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THE NUMBER OF OPEN PATHS IN ORIENTED PERCOLATION

OLIVIER GARET, JEAN-BAPTISTE GOUÉRE AND RÉGINE MARCHAND

ABSTRACT. We study the number N_n of open paths of length n in supercritical oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$, with $d \geq 1$. We prove that on the percolation event $\{\inf N_n > 0\}$, $N_n^{1/n}$ almost surely converges to a deterministic constant. The proof relies on the introduction of adapted sequences of regenerating times and on subadditive arguments.

1. INTRODUCTION

Consider supercritical oriented percolation. It seems natural to think that, on the percolation event "the cluster of the origin is infinite", the number N_n of open paths with length n starting from the origin should grow exponentially fast in n . The present paper aims to prove that $N_n^{1/n}$ has an almost sure limit on the percolation event. Let us first define precisely the oriented percolation model we work with.

Oriented percolation in dimension $d + 1$. Let $d \geq 1$ be fixed, and let $\|\cdot\|$ be the norm on \mathbb{R}^d defined by

$$\|x\|_1 = \sum_{i=1}^d |x_i|.$$

We consider the oriented graph whose set of sites is $\mathbb{Z}^d \times \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$, and we put an oriented edge from (z_1, n_1) to (z_2, n_2) if and only if

$$n_2 = n_1 + 1 \text{ and } \|z_2 - z_1\|_1 \leq 1;$$

the set of these edges is denoted by $\vec{\mathbb{E}}_{\text{alt}}^{d+1}$. We say that $\gamma = (\gamma_i, i)_{0 \leq i \leq n} \in (\mathbb{Z}^d \times \mathbb{N})^{n+1}$ is a *path* if and only if

$$\forall i \in \{0, \dots, n-1\} \quad \|\gamma_{i+1} - \gamma_i\|_1 \leq 1.$$

Fix now a parameter $p \in [0, 1]$, and open independently each edge with probability p . More formally, consider the probability space $\Omega = \{0, 1\}^{\vec{\mathbb{E}}_{\text{alt}}^{d+1}}$, endowed with its Borel σ -algebra and the probability

$$\mathbb{P}_p = (\text{Ber}(p))^{\otimes \vec{\mathbb{E}}_{\text{alt}}^{d+1}},$$

where $\text{Ber}(p)$ stands for the Bernoulli law of parameter p . For a configuration $\omega = (\omega_e)_{e \in \vec{\mathbb{E}}_{\text{alt}}^{d+1}} \in \Omega$, say that the edge $e \in \vec{\mathbb{E}}_{\text{alt}}^{d+1}$ is open if $\omega_e = 1$ and closed otherwise. A path $\gamma = (\gamma_i, i)_{0 \leq i \leq n} \in (\mathbb{Z}^d \times \mathbb{N})^{n+1}$ is said *open* in the configuration ω if all its edges are open in ω . For two sites $(v, m), (w, n)$ in $\mathbb{Z}^d \times \mathbb{N}$, we denote by $\{(v, m) \rightarrow (w, n)\}$ the existence of an open path from (v, m) to (w, n) . By extension, we denote by $\{(v, m) \rightarrow +\infty\}$ the percolation

event, *i.e.* the event that there exists an infinite open path starting from (v, m) . There exists a critical probability $\bar{p}_c^{\rightarrow \text{alt}}(d+1) \in (0, 1)$ such that:

- if $p \leq \bar{p}_c^{\rightarrow \text{alt}}(d+1)$, then $\mathbb{P}_p((0, 0) \rightarrow +\infty) = 0$,
- if $p > \bar{p}_c^{\rightarrow \text{alt}}(d+1)$, then $\mathbb{P}_p((0, 0) \rightarrow +\infty) > 0$.

In the following, we assume $p > \bar{p}_c^{\rightarrow \text{alt}}(d+1)$, and we will mainly work under the following conditional probability:

$$\bar{\mathbb{P}}_p(\cdot) = \mathbb{P}_p(\cdot | (0, 0) \rightarrow +\infty).$$

Finally, we denote by N_n the random variable giving the number of open paths starting from $(0, 0)$ with length n . We prove the following convergence result:

Theorem 1.1. *For every $p > \bar{p}_c^{\rightarrow \text{alt}}(d+1)$, there exists $\alpha_p \in (0, \log((2d+1)p))$ such that $\bar{\mathbb{P}}_p$ -almost surely, and in $L^1(\bar{\mathbb{P}}_p)$,*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log N_n = \alpha_p.$$

We are currently working to prove, with the same methods, the existence of directional limits, when one counts paths with a prescribed slope.

Previous results. The problem of the existence of a limit for $N_n^{1/n}$ is related to some questions that we recall now.

First, note that $\mathbb{E}_p(N_n) = ((2d+1)p)^n$. As noticed by Darling [2], the sequence $\left(\frac{N_n}{(2d+1)^n p^n}\right)_{n \geq 0}$ is a non-negative martingale, so there exists a non-negative random variable such that

$$\mathbb{P}_p - a.s. \quad \frac{N_n}{(2d+1)^n p^n} \rightarrow W \text{ and } \mathbb{E}_p[W] \leq 1.$$

Therefore, it is easy to see that

$$\frac{1}{n} \log N_n \rightarrow \log((2d+1)p) \text{ on the event } \{W > 0\}.$$

In that case, N_n has thus the same growth rate as its expectation.

In his paper [2], Darling was seeking for conditions implying that $W > 0$. It seems that these questions have been forgotten for a while, but there is currently an increasing activity due to the links with random polymers – see for example Lacoïn [7] and Yoshida [8].

Actually, it is not always the case that $W > 0$. Let us summarize some known results:

- $\bar{\mathbb{P}}_p(W > 0) \in \{0, 1\}$. The random variable $\chi = \overline{\lim}_{n \rightarrow +\infty} \frac{1}{n} \log N_n$ is $\bar{\mathbb{P}}_p$ -almost surely constant (see Lacoïn [7]). Note that a simple Borel-Cantelli argument ensures that $\chi \leq \log((2d+1)p)$.
- $W = 0$ a.s. if $d = 1$ or $d = 2$ (see Yoshida [8]).
- for $d \geq 3$, there exists $\bar{p}_{c,2}^{\rightarrow \text{alt}}(d+1) \in [\bar{p}_c^{\rightarrow \text{alt}}(d+1), 1)$ such that $\bar{\mathbb{P}}_p(W > 0) = 1$ when $p > \bar{p}_{c,2}^{\rightarrow \text{alt}}(d+1)$ (see remark 2.7 in Lacoïn [7]).
- It is believed that $\bar{p}_{c,2}^{\rightarrow \text{alt}}(d+1) > \bar{p}_c^{\rightarrow \text{alt}}(d+1)$. Lacoïn [7] proved that the inequality is indeed strict for L -spread-out percolation for $d \geq 5$ and L large.

Then, it is clear that we need a proof of the existence of a limit for $\frac{1}{n} \log N_n$ that would not require that $W > 0$.

Note that if one does not count open paths with length n but rather paths with length n using at least θn open edges with $\theta \in [0, 1)$, the existence of the limit has been proved in Comets–Popov–Vachkovskaia [1] and in Kesten–Sidoravicius [6] by different methods.

A natural idea to prove the existence of such a limit is to use Kingman’s subadditive ergodic theorem. Obviously, if $N(a, b)$ denotes the number of open paths from a to b , by concatenation of paths we see that $N(a, c) \geq N(a, b)N(b, c)$, and thus we have the superadditivity property:

$$\log N(a, c) \geq \log N(a, b) + \log N(b, c).$$

The problem is naturally that $\log N(\cdot, \cdot)$ may be infinite, and therefore not integrable. Those kind of problem may be solved by using convenient subsequences that lead to integrable variables. To this aim, we use the technics of essential hitting times introduced in Garet–Marchand [4] to establish a shape theorem for the contact process in random environment.

Notation. For $n \geq 1$ and $x \in \mathbb{Z}^d$, we denote by

- N_n the number of open paths from $(0, 0)$ to $\mathbb{Z}^d \times \{n\}$,
- \overline{N}_n the number of open paths from $(0, 0)$ to $\mathbb{Z}^d \times \{n\}$ that are the beginning of an infinite open path,
- $N_{(x, n)}$ the number of open paths from $(0, 0)$ to (x, n) ,

2. PRELIMINARY RESULTS

Oriented percolation is known as the analogue in discrete time for the contact process. Usually, results are proved for one model, and it is commonly admitted that the proofs could easily be adapted to the other one. For the results concerning supercritical oriented percolation we recall now, we will thus sometimes give the reference for the property concerning the contact process without any further explanation.

2.1. Exponential estimates for supercritical oriented percolation.

We work on the graph $\mathbb{Z}^d \times \mathbb{N}$, as defined in the introduction. We set, for $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$,

$$\begin{aligned} \xi_n^x &= \{y \in \mathbb{Z}^d : (x, 0) \rightarrow (y, n)\}, \\ \xi_n^{\mathbb{Z}^d} &= \bigcup_{x \in \mathbb{Z}^d} \xi_n^x, \\ \tau^x &= \min\{n \in \mathbb{N} : \xi_n^x = \emptyset\}, \\ H_n^x &= \bigcup_{0 \leq k \leq n} \xi_k^x, \\ K_n^x &= \bigcap_{k \geq n} (\xi_k^x \Delta \xi_k^{\mathbb{Z}^d})^c. \end{aligned}$$

To simplify, we often write ξ_n, τ, H_n, K_n^x instead of $\xi_n^0, \tau^0, H_n^0, K_n^{x0}$.

For instance, τ is the length of the longest open path starting from the origin, and the percolation event is equal to $\{\tau = +\infty\}$. First, finite open

paths cannot be too long (see Durrett [3]):

$$(1) \quad \forall p > \vec{p}_c^{\text{alt}}(d+1) \quad \exists A, B > 0 \quad \forall n \in \mathbb{N} \quad \mathbb{P}_p(n \leq \tau < +\infty) \leq Ae^{-Bn}.$$

The set $K'_n \cap H_n$ is called the coupled zone. As for the contact process, the growth of the sets $(H_n)_{n \geq 0}$ and the coupled zones $(K'_n \cap H_n)_{n \geq 0}$ is governed by a shape theorem (see, for the supercritical contact process, Durrett [3] or Garet-Marchand [4]):

Proposition 2.1 (Shape theorem). *We consider independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$. For every $p > \vec{p}_c^{\text{alt}}(d+1)$, there exists a norm μ_p on \mathbb{R}^d which rules the growth of the oriented percolation conditioned to survive: for every $\varepsilon > 0$, $\overline{\mathbb{P}}_p$ almost surely, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,*

$$B_{\mu_p}(0, (1-\varepsilon)n) \subset (K'_n \cap H_n) + [0, 1]^d \subset H_n + [0, 1]^d \subset B_{\mu_p}(0, (1+\varepsilon)n),$$

where $B_{\mu_p}(x, r) = \{y \in \mathbb{R}^d : \mu_p(y-x) \leq r\}$.

We even have large deviations inequalities (see Garet-Marchand [5] for the contact process):

Proposition 2.2 (Large deviations inequalities). *We consider independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$. For every $p > \vec{p}_c^{\text{alt}}(d+1)$, for every $\varepsilon > 0$ there exist positive constants A, B such that, for every $n \geq 1$,*

$$\overline{\mathbb{P}}_p \left(\begin{array}{l} B_{\mu_p}(0, (1-\varepsilon)n) \subset (K'_n \cap H_n) + [0, 1]^d \\ \subset H_n + [0, 1]^d \subset B_{\mu_p}(0, (1+\varepsilon)n) \end{array} \right) \geq 1 - Ae^{-Bn}.$$

2.2. Essential hitting times and associated translations. We now introduce the analogues, in the discrete setting of oriented percolation, of the essential hitting times used by Garet–Marchand to study the supercritical contact process conditioned to survive in [4] and [5]. As said before, the proofs of the following lemmas are all direct adaptations of the corresponding lemmas for the contact process. We thus just give precise references for the contact process case.

We define a set of oriented edges $\vec{\mathbb{E}}^d$ of \mathbb{Z}^d in the following way: in $(\mathbb{Z}^d, \vec{\mathbb{E}}^d)$, there is an oriented edge between two points z_1 and z_2 in \mathbb{Z}^d if and only if $\|z_1 - z_2\|_1 \leq 1$. The oriented edge in $\vec{\mathbb{E}}_{\text{alt}}^{d+1}$ from (z_1, n_1) to (z_2, n_2) can be identified with the couple $((z_1, z_2), n_2) \in \vec{\mathbb{E}}^d \times \mathbb{N}^*$. Thus, we identify $\vec{\mathbb{E}}_{\text{alt}}^{d+1}$ and $\vec{\mathbb{E}}^d \times \mathbb{N}^*$. We also define, for $(y, h) \in \mathbb{Z}^d \times \mathbb{N}$, the translation $\theta_{(y,h)}$ on Ω by:

$$\theta_{(y,h)}((\omega_{(e,k)})_{e \in \vec{\mathbb{E}}^d, k \geq 1}) = (\omega_{(e+y, k+h)})_{e \in \vec{\mathbb{E}}^d, k \geq 1}.$$

At some point, we will also need to look backwards in time. So we replace $\mathbb{Z}^d \times \mathbb{N}$ by $\mathbb{Z}^d \times \mathbb{Z}$ as set of sites, and we introduce the following reversed time translation defined on $\{0, 1\}^{\mathbb{Z}^d \times \mathbb{Z}}$ by

$$\theta_{(y,h)}^\downarrow((\omega_{(e,k)})_{e \in \vec{\mathbb{E}}^d, k \in \mathbb{Z}}) = (\omega_{(e+y, h-k)})_{e \in \vec{\mathbb{E}}^d, k \in \mathbb{Z}}.$$

Fix $p > \vec{p}_c^{\text{alt}}(d+1)$.

We now recall the construction of the essential hitting times $\sigma(x)$, for $x \in \mathbb{Z}^d$, and the associated translations introduced in [4]. Fix $x \in \mathbb{Z}^d$. The essential hitting time $\sigma(x)$ is a random time $T \geq 1$ such that $(0, 0) \rightarrow$

$(x, T) \rightarrow +\infty$. It is defined through a family of stopping times as follows: we set $u_0 = v_0 = 0$ and we define recursively two increasing sequences of stopping times $(u_n)_{n \geq 0}$ and $(v_n)_{n \geq 0}$ with $u_0 = v_0 < u_1 < v_1 < u_2 \dots$ as follows:

- Assume that v_k is defined. We set $u_{k+1} = \inf\{t > v_k : x \in \xi_t^0\}$.
If $v_k < +\infty$, then u_{k+1} is the first time after v_k where x is once again infected; otherwise, $u_{k+1} = +\infty$.
- Assume that u_k is defined, with $k \geq 1$. We set $v_k = u_k + \tau^0 \circ \theta_{(x, u_k)}$.
If $u_k < +\infty$, the time $\tau^0 \circ \theta_{(x, u_k)}$ is the length of the oriented percolation cluster starting from (x, u_k) ; otherwise, $v_k = +\infty$.

We then set

$$K(x) = \min\{n \geq 0 : v_n = +\infty \text{ or } u_{n+1} = +\infty\}.$$

This quantity represents the number of steps before the success of this process: either we stop because we have just found an infinite v_n , which corresponds to a time u_n when x is occupied and has infinite progeny, or we stop because we have just found an infinite u_{n+1} , which says that after v_n , site 0 is never infected anymore. It is not difficult to see that

$$\mathbb{P}_p(K(x) > n) \leq \mathbb{P}_p(\tau^0 < +\infty)^n,$$

and thus $K(x)$ is \mathbb{P}_p almost surely finite. We define the essential hitting time $\sigma(x)$ by setting

$$\sigma(x) = u_{K(x)} \in \mathbb{N} \cup \{+\infty\}.$$

By construction $(0, 0) \rightarrow (x, \sigma(x)) \rightarrow +\infty$ on the event $\{\tau = +\infty\}$. Note however that $\sigma(x)$ is not necessarily the first positive time when x is occupied and has infinite progeny: for instance, such an event can occur between u_1 and v_1 , being ignored by the recursive construction. It can be checked that conditionally to the event $\{\tau^0 = \infty\}$, the process necessarily stops because of an infinite v_n , and thus $\sigma(x) < +\infty$. At the same time, we define the operator $\tilde{\theta}$ on Ω , which is a random translation, by:

$$\tilde{\theta}_x(\omega) = \begin{cases} \theta_{(x, \sigma(x))}\omega & \text{if } \sigma(x) < +\infty, \\ \omega & \text{otherwise.} \end{cases}$$

If (x_1, \dots, x_m) is a sequence of points in \mathbb{Z}^d , we also introduce the shortened notation

$$\tilde{\theta}_{x_1, \dots, x_m} = \tilde{\theta}_{x_m} \circ \tilde{\theta}_{x_{m-1}} \cdots \circ \tilde{\theta}_{x_1}.$$

For $n \geq 1$, we denote by \mathcal{F}_n the σ -field generated by the maps $(\omega \mapsto \omega_{(e, k)})_{e \in \vec{\mathbb{E}}^d, 1 \leq k \leq n}$. We denote by \mathcal{F} the σ -field generated by the maps $(\omega \mapsto \omega_{(e, k)})_{e \in \vec{\mathbb{E}}^d, k \geq 1}$.

Proposition 2.3. *We consider independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$ with parameter $p > \vec{p}_c^{\text{alt}}(d+1)$. Fix $x_1, \dots, x_m \in \mathbb{Z}^d$.*

- a. *Suppose $A \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{F}$. Then for each $x \in \mathbb{Z}^d$,*

$$\overline{\mathbb{P}}_p(\sigma(x) \in A, \tilde{\theta}_x^{-1}(B)) = \overline{\mathbb{P}}_p(\sigma(x) \in A) \overline{\mathbb{P}}_p(B).$$

- b. *The probability measure $\overline{\mathbb{P}}_p$ is invariant under $\tilde{\theta}_{x_1, \dots, x_m}$.*

- c. The random variables $\sigma(x_1), \sigma(x_2) \circ \tilde{\theta}_{x_1}, \sigma(x_3) \circ \tilde{\theta}_{x_1, x_2}, \dots, \sigma(x_m) \circ \tilde{\theta}_{x_1, \dots, x_{m-1}}$ are independent under $\overline{\mathbb{P}}_p$.
- d. Suppose $t \leq m$, $A \in \mathcal{F}_t$, $B \in \mathcal{F}$

$$\overline{\mathbb{P}}_p(A, \tilde{\theta}_{x_1, \dots, x_m}^{-1}(B)) = \overline{\mathbb{P}}_p(A) \overline{\mathbb{P}}_p(B).$$

- e. For every $x \in \mathbb{Z}^d$, $\mu_p(x) = \lim_{n \rightarrow +\infty} \frac{\overline{\mathbb{E}}_p(\sigma(nx))}{n} = \inf_{n \geq 1} \frac{\overline{\mathbb{E}}_p(\sigma(nx))}{n}$.
- f. There exists $\alpha, \beta > 0$ such that

$$\forall x \in \mathbb{Z}^d \quad \overline{\mathbb{E}}_p(\exp(\alpha \sigma(x))) \leq \exp(\beta(\|x\|_1 \vee 1)).$$

Proof. To prove a.-d., it is sufficient to mimic the proofs of Lemma 8 and Corollary 9 in [4]. The convergence e. has been proved for the contact process in [4], Theorem 22. The existence of exponential moments for σ has been proved for the contact process in [5], Theorem 2. \square

3. DIRECTIONAL LIMITS ALONG SUBSEQUENCES OF REGENERATING TIMES

We next define, for $(y, h) \in \mathbb{Z}^d \times \mathbb{N}$ a regenerating time $s(y, h)$ by setting

$$s(y, h) = \sigma(y) + \sum_{i=1}^h \sigma(0) \circ \tilde{\theta}^{i-1}(0) \circ \tilde{\theta}(y),$$

and the associated translation:

$$\hat{\theta}_{(y, h)}(\omega) = \begin{cases} \theta_{(y, s(y, h))} \omega & \text{if } s(y, h) < +\infty, \\ \omega & \text{otherwise.} \end{cases}$$

Note that on $\{\tau = +\infty\}$, $(0, 0) \rightarrow (y, s(y, h)) \rightarrow +\infty$ and $\hat{\theta}_{(y, h)} = \tilde{\theta}_{y, 0, \dots, 0}$ (with h zeros). We can easily deduce from Proposition 2.3 the following properties of the time $s(y, h)$ under $\overline{\mathbb{P}}_p$:

Lemma 3.1. *We consider independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$ with parameter $p > \overline{p}_c^{\text{alt}}(d+1)$, and we fix $(y, h) \in \mathbb{Z}^d \times \mathbb{N}$.*

- The probability measure $\overline{\mathbb{P}}_p$ is invariant under the translation $\hat{\theta}_{(y, h)}$.
- The random variables $(s(y, h) \circ (\hat{\theta}_{(y, h)})^j)_{j \geq 0}$ are independent and identically distributed under $\overline{\mathbb{P}}_p$.
- The measure-preserving dynamical system $(\Omega, \mathcal{F}, \overline{\mathbb{P}}_p, \hat{\theta}_{(y, h)})$ is mixing.
- There exists $\alpha, \beta > 0$ such that

$$\forall y \in \mathbb{Z}^d \quad \forall h \in \mathbb{N} \quad \overline{\mathbb{E}}_p(\exp(\alpha s(y, h))) \leq \exp(\beta(\|y\| \vee 1 + h)).$$

We fix $(y, h) \in \mathbb{Z}^d \times \mathbb{N}$. We work under $\overline{\mathbb{P}}_p$, and we set, for every $n \geq 1$,

$$S_n = S_n(y, h) = \sum_{k=0}^{n-1} s(y, h) \circ \hat{\theta}_{(y, h)}^k.$$

The points $(ny, S_n(y, h))_{n \geq 1}$ are the sequence of regenerating points associated to (y, h) along which we are going to look for subadditivity properties. As, under $\overline{\mathbb{P}}_p$, the random variables $(s(y, h) \circ \hat{\theta}_{(y, h)}^j)_{j \geq 0}$ are independent and

identically distributed with finite first moment (see Lemma 3.1), the strong law of large numbers ensures that $\overline{\mathbb{P}}_p$ -almost surely

$$(2) \quad \lim_{n \rightarrow +\infty} \frac{S_n(y, h)}{n} = \overline{\mathbb{E}}_p(s(y, h)) = \overline{\mathbb{E}}_p(\sigma(y)) + h \overline{\mathbb{E}}_p(\sigma(0)).$$

Thus, for large n , the point $(ny, S_n(y, h))$ is not far from the line $\mathbb{R}(y, \overline{\mathbb{E}}_p(s(y, h)))$.

To apply directional limits along subsequences, we first apply Kingman's subadditive ergodic theorem to $f_n = -\log N_{(ny, S_n(y, h))}$ for a fixed $(y, h) \in \mathbb{Z}^d \times \mathbb{N}$.

Lemma 3.2. *Let us consider independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$ with parameter $p > \overrightarrow{p}_c^{\text{alt}}(d+1)$. Fix $(y, h) \in \mathbb{Z}^d \times \mathbb{N}$. There exists $\alpha_p(y, h) \in (0, \log(2d+1)]$ such that $\overline{\mathbb{P}}_p$ -almost surely and in $L^1(\overline{\mathbb{P}}_p)$,*

$$\lim_{n \rightarrow +\infty} \frac{1}{S_n(y, h)} \log N_{(ny, S_n(y, h))} = \alpha_p(y, h).$$

Proof. Fix $(y, h) \in \mathbb{Z}^d \times \mathbb{N}$. To avoid heavy notations, we omit all the dependence in (y, h) . For instance $S_n = S_n(y, h)$ and $\hat{\theta} = \hat{\theta}_{(y, h)}$. Note that by definition, $\overline{\mathbb{P}}_p$ -almost surely, for every $n \geq 1$, $(0, 0) \rightarrow (ny, S_n) \rightarrow +\infty$ and consequently, $N_{(ny, S_n)} \geq 1$. For $n \geq 1$, we set

$$f_n = -\log N_{(ny, S_n)}.$$

Let $n, p \geq 1$. Note that $S_n + S_p \circ \hat{\theta}_{(y, h)}^n = S_{n+p}$. As $N_{(py, S_p)} \circ \hat{\theta}^n$ counts the number of open paths from (ny, S_n) to $((n+p)y, S_n + S_p \circ \hat{\theta}^n)$, concatenation of paths ensures that $N_{(ny, S_n)} \times N_{(py, S_p)} \circ \hat{\theta}^n \leq N_{((n+p)y, S_{n+p})}$ which implies that

$$\forall n, p \geq 1 \quad f_{n+p} \leq f_n + f_p \circ \hat{\theta}^n.$$

As $1 \leq N_{(ny, S_n)} \leq (2d+1)^{S_n}$,

$$-S_n \log(2d+1) \leq f_n \leq 0.$$

The integrability of s thus implies the integrability of every f_n . So we can apply Kingman's subadditive ergodic theorem. By property c. in 3.1, the dynamical system $(\Omega, \mathcal{F}, \overline{\mathbb{P}}, \hat{\theta})$ is mixing. Particularly, it is ergodic, so the limit is deterministic: if we define

$$-\alpha'_p(y, h) = \inf_{n \geq 1} \frac{\overline{\mathbb{E}}_p(f_n)}{n},$$

we have $\overline{\mathbb{P}}_p$ -almost surely and in $L^1(\overline{\mathbb{P}}_p)$: $\lim_{n \rightarrow +\infty} \frac{f_n}{n} = -\alpha'_p(y, h)$.

The limit of the lemma follows then directly from (2) by setting

$$\alpha_p(y, h) = \frac{\alpha'_p(y, h)}{\overline{\mathbb{E}}_p s(y, h)}.$$

Finally $\alpha'_p(y, h) \geq \overline{\mathbb{E}}_p(-f_1) = \overline{\mathbb{E}}_p(\log N_{(y, S_1)})$. Since $N_{(y, S_1)} \geq 1$ $\overline{\mathbb{P}}_p$ -a.s. and $N_{(y, S_1)} \geq 2$ with positive probability, it follows that $\alpha'_p(y, h) > 0$, and consequently $\alpha_p(y, h) > 0$.

As $N_{(ny, S_n)} \leq (2d+1)^{S_n}$, we see that $\alpha_p(y, h) \leq \log(2d+1)$ and that the convergence also holds in $L^1(\overline{\mathbb{P}}_p)$. \square

We can now introduce a natural candidate for the limit in Theorem 1.1:

$$(3) \quad \alpha_p = \sup \left\{ \alpha_p(y, h) : (y, h) \in \mathbb{Z}^d \times \mathbb{N} \right\} < +\infty.$$

Indeed, at the logarithmic scale we are working with, we can expect that the dominant contribution to the number N_n of open paths to level n will be due to the number $N_{(nz, n)}$ of open paths to level n in the direction $(z, 1)$ that optimizes the previous limit. Note however that in our construction, (y, h) has no real geometrical signification, but it is just a useful encoding: as said before, the asymptotic direction of the regenerating point $(ny, S_n(y, h))$ in $\mathbb{Z}^d \times \mathbb{N}$ is

$$\left(\frac{y}{\overline{\mathbb{E}}_p(s(y, h))}, 1 \right).$$

To skip from the subsequences to the full limit, we approximate $B_{\mu_p}(0, 1)$ with a finite number of points:

Lemma 3.3. *We have the inclusion $B_{\mu_p}(0, 1) \subset \overline{\left(\frac{y}{\overline{\mathbb{E}}_p(s(y, h))} \right)_{y \in \mathbb{Z}^d, h \in \mathbb{N}}}$.*

Proof. Note that the set $\{ \frac{z}{l} : (z, l) \in \mathbb{Z}^d \times \mathbb{N}^* \text{ and } \mu_p(z) < l \}$ is dense in $B_{\mu_p}(0, 1)$. Thus fix $(z, l) \in \mathbb{Z}^d \times \mathbb{N}^*$ such that $\mu_p(z) < l$ and consider

$$(y_n, h_n) = \left(nz, \left\lfloor \frac{n(l - \mu_p(z))}{\overline{\mathbb{E}}_p(\sigma(0))} \right\rfloor \right).$$

Then

$$\frac{y_n}{\overline{\mathbb{E}}_p(s(y_n, h_n))} = \frac{nz}{\overline{\mathbb{E}}_p(\sigma(y_n)) + h_n \overline{\mathbb{E}}_p(\sigma(0))} \rightarrow \frac{z}{l}$$

as n goes to $+\infty$. □

Finally, for $(y, h) \in \mathbb{Z}^d \times \mathbb{N}$, we denote by

$$(4) \quad \forall n \in \mathbb{N} \quad \varphi(n) = \varphi_{(y, h)}(n) = \inf \{ k \in \mathbb{N} : S_k(y, h) \geq n \}.$$

Thus, for large n , $(\varphi(n).y, S_{\varphi(n)})$ is the first point among the sequence of renewal points associated to (y, h) to be above level n . By the renewal theory, $\overline{\mathbb{P}}_p$ almost surely,

$$(5) \quad \lim_{n \rightarrow +\infty} \frac{\varphi_{(y, h)}(n)}{n} = \frac{1}{\overline{\mathbb{E}}_p(s(y, h))} \text{ and } \lim_{n \rightarrow +\infty} \frac{S_{\varphi_{(y, h)}(n)}(y, h)}{n} = 1.$$

It is also not too far above level n :

Lemma 3.4. *There exist positive constants A, B such that*

$$\forall n \in \mathbb{N} \quad \overline{\mathbb{P}}_p(S_{\varphi(n)} - n \geq n) \leq A \exp(-Bn).$$

Proof. As we work in discrete time, $\varphi(n) \leq n$. So

$$\overline{\mathbb{P}}_p(S_{\varphi(n)} - n \geq n) \leq \overline{\mathbb{P}}_p(\exists k \leq n : s(y, h) \circ \hat{\theta}_{(y, h)}^k \geq n) \leq n \overline{\mathbb{P}}_p(s(y, h) \geq n).$$

As $s(y, h)$ admits exponential moments thanks to Lemma 3.1, we can conclude with the Markov inequality. □

4. PROOF OF THEOREM 1.1

Let $d \geq 1$ be fixed. Consider independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$ with fixed parameter $p > \vec{p}_c^{\text{alt}}(d+1)$. The proof of Theorem 1.1 is a direct consequence of the forthcoming Lemmas 4.1, 4.2 and 4.3. Remember that α_p is defined in (3).

Lemma 4.1. $\bar{\mathbb{P}}_p$ -almost surely, $\liminf_{n \rightarrow +\infty} \frac{1}{n} \log \bar{N}_n \geq \alpha_p$.

Proof. Take $(y, h) \in \mathbb{Z}^d \times \mathbb{N}$. Note that $(\bar{N}_n)_{n \geq 1}$ is non-decreasing, and considering the increasing sequence $S_k = S_k(y, h)$, we see that, $\bar{\mathbb{P}}_p$ almost surely, for every integer n such that $S_k \leq n \leq S_{k+1}$,

$$\frac{1}{n} \log \bar{N}_n \geq \frac{1}{S_{k+1}} \log \bar{N}_{S_k} \geq \frac{S_k}{S_{k+1}} \frac{\log \bar{N}_{(ky, S_k)}}{S_k}.$$

With (2) and Lemma 3.2, we deduce that $\bar{\mathbb{P}}_p$ almost surely,

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \log \bar{N}_n \geq \alpha_p(y, h),$$

which completes the proof. \square

Lemma 4.2. $\bar{\mathbb{P}}_p$ -almost surely, $\overline{\lim}_{n \rightarrow +\infty} \frac{1}{n} \log \bar{N}_n \leq \alpha_p$.

Proof. Fix $\varepsilon > 0$ and $\eta \in (0, 1)$. We first approximate $B_{\mu_p}(0, 1)$ with a finite number of points: with Lemma 3.3, we can find a finite set $F \subset \mathbb{Z}^d \times \mathbb{N}$ such that

$$B_{\mu_p}(0, 1 + \varepsilon) \subset \bigcup_{(y, h) \in F} B_{\mu_p} \left(\frac{(1 + \varepsilon)y}{\bar{\mathbb{E}}_p(s(y, h))}, (1 - \eta)\varepsilon/2 \right).$$

Then, for n large, we will control the number \bar{N}_n using these directions. We define $M_n(y, h)$ as the first point in the sequence $(ky, S_{(y, h)}(k))_{k \geq 1}$ of regenerating points associated to (y, h) to be above level $n(1 + \varepsilon)$. Using the notation introduced in (4), we set

$$\begin{aligned} \forall (y, h) \in F \quad k_n = k_n(y, h) &= \varphi_{(y, h)}(n(1 + \varepsilon)), \\ Z_n = Z_n(y, h) &= k_n \cdot y \in \mathbb{Z}^d \quad \text{and} \quad V_n = V_n(y, h) = S_{k_n}(y, h) \in \mathbb{N}, \\ M_n = M_n(y, h) &= (Z_n, V_n). \end{aligned}$$

For a given $(y, h) \in F$, the law of large numbers (5) says that

$$(6) \quad k_n(y, h) \sim \frac{n(1 + \varepsilon)}{\bar{\mathbb{E}}_p(s(y, h))} \quad \text{and} \quad V_n(y, h) \sim n(1 + \varepsilon).$$

So $\bar{\mathbb{P}}_p$ almost surely, for all n large enough

$$\forall (y, h) \in F \quad B_{\mu_p} \left(\frac{(1 + \varepsilon)ny}{\bar{\mathbb{E}}_p(s(y, h))}, (1 - \eta)\varepsilon n/2 \right) \subset B_{\mu_p}(Z_n(y, h), (1 - \eta)\varepsilon n),$$

and so, $\bar{\mathbb{P}}_p$ almost surely, for all n large enough

$$(7) \quad \xi_n \subset B_{\mu_p}(0, (1 + \varepsilon)n) \subset \bigcup_{(y, h) \in F} B_{\mu_p}(Z_n(y, h), (1 - \eta)\varepsilon n).$$

The strategy is to prove that for n large enough, then for each $x \in B_{\mu_p}(0, n(1+\varepsilon))$ the number of open paths from $(0, 0)$ to (x, n) that contribute to \bar{N}_n is not far from $N_{M_n(y, h)}$ for the $(y, h) \in F$ such that $x \in B_{\mu_p}(Z_n(y, h), (1-\eta)\varepsilon n)$. To do so, we use the coupled zone. Note

$$G_n = \bigcap_{M \in \{-2n, \dots, 2n\}^d \times \{0, \dots, 2n\}} \left\{ \begin{array}{l} \tau < n(1+\varepsilon) \\ \text{or } K'_{n\varepsilon} \supset B_{\mu_p}(0, (1-\eta)\varepsilon n) \cap \mathbb{Z}^d \end{array} \right\} \circ \theta_M^\downarrow.$$

Since θ_M^\downarrow preserves \mathbb{P}_p , we easily deduce from (1), Proposition 2.2 and a Borel–Cantelli argument that $\bar{\mathbb{P}}_p$ almost surely, G_n holds for n large enough.

Now take n large enough such that (7) holds, G_n hold, together with $V_n(y, h) \leq 2n$ for each $(y, h) \in F$ (which is possible thanks to (6)) and $\xi_n \subset B_{\mu_p}(0, n(1+\varepsilon))$.

Fix $x \in \xi_n$ such that $(x, n) \rightarrow \infty$. As $\xi_n \subset B_{\mu_p}(0, n(1+\varepsilon))$ and (7) holds, choose $(y, h) \in F$ such that $x \in B_{\mu_p}(Z_n(y, h), (1-\eta)\varepsilon n)$.

Since $(0, 0) \rightarrow M_n$ and $V_n \geq n(1+\varepsilon)$, we know that $\tau \circ \theta_{M_n}^\downarrow \geq n(1+\varepsilon)$. Since $M_n \in \{-2n, \dots, 2n\}^d \times \{0, \dots, 2n\}$, $\mu_p(x - Z_n) \leq (1-\eta)\varepsilon n$ and G_n holds, we have $x - Z_n \in K'_{n\varepsilon} \circ \theta_{M_n}^\downarrow$. Note that $V_n(y, h) \geq n(1+\varepsilon)$, so $V_n(y, h) - n \geq \varepsilon n$. Note that $(x, n) \rightarrow \infty$ implies that $x - Z_n \in \xi_{V_n(y, h) - n}^{\mathbb{Z}^d} \circ \theta_{M_n}^\downarrow$. By definition of the coupled zone, we have $x - Z_n \in \xi_{V_n(y, h) - n}^0 \circ \theta_{M_n}^\downarrow$. Going back to the initial orientation, it means that $(x, n) \rightarrow M_n$. So

$$\bar{N}_n \leq \sum_{(y, h) \in F} \bar{N}_{M_n(y, h)}.$$

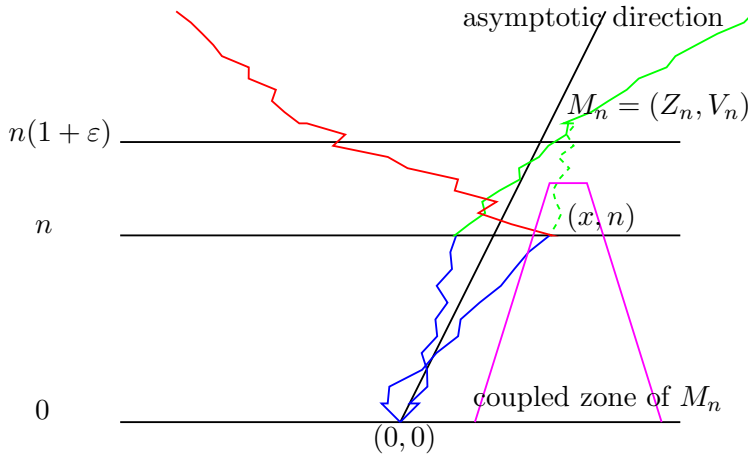


FIGURE 1. Red paths are the part above n of paths from $(0, 0)$ to infinity. Green paths are the part above n of paths from $(0, 0)$ to infinity that meet some M_n . To bound \bar{N}_n , we prove that each start of a red path is the start of a green path.

Next, we use the directional limits given by Lemma 3.2: $\overline{\mathbb{P}}_p$ almost surely,

$$\forall (y, h) \in F \quad \lim_{n \rightarrow +\infty} \frac{1}{V_n(y, h)} \log \overline{N}_{M_n(y, h)} = \alpha_p(y, h).$$

As $V_n(y, h) \sim n(1 + \varepsilon)$, we obtain from the Shape Theorem (Proposition 2.1) that $\overline{\mathbb{P}}_p$ almost surely, for all n large enough

$$\forall (y, h) \in F \quad \frac{1}{n(1 + \varepsilon)} \log \overline{N}_{M_n(y, h)} \leq \alpha_p(y, h) + \varepsilon \leq \alpha_p + \varepsilon.$$

So, for n large enough, we have $\overline{\mathbb{P}}_p$ almost surely,

$$\overline{N}_n \leq \sum_{(y, h) \in F} \overline{N}_{M_n(y, h)} \leq |F| \exp((\alpha_p + \varepsilon)n(1 + \varepsilon)),$$

$$\text{so } \overline{\lim}_{n \rightarrow +\infty} \frac{1}{n} \log(\overline{N}_n) \leq (1 + \varepsilon)(\alpha_p + \varepsilon).$$

We complete the proof by letting ε go to 0. \square

Finally, we prove that working with open paths or with open paths that are the beginning of an infinite open path is essentially the same:

Lemma 4.3. $\overline{\mathbb{P}}_p$ -almost surely,

$$\overline{\lim}_{n \rightarrow +\infty} \frac{\log N_n}{n} = \overline{\lim}_{n \rightarrow +\infty} \frac{\log \overline{N}_n}{n} \quad \text{and} \quad \underline{\lim}_{n \rightarrow +\infty} \frac{\log N_n}{n} = \underline{\lim}_{n \rightarrow +\infty} \frac{\log \overline{N}_n}{n}.$$

Proof. Fix $0 < \varepsilon < 1$ and define, for $n \geq 1$, the following event

$$E_n = \bigcap_{\|z\| \leq n} \{\tau < \varepsilon n \text{ or } \tau = +\infty\} \circ \theta_{(z, \lfloor n(1-\varepsilon) \rfloor)}.$$

Assume that E_n occurs. Consider a path $\gamma = (\gamma_i, i)_{0 \leq i \leq n}$ from $(0, 0)$ to $\mathbb{Z}^d \times \{n\}$ and set $z = \gamma_{\lfloor n(1-\varepsilon) \rfloor}$: as $\tau \circ \theta_{(z, \lfloor n(1-\varepsilon) \rfloor)} \geq \varepsilon n$, the event E_n implies that $\tau \circ \theta_{(z, \lfloor n(1-\varepsilon) \rfloor)} = +\infty$. So $(\gamma_i, i)_{0 \leq i \leq \lfloor n(1-\varepsilon) \rfloor}$ contributes to $\overline{N}_{\lfloor n(1-\varepsilon) \rfloor}$ and thus, on E_n ,

$$\begin{aligned} N_n &\leq (2d + 1)^{\varepsilon n + 1} \overline{N}_{\lfloor n(1-\varepsilon) \rfloor}, \\ \text{so } \frac{1}{n} \log N_n &\leq \left(\varepsilon + \frac{1}{n} \right) \log(2d + 1) + \frac{1}{n} \log \overline{N}_{\lfloor n(1-\varepsilon) \rfloor} \\ &\leq \left(\varepsilon + \frac{1}{n} \right) \log(2d + 1) + \frac{1}{\lfloor n(1-\varepsilon) \rfloor} \log \overline{N}_{\lfloor n(1-\varepsilon) \rfloor}. \end{aligned}$$

The exponential estimate (1) ensures that

$$\forall n \geq 1 \quad \mathbb{P}_p(E_n^c) \leq C_d A n^d \exp(-B\varepsilon n) \leq A' \exp(-B'n).$$

With the Borel–Cantelli lemma, this leads to:

$$\overline{\lim}_{n \rightarrow +\infty} \frac{1}{n} \log N_n \leq \varepsilon \log(2d + 1) + \overline{\lim}_{n \rightarrow +\infty} \frac{1}{n} \log \overline{N}_n.$$

By taking ε to 0, we obtain

$$\overline{\lim}_{n \rightarrow +\infty} \frac{\log N_n}{n} \leq \overline{\lim}_{n \rightarrow +\infty} \frac{\log \overline{N}_n}{n}.$$

The proof for the inequality with $\underline{\lim}$ instead of $\overline{\lim}$ is identical. Since $\overline{N}_n \leq N_n$, the reversed inequalities are obvious. \square

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