The Number of Open Paths in Oriented Percolation

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Abstract. We study the number N_n of open paths of length n in supercritical oriented percolation. We prove that on the percolation event $\{\inf N_n > 0\}$, $N_n^{1/n}$ almost surely converges to a deterministic constant. The proof relies on subadditive arguments and on a recent result of Birkner, Cerny, Depperschmidt and Gantert about the behaviour of the random walk on the backbone of the infinite cluster of oriented percolation.

1. Introduction

Consider supercritical oriented percolation. It seems natural to think that, on the percolation event "the cluster of the origin is infinite", the number N_n of open paths with length n starting from the origin should grow exponentially fast in n. The present paper aims to prove that $N_n^{1/n}$ (or rather $\frac{1}{n}\log N_n$) has an almost sure limit on the percolation event. Surprisingly, this result does not seem to be available in the litterature. More precisely, denoting by $\overline{\mathbb{P}}_p$ the probability of Bernoulli percolation conditioned by the fact that the cluster of $(0,0)$ is infinite, we proved

Theorem 1.1. *There exists $\alpha > 0$ such that $\overline{\mathbb{P}}_p$-almost surely,*

$$\lim_{n\to+\infty} \frac{\log N_n}{n} = \alpha.$$

We now define precisely the oriented percolation model we work with.

Oriented percolation in dimension $d+1$. Let $d \geq 1$ be fixed, and let $\|.\|$ be the norm on $\mathbb{R}^d$ defined by

$$\|x\| = \sum_{i=1}^d |x_i|.$$

We note $B(x,R)$ the associated balls. We consider the oriented graph whose set of sites is

$$\{(z,n) \in \mathbb{Z}^d \times \mathbb{N}\},$$

where $\mathbb{N} = \{0,1,2,\dots\}$, and we put an oriented edge from (z_1,n_1) to (z_2,n_2) if and only if

$$n_2 = n_1 + 1 \text{ and } \|z_2 - z_1\| \leq 1;$$

the set of these edges is denoted by $\overrightarrow{\mathbb{E}}^{d+1}_{\text{alt}}$. We say that $\gamma = (\gamma_i, i)_{0\leq i\leq n} \in (\mathbb{Z}^d \times \mathbb{N})^{n+1}$ is a *path* if and only if

$$\forall i \in \{0,\dots,n-1\} \quad \|\gamma_{i+1} - \gamma_i\| \leq 1.$$

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Fix now a parameter $p \in [0,1]$, and open independently each edge with probability p. More formally, consider the probability space $\Omega = \{0,1\}^{\overrightarrow{\mathbb{E}}^{d+1}_{\text{alt}}}$, endowed with its Borel σ-algebra and the probability

$$\mathbb{P}_p = (\text{Ber}(p))^{\otimes \overrightarrow{\mathbb{E}}^{d+1}_{\text{alt}}},$$

where Ber(p) stands for the Bernoulli law of parameter p. For a configuration $\omega = (\omega_e)_{e \in \overrightarrow{\mathbb{E}}^{d+1}_{\text{alt}}} \in \Omega$, say that the edge $e \in \overrightarrow{\mathbb{E}}^{d+1}_{\text{alt}}$ is open if $\omega_e = 1$ and closed otherwise. A path $\gamma = (\gamma_i, i)_{0 \le i \le n} \in (\mathbb{Z}^d \times \mathbb{N})^{n+1}$ is said *open* in the configuration ω if all its edges are open in ω. For two sites $(v,m), (w,n)$ in $\mathbb{Z}^d \times \mathbb{N}$, we denote by $\{(v,m) \to (w,n)\}$ the existence of an open path from (v,m) to (w,n). By extension, we denote by $\{(0,0) \to +\infty\}$ the percolation event, *i.e.* the event that there exists an infinite open path starting from the origin.

There exists a critical propability $\overrightarrow{p_c}^{\text{alt}}(d+1) \in (0,1)$ such that:

- if $p \le \overrightarrow{p_c}^{\text{alt}}(d+1)$, then $\mathbb{P}_p((0,0) \to +\infty) = 0$,
- if $p > \overrightarrow{p_c}^{\text{alt}}(d+1)$, then $\mathbb{P}_p((0,0) \to +\infty) > 0$.

In the following, we assume $p > \overrightarrow{p_c}^{\text{alt}}(d+1)$, and we will mainly work under the following conditional probability:

$$\overline{\mathbb{P}}_p(.) = \mathbb{P}_p(.|(0,0) \to +\infty).$$

Finally, we denote by N_n the random variable giving the number of open paths starting from $(0,0)$ with length n. Note that

$$\mathbb{E}_p(N_n) = ((2d+1)p)^n.$$

Previous results. The problem of the existence of a limit for $N_n^{1/n}$ is related to some questions that we recall now.

First, as noticed by Darling [3], the sequence $\left(\frac{N_n}{(2d+1)^n p^n}\right)_{n \ge 0}$ is a non-negative martingale, so there exists a non-negative random variable such that

$$\mathbb{P}_p - a.s. \quad \frac{N_n}{(2d+1)^n p^n} \longrightarrow W \text{ and } \mathbb{E}_p[W] \le 1.$$

Therefore, it is easy to see that

$$\frac{1}{n} \log N_n \to \log((2d+1)p) \text{ on the event } \{W > 0\}.$$

In that case, N_n has thus the same growth rate as its expectation.

In his paper [3], Darling was seeking for conditions implying that $W > 0$. It seems that these questions have been forgotten for a while, but there is currently an increased activity due to the links with random polymers – see for example Lacoin [7] and Yoshida [8].

Actually, it is not always the case that $W > 0$. Let us summarize some known results:

- $\overline{\mathbb{P}}_p(W > 0) \in \{0,1\}$ (see Lacoin [7]).
- $W = 0$ a.s. if $d = 1$ or $d = 2$ (see Yoshida [8]).
- for $d \ge 3$, there exists $\overrightarrow{p_{c,2}}^{\text{alt}}(d+1) \in [\overrightarrow{p_c}^{\text{alt}}(d+1), 1)$ such that $W > 0$ on $\{(0,0) \to \infty\}$ when $p > \overrightarrow{p_{c,2}}^{\text{alt}}(d+1)$ (see remark 2.7 in Lacoin [7]).

- The random variable $\chi = \overline{\lim_{n\to+\infty}} \frac{1}{n}\log N_n$ is $\overline{\mathbb{P}}_p$-almost surely constant (see Lacoin [7]). Note that a simple Borel-Cantelli argument ensures that $\chi \le \log((2d+1)p)$.

It is believed that $\overrightarrow{p_{c,2}}^{\text{alt}}(d+1) > \overrightarrow{p_c}^{\text{alt}}(d+1)$. Lacoin [7] proved that the inequality is indeed strict for L-spread-out percolation for $d \ge 5$ and L large. Then, it is clear that we need a proof of the existence of a limit for $\frac{1}{n}\log N_n$ that would not require that $W > 0$.

A natural idea to prove the existence of such a limit is to use Kingman's subadditive ergodic theorem. Obviously, if $N(a,b)$ denotes the number of open paths from a to b, we have the inequality

$$\log N(a,c) \ge \log N(a,b) + \log N(b,c).$$

The problem is naturally that $\log N(\cdot,\cdot)$ may be infinite, and therefore not integrable. Those kind of problem may be solved by using convenient subsequences that lead to integrable variables. To this aim, we use the technics of essential hitting times we introduced in Garet–Marchand [6] to establish a shape theorem for the contact process in random environment.

To recover the convergence along the full sequence, we need some kind of continuity. This continuity is obtained as a consequence of the following fact: most open paths from $(0,0)$ to the n-th level have their end in the ball $B(0,\varepsilon n)$. This fact is an easy consequence of a recent result of Birkner et al. [2] about the behaviour of the random walk on the backbone of the infinite cluster of oriented percolation.

2. Preliminary results

Exponential estimates for supercritical oriented percolation. We work on the graph $\mathbb{Z}^d \times \mathbb{N}$, as defined in the introduction. We set, for $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$,

$$\begin{aligned}
\xi_n^x &= \{y \in \mathbb{Z}^d :\ (x,0) \to (y,n)\},\\
\xi_n^{\mathbb{Z}^d} &= \bigcup_{x\in\mathbb{Z}^d} \xi_n^x,\\
\tau^x &= \min\{n \in \mathbb{N} :\ \xi_n^x = \varnothing\},\\
H_n^x &= \bigcup_{0\le k\le n} \xi_k^x,\\
K_n^x &= (\xi_n^x \Delta \xi_n^{\mathbb{Z}^d})^c = \xi_n^x \cup (\mathbb{Z}^d \backslash \xi_n^{\mathbb{Z}^d}).
\end{aligned}$$

For instance, τ^0 is the length of the longest open path starting from the origin, and the percolation event is equal to $\{\tau^0 = +\infty\}$. The set $K_n^x \cap H_n^x$ is called the coupled zone. As for the contact process, the sets $(H_n^x)_{n\ge0}$ and the coupled zones $(K_n^x \cap H_n^x)_{n\ge0}$ grow as least linearly in case of survival and the length of finite oriented paths has exponential moments:

Lemma 2.1. *We consider independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$. For every $p > \overrightarrow{p_c}^{alt}(d+1)$, there exist strictly positive constants A, B, C such that for every $x \in \mathbb{Z}^d$, for every $L, n > 0$:*

$$\mathbb{P}_p(\tau^x = +\infty, B(x,L) \not\subset K_{CL+n}^x) \le Ae^{-Bn}, \tag{1}$$

$$\mathbb{P}_p(\tau^x = +\infty, B(x,L) \not\subset H_{CL+n}^x) \le Ae^{-Bn}, \tag{2}$$

$$\mathbb{P}_p(n \le \tau^x < +\infty) \le Ae^{-Bn}. \tag{3}$$

Proof. For the contact process, Durrett [4] showed how to deduce an analogous result from the construction of Bezuidenhout–Grimmett [1]. As explained in [1], the proofs remain valid for oriented percolation, which is the discrete-time analogous of the contact process. □

Essential hitting time and associated translation. Define $\overrightarrow{\mathbb{E}}^d$ in the following way: in $\overrightarrow{\mathbb{E}}^d$, there is an oriented edge between two points z_1 and z_2 in $\mathbb{Z}^d$ if and only if $\|z_1 - z_2\|_1 \leq 1$. The oriented edge in $\overrightarrow{\mathbb{E}}^{d+1}_{\mathrm{alt}}$ from (z_1, n_1) to (z_2, n_2) can be identified with the couple $((z_1, z_2), n_2) \in \overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*$. Thus, we identify $\overrightarrow{\mathbb{E}}^{d+1}_{\mathrm{alt}}$ and $\overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*$. We also define, for $n \geq 0$, the time translation θ_n on Ω by:

$$\theta_n\big((\omega_{(e,k)})_{e\in\overrightarrow{\mathbb{E}}^d, k\geq 1}\big) = (\omega_{(e,k+n)})_{e\in\overrightarrow{\mathbb{E}}^d, k\geq 1}.$$

We now introduce the (next) essential hitting time σ: it is a random time $T \geq 1$ such that $(0,0) \to (0,T) \to +\infty$. It is defined through a family of stopping times as follows: we set $u_0 = v_0 = 0$ and we define recursively two increasing sequences of stopping times $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ with $u_0 = v_0 < u_1 < v_1 < u_2 \dots$ as follows:

- Assume that v_k is defined. We set $u_{k+1} = \inf\{t > v_k :\ 0 \in \xi_t^0\}$.
  If $v_k < +\infty$, then u_{k+1} is the first time after v_k where 0 is once again infected; otherwise, $u_{k+1} = +\infty$.
- Assume that u_k is defined, with $k \geq 1$. We set $v_k = u_k + \tau^0 \circ \theta_{u_k}$.
  If $u_k < +\infty$, the time $\tau^0 \circ \theta_{u_k}$ is the length of the oriented percolation cluster starting from $(0, u_k)$; otherwise, $v_k = +\infty$.

We then set

$$K = \min\{n \geq 0 :\ v_n = +\infty \text{ or } u_{n+1} = +\infty\}. \tag{4}$$

This quantity represents the number of steps before the success of this process: either we stop because we have just found an infinite v_n, which corresponds to a time u_n when 0 is occupied and has infinite progeny, or we stop because we have just found an infinite u_{n+1}, which says that after v_n, site 0 is never infected anymore. It is not difficult to see that

$$\mathbb{P}_p(K > n) \leq \mathbb{P}_p(\tau^0 < +\infty)^n,$$

and thus K is $\mathbb{P}_p$ almost surely finite. We define the next essential hitting time σ by setting

$$\sigma = u_K \in \mathbb{N} \cup \{+\infty\}.$$

Note however that σ is not necessary the first positive time when 0 is occupied and has infinite progeny: for instance, such an event can occur between u_1 and v_1, being ignored by the recursive construction. Note moreover that (2) implies that, conditionally to the event $\{\tau^0 = \infty\}$, the process necessarily stops because of an infinite v_n, and thus $\sigma < +\infty$. At the same time, we define the operator $\tilde{\theta}$ on Ω, which is a random translation, by:

$$\tilde{\theta}(\omega) = \begin{cases} \theta_{\sigma(\omega)}\omega & \text{if } \sigma(\omega) < +\infty, \\ \omega & \text{otherwise.} \end{cases}$$

The essential hitting time σ can be seen as a regeneration time for the process under $\overline{\mathbb{P}}_p$:

Lemma 2.2. *We consider independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$ with parameter $p > \overrightarrow{p_c}^{alt}(d+1)$.*

(1) *The probability measure* $\overline{\mathbb{P}}_p$ *is invariant under the translation* $\tilde{\theta}$.
(2) *The random variables* $(\sigma \circ (\tilde{\theta})^j)_{j\geq 0}$ *are independent and identically distributed under* $\overline{\mathbb{P}}_p$.
(3) *The measure-preserving dynamical system* $(\Omega, \mathcal{F}, \overline{\mathbb{P}}_p, \tilde{\theta})$ *is mixing.*

Proof. It is sufficient to mimic the proof of the lemmas and corollaries numbered from 6 to 10 in Garet-Marchand [5]. □

We also have a good control of the tail of the essential hitting time σ :

Lemma 2.3. *There exist positive constants* A, B *such that*

$$\forall t \geq 0 \quad \overline{\mathbb{P}}(\sigma > t) \leq A \exp(-B\sqrt{t}).$$

Proof. Again, we follow closely Garet–Marchand [5]. Proceeding as in lemmas 14 and 15 in Garet–Marchand [5], we show that

$$\overline{\mathbb{P}}(\sigma > Kt) \leq A \exp(-Bt).$$

Since K has exponential tail, this gives the desired result. □

Random walk on the backbone of oriented percolation. Consider independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$ with parameter $p > \overrightarrow{p_c}^{\mathrm{alt}}(d+1)$. Birkner et al. [2] study the random walk on the backbone of the oriented percolation cluster, and prove in particular that it satisfies a strong law of large numbers.

This random walk $(X_n)_{n\geq 0}$ is only defined on the event $\{\tau^0 = +\infty\}$: it starts at 0, and knowing that $X_n = x$, X_{n+1} is uniformly distributed among the sites

$$\{y \in \mathbb{Z}^d : \ \|y - x\| \leq 1 \text{ and } (y, n+1) \to +\infty\}.$$

Let us define this walk more precisely. We choose an arbitrary order, for example the lexicographic order, on

$$V = \{x \in \mathbb{Z}^d : \ \|x\| \leq 1\}.$$

We consider the enlarged set $\tilde{\Omega} = \Omega \times [0, 1]^{\mathbb{N}^*}$, endowed with the probability

$$\tilde{\mathbb{P}}_p = \overline{\mathbb{P}}_p \otimes \mathcal{U}([0, 1])^{\otimes \mathbb{N}},$$

where $\mathcal{U}([0, 1])$ denotes the uniform law on $[0, 1]$. We denote by (ω, η) the generic element of $\tilde{\Omega}$: $\omega = (\omega_e)_{e\in \overrightarrow{\mathbb{E}}^{d+1}_{\mathrm{alt}}}$ is the environment (oriented percolation) where the random walk lives while $\eta = (\eta_n)_{n\geq 1}$ encodes the randomness of the steps of the walk.

Proposition 2.4 (Birkner et al. [2])**.** *Define* $(X_n)_{n\geq 0}$ *by setting* $X_0 = 0$ *and, for* $n \geq 0$

$$\begin{aligned} V_n &= \{k \in V : \ (X_n + k, n+1) \to \infty\}, \\ D_n &= \inf \left\{ k \in V : \ \eta_n < \frac{\sum_{i\leq k} \mathbb{1}_{\{i\in V_n\}}}{\sum_{i\in V} \mathbb{1}_{\{i\in V_n\}}} \right\}, \\ X_{n+1} &= X_n + D_n. \end{aligned}$$

Then $\tilde{\mathbb{P}}_p$*-almost surely,* $(X_n/n)_{n\geq 0}$ *converges to* 0 *when* n *goes to* $+\infty$.

3. Proof of Theorem 1.1

Let $d \geq 1$ be fixed. Consider independent oriented percolation on $\mathbb{Z}^d \times \mathbb{N}$ with parameter $p > \overrightarrow{p_c}^{\text{alt}}(d+1)$. Remember that C has been defined in Lemma 2.1. We set

$$D = 2(C+1). \tag{5}$$

Notation. For $n \geq 1$, $0 < \varepsilon < 1/D$ and $x \in \mathbb{Z}^d$, we denote by

- N_n the number of open paths from $(0,0)$ to $\mathbb{Z}^d \times \{n\}$,
- $\overline{N}_n$ the number of open paths from $(0,0)$ to $\mathbb{Z}^d \times \{n\}$ that are the beginning of an infinite open path,
- N_n^x the number of open paths from $(0,0)$ to (x,n),
- $\overline{N}_n^x$ the number of open paths from $(0,0)$ to (x,n) that are the beginning of an infinite open path,
- N_n^ε the number of open paths $\gamma = ((\gamma_i, i)_{0 \leq i \leq n})$ from $(0,0)$ to $(B(0,\varepsilon n) \cap \mathbb{Z}^d) \times \{n\}$ such that, moreover, $\gamma_{\lfloor (1-D\varepsilon)n \rfloor} \in B(0,\varepsilon n)$,
- $\overline{N}_n^\varepsilon$ the number of open paths $\gamma = ((\gamma_i, i)_{0 \leq i \leq n})$ from $(0,0)$ to $(B(0,\varepsilon n) \cap \mathbb{Z}^d) \times \{n\}$ such that, moreover, $\gamma_{\lfloor (1-D\varepsilon)n \rfloor} \in B(0,\varepsilon n)$ and that are the beginning of an infinite open path,
- $N_n^{x,\varepsilon}$ the number of open paths $\gamma = ((\gamma_i, i)_{0 \leq i \leq n})$ from $(0,0)$ to (x,n) such that, moreover, $\gamma_{\lfloor (1-D\varepsilon)n \rfloor} \in B(0,\varepsilon n)$
- $\overline{N}_n^{x,\varepsilon}$ the number of open paths $\gamma = ((\gamma_i, i)_{0 \leq i \leq n})$ from $(0,0)$ to (x,n) such that, moreover, $\gamma_{\lfloor (1-D\varepsilon)n \rfloor} \in B(0,\varepsilon n)$ and that are the beginning of an infinite open path.

In fact, we are going to prove the following stronger theorem:

Theorem 3.1. *There exists $\alpha > 0$ such that $\overline{\mathbb{P}}_p$-almost surely,*

$$\lim_{n \to +\infty} \frac{\log N_n}{n} = \lim_{n \to +\infty} \frac{\log \overline{N}_n}{n} = \alpha.$$

We work under $\overline{\mathbb{P}}_p$, and we set, for every $n \geq 1$,

$$S_n = \sum_{k=0}^{n-1} \sigma \circ \tilde{\theta}^k. \tag{6}$$

S_n is the n-th essential hitting time of 0.

We recall that under $\overline{\mathbb{P}}_p$, the random variables $(\sigma \circ \tilde{\theta}^j)_{j \geq 0}$ are independent and identically distributed (see Lemma 2.2) with finite first moment (by Lemma 2.3), thus the strong law of large numbers ensures that $\overline{\mathbb{P}}_p$-almost surely

$$\lim_{n \to +\infty} \frac{S_n}{n} = \overline{\mathbb{E}}_p(\sigma). \tag{7}$$

Applying Kingman's subadditive ergodic theorem to $f_n = -\log N_{S_n}^0$, we can prove the following limits:

Lemma 3.2. *There exists $\alpha' > 0$ such that $\overline{\mathbb{P}}_p$-almost surely,*

$$\lim_{n \to +\infty} \frac{1}{n} \log N_{S_n}^0 = \lim_{n \to +\infty} \frac{1}{n} \log \overline{N}_{S_n}^0 = \alpha'.$$

Set $\alpha = \alpha'/\overline{\mathbb{E}}_p(\sigma)$; then $\overline{\mathbb{P}}_p$-almost surely,

$$\lim_{n\to+\infty} \frac{1}{S_n} \log N^0_{S_n} = \lim_{n\to+\infty} \frac{1}{S_n} \log \overline{N}^0_{S_n} = \alpha.$$

Proof. Note that by definition of σ, $\overline{\mathbb{P}}_p$-almost surely, for every $n \geq 1$, $(0,0) \to (0,S_n) \to +\infty$ and consequently, $\overline{N}^0_{S_n} = N^0_{S_n} \geq 1$.
For $n \geq 1$, we set

$$f_n = -\log \overline{N}^0_{S_n} = -\log N^0_{S_n}.$$

Let $n,p \geq 1$. Note that $S_n + S_p \circ \tilde{\theta}^n = S_{n+p}$. As $N^0_{S_p} \circ \tilde{\theta}^n$ counts the number of open paths from $(0,S_n)$ to $(0,S_n + S_p \circ \tilde{\theta}^n)$, concatenation of paths ensures that $N^0_{S_n} \times N^0_{S_p} \circ \tilde{\theta}^n \leq N^0_{S_{n+p}}$, which implies that

$$\forall n,p \geq 1 \quad f_{n+p} \leq f_n + f_p \circ \tilde{\theta}^n.$$

As $1 \leq N^0_{S_n} \leq (2d+1)^{S_n}$,

$$-S_n \log(2d+1) \leq f_n \leq 0.$$

The integrability of σ thus implies the integrability of every f_n: so we can apply Kingman's subadditive ergodic theorem: if we define

$$-\alpha' = \inf_{n\geq 1} \frac{\overline{\mathbb{E}}_p(f_n)}{n},$$

we have $\overline{\mathbb{P}}_p$-almost surely and in $L^1(\overline{\mathbb{P}}_p)$: $\lim\limits_{\to n+\infty} \dfrac{f_n}{n} = -\alpha'$.
The two last limits follow then directly from (7).

Finally $\alpha' \geq \overline{\mathbb{E}}_p(-f_1) = \overline{\mathbb{E}}_p(\log N^0_\sigma)$. Since $N^0_\sigma \geq 1$ $\overline{\mathbb{P}}_p$-a.s. and $N^0_\sigma \geq 2$ with positive probability, it follows that $\alpha' > 0$, and consequently $\alpha > 0$. □

In the next step, we prove that open paths from $(0,0)$ to $\mathbb{Z}^d \times \{n\}$ are in some sense concentrated around the vertical axis. This Lemma follows from the recent result of Birkner et al. [2] concerning the random walk in the random environment given by the backbone of the supercritical oriented percolation cluster. This explains why we first work with open paths that are the beginnings of infinite open paths:

Lemma 3.3. *For every $\varepsilon > 0$, $\overline{\mathbb{P}}_p$-almost surely,* $\lim\limits_{n\to+\infty} \dfrac{\overline{N}^\varepsilon_n}{\overline{N}_n} = 1$.

Proof. As in Proposition 2.4, we work on $\tilde{\Omega} = \Omega \times [0,1]^{\mathbb{N}^*}$ with $\tilde{\mathbb{P}}_p = \overline{\mathbb{P}}_p \otimes \mathcal{U}([0,1])^{\otimes\mathbb{N}}$ and consider the random walk $(X_n)_{n\geq 0}$ in the random environment given by the backbone of the supercritical oriented percolation cluster. By Proposition 2.4, $\tilde{\mathbb{P}}_p$-almost surely, $(X_n/n)_{n\geq 1}$ converges to 0 as n goes to $+\infty$. So if we set

$$Y_n = \mathbb{1}_{\{\|X_n\| \leq \varepsilon n\} \cap \{\|X_{\lfloor(1-D\varepsilon)n\rfloor}\| \leq \varepsilon n\}},$$

then Y_n goes $\tilde{\mathbb{P}}_p$-almost surely to 1. Denote by $\mathcal{E}$ the σ-field generated by the oriented percolation. The conditional version of the dominated convergence theorem ensures that $\tilde{\mathbb{E}}_p[Y_n|\mathcal{E}]$ goes $\tilde{\mathbb{P}}_p$-almost surely to 1. But after a short moment of thought, we see that

$$\tilde{\mathbb{E}}_p[Y_n|\mathcal{E}] = \frac{\overline{N}^\varepsilon_n}{\overline{N}_n},$$

so $\frac{\overline{N}_n^\varepsilon}{\overline{N}_n}$ tends $\tilde{\mathbb{P}}_p$-almost surely to 1, which also means that $\frac{\overline{N}_n^\varepsilon}{\overline{N}_n}$ tends $\overline{\mathbb{P}}_p$-almost surely to 1. □

Now, we prove that working with open paths or with open paths that are the beginning of an infinite open path is essentially the same:

Lemma 3.4. *$\overline{\mathbb{P}}_p$-almost surely,*

$$\varlimsup_{n\to+\infty} \frac{\log N_n}{n} = \varlimsup_{n\to+\infty} \frac{\log \overline{N}_n}{n} \quad \textit{and} \quad \varliminf_{n\to+\infty} \frac{\log N_n}{n} = \varliminf_{n\to+\infty} \frac{\log \overline{N}_n}{n}.$$

Proof. Fix $0 < \varepsilon < 1$ and define, for $n \geq 1$, the following event

$$E_n = \bigcap_{\|z\|\leq n} \{\tau^z \circ \theta_{\lfloor n(1-\varepsilon)\rfloor)} < \varepsilon n \text{ or } \tau^z \circ \theta_{\lfloor n(1-\varepsilon)\rfloor)} = +\infty\}.$$

Assume that E_n occurs. Consider a path $\gamma = (\gamma_i, i)_{0\leq i\leq n}$ from $(0,0)$ to $\mathbb{Z}^d\times\{n\}$ and set $z = \gamma_{\lfloor n(1-\varepsilon)\rfloor}$: as $\tau^z \circ \theta_{\lfloor n(1-\varepsilon)\rfloor)} \geq \varepsilon n$, the event E_n implies that $\tau^z \circ \theta_{\lfloor n(1-\varepsilon)\rfloor)} = +\infty$. So $(\gamma_i, i)_{0\leq i\leq \lfloor n(1-\varepsilon)\rfloor}$ contributes to $\overline{N}_{\lfloor n(1-\varepsilon)\rfloor}$ and thus, on E_n,

$$\begin{aligned} N_n &\leq (2d+1)^{\varepsilon n+1} \overline{N}_{\lfloor n(1-\varepsilon)\rfloor}, \\ \text{so } \frac{\log N_n}{n} &\leq (\varepsilon + \frac{1}{n}) \log(2d+1) + \frac{\log \overline{N}_{\lfloor n(1-\varepsilon)\rfloor}}{n} \\ &\leq (\varepsilon + \frac{1}{n}) \log(2d+1) + \frac{\log \overline{N}_{\lfloor n(1-\varepsilon)\rfloor}}{\lfloor n(1-\varepsilon)\rfloor}. \end{aligned}$$

The exponential estimate (3) on the long but finite lifetimes in supercritical oriented percolation ensures that

$$\forall n \geq 1 \quad \mathbb{P}_p(E_n^c) \leq C_d A n^d \exp(-B\varepsilon n) \leq A' \exp(-B'n).$$

With the Borel–Cantelli lemma, this leads to:

$$\varlimsup_{n\to+\infty} \frac{\log N_n}{n} \leq \varepsilon \log(2d+1) + \varlimsup_{n\to+\infty} \frac{\log \overline{N}_n}{n}.$$

By taking ε to 0, we obtain

$$\varlimsup_{n\to+\infty} \frac{\log N_n}{n} \leq \varlimsup_{n\to+\infty} \frac{\log \overline{N}_n}{n}.$$

The proof for the inequality with $\varliminf$ instead of $\varlimsup$ is identical. Since $\overline{N}_n \leq N_n$, the reversed inequalities are obvious. □

The next step consists in proving a continuity property which says that close points linked to $(0,0)$ have close number of open paths linking them to $(0,0)$. To prove this, we use the at least linear growth (1) of the coupled zone in supercritical oriented percolation, and must thus consider paths close to the time axis:

Lemma 3.5 (Continuity Lemma)**.** *Let $0 < \varepsilon < 1/D$ be fixed. There exist constants $A, B > 0$ such that for every $n \geq 1$,*

$$\mathbb{P}_p\left(\begin{array}{c} \exists y \in B(0,\varepsilon n) \textit{ such that } (0,0) \to (y,n) \textit{ and} \\ N_n^\varepsilon > N_n^{y,\varepsilon} \times (2d+1)^{D\varepsilon n} \end{array} \right) \leq A\exp(-Bn). \tag{8}$$

Proof. We recall that D is defined in (5). Define, for $n \geq 1$,

$$G_n = \left\{ \begin{array}{c} \forall z \in B(0,\varepsilon n) \quad (\tau^z \leq D\varepsilon n \text{ or } \tau^z = +\infty); \\ \forall z \in B(0,\varepsilon n) \quad (\tau^z = +\infty) \Rightarrow B(0,n\varepsilon) \subset K^z_{D\varepsilon n} \end{array} \right\}. \tag{9}$$

Since $B(0,\varepsilon n) \subset B(z,2n\varepsilon)$ for each $z \in B(0,\varepsilon n)$, lemma 2.1 gives

$$\forall n \geq 1 \quad \mathbb{P}(G_n^c) \leq Cn^d 2A \exp(-BD\varepsilon n) \leq A' \exp(-B'n). \tag{10}$$

Let $y \in B(0,\varepsilon n)$. Let us show that the inequality

$$N_n^\varepsilon \leq N_n^{y,\varepsilon} \times (2d+1)^{D\varepsilon n} \tag{11}$$

holds on the event $\{G_n \circ \theta_{\lfloor (1-D\varepsilon)n \rfloor}\} \cap \{(0,0) \to (y,n)\}$: this will end the proof of the lemma.

Assume that $\{G_n \circ \theta_{\lfloor (1-D\varepsilon)n \rfloor}\} \cap \{(0,0) \to (y,n) \to \infty\}$ holds. Denote by $\mathcal{N}_n^\varepsilon$ the set of open paths $\gamma = ((\gamma_i, i)_{0 \leq i \leq n})$ from $(0,0)$ to some (γ_n, n) with $\gamma_n \in B(0,\varepsilon n)$ and $\gamma_{\lfloor (1-D\varepsilon)n \rfloor} \in B(0,\varepsilon n)$: it has cardinality N_n^ε. Note also $\mathcal{N}_n^{y,\varepsilon}$ the set of $\gamma \in \mathcal{N}_n^\varepsilon$ with $\gamma_n = y$.

Let γ be a path in $\mathcal{N}_n^\varepsilon$ and note $z = \gamma_{\lfloor (1-D\varepsilon)n \rfloor} \in B(0,\varepsilon n)$. As $\tau^z \circ \theta_{\lfloor (1-D\varepsilon)n \rfloor} \geq D\varepsilon n$, the occurrence of $G_n \circ \theta_{\lfloor (1-D\varepsilon)n \rfloor}$ ensures that $\tau^z \circ \theta_{\lfloor (1-D\varepsilon)n \rfloor} = +\infty$, and that

$$y \in B(0,\varepsilon n) \subset K^z_{D\varepsilon n} \circ \theta_{\lfloor (1-D\varepsilon)n \rfloor}.$$

Since there exists a path from $(0,0)$ to (y,n), y also belongs to $\xi^{\mathbb{Z}^d}_{D\varepsilon n} \circ \theta_{\lfloor (1-D\varepsilon)n \rfloor}$, so $y \in \xi^z_{D\varepsilon n} \circ \theta_{\lfloor (1-D\varepsilon)n \rfloor}$. Then, there exists an open path γ' from $(z, \lfloor (1-D\varepsilon)n \rfloor)$ to (y,n). Now, we associate to the path $\gamma \in \mathcal{N}_n^\varepsilon$ the couple $\Theta(\gamma) = (\Theta_1(\gamma), \Theta_2(\gamma))$, where

- $\Theta_1(\gamma)$ is the concatenation of the portion of γ between $(0,0)$ and $(z, \lfloor (1-D\varepsilon)n \rfloor)$ and of γ': it is thus in $\mathcal{N}_n^{y,\varepsilon}$;
- $\Theta_2(\gamma)$ is the portion of γ between $(z, \lfloor (1-D\varepsilon)n \rfloor)$ and (x,n).

As we can recover γ from $\Theta(\gamma)$, Θ is an injection from $\mathcal{N}_n^\varepsilon$ into a set of cardinality $N_n^{y,\varepsilon} \times (2d+1)^{D\varepsilon n}$, which gives (11). □

Proof of Theorem 3.1. Let $0 < \varepsilon < 1/D$ be fixed. Remember that $(S_n)_{n \geq 1}$ is the random subsequence of times defined by (6). With a Borel-Cantelli argument, Lemma 3.5 gives, $\overline{\mathbb{P}}_p$-almost surely, for all n large enough,

$$(0,0) \to (0,n) \Longrightarrow N_n^\varepsilon \leq N_n^{0,\varepsilon} \times (2d+1)^{D\varepsilon n}.$$

But $\overline{N}^\varepsilon_{S_n} \leq N^\varepsilon_{S_n}$, $N^{0,\varepsilon}_{S_n} = \overline{N}^{0,\varepsilon}_{S_n} \leq \overline{N}^0_{S_n}$ and $\{(0,0) \to (0,S_n)\}$ always hold, so $\overline{\mathbb{P}}_p$-almost surely, for large n, we have

$$\overline{N}^\varepsilon_{S_n} \leq \overline{N}^0_{S_n} \times (2d+1)^{D\varepsilon S_n},$$

$$\text{or, equivalently,} \quad \log \overline{N}^\varepsilon_{S_n} \leq \log \overline{N}^0_{S_n} + D\varepsilon S_n \log(2d+1).$$

As $N^0_{S_n} = \overline{N}^0_{S_n} \leq \overline{N}_{S_n}$ and $\log \overline{N}_{S_n} = \log \dfrac{\overline{N}_{S_n}}{\overline{N}^\varepsilon_{S_n}} + \log \overline{N}^\varepsilon_{S_n}$, we get that $\overline{\mathbb{P}}_p$-almost surely, for large n

$$\log N^0_{S_n} \leq \log \overline{N}_{S_n} \leq \log \frac{\overline{N}_{S_n}}{\overline{N}^\varepsilon_{S_n}} + \log N^0_{S_n} + D\varepsilon S_n \log(2d+1).$$

Lemma 3.2, Lemma 3.3 and the strong law of large numbers (7) for S_n ensure that

$$\alpha' \leq \underline{\lim}_{n\to+\infty} \frac{1}{n}\log \overline{N}_{S_n} \leq \overline{\lim}_{n\to+\infty} \frac{1}{n}\log \overline{N}_{S_n} \leq \alpha' + D\varepsilon\overline{\mathbb{E}}_p(\sigma)\log(2d+1).$$

By taking ε to 0, this implies that $\overline{\mathbb{P}}_p$-almost surely

$$\lim_{n\to+\infty} \frac{1}{n}\log \overline{N}_{S_n} = \alpha'.$$

Equation (7) ensures that $S_n \sim n\overline{\mathbb{E}}_p(\sigma)$, and $(\overline{N}_n)_{n\geq 1}$ is non decreasing, thus $\overline{\mathbb{P}}_p$-almost surely

$$\lim_{n\to+\infty} \frac{1}{n}\log \overline{N}_n = \lim_{n\to+\infty} \frac{1}{S_n}\log \overline{N}_{S_n} = \frac{\alpha'}{\overline{\mathbb{E}}_p(\sigma)} = \alpha.$$

Finally, the limit for $\frac{1}{n}\log N_n$ follows from Lemma 3.4. □

Corollary 3.6. *Let (a_n) be a non-negative sequence with $\lim_{n\to+\infty} \frac{a_n}{n} = 0$. For each $\eta > 0$, $\mathbb{P}_p$-almost surely, there exists n_0 such that for $n \geq n_0$:*

$$\forall x \in \mathbb{Z}^d; \|x\| \leq a_n \text{ and } (0,0) \to (x,n) \Longrightarrow \alpha - \eta \leq \frac{\log N((0,0),(x,n))}{n} \leq \alpha + \eta.$$

Proof. The upper bound obviously follows from the inequality

$$\sup_{\|x\|\leq a_n} N_n^x) \leq \sum_{\|x\|\leq a_n} N((0,0),(x,n)) \leq N_n.$$

Take $\varepsilon > 0$ with $D\varepsilon\log(2d+1) < \eta/2$. Putting Lemma 3.3 and Theorem 3.1 together, we get that $\overline{\mathbb{P}}_p$ almost surely, $\frac{\log N_n^\varepsilon}{n} > \alpha - \eta/2$ for large n.

For large n, $B(0,a_n) \subset B(0,\varepsilon n)$, by Lemma 3.5, we see that for large n, $(0,0) \to (x,n) \Longrightarrow \frac{\log N_n^{y,\varepsilon}}{n} \geq \frac{\log N_n^\varepsilon}{n} - D\varepsilon\log(2d+1)$, which gives the result. □

4. Possible extensions

There are several directions of extension of the results presented here.

A first way is to change the percolating structure. Here, the bond between (x,n) and $(y,n+1)$ is open with probability $p1\!\!1_{\{\|x-y\|\leq 1\}}$. A natural extension is to take the model in Lacoin [7]: then, the probability of being open is $p\varphi(x-y)$, where φ is a non-negative function with finite support. Some discussions with the authors of [2] let think that an extension of Proposition 2.4 is possible. Also, the change of percolating structure (with possible jumps) creates some difficulties, but they should be tractable.

Another interesting question is to observe the number of path along precise directions. We have already seen in corollary 3.6 what happens in the main direction. We can for example imagine the following extension:

Conjecture 1. *Let $y \in \mathbb{R}^d$. There exists $\alpha(y) \geq 0$ such that, for each non-negative sequence (a_n) with $\lim_{n\to+\infty} \frac{a_n}{n} = 0$, for each $\eta > 0$, $\mathbb{P}_p -$ a.s., there exists n_0 such that for $n \geq n_0$:*

$$\forall x \in \mathbb{Z}^d; x \in B(ny,a_n) \text{ and } (0,0) \to (x,n) \Longrightarrow \alpha(y) - \eta \leq \frac{\log N_n^x}{n} \leq \alpha(y) + \eta.$$

Presumably, $\alpha(y) > 0$ if and only if y belongs to the open cone of oriented percolation.

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