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# Chapter 1

## On Landau's Function $g(n)$

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### 1.1 Introduction

Let  $S_n$  be the symmetric group of  $n$  letters. Landau considered the function  $g(n)$  defined as the maximal order of an element of  $S_n$ ; Landau observed that (cf. [9])

$$g(n) = \max \operatorname{lcm}(m_1, \dots, m_k) \quad (1.1)$$

where the maximum is taken on all the partitions  $n = m_1 + m_2 + \dots + m_k$  of  $n$  and proved that, when  $n$  tends to infinity

$$\log g(n) \sim \sqrt{n \log n}. \quad (1.2)$$

More precise asymptotic estimates have been given in [22, 25, 11]. In [25] and [11] one also can find asymptotic estimates for the number of prime factors of  $g(n)$ . In [8] and [3], the largest prime factor  $P^+(g(n))$  of  $g(n)$  is investigated. In [10] and [12], effective upper and lower bounds of  $g(n)$  are given. In [17], it is proved that  $\lim_{n \rightarrow \infty} g(n+1)/g(n) = 1$ . An algorithm able to calculate  $g(n)$  up to  $10^{15}$  is given in [2] (see also [26]). The sequence of distinct values of  $g(n)$  is entry A002809 of [24]. A nice survey paper was written by W. Miller in 1987 (cf. [13]).

My very first mathematical paper [15] was about Landau's function, and the main result was that  $g(n)$ , which is obviously non decreasing, is constant on arbitrarily long intervals (cf. also [16]). First time I met A. Schinzel in Paris in May 1967. He told me that he was interested in my results, but that P. Erdős would be more interested than himself. Then I wrote my first letter to Paul

with a copy of my work. I received an answer dated of June 12 1967 saying " I sometimes thought about  $g(n)$  but my results were very much less complete than yours". Afterwards, I met my advisor, the late Professor Pisot, who, in view of this letter, told me that my work was good for a thesis.

The main idea of my work about  $g(n)$  was to use the tools introduced by S. Ramanujan to study highly composite numbers (cf. [19, 20]). P. Erdős was very well aware of this paper of Ramanujan (cf. [1, 4, 6, 5]) as well as of the symmetric group and the order of its elements, (cf. [7]) and I think that he enjoyed the connection between these two areas of mathematics. Anyway, since these first letters, we had many occasions to discuss Landau's function.

Let us define  $n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4, n_5 = 5, n_6 = 7$ , etc  $\dots, n_k$  (see a table of  $g(n)$  in [16, p. 187]), such that

$$g(n_k) > g(n_k - 1). \quad (1.3)$$

The above mentioned result can be read:

$$\overline{\lim} (n_{k+1} - n_k) = +\infty. \quad (1.4)$$

Here, I shall prove the following result:

**Theorem 1.**

$$\underline{\lim} (n_{k+1} - n_k) < +\infty. \quad (1.5)$$

Let us set  $p_1 = 2, p_2 = 3, p_3 = 5, \dots, p_k$  = the  $k$ -th prime. It is easy to deduce Theorem 1 from the twin prime conjecture (i.e.  $\underline{\lim} (p_{k+1} - p_k) = 2$ ) or even from the weaker conjecture  $\underline{\lim} (p_{k+1} - p_k) < +\infty$ . (cf. §1.4 below). But I shall prove Theorem 1 independently of these deep conjectures. Moreover I shall explain below why it is reasonable to conjecture that the mean value of  $n_{k+1} - n_k$  is 2; in other terms one may conjecture that

$$n_k \sim 2k \quad (1.6)$$

and that  $n_{k+1} - n_k = 2$  has infinitely many solutions. Due to a parity phenomenon,  $n_{k+1} - n_k$  seems to be much more often even than odd; nevertheless, I conjecture that:

$$\underline{\lim} (n_{k+1} - n_k) = 1. \quad (1.7)$$

The steps of the proof of Theorem 1 are first to construct the set  $G$  of values of  $g(n)$  corresponding to the so called superior highly composite numbers introduced by S. Ramanujan, and then, when  $g(n) \in G$ , to build the table of  $g(n+d)$  when  $d$  is small. This will be done in §1.4 and §1.5. Such values of  $g(n+d)$  will be linked with the number of distinct differences of the form  $P-Q$  where  $P$  and  $Q$  are primes satisfying  $x - x^\alpha \leq Q \leq x < P \leq x + x^\alpha$ , where  $x$  goes to infinity and  $0 < \alpha < 1$ . Our guess is that these differences  $P-Q$  represent almost all even numbers between 0 and  $2x^\alpha$ , but we shall only prove in §1.3 that the number of these differences is of the order of magnitude of  $x^\alpha$ , under certain strong hypothesis on  $x$  and  $\alpha$ , and for that a result due to Selberg about the primes between  $x$  and  $x + x^\alpha$  will be needed (cf. §1.2).

To support conjecture (1.6), I think that what has been done here with  $g(n) \in G$  can also be done for many more values of  $g(n)$ , but, unfortunately, even assuming strong hypotheses, I do not see for the moment how to manage it.

I thank very much E. Fouvry who gave me the proof of Proposition 2.

### 1.1.1 Notation

$p$  will denote a generic prime,  $p_k$  the  $k$ -th prime;  $P, Q, P_i, Q_j$  will also denote primes. As usual  $\pi(x) = \sum_{p \leq x} 1$  is the number of primes up to  $x$ .

$|S|$  will denote the number of elements of the set  $S$ . The sequence  $n_k$  is defined by (1.3).

## 1.2 About the distribution of primes

**Proposition 1.** *Let us define  $\pi(x) = \sum_{p \leq x} 1$ , and let  $\alpha$  be such that  $\frac{1}{6} < \alpha < 1$ , and  $\varepsilon > 0$ . When  $\xi$  goes to infinity, and  $\xi' = \xi + \xi/\log \xi$ , then for all  $x$  in the interval  $[\xi, \xi']$  but a subset of measure  $O((\xi' - \xi)/\log^3 \xi)$  we have:*

$$\left| \pi(x + x^\alpha) - \pi(x) - \frac{x^\alpha}{\log x} \right| \leq \varepsilon \frac{x^\alpha}{\log x} \quad (1.8)$$

$$\left| \pi(x) - \pi(x - x^\alpha) - \frac{x^\alpha}{\log x} \right| \leq \varepsilon \frac{x^\alpha}{\log x} \quad (1.9)$$

$$\left| \frac{x}{\log x} - \frac{Q^k - Q^{k-1}}{\log Q} \right| \geq \frac{\sqrt{x}}{\log^4 x} \text{ for all primes } Q, \text{ and } k \geq 2. \quad (1.10)$$

*Proof.* This proposition is an easy extension of a result of Selberg (cf. [21]) who proved that (1.8) holds for most  $x$  in  $(\xi, \xi')$ . In [18], I gave a first extension of Selberg's result by proving that (1.8) and (1.9) hold simultaneously for all  $x$  in  $(\xi, \xi')$  but for a subset of measure  $O((\xi' - \xi)/\log^3 \xi)$ . So, it suffices to prove that the measure of the set of values of  $x$  in  $(\xi, \xi')$  for which (1.10) does not hold is  $O((\xi' - \xi)/\log^3 \xi)$ .

We first count the number of primes  $Q$  such that for one  $k$  we have:

$$\frac{\xi}{\log \xi} \leq \frac{Q^k - Q^{k-1}}{\log Q} \leq \frac{\xi'}{\log \xi'}. \quad (1.11)$$

If  $Q$  satisfies (1.11), then  $k \leq \frac{\log \xi'}{\log 2}$  for  $\xi'$  large enough. Further, for  $k$  fixed, (1.11) implies that  $Q \leq (\xi')^{1/k}$ , and the total number of solutions of (1.11) is

$$\leq \sum_{k=2}^{\log \xi' / \log 2} (\xi')^{1/k} = O(\sqrt{\xi'}) = O(\sqrt{\xi}).$$

With a more careful estimation, this upper bound could be improved, but this crude result is enough for our purpose. Now, for all values of  $y = \frac{Q^k - Q^{k-1}}{\log Q}$  satisfying (1.11), we cross out the interval  $\left(y - \frac{\sqrt{\xi'}}{\log^4 \xi'}, y + \frac{\sqrt{\xi'}}{\log^4 \xi'}\right)$ . We also cross out this interval whenever  $y = \frac{\xi}{\log \xi}$  and  $y = \frac{\xi'}{\log \xi'}$ . The total sum of the lengths of the crossed out intervals is  $O\left(\frac{\xi}{\log^4 \xi}\right)$ , which is smaller than the length of the interval  $\left(\frac{\xi}{\log \xi}, \frac{\xi'}{\log \xi'}\right)$  and if  $\frac{x}{\log x}$  does not fall into one of these forbidden intervals, (1.10) will certainly hold. Since the derivative of the function  $\varphi(x) = x/\log x$  is  $\varphi'(x) = \frac{1}{\log x} - \frac{1}{\log^2 x}$  and satisfies  $\varphi'(x) \sim \frac{1}{\log \xi}$  for all  $x \in (\xi, \xi')$ , the measure of the set of values of  $x \in (\xi, \xi')$  such that  $\varphi(x)$  falls into one of the above forbidden intervals is, by the mean value theorem  $O\left(\frac{\xi}{\log^3 \xi}\right)$ , and the proof of Proposition 1 is completed.

### 1.3 About the differences between primes

**Proposition 2.** *Suppose that there exists  $\alpha, 0 < \alpha < 1$ , and  $x$  large enough such that the inequalities*

$$\pi(x + x^\alpha) - \pi(x) \geq (1 - \varepsilon)x^\alpha / \log x \quad (1.12)$$

$$\pi(x) - \pi(x - x^\alpha) \geq (1 - \varepsilon)x^\alpha / \log x \quad (1.13)$$

hold. Then the set

$$E = E(x, \alpha) = \{P - Q; P, Q \text{ primes}, x - x^\alpha < Q \leq x < P \leq x + x^\alpha\}$$

satisfies:

$$|E| \geq C_2 x^\alpha$$

where  $C_2 = C_1 \alpha^4 (1 - \varepsilon)^4$  and  $C_1$  is an absolute constant ( $C_1 = 0.00164$  works).

*Proof.* The proof is a classical application of the sieve method that Paul Erdős enjoys very much. Let us set, for  $d \leq 2x^\alpha$ ,

$$r(d) = |\{(P, Q); x - x^\alpha < Q \leq x < P \leq x + x^\alpha, P - Q = d\}|.$$

Clearly we have

$$|E| = \sum_{\substack{0 < d \leq 2x^\alpha \\ r(d) \neq 0}} 1 \quad (1.14)$$

and

$$\sum_{0 < d \leq 2x^\alpha} r(d) = (\pi(x + x^\alpha) - \pi(x))(\pi(x) - \pi(x - x^\alpha)) \geq (1 - \varepsilon)^2 x^{2\alpha} / \log^2 x. \quad (1.15)$$

Now to get an upper bound for  $r(d)$ , we sift the set

$$A = \{n; x - x^\alpha < n \leq x\}$$

with the primes  $p \leq z$ . If  $p$  divides  $d$ , we cross out the  $n$ 's satisfying  $n \equiv 0 \pmod{p}$ , and if  $p$  does not divide  $d$ , the  $n$ 's satisfying

$$n \equiv 0 \pmod{p} \quad \text{or} \quad n \equiv -d \pmod{p}$$

so that we set for  $p \leq z$ :

$$w(p) = \begin{cases} 1 & \text{if } p \text{ divides } d \\ 2 & \text{if } p \text{ does not divide } d. \end{cases}$$

By applying the large sieve (cf. [14, Corollary 1]), we have

$$r(d) \leq \frac{|A|}{L(z)}$$

with

$$L(z) = \sum_{n \leq z} \left(1 + \frac{3}{2}n|A|^{-1}z\right)^{-1} \mu(n)^2 \left(\prod_{p|n} \frac{w(p)}{p-w(p)}\right)$$

( $\mu$  is the Möbius function), and with the choice  $z = (\frac{2}{3}|A|)^{1/2}$ , it is proved in [23] that

$$\frac{|A|}{L(z)} \leq 16 \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right) \frac{|A|}{\log^2(|A|)} \prod_{\substack{p|d \\ p > 2}} \frac{p-1}{p-2}.$$

The value of the above infinite product is  $0.6602\dots < 2/3$ . We set  $f(d) = \prod_{\substack{p|d \\ p > 2}} \frac{p-1}{p-2}$ , and we observe that  $|A| \geq x^\alpha - 1$ , so that for  $x$  large enough

$$r(d) \leq \frac{32}{3\alpha^2} \frac{|A|}{\log^2 x} f(d). \quad (1.16)$$

Now, for the next step, we shall need an upper bound for  $\sum_{n \leq x} f^2(n)$ . By using the convolution method and defining

$$h(n) = \sum_{a|n} \mu(a) f^2(n/a)$$

one gets  $h(2) = h(2^2) = h(2^3) = \dots = 0$  and, for  $p \geq 3$ ,  $h(p) = \frac{2p-3}{(p-2)^2}$ ,  $h(p^2) = h(p^3) = \dots = 0$ , so that

$$\begin{aligned} \sum_{n \leq x} f^2(n) &= \sum_{n \leq x} \sum_{a|n} h(a) = \sum_{a \leq x} h(a) \left\lfloor \frac{x}{a} \right\rfloor \\ &\leq x \sum_{a=1}^{\infty} \frac{h(a)}{a} = x \prod_{p \geq 3} \left(1 + \frac{2p-3}{p(p-2)^2}\right) \\ &= 2.63985\dots x \leq \frac{8}{3}x. \end{aligned} \quad (1.17)$$

From (1.15) and (1.16), one can deduce

$$\frac{(1-\varepsilon)^2 x^{2\alpha}}{\log^2 x} \leq \sum_{\substack{0 < d \leq 2x^\alpha \\ r(\bar{d}) \neq 0}} r(d) \leq \frac{32}{3\alpha^2} \frac{|A|}{\log^2 x} \sum_{\substack{0 < d \leq 2x^\alpha \\ r(\bar{d}) \neq 0}} f(d).$$

which implies

$$\sum_{\substack{0 < d \leq 2x^\alpha \\ r(\bar{d}) \neq 0}} f(d) \geq \frac{3\alpha^2 x^{2\alpha} (1-\varepsilon)^2}{32|A|}.$$

By Cauchy-Schwarz's inequality, one has

$$\left( \sum_{\substack{0 < d \leq 2x^\alpha \\ r(\bar{d}) \neq 0}} 1 \right) \left( \sum_{\substack{0 < d \leq 2x^\alpha \\ r(\bar{d}) \neq 0}} f^2(d) \right) \geq \frac{9\alpha^4 x^{4\alpha} (1-\varepsilon)^4}{1024|A|^2}$$

and, by (1.14) and (1.17)

$$|E| \geq \frac{9\alpha^4 x^{4\alpha} (1-\varepsilon)^4}{1024|A|^2} \bigg/ \frac{8}{3} (2x^\alpha) = \frac{27}{16384} \frac{x^{3\alpha} (1-\varepsilon)^4}{|A|^2}.$$

Since  $|A| \leq x^\alpha + 1$ , and  $x$  has been supposed large enough, proposition 2 is proved.

## 1.4 Some properties of $g(n)$

Here, we recall some known properties of  $g(n)$  which can be found for instance in [16]. Let us define the arithmetic function  $\ell$  in the following way:  $\ell$  is additive, and, if  $p$  is a prime and  $k \geq 1$ , then  $\ell(p^k) = p^k$ . It is not difficult to deduce from (1.1) (cf. [13] or [16]) that

$$g(n) = \max_{\ell(M) \leq n} M. \quad (1.18)$$

Now the relation (cf. [16], p. 139)

$$M \in g(\mathbb{N}) \iff (M' > M \implies \ell(M') > \ell(M)) \quad (1.19)$$

easily follows from (1.18), and shows that the values of the Landau function  $g$  are the "champions" for the small values of  $\ell$ . So the methods introduced by Ramanujan (cf. [19]) to study highly composite numbers can also be used for  $g(n)$ . Indeed  $M$  is highly composite, if it is a "champion" for the divisor function  $d$ , that is to say if

$$M' < M \implies d(M') < d(M).$$

Corresponding to the so-called superior highly composite numbers, one introduces the set  $G : N \in G$  if there exists  $\rho > 0$  such that

$$\forall M \geq 1, \quad \ell(M) - \rho \log M \geq \ell(N) - \rho \log N. \quad (1.20)$$

(1.19) and (1.20) easily imply that  $G \subset g(\mathbb{N})$ . Moreover, if  $\rho > 2/\log 2$ , let us define  $x > 4$  such that  $\rho = x/\log x$  and

$$N_\rho = \prod_{p \leq x} p^{\alpha_p} = \prod_p p^{\alpha_p} \quad (1.21)$$

with

$$\alpha_p = \begin{cases} 0 & \text{if } p > x \\ 1 & \text{if } \frac{p}{\log p} \leq \rho < \frac{p^2-p}{\log p} \\ k \geq 2 & \text{if } \frac{p^k-p^{k-1}}{\log p} \leq \rho < \frac{p^{k-1}-p^k}{\log p} \end{cases}$$

then  $N_\rho \in G$ . With the above definition, since  $x \geq 4$ , it is not difficult to show that (cf. [11, (5)])

$$p^{\alpha_p} \leq x \quad (1.22)$$

holds for  $p \leq x$ , whence  $N_\rho$  is a divisor of the l.c.m. of the integers  $\leq x$ . Here we can prove

**Proposition 3.** *For every prime  $p$ , there exists  $n$  such that the largest prime factor of  $g(n)$  is equal to  $p$ .*

*Proof.* We have  $g(2) = 2, g(3) = 3$ . If  $p \geq 5$ , let us choose  $\rho = p/\log p > 2/\log 2$ .  $N_\rho$  defined by (1.21) belongs to  $G \subset g(\mathbb{N})$ , and its largest prime factor is  $p$ , which proves Proposition (3).

From Proposition 3, it is easy to deduce a proof of Theorem 1, under the twin prime conjecture. Let  $P = p + 2$  be twin primes, and  $n$  such that the largest prime factor of  $g(n)$  is  $p$ . The sequence  $n_k$  being defined by (1.3), we define  $k$  in terms of  $n$  by  $n_k \leq n < n_{k+1}$ , so that  $g(n_k) = g(n)$  has its largest prime factor equal to  $p$ . Now, from (1.18) and (1.19),

$$\ell(g(n_k)) = n_k$$

and  $g(n_{k+2}) > g(n_k)$  since  $M = \frac{P}{p}g(n_k)$  satisfies  $M > g(n_k)$  and  $\ell(M) = n_k + 2$ . So  $n_{k+1} \leq n_k + 2$ , and Theorem 1 is proved under this strong hypothesis.

Let us introduce now the so-called benefit method. For a fixed  $\rho > 2/\log 2$ ,  $N = N_\rho$  is defined by (1.21), and for any integer  $M$ ,

$$M = \prod_p p^{\beta_p},$$

one defines the benefit of  $M$ :

$$\text{ben}(M) = \ell(M) - \ell(N) - \rho \log M/N. \quad (1.23)$$



Clearly, from (1.20),  $\text{ben}(M) \geq 0$  holds, and from the additivity of  $\ell$  one has

$$\text{ben}(M) = \sum_p (\ell(p^{\beta_p}) - \ell(p^{\alpha_p}) - \rho(\beta_p - \alpha_p) \log p). \quad (1.24)$$

In the above formula, let us observe that  $\ell(p^\beta) = p^\beta$  if  $\beta \geq 1$ , but that  $\ell(p^\beta) = 0 \neq p^\beta = 1$  if  $\beta = 0$ , and, due to the choice of  $\alpha_p$  in (1.21), that, in the sum (1.24), all the terms are non negative: for all  $p$  and for  $\beta \geq 0$ , we have

$$\ell(p^\beta) - \ell(p^{\alpha_p}) - \rho(\beta - \alpha_p) \log p \geq 0 \quad (1.25)$$

Indeed, let us consider the set of points  $(0,0)$  and  $(\beta, p^\beta / \log p)$  for  $\beta$  integer  $\geq 1$ . For all  $p$ , the piecewise linear curve going through these points is convex, and for a given  $\rho$ ,  $\alpha_p$  is chosen so that the straight line  $L$  of slope  $\rho$  going through  $(\alpha_p, \frac{p^{\alpha_p}}{\log p})$  does not cut that curve. The left-hand side of (1.25), (which is  $\text{ben}(Np^{\beta - \alpha_p})$ ) can be seen as the product of  $\log p$  by the vertical distance of the point  $(\beta, \frac{p^\beta}{\log p})$  to the straight line  $L$ , and because of convexity, we shall have for all  $p$ ,

$$\text{ben}(Np^t) \geq t \text{ben}(Np), \quad t \geq 1 \quad (1.26)$$

and for  $p \leq x$ ,

$$\text{ben}(Np^{-t}) \geq t \text{ben}(Np^{-1}), \quad 1 \leq t \leq \alpha_p. \quad (1.27)$$

## 1.5 Proof of Theorem 1

First the following proposition will be proved:

**Proposition 4.** *Let  $\alpha < 1/2$ , and  $x$  large enough such that (1.10) holds. Let us denote the primes surrounding  $x$  by:*

$$\dots < Q_j < \dots < Q_2 < Q_1 \leq x < P_1 < P_2 < \dots < P_i < \dots$$

*Let us define  $\rho = x / \log x$ ,  $N = N_\rho$  by (1.21),  $n = \ell(N)$ . Then for  $n \leq m \leq n + 2x^\alpha$ ,  $g(m)$  can be written*

$$g(m) = N \frac{P_{i_1} P_{i_2} \dots P_{i_r}}{Q_{j_1} Q_{j_2} \dots Q_{j_r}} \quad (1.28)$$

*with  $r \geq 0$  and  $i_1 < \dots < i_r, j_1 < \dots < j_r, P_{i_r} \leq x + 4x^\alpha, Q_{j_r} \geq x - 4x^\alpha$ .*

*Proof.* First, from (1.18), one has  $\ell(g(m)) \leq m$ , and from (1.23) and (1.18)

$$\text{ben}(g(m)) = \ell(g(m)) - \ell(N) - \rho \log \frac{g(m)}{N} \leq m - n \leq 2x^\alpha \quad (1.29)$$

for  $n \leq m \leq 2x^\alpha$ .

Further, let  $Q \leq x$  be a prime, and  $k = \alpha_Q \geq 1$  the exponent of  $Q$  in the standard factorization of  $N$ . Let us suppose that for a fixed  $m$ ,  $Q$  divides  $g(m)$

with the exponent  $\beta_Q = k + t, t > 0$ . Then, from (1.24), (1.25), and (1.26), one gets

$$\text{ben}(g(m)) \geq \text{ben}(NQ^t) \geq \text{ben}(NQ) \quad (1.30)$$

and

$$\begin{aligned} \text{ben}(NQ) &= Q^{k+1} - Q^k - \rho \log Q \\ &= \log Q \left( \frac{Q^{k+1} - Q^k}{\log Q} - \rho \right). \end{aligned}$$

From (1.21), the above parenthesis is non negative, and from (1.10), one gets:

$$\text{ben}(NQ) \geq \log 2 \frac{\sqrt{x}}{\log^4 x}. \quad (1.31)$$

For  $x$  large enough, there is a contradiction between (1.29), (1.30) and (1.31), and so,  $\beta_Q \leq \alpha_Q$ .

Similarly, let us suppose  $Q \leq x, k = \alpha_Q \geq 2$  and  $\beta_Q = k - t, 1 \leq t \leq k$ . One has, from (1.24), (1.25) and (1.27),

$$\text{ben}(g(m)) \geq \text{ben}(NQ^{-t}) \geq \text{ben}(NQ^{-1})$$

and

$$\begin{aligned} \text{ben}(NQ^{-1}) &= Q^{k-1} - Q^k + \rho \log Q \\ &= \log Q \left( \rho - \frac{Q^k - Q^{k-1}}{\log Q} \right) \geq \log 2 \frac{\sqrt{x}}{\log^4 x} \end{aligned}$$

which contradicts (1.29), and so, for such a  $Q, \beta_Q = \alpha_Q$ .

Now, let us suppose  $Q \leq x, \alpha_Q = 1$ , and  $\beta_Q = 0$  for some  $m, n \leq m \leq n + 2x^\alpha$ . Then

$$\text{ben}(g(m)) \geq \text{ben}(NQ^{-1}) = -Q + \rho \log Q = y(Q)$$

by setting  $y(t) = \rho \log t - t$ . From the concavity of  $y(t)$  for  $t > 0$ , for  $x \geq e^2$ , we get

$$\begin{aligned} y(Q) \geq y(x) + (Q - x)y'(x) &= (Q - x) \left( \frac{\rho}{x} - 1 \right) \\ &= (x - Q) \left( 1 - \frac{1}{\log x} \right) \geq \frac{1}{2}(x - Q) \end{aligned}$$

and so,

$$\text{ben}(g(m)) \geq \frac{1}{2}(x - Q)$$

which, from (1.29) yields

$$x - Q \leq 4x^\alpha.$$

In conclusion, the only prime factors allowed in the denominator of  $\frac{g(m)}{N}$  are the  $Q$ 's, with  $x - 4x^\alpha \leq Q \leq x$ , and  $\alpha_Q = 1$ .

What about the numerator? Let  $P > x$  be a prime number and suppose that  $P^t$  divides  $g(m)$  with  $t \geq 2$ . Then, from (1.26) and (1.23),

$$\text{ben}(Np^t) \geq \text{ben}(Np^2) = P^2 - 2\rho \log P.$$

But the function  $t \mapsto t^2 - 2\rho \log t$  is increasing for  $t \geq \sqrt{\rho}$ , so that,

$$\text{ben}(NP^t) \geq x^2 - 2x > 2x^\alpha$$

for  $x$  large enough, which contradicts (1.29). The only possibility is that  $P$  divides  $g(m)$  with exponent 1. In that case, from the convexity of the function  $z(t) = t - \rho \log t$ , inequality (1.26) yields

$$\begin{aligned} \text{ben}(g(m)) &\geq \text{ben}(NP) = z(P) \geq z(x) + (P-x)z'(x) \\ &= (P-x) \left(1 - \frac{1}{\log x}\right) \geq \frac{1}{2}(P-x) \end{aligned}$$

for  $x \geq e^2$ , which, with (1.29), implies

$$P - x \leq 4x^\alpha.$$

Up to now, we have shown that

$$g(m) = N \frac{P_{i_1} \cdots P_{i_r}}{Q_{j_1} \cdots Q_{j_s}}$$

with  $P_{i_r} \leq x + 4x^\alpha$ ,  $Q_{j_s} \geq x - 4x^\alpha$ . It remains to show that  $r = s$ . First, since  $n \leq m \leq n + 2x^\alpha$ , and  $N$  belongs to  $G$ , we have from (1.18) and (1.19)

$$n \leq \ell(g(m)) \leq n + 2x^\alpha. \quad (1.32)$$

Further,

$$\ell(g(m)) - n = \sum_{t=1}^r P_{i_t} - \sum_{t=1}^s Q_{j_t}$$

and since  $r \leq 4x^\alpha$ , and  $s \leq 4x^\alpha$ ,

$$\begin{aligned} \ell(g(m)) - n &\leq r(x + 4x^\alpha) - s(x - 4x^\alpha) \\ &\leq (r-s)x + 32x^{2\alpha}. \end{aligned}$$

From (1.32),  $\ell(g(m)) - n \geq 0$  holds and as  $\alpha < 1/2$ , this implies that  $r \geq s$  for  $x$  large enough. Similarly,

$$\ell(g(m)) - n \geq (r-s)x,$$

so, from (1.32),  $(r-s)x$  must be  $\leq 2x^\alpha$ , which, for  $x$  large enough, implies  $r \leq s$ ; finally  $r = s$ , and the proof of Proposition 4 is completed.

**Lemma 1.** *Let  $x$  be a positive real number,  $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k$  be real number such that*

$$b_k \leq b_{k-1} \leq \dots \leq b_1 \leq x < a_1 \leq a_2 \leq \dots \leq a_k$$

and  $\Delta$  be defined by  $\Delta = \sum_{i=1}^k (a_i - b_i)$ . Then the following inequalities

$$\frac{x + \Delta}{x} \leq \prod_{i=1}^k \frac{a_i}{b_i} \leq \exp\left(\frac{\Delta}{x}\right)$$

hold.

*Proof.* It is easy, and can be found in [16], p. 159.

Now it is time to prove Theorem 1. With the notation and hypothesis of Proposition 4, let us denote by  $B$  the set of integers  $M$  of the form

$$M = N \frac{P_{i_1} P_{i_2} \dots P_{i_r}}{Q_{j_1} Q_{j_2} \dots Q_{j_r}}$$

satisfying

$$\ell(M) - \ell(N) = \sum_{t=1}^r (P_{i_t} - Q_{j_t}) \leq 2x^\alpha.$$

From Proposition 4, for  $n \leq m \leq 2x^\alpha$ ,  $g(m) \in B$ , and thus, from (1.18),

$$g(m) = \max_{M \in B} \ell(M) \leq m. \quad (1.33)$$

Further, for  $0 \leq d \leq 2x^\alpha$ , define

$$B_d = \{M \in B; \ell(M) - \ell(N) = d\}.$$

I claim that, if  $d < d'$  (which implies  $d \leq d' - 2$ ), any element of  $B_d$  is smaller than any element of  $B_{d'}$ . Indeed, let  $M \in B_d$ , and  $M' \in B_{d'}$ . From Lemma 1, one has

$$\frac{M}{N} \leq \exp\left(\frac{d}{x}\right) \quad \text{and} \quad \frac{M'}{N} \geq \frac{x + d'}{x} \geq \frac{x + d + 2}{x}.$$

Since  $d < 2x^\alpha < x$ , and  $e^t \leq \frac{1}{1-t}$  for  $0 \leq t < 1$ , one gets

$$\frac{M}{N} \leq \frac{1}{1 - d/x} = \frac{x}{x - d}.$$

This last quantity is smaller than  $\frac{x+d+2}{x}$  if  $(d+1)^2 < 2x+1$ , which is true for  $x$  large enough, because  $d \leq 2x^\alpha$  and  $\alpha < 1/2$ .

From the preceding claim, and from (1.33), it follows that, if  $B_d$  is non empty, then

$$g(n+d) = \max B_d.$$

Further, since  $N \in G$ , we know that  $n = \ell(N)$  belongs to the sequence  $(n_k)$  where  $g$  is increasing, and so,  $n = n_{k_0}$ . If  $0 < d_1 < d_2 < \dots < d_s \leq 2x^\alpha$  denote the values of  $d$  for which  $B_d$  is non empty, then one has

$$n_{k_0+i} = n + d_i, 1 \leq i \leq s. \quad (1.34)$$

Suppose now that  $\alpha < 1/2$  and  $x$  have been chosen in such a way that (1.12) and (1.13) hold. With the notation of Proposition 2, the set  $E(x, \alpha)$  is certainly included in the set  $\{d_1, d_2, \dots, d_s\}$ , and from Proposition 2,

$$s \geq C_2 x^\alpha \quad (1.35)$$

which implies that for at least one  $i$ ,  $d_{i+1} - d_i \leq \frac{2}{C_2}$ , and thus

$$n_{k_0+i+1} - n_{k_0+i} \leq \frac{2}{C_2}.$$

Finally, for  $\frac{1}{6} < \alpha < \frac{1}{2}$ , Proposition 1 allows us to choose  $x$  as wished, and thus, the proof of Theorem 1 is completed. With  $\varepsilon$  very small, and  $\alpha$  close to  $1/2$ , the values of  $C_1$  and  $C_2$  given in Proposition 2 yield that for infinitely many  $k$ 's,

$$n_{k+1} - n_k \leq 20000.$$

To count how many such differences we get, we define

$$\gamma(n) = \text{Card}\{m \leq n; g(m) > g(m-1)\}.$$

Therefore, with the notation (1.3), we have  $n_{\gamma(n)} = n$ .

In [16, 162–164], it is proved that

$$n^{1-\tau/2} \ll \gamma(n) \leq n - c \frac{n^{3/4}}{\sqrt{\log n}}$$

where  $\tau$  is such that the sequence of consecutive primes satisfies  $p_{i+1} - p_i \ll p_i^\tau$ . Without any hypothesis, the best known  $\tau$  is  $> 1/2$ .

**Proposition 5.** *We have  $\gamma(n) \geq n^{3/4-\varepsilon}$  for all  $\varepsilon > 0$ , and  $n$  large enough.*

*Proof.* With the definition of  $\gamma(n)$ , (1.34) and (1.35) give

$$\gamma(n + 2x^\alpha) - \gamma(n) \geq s \gg x^\alpha \quad (1.36)$$

whenever  $n = \ell(N)$ ,  $N = N_\rho$ ,  $\rho = x/\log x$ , and  $x$  satisfies Proposition 1. But, from (1.21), two close enough distinct values of  $x$  can yield the same  $N$ .

I now claim that, with the notation of Proposition 1, the number of primes  $p_i$  between  $\xi$  and  $\xi'$  such that there is at least one  $x \in [p_i, p_{i+1})$  satisfying (1.8), (1.9) and (1.10) is bigger than  $\frac{1}{2}(\pi(\xi') - \pi(\xi))$ . Indeed, for each  $i$  for which  $[p_i, p_{i+1})$  does not contain any such  $x$ , we get a measure  $p_{i+1} - p_i \geq 2$ , and if there are more than  $\frac{1}{2}(\pi(\xi') - \pi(\xi))$  such  $i$ 's, the total measure will be greater than  $\pi(\xi') - \pi(\xi) \sim \xi/\log^2 \xi$ , which contradicts Proposition 1.

From the above claim, there will be at least  $\frac{1}{2}(\pi(\xi') - \pi(\xi))$  distinct  $N$ 's, with  $N = N_\rho$ ,  $\rho = x/\log x$ , and  $\xi \leq x \leq \xi'$ . Moreover, for two such distinct  $N$ , say  $N' < N''$ , we have from (1.21),  $\ell(N'') - \ell(N') \geq \xi$ .

Let  $N^{(1)}$  and  $N^{(0)}$  the biggest and the smallest of these  $N$ 's, and  $n^{(1)} = \ell(N^{(1)})$ ,  $n^{(0)} = \ell(N^{(0)})$ , then from (1.36),

$$\gamma(n^{(1)}) \geq \gamma(n^{(1)}) - \gamma(n^{(0)}) \geq \frac{1}{2} (\pi(\xi') - \pi(\xi)) \xi^\alpha \gg \frac{\xi^{1+\alpha}}{\log^2 \xi}. \quad (1.37)$$

But from (1.21) and (1.22),  $x \sim \log N_\rho$ , and from (1.2),

$$x \sim \log N_\rho \sim \sqrt{n \log n} \quad \text{with} \quad n = \ell(N_\rho)$$

so

$$\xi \sim \sqrt{n^{(1)} \log n^{(1)}}$$

and since  $\alpha$  can be chosen in (1.37) as close as wished of  $1/2$ , this completes the proof of Proposition 5.



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