-^{sub}(\alpha_0) = g_+^{sub}(\alpha_0)$ .*

Suppose that the upper extremity  $(\rho_{L,ex}, g_-^{sub}(c\rho_{L,ex}))$  of  $\Gamma_-^{sub}$  does not belong to  $\Gamma_+^{sub}$ . Suppose that the lower extremity  $(\rho_{R,ex}, g_+^{sub}(-c\rho_{R,ex}))$  does not belong to  $\Gamma_-^{sub}$ . We then have

$$g_-^{sub}(c\rho_{L,ex}) < g_+^{sub}(c\rho_{L,ex}) \quad \text{and} \quad g_+^{sub}(-c\rho_{R,ex}) < g_-^{sub}(-c\rho_{R,ex}),$$

and we conclude by continuity of  $g_-^{sub} - g_+^{sub}$ .

*Step 3: Conclusion.*

If  $\alpha_0 > 0$ , as  $g_-^{sub} - g_+^{sub}$  is decreasing, we have  $g_-^{sub} - g_+^{sub} \geq 0$  on  $[0, \alpha_0]$ . Moreover  $\Delta(\alpha_0) = -\lambda\alpha_0 < 0$  and  $\Delta(0) = c^2(g_-^{sub}(0) - g_+^{sub}(0)) > 0$ . Lemma 3.8 gives that there exists a unique  $\alpha \in [0, \alpha_0]$  such that  $\Delta(\alpha) = 0$ . But  $g_-^{sub} - g_+^{sub} < 0$  if  $\alpha > \alpha_0$ , so the first step exactly says that any solution is on the interval  $\alpha \in [0, \alpha_0]$ . If  $\alpha_0 < 0$ , the same reasoning holds on  $[\alpha_0, 0]$ .  $\square$

### 3.3 Resolution of the Riemann problem in the supersonic case

If  $\alpha_L > c\rho_L$  or  $\alpha_R < -c\rho_R$ , Lemma 3.7 does not hold anymore. Without loss of generality, let us assume that  $\alpha_L > c\rho_L$ ; the case  $\alpha_R < -c\rho_L$  may be treated in a symmetrical way. We must study in detail the case where  $(\rho_L, \alpha_L)$  belongs to  $V_+(\rho_R, \alpha_R, v)$ . For this purpose, we introduce some notation, summarized on Figure 5, which also recall the notation introduced in Lemmas 3.4 and 3.5. In the sequel we denote by:

- for any subscript  $i$  and for any point  $(\rho_i, \alpha_i)$ ,  $\tilde{\rho}_i = \frac{\alpha_i^2}{c^2\rho_i}$ . Remark that  $\tilde{\tilde{\rho}}_i = \frac{\alpha_i^2}{c^2\tilde{\rho}_i} = \rho_i$ ;
- $\rho_{L,ex}$  the extremity of the curve  $g_-^{sub}$ . Lemma 3.4 shows that when  $u_L - v > c$ ,  $\rho_{L,ex} = \tilde{\rho}_L$ , and that

$$\forall \alpha < 0, (g_-^{sup}(\alpha), \alpha) = (\widetilde{g_-^{sub}(\alpha)}, \alpha) \quad \text{and} \quad \forall \alpha > 0, (g_+^{sup}(\alpha), \alpha) = (\widetilde{g_+^{sub}(\alpha)}, \alpha).$$

- $\rho_E = g_+^{sup}(\alpha_L)$  the intersection of the line  $\alpha = \alpha_L$  with  $\Gamma_+^{sup}$ . Note that  $\tilde{\rho}_E = g_+^{sub}(\alpha_L)$ .
- $(\rho_{R,ex}, \alpha_{R,ex})$  the extremity of the curve  $g_+^{sub}$ . We recall that, by Lemma 3.5, if  $\alpha_R < -c\rho_R$ ,  $\alpha_{R,ex} = \alpha_R$  and  $\rho_{R,ex} = \frac{\alpha_R^2}{c^2\rho_R}$ ; and that if  $\alpha_R \geq -c\rho_R$ ,  $\rho_{R,ex} \leq \rho_R$  and  $\alpha_{R,ex} = -c\rho_{R,ex}$ .
- $\rho_F = g_-^{sup}(\alpha_{R,ex})$  the intersection of the line  $\alpha = \alpha_{R,ex}$  with  $\Gamma_{sup}^-$ . Note that  $\tilde{\rho}_F = g_+^{sub}(\alpha_{R,ex})$ .

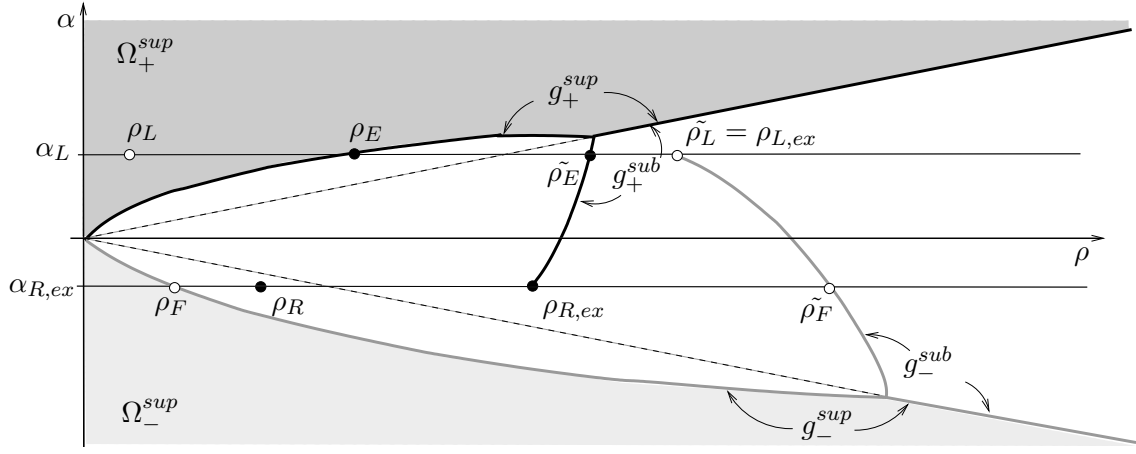


Figure 5: Notation for the supersonic case  $\alpha_L > c\rho_L$ .  $V_+(v, \rho_R, \alpha_R)$  is the union of the open set  $\Omega_+^{sub}$  above the graph of  $g_+^{sup}$  and of the graph of  $g_+^{sub}$ ;  $V_-(v, \rho_L, \alpha_L)$  is the union of the open set  $\Omega_-^{sub}$  below the graph of  $g_-^{sup}$  and of the graph of  $g_-^{sub}$ .

We first exhibit a link between the position of  $(\rho_L, \alpha_L)$  and the position of  $(\rho_R, \alpha_R)$ .

**Lemma 3.10.** *Let  $(\rho_L, \alpha_L) \in \mathbb{R}_+^* \times \mathbb{R}$  and  $(\rho_R, \alpha_R) \in \mathbb{R}_+^* \times \mathbb{R}$  such that  $\alpha_L > c\rho_L$  and  $\alpha_R < -c\rho_R$ . It is not possible to have*

$$(\rho_L, \alpha_L) \in \Omega_+^{sup} \quad \text{and} \quad (\rho_R, \alpha_R) \in \Omega_-^{sup}.$$

*Proof.* The hypothesis  $\alpha_R < -c\rho_R$  implies that  $\rho_{R,ex} = \tilde{\rho}_R$ . Suppose that  $(\rho_L, \alpha_L)$  belongs to  $\Omega_+^{sup}$ . Then  $\rho_L < \rho_E$ , and we have  $\tilde{\rho}_E < \tilde{\rho}_L$ . The monotonicity of  $g_+^{sub}$  and  $g_-^{sub}$  gives  $\tilde{\rho}_R = \rho_{R,ex} < \tilde{\rho}_F$ . Thus  $\rho_R > \rho_F$ , which means that  $(\rho_R, \alpha_R)$  does not belong to  $\Omega_-^{sup}$ .  $\square$

It allows us to exclude the case  $(\rho_R, \alpha_R) \in \Omega_-^{sup}$  of our study. Indeed, if  $(\rho_R, \alpha_R) \in \Omega_-^{sup}$ , then  $\alpha_R < -c\rho$  and  $(\rho_L, \alpha_L)$  does not belong to  $\Omega_-^{sup}$ , and we treat that case by symmetry. We now state the result in the supersonic case.

**Theorem 3.11.** *Let  $(\rho_L, \alpha_L) \in \mathbb{R}_+^* \times \mathbb{R}$  and  $(\rho_R, \alpha_R) \in \mathbb{R}_+^* \times \mathbb{R}$  such that  $\alpha_L > c\rho_L$  and  $(\rho_R, \alpha_R) \notin \Omega_-^{sup}$ . Then the set*

$$\mathcal{G}_v \cap (V_-(\rho_L, \alpha_L, v) \times V_+(\rho_R, \alpha_R, v))$$

*is reduced to a unique element  $((\rho_-, \alpha_-), (\rho_+, \alpha_+))$ . Two cases arise:*

- *either both traces are subsonic:  $-c\rho_- \leq \alpha_- \leq c\rho_-$  and  $-c\rho_+ \leq \alpha_+ \leq c\rho_+$ ;*
- *or both traces are supersonic:  $(\rho_-, \alpha_-) = (\rho_L, \alpha_L)$  and  $\alpha_+ \geq c\rho_+$ .*

*Proof.* The proof relies on the relative positions of  $\rho_L$  and  $\rho_E$ , and on the fact that the transformation  $(\rho, \alpha) \mapsto (\tilde{\rho} = \frac{\alpha^2}{c^2\rho}, \alpha)$  reverses the positions of points around the point  $(\alpha/c, \alpha)$ , as depicted on Figure 5.

*Case 1:  $\rho_L \leq \rho_E$ , or equivalently  $\rho_L \in \Omega_+^{sup}$ .*

In that case, we can chose  $(\rho_-, \alpha_-) = (\rho_L, \alpha_L)$  if and only if

$$\left( \frac{\alpha_L^2}{\rho_L} + c^2\rho_L \right) - \left( \frac{\alpha_L^2}{\rho_E} + c^2\rho_E \right) > \lambda\alpha_L.$$

Assume that this inequality is fulfilled. Then, as  $f_{\alpha_L} : \rho \mapsto \frac{\alpha_L^2}{\rho} + c^2\rho$  decreases on  $(0, \frac{\alpha}{c})$ , there exists a unique  $\rho_+ \in (\rho_L, \rho_E)$  such that

$$\left( \frac{\alpha_L^2}{\rho_L} + c^2\rho_L \right) - \left( \frac{\alpha_L^2}{\rho_+} + c^2\rho_+ \right) = \lambda\alpha_L.$$

Therefore we obtain the solution

$$((\rho_L, \alpha_L), (\rho_+, \alpha_L)) \in \mathcal{G}_v \cap (V_-(\rho_L, \alpha_L, v) \times V_+(\rho_R, \alpha_R, v)).$$

In that case we have

$$\left(\frac{\alpha_L^2}{\tilde{\rho}_L} + c^2 \tilde{\rho}_L\right) - \left(\frac{\alpha_L^2}{\tilde{\rho}_E} + c^2 \tilde{\rho}_E\right) > \lambda \alpha_L,$$

so  $\Delta(\alpha_L) > 0$ . The function  $g_-^{sub} - g_+^{sub}$  decreases, so it remain positive on  $(0, \alpha_L)$ . Therefore, Lemma (3.8) shows that there is no other solution. Suppose now that

$$\left(\frac{\alpha_L^2}{\rho_L} + c^2 \rho_L\right) - \left(\frac{\alpha_L^2}{\rho_E} + c^2 \rho_E\right) < \lambda \alpha_L.$$

We can conclude as in the subsonic case. Indeed, we have that  $\Delta(\alpha_L) < 0$  and  $\Delta(0) > 0$  and we conclude by monotonicity of  $\Delta$  as in Lemma 3.8, while the uniqueness follows from the Step 1.

*Case 2:  $\rho_L > \rho_E$ , or equivalently  $\rho_L \notin \Omega_+^{sup}$ .*

It implies that  $\tilde{\rho}_E > \tilde{\rho}_L$ . Moreover,  $(\rho_R, \alpha_R)$  does not belong to  $\Omega_-^{sup}$ . Therefore,  $\rho_F < \rho_R$  and  $\tilde{\rho}_F \geq \rho_{R,ex}$ . Thus, there exists a unique  $\alpha_0$  such that  $g_+^{sub}(\alpha_0) = g_-^{sub}(\alpha_0)$  and we conclude exactly as in the subsonic case.  $\square$

We recover the waited asymptotics when the friction parameter  $\lambda$  becomes very small or very large.

**Proposition 3.12.** *When  $\lambda$  tends to infinity, the solution of (14) tends to the solution of the Riemann problem with the same initial data for the Euler equation with a solid wall along  $x = vt$ . When  $\lambda$  vanishes, the solution of (14) tends to the solution of the Riemann problem with the same initial data for the Euler equation without particle.*

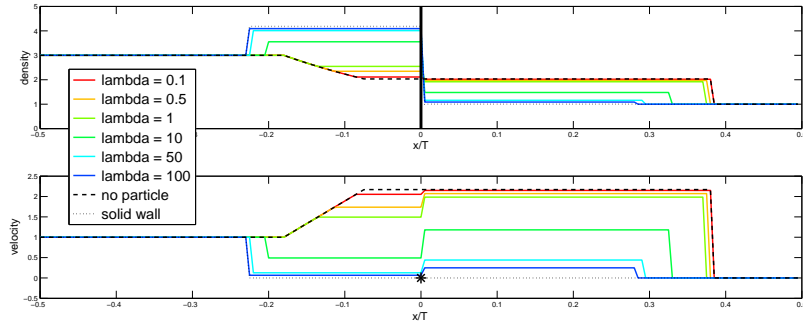


Figure 6: Behavior of the solutions of a subsonic Riemann problem when  $\lambda$  tends to zero or to infinity.

*Proof.* Let us begin by the case  $\lambda \rightarrow +\infty$ . Then in the supersonic case, it becomes impossible to chose  $(\rho_-, \alpha_-) = (\rho_L, \alpha_L)$  because the inequality

$$\left(\frac{\alpha_L^2}{\rho_L} + c^2 \rho_L\right) - \left(\frac{\alpha_L^2}{\rho_E} + c^2 \rho_E\right) > \lambda \alpha_L$$

will always fail. It can be seen on Figure 7, where this property holds for  $\lambda = 1$ , while it is lost for  $\lambda = 20$ . In the subsonic case, as  $g_-^{sub}$  and  $g_+^{sub}$  take value in a bounded interval, for all non null  $\alpha$ ,  $\Delta(\alpha)$  tends to  $-\infty$  as  $\lambda$  tends to infinity. Therefore,  $(\rho_-, \alpha_-)$  tends to  $(g_-^{sub}(0), 0)$  while  $(\rho_+, \alpha_+)$  tends to  $(g_+^{sub}(0), 0)$ . This is exactly the solution for the Riemann problem with a solid wall. When  $\lambda$  tends to zero, remark that in the subsonic case (or the supersonic case when  $(\rho_L, \alpha_L)$  is not in  $\Omega_+^{sup}$ ) the solution of  $\Delta(\alpha) = 0$  tends to the crossing point  $\alpha_0$ . In the supersonic case when  $(\rho_L, \alpha_L) \in \Omega_+^{sup}$ , the inequality

$$\left(\frac{\alpha_L^2}{\rho_L} + c^2 \rho_L\right) - \left(\frac{\alpha_L^2}{\rho_E} + c^2 \rho_E\right) > \lambda \alpha_L$$

is verified for small enough  $\lambda$ , and  $(\rho_-, \alpha_-)$  is  $(\rho_L, \alpha_L)$  while  $(\rho_+, \alpha_+)$  tends to  $(\rho_L, \alpha_L)$ . It corresponds to the value of the solution of the Euler equation without particle on the line  $x = vt$ . This two asymptotics behaviors are depicted, in the subsonic case, on Figure 6.  $\square$

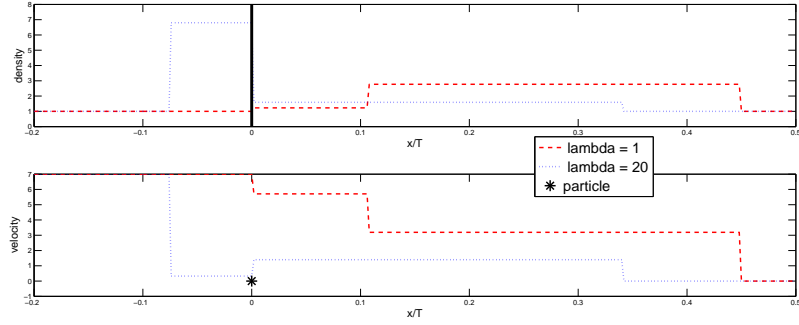


Figure 7: Two types of solutions when  $\alpha_L > c\rho_L$ . For small enough  $\lambda$  there is no wave on the left of the particle. As  $\lambda$  becomes greater, we recover the subsonic solution.

## 4 Other types of friction

The purpose of this section is to exhibit some sufficient conditions on the drag force  $D$  for the Riemann problem for a particle moving at constant speed  $v$ ,

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{\rho} + c^2 \rho \right) = -D(\rho, u - v) \delta_{vt}, \\ \rho(0, x) = \rho_L \mathbf{1}_{x < 0} + \rho_R \mathbf{1}_{x > 0}, \\ q(0, x) = q_L \mathbf{1}_{x < 0} + q_R \mathbf{1}_{x > 0}, \end{cases} \quad (15)$$

to have a unique self-similar solution. With such a general drag force, the germ is more complicated because each discontinuity in the solution of Problem (8) with a thick particle yields to different traces on the extremities of the particle, unlike in Theorem 2.5. This raises new issues in the supersonic case, which requires some extra attention. Actually, depending on the choice of the drag force  $D$ , it is possible to lose uniqueness and to obtain up to three solutions. This is not surprising: the choice  $D(\rho, u - v) = \rho$  is, up to a change of variable and pressure law, similar to the problem of the shallow water with a discontinuous topography, where these three solutions arise. In a more general framework, hyperbolic systems with resonant source term like (15) have been investigated in [IT92] and [GL04], where the possible coexistence of three solutions is proved.

### 4.1 Description of the germ

We assume that the drag force  $D$  has the same sign as  $u - h'(t)$ , and is null only when  $u = h'(t)$ . This seems to be reasonable assumptions for a drag force. Following what was done in Section 2.2 to define the germ, we fix the velocity of the particle equals to  $v$  and a regularization  $H^\varepsilon$  of the Heaviside function fulfilling the hypothesis of Theorem 2.5. In the sequel, we denote by  $F_\alpha$ , for  $\alpha \neq 0$ , the antiderivative of the function

$$\rho \mapsto \frac{1}{|D(\rho, \alpha)|} \left( -\frac{\alpha^2}{\rho^2} + c^2 \right)$$

which is null in  $\frac{|\alpha|}{c}$ . The first result consists in the generalization of Lemma 2.3 and Proposition 2.5 for all drag forces.

**Theorem 4.1.** *Suppose that the pair of states  $(\rho_-, q_-)$  and  $(\rho_+, q_+)$  in  $\mathbb{R}_+ \times \mathbb{R}$  verify the two relations below.*

- First,

$$q_- - v\rho_- = q_+ - v\rho_+.$$

*We denote by  $\alpha$  this quantity;*

- Second,

$$- \text{ Either } \alpha = 0, \rho_- = \rho_+ \text{ and } q_- = q_+;$$

– Or  $0 < \alpha$ ,  $(\frac{\alpha}{c} - \rho_+)(\frac{\alpha}{c} - \rho_-) \geq 0$  and

$$F_\alpha(\rho_-) - F_\alpha(\rho_+) = 1;$$

– Or  $\alpha < 0$ ,  $(\frac{|\alpha|}{c} - \rho_+)(\frac{|\alpha|}{c} - \rho_-) \geq 0$  and

$$F_\alpha(\rho_+) - F_\alpha(\rho_-) = 1;$$

– Or  $c\rho_- < \alpha$ ,  $\rho_+ \geq \frac{\alpha}{c}$ , and there exists  $\rho \in (\rho_-, \frac{\alpha}{c})$  and  $\theta \in [0, 1]$  such that

$$\begin{cases} F_\alpha(\rho_-) - F_\alpha(\rho) &= \theta, \\ F_\alpha(\frac{\alpha^2}{c^2\rho}) - F_\alpha(\rho_+) &= (1 - \theta), \end{cases}$$

– Or  $\alpha < -c\rho_+$ ,  $\rho_- \geq \frac{|\alpha|}{c}$  and there exists  $\rho \in (\rho_+, \frac{|\alpha|}{c})$  and  $\theta \in [0, 1]$  such that

$$\begin{cases} F_\alpha(\rho_+) - F_\alpha(\rho) &= \theta, \\ F_\alpha(\frac{\alpha^2}{c^2\rho}) - F_\alpha(\rho_-) &= (1 - \theta), \end{cases}$$

Then, for all positive  $\varepsilon$  and for all regularization  $(H^\varepsilon)$  fulfilling the hypothesis of Theorem 2.5, there exists a piecewise  $C^1$  Lax solution of

$$\begin{cases} \partial_t \rho^\varepsilon + \partial_x q^\varepsilon &= 0, \\ \partial_t q^\varepsilon + \partial_x \left( \frac{(q^\varepsilon)^2}{\rho^\varepsilon} + c^2 \rho^\varepsilon \right) &= -D(\rho^\varepsilon, \rho^\varepsilon(u^\varepsilon - v))(H^\varepsilon)'(x - vt), \end{cases} \quad (16)$$

only depending on  $\xi = x - vt$ , such that  $(q^\varepsilon(-\varepsilon/2), \rho^\varepsilon(-\varepsilon/2)) = (q_-, \rho_-)$  and  $(q^\varepsilon(\varepsilon/2), \rho^\varepsilon(\varepsilon/2)) = (q_+, \rho_+)$ . Conversely, if  $(\rho_-, q_-)$  and  $(\rho_+, q_+)$  are the values in  $-\varepsilon/2$  and  $\varepsilon/2$  of such a solution of (16), then they verify the two relations stated above.

*Proof.* We mimic the proof of Theorem 2.5. If a solution of (16) has a discontinuity at a point  $\xi_0$ , it verifies the Rankine-Hugoniot relations for a shock at speed  $v$ :

$$q^\varepsilon(\xi_0^+) - v\rho^\varepsilon(\xi_0^+) = q^\varepsilon(\xi_0^-) - v\rho^\varepsilon(\xi_0^-) := \alpha \quad \text{and} \quad \rho^\varepsilon(\xi_0^-)\rho^\varepsilon(\xi_0^+) = \frac{(\alpha^\varepsilon)^2}{c^2}.$$

On its intervals of smoothness, the solutions of (16) verify

$$\begin{cases} -v(\rho^\varepsilon)' + (q^\varepsilon)' &= 0, \\ -v(q^\varepsilon)' + \left( \frac{(q^\varepsilon)^2}{\rho^\varepsilon} + c^2 \rho^\varepsilon \right)' &= -D(\rho^\varepsilon, \rho^\varepsilon(u^\varepsilon - v))(H^\varepsilon)'. \end{cases} \quad (17)$$

Therefore  $\alpha^\varepsilon = q^\varepsilon - v\rho^\varepsilon$  remains constant on the whole interval  $(-\varepsilon/2, \varepsilon/2)$ . We treat the case where this conserved quantity is strictly positive, and denote it by  $\alpha_-$ . We fix the left state

$$\begin{cases} \rho^\varepsilon(-\varepsilon/2) = \rho_-, \\ q^\varepsilon(-\varepsilon/2) = q_-, \end{cases}$$

with  $q_- - v\rho_- = \alpha_-$ . Replacing the momentum  $q$  by  $\alpha + v\rho$  in the second line of (17) yields

$$\left( \frac{\alpha^2}{\rho^\varepsilon} + c^2 \rho^\varepsilon \right)' = -D(\rho^\varepsilon, \alpha)(H^\varepsilon)'$$

which rewrites, by definition of  $F_\alpha$ ,

$$[F_{\alpha_-}(\rho^\varepsilon(\xi))] = \frac{1}{|D(\rho^\varepsilon, \alpha)|} \left( -\frac{\alpha^2}{(\rho^\varepsilon)^2} + c^2 \right) (\rho^\varepsilon)' = -\text{sign}(\alpha)(H^\varepsilon)'. \quad (18)$$

The function  $F_{\alpha_-}$  decreases on  $(0, \frac{\alpha_-}{c})$  and increases on  $(\frac{\alpha_-}{c}, +\infty)$ . Its minimum is reached for  $\rho = \frac{\alpha_-}{c}$ . Moreover, the regularization  $H^\varepsilon$  of the Heaviside function is increasing. As a consequence,  $F_{\alpha_-}(\rho^\varepsilon)$  decreases, and  $\rho^\varepsilon$  cannot cross continuously  $\alpha_-/c$ : on its interval of smoothness, a solution of (16) is always subsonic (i.e  $|u^\varepsilon - v| \leq c$ ) or always supersonic (i.e  $|u^\varepsilon - v| \geq c$ ). On the other hand, a discontinuity at a point  $\xi_0$  is entropy satisfying if and only if  $u^\varepsilon(\xi_0^-) - v > c$ . Therefore a solution

has no discontinuity is  $u_- - v \leq c$  (which rewrites  $\alpha_- \leq c\rho_-$ ) and has at most one discontinuity if  $u_- - v > c$ . When the solution is continuous on the whole interval  $(-\varepsilon/2, \varepsilon/2)$ , we integrate (18) to obtain

$$F_{\alpha_-}(\rho_-) - F_{\alpha_-}(\rho_+) = 1.$$

Suppose now that the solution has a discontinuity in  $\xi_0 \in (-\varepsilon/2, \varepsilon/2)$ . Integrating (18) on  $(-\varepsilon/2, \xi_0)$  yields

$$F_{\alpha_-}(\rho_-) - F_{\alpha_-}(\rho(\xi_0^-)) = H^\varepsilon(\xi_0)$$

while integrating (18) on  $(\xi_0, \varepsilon/2)$  yields

$$F_{\alpha_-}(\rho(\xi_0^+)) - F_{\alpha_-}(\rho_+) = (1 - H^\varepsilon(\xi_0)).$$

The Rankine-Hugoniot relations imply that

$$\rho^\varepsilon(\xi_0^-)\rho^\varepsilon(\xi_0^+) = \frac{(\alpha_-)^2}{c^2}$$

and we obtain the third point with  $\rho = \rho^\varepsilon(\xi_0^-)$  and  $\theta = H^\varepsilon(\xi_0)$ . The case with a non-positive  $\alpha_-$  can be treated in the exact same way from right to left.  $\square$

*Remark 4.2.* Unlike what was happening with the linear drag force studied in Section 3, we now had that

$$F_\alpha\left(\frac{\alpha^2}{c^2\rho}\right) \neq F_\alpha(\rho).$$

As a consequence, when  $(\rho_-, q_-)$  is fixed such that  $q_- - v\rho_- > 0$  and  $\alpha_- > c\rho_-$ , each choice of  $\theta \in [0, 1]$  in Theorem 4.1 yields to a different right state  $(\rho_+, q_+)$ , as depicted on Figure 8 below.

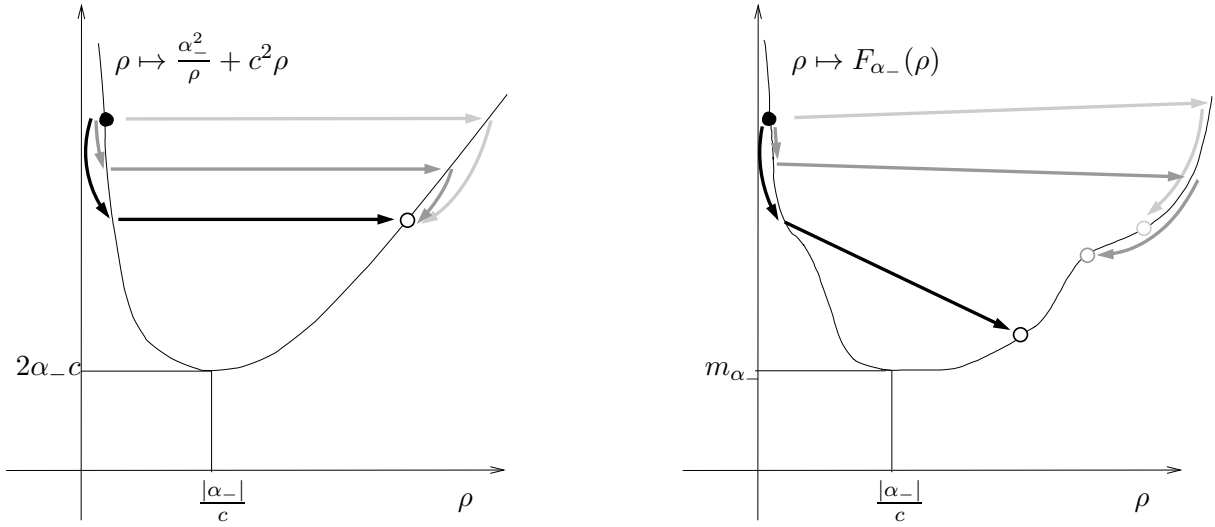


Figure 8: Densities  $\rho_+$  (white dots) accessible from  $\rho_-$  with  $\theta = 0$  (light grey)  $0 < \theta < 1$  (medium grey) and  $\theta = 1$  (dark grey), for the linear drag force  $D(\rho, u - v) = \rho(u - v)$  on the left, and for a general drag force on the right. The horizontal arrows are entropy satisfying shocks.

## 4.2 Riemann problem for a particle moving at constant speed

Let us now focus on the solution of the Riemann problem (2). In Theorem 4.3 below, we state an existence and uniqueness result under two hypothesis. The first one allows us to treat the subsonic case as we did in Section 3. The second one ensures that the germ has a particular form, which yields good properties in the supersonic case.

**Theorem 4.3.** *Suppose that the following properties hold:*

- The function

$$\alpha \mapsto F_\alpha(g_-^{sub}(\alpha)) - F_\alpha(g_+^{sub}(\alpha))$$

is strictly decreasing on any interval included in  $(0, +\infty)$  where  $g_-^{sub} \geq g_+^{sub}$  and on any interval included in  $(-\infty, 0)$  where  $g_-^{sub} \leq g_+^{sub}$ ;

- For every  $\alpha \in \mathbb{R}$ , for all states  $(\rho_1, \alpha)$  and  $(\rho_2, \alpha)$  in  $\mathbb{R}_+^* \times \mathbb{R}$  with  $\rho_1 < \rho_2 \leq \frac{|\alpha|}{c}$  we have

$$F_\alpha(\rho_1) - F_\alpha(\rho_2) \leq F_\alpha\left(\frac{\alpha^2}{c^2\rho_1}\right) - F_\alpha\left(\frac{\alpha^2}{c^2\rho_2}\right). \quad (19)$$

Then for all states  $(\rho_L, q_L)$  and  $(\rho_R, q_R)$  in  $\mathbb{R}_+^* \times \mathbb{R}$  and for every particle velocity  $v$  in  $\mathbb{R}$ , the Riemann problem

$$\begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x \left( \frac{q^2}{\rho} + c^2 \rho \right) = -D(\rho, u - v) \delta_{vt} \\ \rho(0, x) = \rho_L \mathbf{1}_{x < 0} + \rho_R \mathbf{1}_{x > 0} \\ q(0, x) = q_L \mathbf{1}_{x < 0} + q_R \mathbf{1}_{x > 0} \end{cases} \quad (20)$$

has a unique solution.

In the sequel we use the notation of Section 3.3 summarized on Figure 5. We recall in particular that for any  $(\rho, \alpha)$ , the state denoting by  $(\tilde{\rho}, \alpha)$  is reached with a shock at speed  $v$ :  $\tilde{\rho} = \frac{\alpha^2}{c^2\rho}$ . Let us first describe the form of the germ imposed by Hypothesis (19).

**Lemma 4.4.** *Suppose that the drag force  $D$  verifies the Hypothesis (19). Fix  $(\rho_-, q_-)$  such that  $\alpha_- = q_- - v\rho_-$  is positive and that  $\rho_- \leq \frac{\alpha_-}{c}$ . Then:*

- If it exists  $\rho_0 \leq \frac{\alpha_-}{c}$  such that  $F_{\alpha_-}(\rho_-) - F_{\alpha_-}(\rho_0) = 1$ , there exists a unique density  $\rho_{1,+}$  greater than  $\frac{\alpha_-}{c}$ , such that  $F_{\alpha_-}(\tilde{\rho}_-) - F_{\alpha_-}(\rho_{1,+}) = 1$ . Then solutions of (17) in  $\xi = \varepsilon/2$  can take the values  $(\rho_+, q_+)$  with

$$\rho_+ \in [\rho_{0,+}, \rho_{1,+}] \quad \text{and} \quad q_+ = \alpha_- + v\rho_+,$$

where we denote by  $\rho_{0,+} = \tilde{\rho}_0$ ;

- If  $\rho_0$  does not exist but  $\rho_{1,+}$  does, the solutions of (17) in  $\xi = \varepsilon/2$  can take the values  $(\rho_+, q_+)$  with

$$\rho_+ \in \left[ \frac{\alpha_-}{c}, \rho_{1,+} \right] \quad \text{and} \quad q_+ = \alpha_- + v\rho_+,$$

and we denote by  $\rho_{0,+} = \frac{\alpha_-}{c}$ ;

- If neither  $\rho_0$  nor  $\rho_{1,+}$  exist, system (17) does not admit any solution on the whole interval  $(-\varepsilon/2, \varepsilon/2)$ .

Those three cases are illustrated in Figure 9 below.

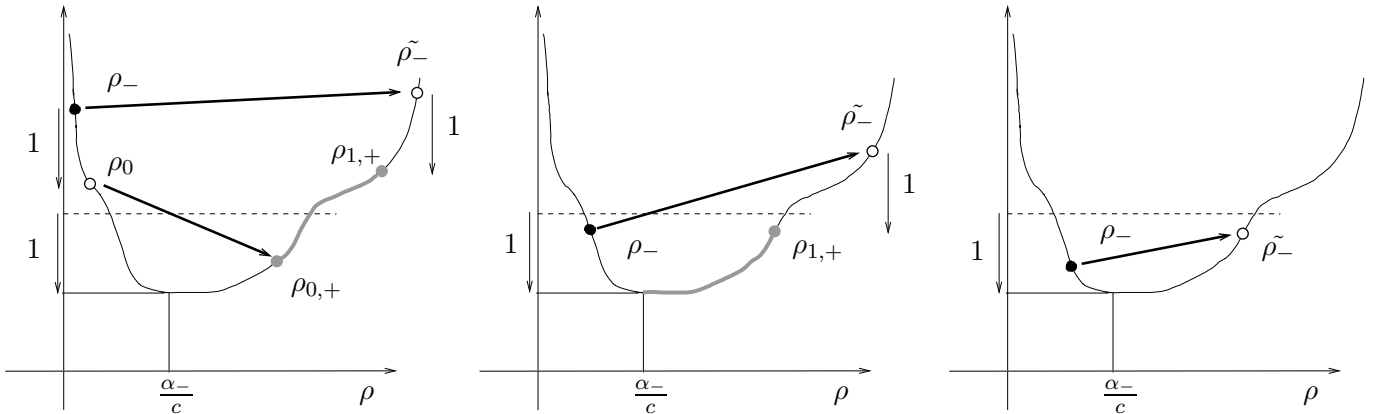


Figure 9: The three cases of Lemma 4.4 from left to right. The bold arrows are entropy shocks at speed  $v$ .



*Proof (Lemma 4.4).* The proof relies on the fact that Hypothesis (19) ensures that the values  $\rho_+(\theta)$  at the right of the particle are ordered toward  $\theta$  exactly as on Figure 8. Let us first suppose that  $\rho_0$  exists. Then by (19),  $F_{\alpha_-}(\tilde{\rho}_-) - F_{\alpha_-}(\tilde{\rho}_0) \geq 1$  and  $\rho_{1,+}$  exists. For each  $\theta \in [0, 1]$ , there exists a unique density  $\rho \in [\rho_-, \rho_0]$  such that

$$F_{\alpha_-}(\rho_-) - F_{\alpha_-}(\rho) = \theta.$$

By (19) we have that  $F_{\alpha_-}(\tilde{\rho}) - F_{\alpha_-}(\rho_{0,+}) \geq (1 - \theta)$ , and there exists a unique  $\bar{\rho} \geq \rho_{0,+}$  such that

$$F_{\alpha_-}(\tilde{\rho}) - F_{\alpha_-}(\bar{\rho}) = (1 - \theta).$$

Moreover we have

$$\begin{aligned} F_{\alpha_-}(\rho_{1,+}) - F_{\alpha_-}(\bar{\rho}) &= (F_{\alpha_-}(\rho_{1,+}) - F_{\alpha_-}(\tilde{\rho}_-)) + (F_{\alpha_-}(\tilde{\rho}_-) - F_{\alpha_-}(\tilde{\rho})) + (F_{\alpha_-}(\tilde{\rho}) - F_{\alpha_-}(\bar{\rho})) \\ &\geq (F_{\alpha_-}(\rho_{1,+}) - F_{\alpha_-}(\tilde{\rho}_-)) + (F_{\alpha_-}(\rho_-) - F_{\alpha_-}(\rho)) + (F_{\alpha_-}(\tilde{\rho}) - F_{\alpha_-}(\bar{\rho})) \\ &\geq -1 + \theta + (1 - \theta) \\ &\geq 0 \end{aligned}$$

and  $\bar{\rho} \leq \rho_{1,+}$ . By continuity, the entire interval  $[\rho_{0,+}, \rho_{1,+}]$  can be reached. If  $\rho_0$  does not exist and  $\rho_{1,+}$  does, the smaller density that we can reach is  $\frac{\alpha_-}{c}$ , because the map  $\rho \mapsto \bar{\rho}$  is continuous and decreasing.  $\square$

We are now in position to prove Theorem 4.3.

*Proof (Theorem 4.3).* Let us first treat the subsonic case where  $u_L - v \leq c$  and  $u_R - v \geq -c$ . It can be treated exactly as we did in Section 3.2 for the linear drag force. Indeed, the second and third points of Theorem 4.1, together with the monotonicity of  $F_\alpha$ , imply that the first, third and last point of Lemma (2.5) hold. Therefore, the first step of the proof of Theorem 3.9 remains true. It states that the traces are both subsonic, and are such that  $\alpha$  and  $\rho_- - \rho_+$  have the same sign. The second step shows the existence of  $\alpha_0$  such that  $g_-^{sub}(\alpha_0) = g_+^{sub}(\alpha_0)$ , and does not use the definition of the germ at all. Let us introduce

$$\Delta(\alpha) = F_\alpha(g_-^{sub}(\alpha)) - F_\alpha(g_+^{sub}(\alpha)) - \text{sign}(\alpha).$$

We have  $\Delta(\alpha_0) = -\text{sign}(\alpha)$ , and by Rolle's theorem there exists  $\rho_I \in |g_-^{sub}(\alpha), g_+^{sub}(\alpha)|$  such that

$$\begin{aligned} \Delta(\alpha) &= (g_-^{sub}(\alpha) - g_+^{sub}(\alpha))F'_\alpha(\rho_I) - \text{sign}(\alpha) \\ &= (g_-^{sub}(\alpha) - g_+^{sub}(\alpha)) \int_{\frac{|\alpha|}{c}}^{\rho_I} \frac{1}{|D(\alpha, r)|} \left( c^2 - \frac{\alpha^2}{r^2} \right) dr - \text{sign}(\alpha). \end{aligned}$$

If  $\alpha_0 > 0$ ,  $g_-^{sub} - g_+^{sub}$  is non-negative on  $(0, \alpha_0)$  and  $\Delta$  decreases between  $+\infty$  and  $-1$  on  $(0, \alpha_0)$ , while if  $\alpha_0 < 0$ ,  $\Delta$  decreases between  $1$  and  $-\infty$  on  $(\alpha_0, 0)$ . Thus there exists a unique  $\alpha_-$  such that  $\Delta(\alpha_-) = 0$  and the corresponding traces around the particle are  $(\rho_- = g_-^{sub}(\alpha_-), \alpha_-)$  and  $(\rho_+ = g_+^{sub}(\alpha_-), \alpha_-)$  give a solution. If  $\alpha_L$  is greater than  $c\rho_L$ , but  $(\rho_L, \alpha_L)$  does not belong to  $\Omega_+^{sup}$ ,  $\alpha_0$  exists and we conclude in a similar way.

Let us now tackle the supersonic case where  $(\rho_L, \alpha_L)$  belongs to  $\Omega_+^{sup}$  (the supersonic case where  $(\rho_R, \alpha_R)$  belongs to  $\Omega_-^{sup}$  can be treated symmetrically). Let us first suppose that, as in top of Figure 10,

$$F_{\alpha_L}(\rho_L) - F_{\alpha_L}(\rho_E) \geq 1.$$

Then, there exists  $\rho_0 \in [\rho_L, \rho_E]$  such that

$$F_{\alpha_L}(\rho_L) - F_{\alpha_L}(\rho_0) = 1,$$

and the state  $(\rho_0, \alpha_L)$  belongs to  $\Omega_+^{sup}$  and provides an admissible state on the right of the particle. Let us prove that it is the unique solution. As  $\rho_0 < \rho_E$ , Hypothesis (19) yields  $\rho_{0,+} > \tilde{\rho}_E$ . We also proved in Lemma 4.4 that  $\rho_{1,+} \geq \rho_{0,+}$ . Therefore, the interval  $[\rho_{0,+}, \rho_{1,+}]$  does not intersect  $\Gamma_+^{sub}$ . Eventually, Hypothesis (19) gives that  $F_\alpha(\tilde{\rho}_L) - F_\alpha(\tilde{\rho}_E) \geq 1$ . Thus we cannot choose any subsonic traces by Lemma 3.8. We now suppose that

$$F_{\alpha_L}(\rho_L) - F_{\alpha_L}(\rho_E) < 1.$$

If  $F_\alpha(\tilde{\rho}_L) - F_\alpha(\tilde{\rho}_E) < 1$  (Figure 10, in the middle), which rewrites  $\Delta(\alpha_L) < 0$ , Lemma 3.8 implies that there exists a unique solution inside the subsonic triangle  $|\alpha| < c$ . Moreover, Hypothesis (19) yield  $F_\alpha(\rho_L) - F_\alpha(\rho_E) < 1$ , and  $[\rho_{0,+}, \rho_{1,+}]$  is included in  $[\frac{\alpha_L}{c}, \tilde{\rho}_E]$ , so there is no solution with  $(\rho_L, \alpha_L)$  for the left trace. Eventually, if  $F_\alpha(\tilde{\rho}_L) - F_\alpha(\tilde{\rho}_E) > 1$  (at the bottom of Figure 10), there is no solution in the subsonic triangle. But in that case,  $\rho_{1,+}$  exists and is greater than  $\tilde{\rho}_E$ , while  $\rho_{0,+}$  is smaller than  $\tilde{\rho}_E$  (and might be equal to  $\frac{\alpha_L}{c}$ ). Therefore we can take  $(\rho_L, \alpha_L)$  for the left trace and  $(\tilde{\rho}_E, \alpha_L)$  for the right trace.

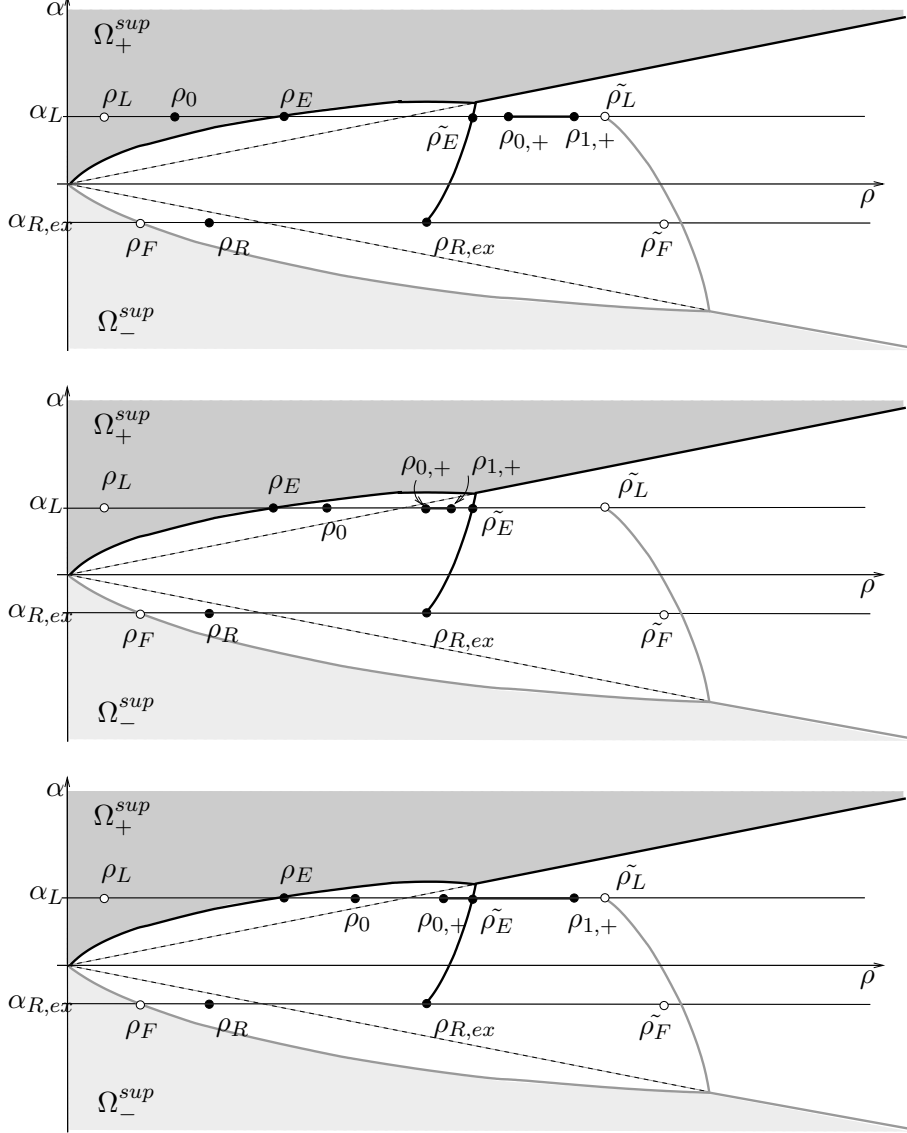


Figure 10: Relative positions of  $\rho_{0,+}$  and  $\rho_{1,+}$ .

The conditions of Theorem 4.3 are not easy to check. In the following Theorem, we exhibit two simpler conditions that imply them.

**Theorem 4.5.** *Consider a drag force*

$$D : \mathbb{R}_+^* \times \mathbb{R} \longrightarrow \mathbb{R} \\ (\rho, \alpha) \longmapsto D(\rho, \alpha)$$

having the same sign as  $\alpha$ . Suppose that  $D$  is an increasing function of  $\alpha$  and that  $|D|$  is a decreasing function of  $\rho$ . Then for all states  $(\rho_L, q_L)$  and  $(\rho_R, q_R)$  in  $\mathbb{R}_+^* \times \mathbb{R}$  and for every particle velocity  $v$  in  $\mathbb{R}$ , the Riemann problem (2) has a unique solution.

*Proof.* Let us show that those two condition on  $D$  imply the two conditions of Theorem 4.3.

$$\begin{aligned} [F_\alpha(g_-^{sub}(\alpha)) - F_\alpha(g_+^{sub}(\alpha))] &= F'_\alpha(g_-^{sub}(\alpha))(g_-^{sub})'(\alpha) + \left[ \frac{\partial}{\partial \alpha} F_\alpha \right] (g_-^{sub}(\alpha)) \\ &\quad - F'_\alpha(g_+^{sub}(\alpha))(g_+^{sub})'(\alpha) - \left[ \frac{\partial}{\partial \alpha} F_\alpha \right] (g_+^{sub}(\alpha)) \end{aligned}$$

On the one hand,  $\alpha \mapsto g_-^{sub}(\alpha)$  decreases,  $\alpha \mapsto g_+^{sub}(\alpha)$  increases, and for any  $\rho \geq \frac{|\alpha|}{c}$

$$F'_\alpha(\rho) = \frac{1}{|D(\rho, \alpha)|} \left( c^2 - \frac{\alpha^2}{\rho^2} \right) \geq 0.$$

It follows that

$$F'_\alpha(g_-^{sub}(\alpha))(g_-^{sub})'(\alpha) - F'_\alpha(g_+^{sub}(\alpha))(g_+^{sub})'(\alpha) \leq 0.$$

On the other hand for any  $\rho$ ,

$$F_\alpha(\rho) = \int_{\frac{|\alpha|}{c}}^{\rho} \frac{1}{|D(r, \alpha)|} \left( c^2 - \frac{\alpha^2}{r^2} \right) dr.$$

It implies that for any  $\frac{|\alpha|}{c} \leq \rho_1$  and  $\frac{|\alpha|}{\rho} \leq \rho_2$ ,

$$\begin{aligned} \left[ \frac{\partial}{\partial \alpha} F_\alpha \right] (\rho_2) - \left[ \frac{\partial}{\partial \alpha} F_\alpha \right] (\rho_1) &= \int_{\rho_1}^{\rho_2} \frac{\partial}{\partial \alpha} \left[ \frac{1}{|D(r, \alpha)|} \left( c^2 - \frac{\alpha^2}{r^2} \right) \right] dr \\ &= \int_{\rho_1}^{\rho_2} \left[ \frac{-\text{sign}(D(r, \alpha)) \partial_\alpha D(r, \alpha)}{|D(r, \alpha)|^2} \left( c^2 - \frac{\alpha^2}{r^2} \right) - \frac{2\alpha}{r^2 |D(r, \alpha)|} \right] dr. \end{aligned}$$

As  $D(r, \alpha)$  and  $\alpha$  have the same sign and  $D$  is an non-decreasing function of  $\alpha$ , this term has the opposite sign as  $\alpha(\rho_2 - \rho_1)$ . In particular, on any interval where  $\alpha$  is positive and  $g_-^{sub} \geq g_+^{sub}$  and on any interval where  $\alpha$  is negative and  $g_-^{sub} \leq g_+^{sub}$ ,

$$\left[ \frac{\partial}{\partial \alpha} F_\alpha \right] (g_-^{sub}(\alpha)) - \left[ \frac{\partial}{\partial \alpha} F_\alpha \right] (g_+^{sub}(\alpha)) \leq 0,$$

which proves the first hypothesis of Theorem 4.3. The second hypothesis states that the function

$$\rho \mapsto F_\alpha(\rho) - F_\alpha \left( \frac{\alpha^2}{c^2 \rho} \right)$$

increases on  $(0, \frac{|\alpha|}{c})$ . We compute the derivative of this function:

$$\begin{aligned} \frac{\partial}{\partial \rho} \left[ F_\alpha(\rho) - F_\alpha \left( \frac{\alpha^2}{c^2 \rho} \right) \right] &= F'_\alpha(\rho) + \frac{\alpha^2}{c^2 \rho^2} F'_\alpha \left( \frac{\alpha^2}{c^2 \rho} \right) \\ &= \frac{1}{|D|(\rho, \alpha)} \left( c^2 - \frac{\alpha^2}{\rho^2} \right) + \frac{\frac{\alpha^2}{c^2 \rho^2}}{|D|(\frac{\alpha^2}{c^2 \rho}, \alpha)} \left( c^2 - \frac{\alpha^2}{\frac{\alpha^4}{c^4 \rho^2}} \right) \\ &= \left[ \frac{1}{|D|(\rho, \alpha)} - \frac{1}{|D|(\frac{\alpha^2}{c^2 \rho}, \alpha)} \right] \left( c^2 - \frac{\alpha^2}{\rho^2} \right). \end{aligned}$$

On  $(0, \frac{|\alpha|}{c})$ , this quantity has same sign as

$$|D|(\rho, \alpha) - |D| \left( \frac{\alpha^2}{c^2 \rho}, \alpha \right),$$

which is positive if  $|D|$  is a decreasing function of  $\rho$ , because  $\rho \leq \frac{\alpha^2}{c^2 \rho}$  on  $(0, \frac{|\alpha|}{c})$ . □

**Example 4.6.** All frictions of the form

$$\Gamma(\rho, u, h') = \rho^n |u - h'|^{m-1} (u - h') = \rho^{n-m} |\alpha|^{m-1} \alpha$$

with  $n \geq 0$ ,  $m \geq 1$  and  $m \geq n$ , verify the two conditions of Theorem 4.5.

**Example 4.7.** The nonconservative product corresponding to the shallow water with discontinuous topography or to the gas dynamics in a nozzle with discontinuous cross section write

$$D(\rho, u - h') = \rho$$

and is increasing in  $\rho$ . Our theorem does not apply and actually, uniqueness is lost (see [LT03] and [LT07]).

### 4.3 Existence of up to three solutions

If Condition (19) of Theorem 4.3 does not hold, we can lose uniqueness and obtain up to three solutions. For example, suppose that this condition is reversed in

$$\forall \alpha, \forall \rho_1 < \rho_2 \leq \frac{|\alpha|}{c}, \quad F_\alpha(\rho_1) - F_\alpha(\rho_2) \geq F_\alpha(\tilde{\rho}_1) - F_\alpha(\tilde{\rho}_2). \quad (21)$$

Fix  $\alpha_L > 0$  and  $\rho_L < \frac{\alpha_L}{c}$ . For  $0 \leq \theta_1 \leq \theta_2 \leq 1$ , consider  $\rho_1$  and  $\rho_2$  such that

$$F_{\alpha_L}(\rho_L) - F_{\alpha_L}(\rho_i) = \theta_i,$$

and  $\bar{\rho}_1$  and  $\bar{\rho}_2$  such that

$$F_{\alpha_L}(\tilde{\rho}_i) - F_{\alpha_L}(\bar{\rho}_i) = 1 - \theta_i.$$

Then we have

$$\begin{aligned} F_{\alpha_L}(\bar{\rho}_1) - F_{\alpha_L}(\bar{\rho}_2) &= (F_{\alpha_L}(\bar{\rho}_1) - F_{\alpha_L}(\tilde{\rho}_1)) + (F_{\alpha_L}(\tilde{\rho}_1) - F_{\alpha_L}(\tilde{\rho}_2)) + (F_{\alpha_L}(\tilde{\rho}_2) - F_{\alpha_L}(\bar{\rho}_2)) \\ &\leq -(1 - \theta_1)\lambda a(\alpha_L) + (F_{\alpha_L}(\rho_1) - F_{\alpha_L}(\rho_2)) + (1 - \theta_2)\lambda a(\alpha_L) \quad \text{with (21)} \\ &\leq 0. \end{aligned}$$

It implies that  $\rho_{0,+} \geq \rho_{1,+}$ . It makes possible the coexistence of three facts that exclude each other under Hypothesis (19).

- First, there exists  $(\rho_0, \alpha_L) \in \Omega_+^{sup}$  such that:

$$F_{\alpha_L}(\rho_L) - F_{\alpha_L}(\rho_0) = 1.$$

And  $(\rho_-, \alpha_-) = (\rho_L, \alpha_L)$  and  $(\rho_+, \alpha_+) = (\rho_0, \alpha_L)$  gives a solution that has no wave on the left of the particle, and two supersonic waves on its right, as depicted in Figure 11;

- Second,

$$F_{\alpha_L}(\tilde{\rho}_L) - F_{\alpha_L}(\tilde{\rho}_E) \leq 1$$

and there exists a pair of subsonic traces thanks to Lemma (3.8) that gives a solution with a 1-wave on the left of the particle, and a 2-wave on its right, as depicted in Figure 13;

- And finally

$$\rho_{1,+} \leq \tilde{\rho}_E \quad \text{while} \quad \rho_{0,+} \geq \tilde{\rho}_E.$$

In that case there exists  $\rho_I \in [\rho_L, \rho_0]$  and  $\theta_I \in [0, 1]$  such that

$$F_{\alpha_L}(\rho_L) - F_{\alpha_L}(\rho_I) = \theta_I \quad \text{and} \quad F_{\alpha_L}(\tilde{\rho}_I) - F_{\alpha_L}(\tilde{\rho}_E) = (1 - \theta_I)$$

and  $(\rho_-, \alpha_-) = (\rho_I, \alpha_L)$  and  $(\rho_+, \alpha_+) = (\tilde{\rho}_E, \alpha_L)$  are admissible traces around the particle. The corresponding solution has no wave on its left, and just a 2-wave on its right, as depicted in Figure 12.  $\square$

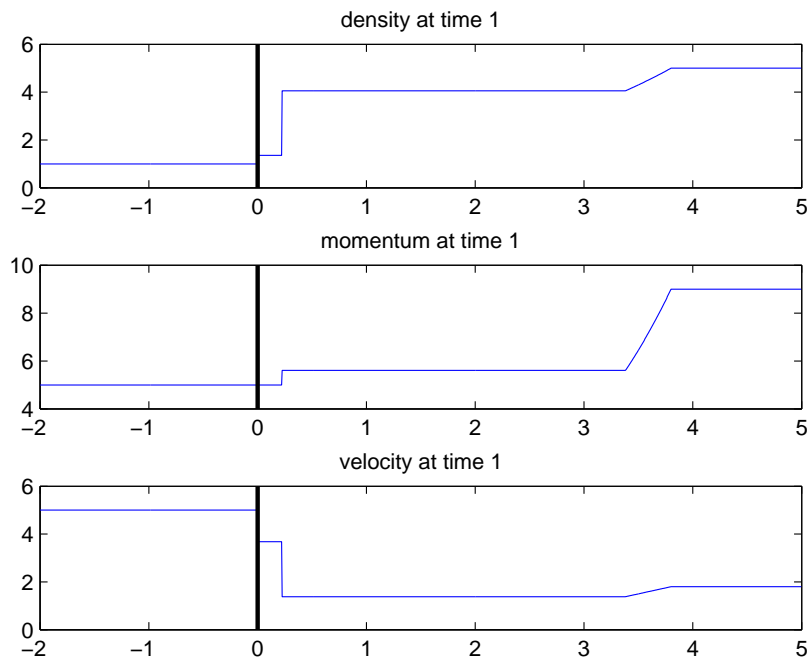


Figure 11: Solution with two supersonic traces

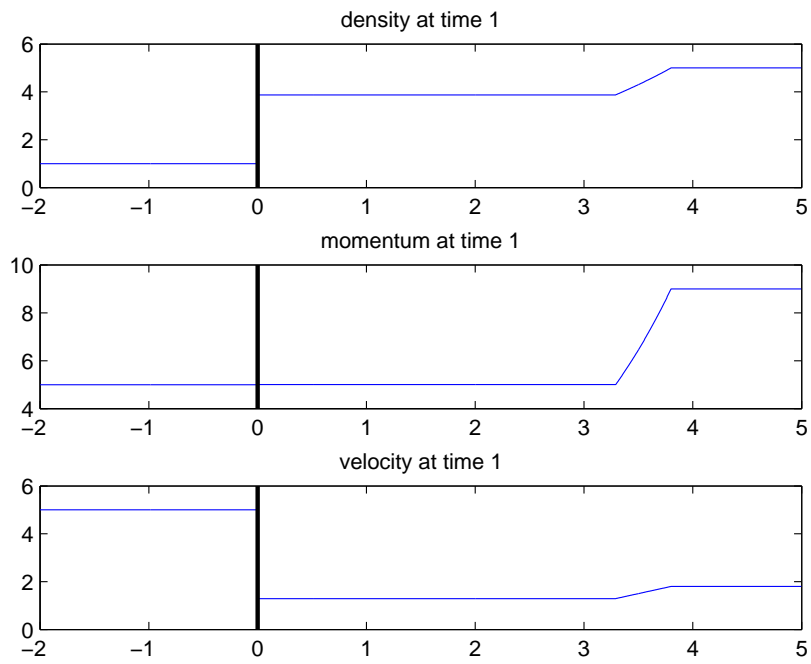


Figure 12: Solution with one supersonic trace (on the left) and one subsonic trace (on the right)

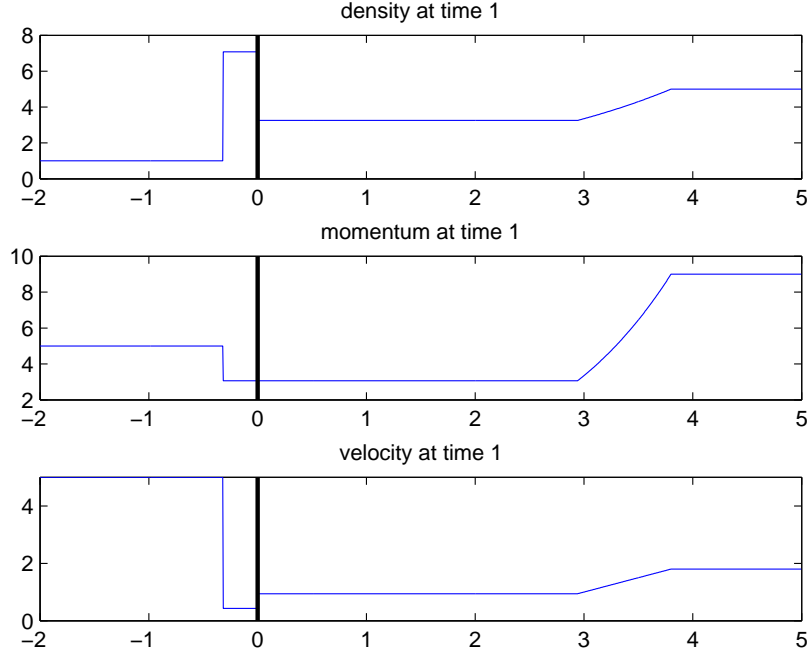


Figure 13: Solution with two subsonic traces

Figures 11, 12 and 13 represent the three solutions at time  $T = 1$  that are obtained for the Riemann problem

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \\ \partial_t q + \partial_x \left( \frac{q^2}{\rho} + 4\rho \right) = -0.9\rho^2 u \delta_0, \\ \rho(0, x) = 1 \mathbf{1}_{x < 0} + 5 \mathbf{1}_{x > 0}, \\ q(0, x) = 5 \mathbf{1}_{x < 0} + 9 \mathbf{1}_{x > 0}. \end{cases} \quad (22)$$

In that case we have  $F_\alpha(\rho) = \frac{\alpha}{2\rho^2} + \frac{c^2}{\alpha} \log(\rho)$ . Another way to see that we can lose uniqueness, and obtain up to three solutions, is depicted on Figure 14. Following [GL04], we introduce a merged 1-wave, which regroups all the state  $(\rho_I, \alpha_I)$  that can be reached from  $(\rho_L, \alpha_L)$  through three successive steps:

- From  $(\rho_L, \alpha_L)$  we reach a state  $(\rho_-, \alpha_-)$  on the left of the particle, by following a 1-wave and a 2-wave, both traveling at a speed smaller than the particle's velocity  $v$ :

$$(\rho_-, \alpha_-) \in V_-(\rho_L, \alpha_L, v);$$

- From  $(\rho_-, \alpha_-)$  we reach, through the particle, a state  $(\rho_+, \alpha_+)$ :

$$((\rho_-, \alpha_-), (\rho_+, \alpha_+)) \in \mathcal{G}_v;$$

- From  $(\rho_+, \alpha_+)$  we reach  $(\rho_I, \alpha_I)$  with a 1-wave traveling faster than  $v$ .

The solutions of the Riemann problem (4.3) are the intersection between this merged 1-wave and the usual curve of 2-waves arriving in  $(\rho_R, \alpha_R)$ . In the case of a supersonic left state the merged 1-wave contains three different types of state:

- Those obtained by taking  $(\rho_-, \alpha_-) = (\rho_L, \alpha_L)$ , then by decreasing continuously the quantity  $F_{\alpha_L}$  of 1 inside the particle. In that case  $(\rho_+, \alpha_+)$  is supersonic, and we can carry on with any 1-rarefaction wave and some 1-shocks to reach  $(\rho_I, \alpha_I)$ . This is part 1 of the black curve on Figures 14 and 15.
- Those obtained by taking  $(\rho_-, \alpha_-) = (\rho_L, \alpha_L)$ , then by decreasing continuously the quantity  $F_{\alpha_L}$  of  $\theta$  for some  $\theta \in [0, 1]$ , making a shock inside the particle, and finally continuously decreasing of  $(1 - \theta)$  along  $F_{\alpha_L}$ . In that case  $(\rho_+, \alpha_+)$  is subsonic and there exists no 1-wave faster than  $v$  starting from  $(\rho_+, \alpha_+)$ . This is part 2 of the black curve on Figures 14 and 15.

- Those obtained by starting from  $(\rho_L, \alpha_L)$  with a 1-shock slower than the particle. There exists no such 1-rarefaction wave and we reach a subsonic state  $(\rho_-, \alpha_-)$ , which lies on the dashed gray line on Figure 14. The state  $(\rho_+, \alpha_+)$  is necessarily obtained by decreasing continuously of 1 along the graph of  $F_{\alpha_-}$ . Therefore,  $(\rho_+, \alpha_+)$  is subsonic and there exists no 1-wave faster than  $v$  starting from  $(\rho_+, q_+)$ . This is part 3 of the black curve on Figures 14 and 15.

As we can see on Figures 14 and 15, the shape of the merged 1-wave depends on the relative positions of the densities  $\rho_{0,+}$  and  $\rho_{1,+}$ . The Hypothesis (19) ensures that  $\rho_{0,+} \leq \rho_{1,+}$ , and the merged 1-wave curve can be parametrized by  $\rho$  as in Figure 15. If this hypothesis does not hold, it becomes possible that  $\rho_{0,+} > \rho_{1,+}$ , in which case the merged 1-wave curve has a Z-shape and can intersect the 2-waves curve up to three times as in Figure 14.

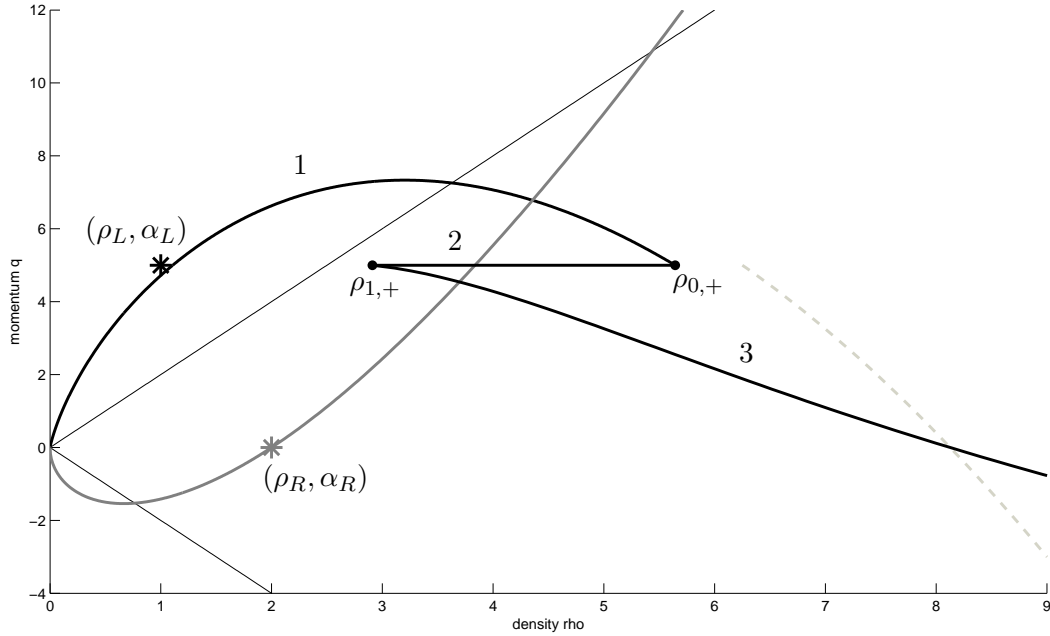


Figure 14: Example of non uniqueness with the friction  $D(\rho, q, h') = \rho^2(u - h') = \rho\alpha$ . The gray line is the usual curve of 2-wave arriving in  $(\rho_R, \alpha_R)$  (gray star). The black line is the merged 1-curve from  $(\rho_L, \alpha_L)$  (black star).

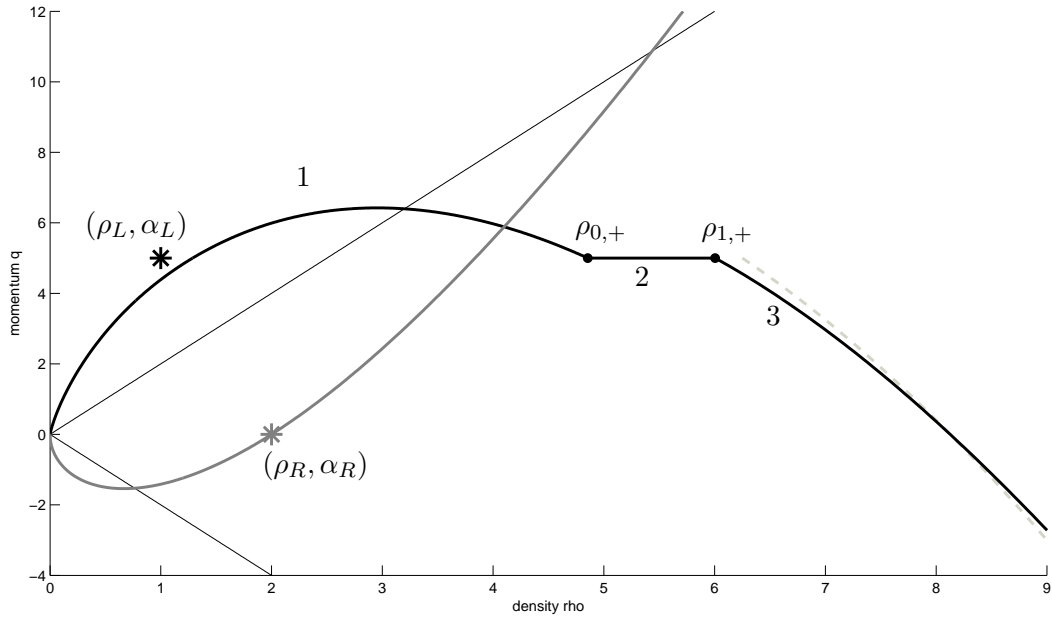


Figure 15: Example of uniqueness with the friction  $\Gamma(\rho, q, h') = u - h' = \frac{\alpha}{\rho}$ . The gray line is the usual curve of 2-wave arriving in  $(\rho_R, \alpha_R)$  (gray star). The black line is the merged 1-curve from  $(\rho_L, \alpha_L)$  (black star).

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