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# LARGE DEVIATIONS FOR THE ORNSTEIN-UHLENBECK PROCESS WITH SHIFT

BERNARD BERCU AND ADRIEN RICHO

ABSTRACT. We investigate the large deviation properties of the maximum likelihood estimators for the Ornstein-Uhlenbeck process with shift. We estimate simultaneously the drift and shift parameters. On the one hand, we establish a large deviation principle for the maximum likelihood estimates of the drift and shift parameters. Surprisingly, we find that the drift estimator shares the same large deviation principle as the one previously established for the Ornstein-Uhlenbeck process without shift. Sharp large deviation principles are also provided. On the other hand, we show that the maximum likelihood estimator of the shift parameter satisfies a large deviation principle with a very unusual implicit rate function.

## 1. INTRODUCTION

Consider the Ornstein-Uhlenbeck process with linear shift  $\gamma \in \mathbb{R}$ , observed over the time interval  $[0, T]$

$$(1.1) \quad dX_t = \theta X_t dt + \gamma dt + dB_t$$

where the drift parameter  $\theta < 0$ , the initial state  $X_0 = 0$  and the driven noise  $(B_t)$  is a standard Brownian motion. The maximum likelihood estimates of the unknown parameters  $\theta$  and  $\gamma$  are given by

$$(1.2) \quad \hat{\theta}_T = \frac{T \int_0^T X_t dX_t - X_T \int_0^T X_t dt}{T \int_0^T X_t^2 dt - \left( \int_0^T X_t dt \right)^2}$$

and

$$(1.3) \quad \hat{\gamma}_T = \frac{X_T \int_0^T X_t^2 dt - \int_0^T X_t dX_t \int_0^T X_t dt}{T \int_0^T X_t^2 dt - \left( \int_0^T X_t dt \right)^2}.$$

A wide range of literature is available on the asymptotic behavior of  $(\hat{\theta}_T)$  and  $(\hat{\gamma}_T)$ . It is well-known [8] that  $\hat{\theta}_T$  and  $\hat{\gamma}_T$  are both strongly consistent estimators of  $\theta$  and  $\gamma$  and their joint asymptotic normality is given by

$$(1.4) \quad \sqrt{T} \begin{pmatrix} \hat{\theta}_T - \theta \\ \hat{\gamma}_T - \gamma \end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, L)$$

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*Key words and phrases.* Ornstein-Uhlenbeck process with shift, Maximum likelihood estimates, Large deviations.

where the limiting matrix

$$L = 2 \begin{pmatrix} \theta & \gamma \\ \gamma & \delta \end{pmatrix}$$

with  $\delta = (2\gamma^2 + \theta)/2\theta$ . Moreover, concentration inequalities for  $(\widehat{\theta}_T)$  and  $(\widehat{\gamma}_T)$  and moderate deviations were established by Gao and Jiang [5], while Jiang [7] recently obtained the joint law of iterated logarithm as well as Berry-Esseen bounds. In the particular case  $\gamma = 0$ , Florens-Landais and Pham [4] proved the large deviation principle (LDP) for  $(\widehat{\theta}_T)$ , while sharp large deviation principles (SLDP) were established in [2]. We also refer the reader to [1] for the sharp large deviations in the non-stationary case  $\theta \geq 0$  and  $\gamma = 0$ .

Our goal is to extend these investigations by establishing the large deviations properties of the maximum likelihood estimators of the drift and shift parameters  $\theta < 0$  and  $\gamma$  in the situation where  $\theta$  and  $\gamma$  are estimated simultaneously.

The paper is organized as follows. In Section 2, we propose three keystone lemmas which are at the core of our analysis. An LDP for the couple

$$(\widehat{\theta}_T, \widehat{\gamma}_T)$$

is given in Section 3. Via the contraction principle, one can realize that  $(\widehat{\theta}_T)$  shares the same LDP as the one previously established for the Ornstein-Uhlenbeck process without shift. One can also observe that  $(\widehat{\gamma}_T)$  satisfies an LDP with a very unusual implicit rate function. Section 4 is devoted to SLDP for the sequence  $(\widehat{\theta}_T)$ . All the technical proofs of Sections 2 to 4 are postponed to Appendices A, B, and C.

## 2. THREE KEYSTONE LEMMAS.

Denote by  $\Lambda_T$  the normalized cumulant generating function of the triplet

$$\left( \frac{X_T}{\sqrt{T}}, \frac{1}{T} \int_0^T X_t^2 dt, \frac{1}{T} \int_0^T X_t dt \right)$$

defined, for all  $(a, b, c) \in \mathbb{R}^3$ , by

$$\Lambda_T(a, b, c) = \frac{1}{T} \log \mathbb{E} \left[ \exp \left( a\sqrt{T}X_T + b \int_0^T X_t^2 dt + c \int_0^T X_t dt \right) \right].$$

Our first lemma deals with the extended real function  $\Lambda$  defined as the pointwise limit of  $\Lambda_T$ .

**Lemma 2.1.** *Let  $\mathcal{D}_\Lambda$  be the effective domain of  $\Lambda$*

$$\mathcal{D}_\Lambda = \left\{ (a, b, c) \in \mathbb{R}^3 / b < \theta^2/2 \right\}$$

and set  $\varphi(b) = \sqrt{\theta^2 - 2b}$ . Then, for all  $(a, b, c) \in \mathcal{D}_\Lambda$ , we have

$$(2.1) \quad \Lambda(a, b, c) = -\frac{1}{2} (\theta + \varphi(b) + \gamma^2) + \frac{1}{2} \left( \frac{a^2}{\varphi(b) - \theta} \right) + \frac{1}{2} \left( \frac{c - \theta\gamma}{\varphi(b)} \right)^2.$$

*Proof.* The proof is given in Appendix A. □

A direct calculation shows that the function  $\Lambda$  is steep, which means that the norm of its gradient goes to infinity for any sequence in the interior of  $\mathcal{D}_\Lambda$  converging to a boundary point of  $\mathcal{D}_\Lambda$ . It will allow us in Section 3 to deduce an LDP for the couple  $(\widehat{\theta}_T, \widehat{\gamma}_T)$ . In order to establish the SLDP for the drift parameter  $\widehat{\theta}_T$ , it is necessary to modify our strategy. To be more precise, we shall now focus our attention on the normalized cumulant generating function  $\mathcal{L}_T$  of the couple

$$\left( \frac{1}{T} \int_0^T (X_t - \bar{X}_T) dX_t, \frac{1}{T} S_T \right)$$

where

$$S_T = \int_0^T (X_t - \bar{X}_T)^2 dt \quad \text{and} \quad \bar{X}_T = \frac{1}{T} \int_0^T X_t dt,$$

which is given, for all  $(a, b) \in \mathbb{R}^2$ , by

$$\mathcal{L}_T(a, b) = \frac{1}{T} \log \mathbb{E} \left[ \exp \left( a \int_0^T (X_t - \bar{X}_T) dX_t + b S_T \right) \right].$$

The reason for this is twofold. On the one hand, it is not possible to deduce an SLDP for  $(\widehat{\theta}_T)$  via  $\Lambda_T$ . On the other hand, it immediately follows from (1.2) that

$$(2.2) \quad \widehat{\theta}_T = \frac{\int_0^T (X_t - \bar{X}_T) dX_t}{S_T}.$$

However, one can observe that for all  $c \in \mathbb{R}$ ,  $\mathbb{P}(\widehat{\theta}_T \geq c) = \mathbb{P}(\mathcal{Z}_T(1, -c) \geq 0)$  where, for all  $(a, b) \in \mathbb{R}^2$ ,  $\mathcal{Z}_T(a, b)$  stands for the random variable

$$(2.3) \quad \mathcal{Z}_T(a, b) = a \int_0^T (X_t - \bar{X}_T) dX_t + b S_T.$$

Our second lemma provides the full asymptotic expansion for  $\mathcal{L}_T$ . Denote by  $\mathcal{L}$  the extended real function defined as the pointwise limit of  $\mathcal{L}_T$ .

**Lemma 2.2.** *Let  $\mathcal{D}_\mathcal{L}$  be the effective domain of  $\mathcal{L}$*

$$\mathcal{D}_\mathcal{L} = \left\{ (a, b) \in \mathbb{R}^2 / \theta^2 - 2b > 0 \quad \text{and} \quad a + \theta < \sqrt{\theta^2 - 2b} \right\}$$

and set  $\varphi(b) = \sqrt{\theta^2 - 2b}$ ,  $\tau(a, b) = \varphi(b) - (a + \theta)$ . Then, for all  $(a, b) \in \mathcal{D}_\mathcal{L}$  and  $T$  large enough, we have

$$(2.4) \quad \mathcal{L}_T(a, b) = \mathcal{L}(a, b) + \frac{1}{T} \mathcal{H}(a, b) + \frac{1}{T^2} \mathcal{R}_T(a, b)$$

where

$$(2.5) \quad \mathcal{L}(a, b) = -\frac{1}{2} \left( a + \theta + \sqrt{\theta^2 - 2b} \right),$$

$$(2.6) \quad \mathcal{H}(a, b) = -\frac{1}{2} \log \left( \frac{\tau(a, b) \theta^2}{2\varphi^3(b)} \right) - \frac{\gamma^2 (a + \theta + \sqrt{\theta^2 - 2b})}{2\theta^2}.$$

Moreover, the remainder  $\mathcal{R}_T(a, b)$  may be explicitly calculated as a rational function

of  $a, b, T$  and  $\exp(-\varphi(b)T)$ . In addition,  $\mathcal{R}_T$  can be extended to the two-dimensional complex plane and it is a bounded analytic function as soon as the real parts of its arguments belong to the interior of  $\mathcal{D}_{\mathcal{L}}$ .

*Proof.* The proof is given in Appendix B.  $\square$

Our third lemma relies on the Karhunen-Loève expansion of the process  $(X_T)$ . Denote by  $\mathcal{F}$  the class of all real-valued continuous functions  $f$  such that  $f(x) = x^2 h(x)$  where  $h$  is continuous. Moreover, let  $g$  be the spectral density of the stationary Ornstein-Uhlenbeck process without shift  $\gamma = 0$  given, for all  $x \in \mathbb{R}$ , by

$$(2.7) \quad g(x) = \frac{1}{\theta^2 + x^2}.$$

**Lemma 2.3.** *One can find two sequences of real numbers  $(\alpha_k^T)$  and  $(\beta_k^T)$  both in  $\ell^2(\mathbb{N})$  such that*

$$(2.8) \quad \mathcal{Z}_T(a, b) = \mathbb{E}[\mathcal{Z}_T(a, b)] + \sum_{k=1}^{\infty} \alpha_k^T (\varepsilon_k^2 - 1) + \beta_k^T \varepsilon_k$$

where  $(\varepsilon_k)$  are independent standard  $\mathcal{N}(0, 1)$  random variables. Moreover, for all  $(a, b) \in \mathcal{D}_{\mathcal{L}}$ , there exist two constants  $A > 0$  and  $B > 0$  that do not depend on  $T$  such that, for  $T$  large enough,  $\alpha_k^T \in [-A, A]$  for all  $k \geq 1$  and

$$(2.9) \quad \sum_{k=1}^{+\infty} (\beta_k^T)^2 < B.$$

Consequently, for all  $(a, b) \in \mathcal{D}_{\mathcal{L}}$  and  $x \in \mathbb{R}$  such that  $|x| < 1/2A$ , and for  $T$  large enough, we have

$$\mathcal{L}_T(xa, xb) = \frac{1}{T} \mathbb{E}[\mathcal{Z}_T(xa, xb)] - \frac{1}{2T} \sum_{k=1}^{\infty} \left( \log(1 - 2x\alpha_k^T) + 2x\alpha_k^T \right) + \frac{1}{2T} \sum_{k=1}^{\infty} \frac{(x\beta_k^T)^2}{1 - 2x\alpha_k^T}.$$

Finally, if  $b \neq 0$ , the empirical spectral measure

$$(2.10) \quad \nu_T = \frac{1}{T} \sum_{k=1}^{\infty} \delta_{\alpha_k^T}$$

satisfies, for any continuous  $f \in \mathcal{F}$  with compact support

$$(2.11) \quad \lim_{T \rightarrow \infty} \langle \nu_T, f \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{k=1}^{\infty} f(\alpha_k^T) = \langle \nu, f \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} f(bg(x)) dx.$$

*Proof.* The proof is given in Appendix C.  $\square$

## 3. LARGE DEVIATIONS RESULTS.

Our large deviations results are as follows.

**Theorem 3.1.** *The couple  $(\widehat{\theta}_T, \widehat{\gamma}_T)$  satisfies an LDP with good rate function*

$$(3.1) \quad I_{\theta, \gamma}(c, d) = \begin{cases} -\frac{(\theta - c)^2}{4c} + \frac{1}{2} \left( \gamma - \frac{d\theta}{c} \right)^2 & \text{if } c \leq \frac{\theta}{3}, \\ (2c - \theta) + \frac{1}{2} \left( \gamma - \frac{d\theta}{c} \right)^2 & \text{if } c \geq \frac{\theta}{3} \text{ and } c \neq 0, \\ -\theta & \text{if } (c, d) = (0, 0), \\ +\infty & \text{if } c = 0 \text{ and } d \neq 0. \end{cases}$$

A direct application of the contraction principle [3] immediately leads to the two following corollaries.

**Corollary 3.2.** *The sequence  $(\widehat{\theta}_T)$  satisfies an LDP with good rate function*

$$(3.2) \quad I_{\theta}(c) = \begin{cases} -\frac{(c - \theta)^2}{4c} & \text{if } c \leq \frac{\theta}{3}, \\ 2c - \theta & \text{if } c \geq \frac{\theta}{3}. \end{cases}$$

**Corollary 3.3.** *The sequence  $(\widehat{\gamma}_T)$  satisfies an LDP with good rate function*

$$(3.3) \quad I_{\gamma}(d) = \inf \left\{ I_{\theta, \gamma}(c, d) / c \in \mathbb{R} \right\}.$$

*Proof.* The proofs are given in Section 5. □

**Remark 3.4.** *On the one hand, one can observe that  $(\widehat{\theta}_T)$  shares exactly the same LDP as the one previously established by Florens-Landaï and Pham [4] for the Ornstein-Uhlenbeck without shift  $\gamma = 0$ . On the other hand,  $(\widehat{\gamma}_T)$  satisfies an LDP with a very unusual rate function. Unfortunately, an explicit expression for this rate function is quite complicated. Its very particular form is given in the special cases  $(\theta, \gamma) = (-2, 2)$  and  $(\theta, \gamma) = -(2, 1)$  in Figure 1 below.*

## 4. SHARP LARGE DEVIATIONS RESULTS.

Our goal is now to improve Corollary 3.2 by a first order SLDP for  $(\widehat{\theta}_T)$ . Via Lemma 2.2, it is possible to establish SLDP of any order for the sequence  $(\widehat{\theta}_T)$ . However, for clearness sake, we have chosen to restrict ourselves to a first order expansion. We already saw from Lemma 2.2 that the effective domain of the limiting cumulant generating function of the auxiliary random variable  $\mathcal{Z}_T(a, -ca)$  is given by

$$\mathcal{D}_{\mathcal{L}} = \left\{ a \in \mathbb{R} / \theta^2 + 2ac > 0 \quad \text{and} \quad \sqrt{\theta^2 + 2ac} > a + \theta \right\}.$$

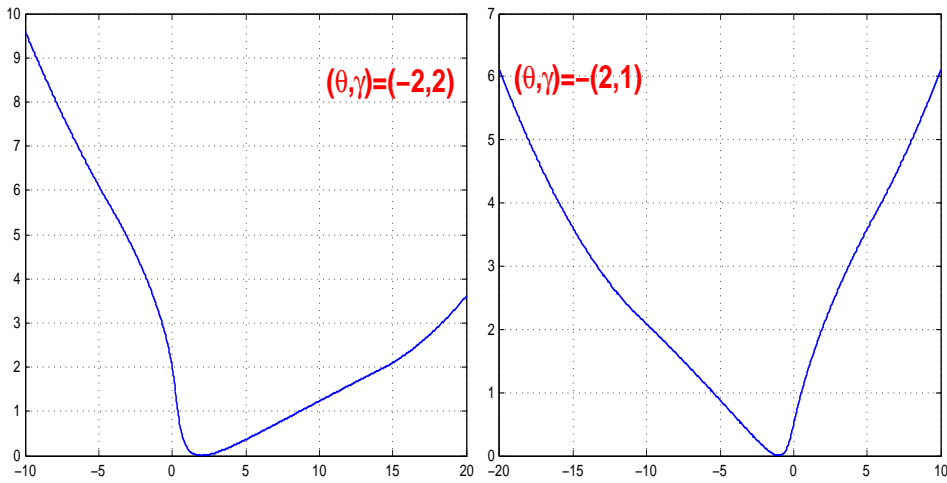


FIGURE 1. Rate functions for the drift parameter.

After some straightforward calculations, it is not hard to see that

$$\mathcal{D}_{\mathcal{L}} = \begin{cases} ]-\infty, \frac{-\theta^2}{2c}[ & \text{if } \theta < c \leq \frac{\theta}{2}, \\ ]-\infty, 2(c - \theta)[ & \text{if } \frac{\theta}{2} < c \leq 0, \\ ]\frac{-\theta^2}{2c}, 2(c - \theta)[ & \text{if } c > 0. \end{cases}$$

In addition, for all  $a \in \mathcal{D}_{\mathcal{L}}$ , let  $\varphi(a) = \sqrt{\theta^2 + 2ac}$  and denote

$$(4.1) \quad L(a) = \mathcal{L}(a, -ac) = -\frac{1}{2} \left( a + \theta + \sqrt{\theta^2 + 2ac} \right),$$

$$(4.2) \quad H(a) = \mathcal{H}(a, -ac) = -\frac{1}{2} \log \left( \frac{(\varphi(a) - a - \theta)\theta^2}{2\varphi^3(a)} \right) - \frac{\gamma^2(a + \theta + \varphi(a))}{2\theta^2}.$$

The function  $L$  is not step as the derivative of  $L$  is finite at the boundary of  $\mathcal{D}_{\mathcal{L}}$ . Moreover,  $L'(a) = 0$  if and only if  $a = a_c$  with

$$a_c = \frac{c^2 - \theta^2}{2c}.$$

Finally, one can observe that  $a_c \in \mathcal{D}_{\mathcal{L}}$  only if  $c < \theta/3$ . Our sharp large deviations results are as follows.

**Theorem 4.1.** *Consider the Ornstein-Uhlenbeck process with shift given by (1.1) where the drift parameter  $\theta < 0$ .*

a) *For all  $\theta < c < \theta/3$ , we have for  $T$  large enough,*

$$(4.3) \quad \mathbb{P}(\widehat{\theta}_T \geq c) = \frac{e^{-TI_{\theta}(c) + J(c)}}{a_c \sigma_c \sqrt{2\pi T}} \left( 1 + o(1) \right)$$

while for  $c < \theta$ ,

$$(4.4) \quad \mathbb{P}(\widehat{\theta}_T \leq c) = -\frac{e^{-TI_\theta(c)+J(c)}}{a_c\sigma_c\sqrt{2\pi T}} \left(1 + o(1)\right)$$

where

$$(4.5) \quad a_c = \frac{c^2 - \theta^2}{2c} \quad \text{and} \quad \sigma_c^2 = -\frac{1}{2c},$$

$$(4.6) \quad J(c) = -\frac{1}{2} \log \left( \frac{\theta^2(c + \theta)(3c - \theta)}{4c^4} \right) + \gamma^2 \frac{(c - \theta)^2}{4c\theta^2}.$$

b) For all  $c > \theta/3$  with  $c \neq 0$ , we have for  $T$  large enough,

$$(4.7) \quad \mathbb{P}(\widehat{\theta}_T \geq c) = \frac{e^{-TI_\theta(c)+K(c)}}{a_c\sigma_c\sqrt{2\pi T}} \left(1 + o(1)\right)$$

where

$$(4.8) \quad a_c = 2(c - \theta) \quad \text{and} \quad \sigma_c^2 = \frac{c^2}{2(2c - \theta)^3}.$$

$$(4.9) \quad K(c) = -\frac{1}{2} \log \left( \frac{\theta^2(c - \theta)(3c - \theta)}{4c^2(2c - \theta)^2} \right) - \frac{\gamma^2}{\theta^2}(2c - \theta).$$

c) For  $c = \theta/3$ , we have for  $T$  large enough,

$$(4.10) \quad \mathbb{P}(\widehat{\theta}_T \geq c) = \frac{e^{-TI_\theta(c)+\gamma^2 b_\theta}}{6\pi T^{1/4}} \frac{\Gamma(1/4)}{\sqrt{2}a_\theta^{3/4}\sigma_\theta} \left(1 + o(1)\right)$$

where

$$(4.11) \quad a_\theta = -\frac{4\theta}{3}, \quad b_\theta = \frac{1}{3\theta}, \quad \sigma_\theta^2 = -\frac{3}{2\theta}.$$

d) For  $c = 0$ , we have for  $T$  large enough,

$$(4.12) \quad \mathbb{P}(\widehat{\theta}_T \geq c) = \frac{\sqrt{2}e^{-TI_\theta(c)+\frac{\gamma^2}{\theta}+2}}{\sqrt{2\pi T}\sqrt{-\theta}} \left(1 + o(1)\right).$$

## 5. PROOFS OF THE LARGE DEVIATIONS RESULTS.

First of all, denote

$$\widehat{V}_T = \begin{pmatrix} \widehat{\theta}_T \\ \widehat{\gamma}_T \end{pmatrix} \quad \text{and} \quad \widetilde{V}_T = \begin{pmatrix} \widetilde{\theta}_T \\ \widetilde{\gamma}_T \end{pmatrix}$$

where

$$(5.1) \quad \widetilde{\theta}_T = \frac{\int_0^T X_t dX_t}{S_T} \quad \text{and} \quad \widetilde{\gamma}_T = -\widetilde{\theta}_T \overline{X}_T.$$

The following lemma shows that the sequences  $(\widehat{V}_T)$  and  $(\widetilde{V}_T)$  share the same LDP. We refer the reader to [3] for the classical notion of exponential approximation.



**Lemma 5.1.** *The sequences of random vectors  $(\widehat{V}_T)$  and  $(\widetilde{V}_T)$  are exponentially equivalent, that is to say, for all  $\varepsilon > 0$ ,*

$$(5.2) \quad \lim_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P} \left( \|\widehat{V}_T - \widetilde{V}_T\| > \varepsilon \right) = -\infty.$$

*In particular, if  $(\widetilde{V}_T)$  satisfies an LDP with good rate function  $I$ , then the same LDP holds for  $(\widehat{V}_T)$ .*

*Proof.* It is easy to see from the very definition of our estimates given in (1.2), (1.3) and (5.1) that

$$\widehat{\theta}_T - \widetilde{\theta}_T = \frac{X_T}{T} \left( \frac{\overline{X}_T}{\Sigma_T} \right) \quad \text{and} \quad \widehat{\gamma}_T - \widetilde{\gamma}_T = \frac{X_T}{T} \left( 1 - \frac{(\overline{X}_T)^2}{\Sigma_T} \right)$$

where

$$\Sigma_T = \frac{S_T}{T}.$$

Consequently, for all  $\varepsilon > 0$ ,

$$(5.3) \quad \mathbb{P} \left( \|\widehat{V}_T - \widetilde{V}_T\| > \varepsilon \right) \leq P_T^\varepsilon + Q_T^\varepsilon$$

where

$$P_T^\varepsilon = \mathbb{P} \left( \left| \frac{X_T}{T} \frac{\overline{X}_T}{\Sigma_T} \right| \geq \frac{\varepsilon}{\sqrt{2}} \right), \quad Q_T^\varepsilon = \mathbb{P} \left( \left| \frac{X_T}{T} \left( 1 - \frac{(\overline{X}_T)^2}{\Sigma_T} \right) \right| \geq \frac{\varepsilon}{\sqrt{2}} \right).$$

For all  $\eta > 0$ , we can upper bound the first term  $P_T^\varepsilon$  by

$$\begin{aligned} P_T^\varepsilon &\leq \mathbb{P} \left( \left| \frac{X_T}{T} \right| \geq \frac{\eta}{\sqrt{2}} \right) + \mathbb{P} \left( \left| \frac{\overline{X}_T}{\Sigma_T} \right| \geq \frac{\varepsilon}{\eta} \right) \\ &\leq \mathbb{P} \left( \left| \frac{X_T}{T} \right| \geq \frac{\eta}{\sqrt{2}} \right) + \mathbb{P} \left( |\overline{X}_T| \geq \sqrt{\frac{\varepsilon}{\eta}} \right) + \mathbb{P} \left( \Sigma_T \leq \sqrt{\frac{\eta}{\varepsilon}} \right) \end{aligned}$$

and the second term  $Q_T^\varepsilon$  by

$$\begin{aligned} Q_T^\varepsilon &\leq \mathbb{P} \left( \left| \frac{X_T}{T} \right| \geq \frac{\varepsilon}{2\sqrt{2}} \right) + \mathbb{P} \left( \left| \frac{X_T}{T} \right| \left| \frac{(\overline{X}_T)^2}{\Sigma_T} \right| \geq \frac{\varepsilon}{2\sqrt{2}} \right) \\ &\leq \mathbb{P} \left( \left| \frac{X_T}{T} \right| \geq \frac{\varepsilon}{2\sqrt{2}} \right) + \mathbb{P} \left( \left| \frac{X_T}{T} \right| \geq \frac{\eta}{2\sqrt{2}} \right) + \mathbb{P} \left( |\overline{X}_T|^2 \geq \sqrt{\frac{\varepsilon}{\eta}} \right) + \mathbb{P} \left( \Sigma_T \leq \sqrt{\frac{\eta}{\varepsilon}} \right). \end{aligned}$$

On the one hand, it is not hard to see that for all  $c > 0$ ,

$$(5.4) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P} \left( \left| \frac{X_T}{T} \right| \geq c \right) = -\infty.$$

As a matter of fact, it is well-known that  $X_T$  is a Gaussian  $\mathcal{N}(m_T, \sigma_T^2)$  random variable with

$$(5.5) \quad m_T = -\frac{\gamma}{\theta}(1 - e^{\theta T}) \quad \text{and} \quad \sigma_T^2 = \frac{1}{2\theta}(e^{2\theta T} - 1).$$

Consequently, for all  $c > 0$ ,

$$\lim_{T \rightarrow \infty} \frac{1}{T^2} \log \mathbb{P}(|X_T| \geq cT) = \theta c^2$$

which immediately leads to (5.4), as  $\theta < 0$ . By the same token,  $\bar{X}_T$  is a Gaussian  $\mathcal{N}(\mu_T, \xi_T^2)$  random variable with

$$(5.6) \quad \mu_T = -\frac{\gamma}{\theta} \left(1 - \frac{1}{\theta T} (e^{\theta T} - 1)\right), \quad \xi_T^2 = \frac{1}{\theta^2 T^2} \left(\frac{1}{2\theta} (e^{2\theta T} - 1) - \frac{2}{\theta} (e^{\theta T} - 1) + T\right).$$

Hence, for all  $c > 0$  such that  $c > |\gamma|/|\theta|$ ,

$$(5.7) \quad \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}(|\bar{X}_T| \geq c) \leq -\frac{\theta^2}{2} \left(c + \left|\frac{\gamma}{\theta}\right|\right)^2,$$

while, if  $c > \gamma^2/\theta^2$ ,

$$(5.8) \quad \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}(|\bar{X}_T|^2 \geq c) \leq -\frac{\theta^2}{2} \left(\sqrt{c} + \left|\frac{\gamma}{\theta}\right|\right)^2.$$

On the other hand, we immediately deduce from Lemma 2.2 together with Gärtner-Ellis's theorem that the sequence  $(\Sigma_T)$  satisfies an LDP with speed  $T$  and good rate function

$$I(c) = \begin{cases} \frac{(2\theta c + 1)^2}{8c} & \text{if } c > 0, \\ +\infty & \text{if } c \leq 0. \end{cases}$$

Therefore, for all  $c > 0$  such that  $-2\theta c < 1$ ,

$$(5.9) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}(\Sigma_T \leq c) = -\frac{(2\theta c + 1)^2}{8c}.$$

Finally, it follows from the conjunction of (5.3), (5.4), (5.7), (5.8) and (5.9) that, for all  $\varepsilon > 0$  and for  $\eta > 0$  small enough,

$$(5.10) \quad \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}(\|\widehat{V}_T - \widetilde{V}_T\| > \varepsilon) \leq -M_{\theta, \gamma}(\varepsilon, \eta)$$

where

$$M_{\theta, \gamma}(\varepsilon, \eta) = \min \left( \frac{\theta^2}{2} \left(\zeta + \left|\frac{\gamma}{\theta}\right|\right)^2, \frac{\theta^2}{2} \left(\sqrt{\zeta} + \left|\frac{\gamma}{\theta}\right|\right)^2, \frac{\zeta}{8} \left(\frac{2\theta}{\zeta} + 1\right)^2 \right)$$

with

$$\zeta = \sqrt{\frac{\varepsilon}{\eta}}.$$

One can observe that if  $\eta$  goes to zero,  $\zeta$  and  $M_{\theta, \gamma}(\varepsilon, \eta)$  tend to infinity, which is exactly what we wanted to prove.  $\square$

We are now in the position to prove our LDP results. Our strategy is to establish an LDP for the triplet

$$\left( \frac{X_T}{\sqrt{T}}, \frac{1}{T} \int_0^T X_t^2 dt, \frac{1}{T} \int_0^T X_t dt \right)$$

and then to make use of the contraction principle [3] in order to prove Theorem 3.1 via Lemma 5.1. The limiting cumulant generating function  $\Lambda$  of the above triplet was already calculated in Lemma 2.1. It is not difficult to check that the function  $\Lambda$  is steep on its effective domain  $\mathcal{D}_\Lambda$ . Consequently, we deduce from Gärtner-Ellis's theorem that the above triplet satisfies an LDP with good rate function  $I$  given by the Fenchel-Legendre transform of  $\Lambda$ ,

$$I(\lambda, \mu, \delta) = \sup_{(a,b,c) \in \mathcal{D}_\Lambda} \left\{ \lambda a + \mu b + \delta c - \Lambda(a, b, c) \right\}.$$

We can prove after some straightforward calculations that

$$(5.11) \quad I(\lambda, \mu, \delta) = \begin{cases} \frac{\theta^2 \mu - \theta \lambda^2}{2} + \frac{\theta + \gamma^2 + 2\theta\gamma\delta}{2} + \frac{(1 + \lambda^2)^2}{8(\mu - \delta^2)} & \text{if } \delta^2 < \mu, \\ +\infty & \text{if } \delta^2 \geq \mu. \end{cases}$$

Hereafter, it follows from the well-known Itô's formula that

$$(5.12) \quad \int_0^T X_t dX_t = \frac{1}{2} (X_T^2 - T)$$

which implies that

$$\tilde{V}_T = f \left( \frac{X_T}{\sqrt{T}}, \frac{1}{T} \int_0^T X_t^2 dt, \frac{1}{T} \int_0^T X_t dt \right)$$

where  $f$  is the continuous function given, for all  $(\lambda, \mu, \delta) \in \mathbb{R}^3$  such that  $\mu > \delta^2$ , by

$$f(\lambda, \mu, \delta) = \begin{pmatrix} \lambda^2 - 1 \\ \frac{2(\mu - \delta^2)}{\delta(\lambda^2 - 1)} \\ -\frac{2(\mu - \delta^2)}{2(\mu - \delta^2)} \end{pmatrix}.$$

Therefore, we infer from the contraction principle given e.g. by Theorem 4.2.1 of [3], together with Lemma 5.1, that the sequences of random vectors  $(\hat{V}_T)$  and  $(\tilde{V}_T)$  share the same LDP with good rate function

$$I_{(\theta, \gamma)}(c, d) = \inf \left\{ I(\lambda, \mu, \delta) / (\lambda, \mu, \delta) \in \mathbb{R}^3, \mu > \delta^2, f(\lambda, \mu, \delta) = \begin{pmatrix} c \\ d \end{pmatrix} \right\},$$

where the infimum over the empty set is equal to  $+\infty$ . If  $c = 0$  and  $d \neq 0$ , we clearly obtain that  $I_{(\theta, \gamma)}(0, d) = +\infty$ . Otherwise, we find that

$$I_{(\theta, \gamma)}(c, d) = \inf_{\lambda^2 > 1} \left\{ \frac{\theta^2}{2} \frac{\lambda^2 - 1}{2c} + \frac{\theta^2}{2} \frac{d^2}{c^2} - \frac{\theta \lambda^2}{2} - \theta \gamma \frac{d}{c} + \frac{\theta + \gamma^2}{2} + \frac{(1 + \lambda^2)^2}{4} \frac{c}{\lambda^2 - 1} \right\}$$

as soon as  $c > 0$ , whereas

$$I_{(\theta, \gamma)}(c, d) = \inf_{\lambda^2 < 1} \left\{ \frac{\theta^2}{2} \frac{\lambda^2 - 1}{2c} + \frac{\theta^2}{2} \frac{d^2}{c^2} - \frac{\theta \lambda^2}{2} - \theta \gamma \frac{d}{c} + \frac{\theta + \gamma^2}{2} + \frac{(1 + \lambda^2)^2}{4} \frac{c}{\lambda^2 - 1} \right\}$$

when  $c < 0$ , and

$$I_{(\theta, \gamma)}(0, 0) = \inf_{\mu > \delta^2} \left\{ \frac{\theta^2}{2} \mu - \frac{\theta}{2} + \theta \gamma \delta + \frac{\theta + \gamma^2}{2} + \frac{1}{2(\mu - \delta^2)} \right\}.$$

Finally, we obtain the rate function given by (3.1) thanks to elementary calculations, which completes the proof of Theorem 3.1.  $\square$

## 6. PROOFS OF THE SHARP LARGE DEVIATIONS RESULTS.

**6.1. Proof of Theorem 4.1, first part.** First of all, we shall focus our attention on the SLDP in the easy case  $\theta < c < \theta/3$ . Denote by  $L_T$  the normalized cumulant generating function of the random variable  $Z_T(a) = \mathcal{Z}_T(a, -ca)$ . We already saw that  $a_c$ , given by (4.5), belongs to the domain  $\mathcal{D}_{\mathcal{L}}$  whenever  $c < \theta/3$ . We can split  $\mathbb{P}(\widehat{\theta}_T \geq c)$  into two terms,  $\mathbb{P}(\theta_T \geq c) = A_T B_T$  with

$$(6.1) \quad A_T = \exp(TL_T(a_c)),$$

$$(6.2) \quad B_T = \mathbb{E}_T \left[ \exp(-Z_T(a_c)) \mathbf{I}_{Z_T(a_c) \geq 0} \right],$$

where  $\mathbb{E}_T$  stands for the expectation after the usual change of probability

$$(6.3) \quad \frac{d\mathbb{P}_T}{d\mathbb{P}} = \exp(Z_T(a_c) - TL_T(a_c)).$$

On the one hand, we can deduce from Lemma 2.2 with  $L(a_c) = \mathcal{L}(a_c, -ca_c)$ ,  $H(a_c) = \mathcal{H}(a_c, -ca_c)$ , and  $R_T(a_c) = \mathcal{R}_T(a_c, -ca_c)$  that

$$(6.4) \quad A_T = \exp(TL(a_c) + H(a_c)) \left(1 + o(1)\right),$$

$$(6.4) \quad A_T = \exp(-TI_{\theta}(c) + J(c)) \left(1 + o(1)\right).$$

It remains to establish an asymptotic expansion for  $B_T$  which can be rewritten as

$$(6.5) \quad B_T = \mathbb{E}_T \left[ \exp(-a_c \sigma_c \sqrt{T} U_T) \mathbf{I}_{U_T \geq 0} \right]$$

where  $\sigma_c^2$  is given by (4.5) and

$$U_T = \frac{Z_T(1)}{\sigma_c \sqrt{T}}.$$

**Lemma 6.1.** *For all  $\theta < c < \theta/3$ , we have*

$$(6.6) \quad B_T = \frac{1}{a_c \sigma_c \sqrt{2\pi T}} \left(1 + o(1)\right).$$

*Proof.* Denote by  $\Phi_T$  the characteristic function of  $U_T$  under  $\mathbb{P}_T$ . As  $\theta < c < \theta/3$ , it follows from (4.5) that  $a_c > 0$  and  $\sigma_c > 0$ . Moreover, (6.3) immediately implies that

$$(6.7) \quad \Phi_T(u) = \exp \left( TL_T \left( a_c + \frac{iu}{\sigma_c \sqrt{T}} \right) - TL_T(a_c) \right).$$

First of all, we deduce from Lemma 2.3 that for  $T$  large enough,  $\Phi_T$  belongs to  $L^2(\mathbb{R})$ . As a matter of fact, as soon as  $1 - 2a_c \alpha_k^T > 0$  for all  $k \geq 1$  and  $T$  large

enough, we obtain from Lemma 2.3 and (6.7) that

$$\begin{aligned}
|\Phi_T(u)|^2 &= \prod_{k=1}^{\infty} \left( 1 + \frac{4u^2(\alpha_k^T)^2}{\sigma_c^2 T(1 - 2a_c \alpha_k^T)^2} \right)^{-1/2} \exp \left( -\frac{(a_c \beta_k^T)^2}{1 - 2a_c \alpha_k^T} \right), \\
(6.8) \quad &\leq \prod_{k=1}^{\infty} \left( 1 + \frac{4u^2(\alpha_k^T)^2}{\sigma_c^2 T(1 - 2a_c \alpha_k^T)^2} \right)^{-1/2}.
\end{aligned}$$

For all  $\varepsilon > 0$  small enough such that  $1 - 2a_c \varepsilon > 0$  we denote

$$q_T(\varepsilon) = \sum_{k=1}^{\infty} \mathbb{I}_{|\alpha_k^T| > \varepsilon}.$$

It follows from Lemma 2.3 that it exists some constants  $\varepsilon > 0$  and  $\eta > 0$ , depending only on  $\varepsilon$ , such that

$$(6.9) \quad \liminf_{T \rightarrow \infty} \frac{q_T(\varepsilon)}{T} \geq 2\eta.$$

Hence, we infer from (6.8) and (6.9) that for  $T$  large enough,

$$(6.10) \quad |\Phi_T(u)|^2 \leq \left( 1 + \frac{\xi^2 u^2}{T} \right)^{-\eta T} \quad \text{where} \quad \xi = \frac{2\varepsilon}{\sigma_c(1 + 2a_c \varepsilon)}$$

which clearly ensures, whenever  $\eta T \geq 1$ , that

$$(6.11) \quad |\Phi_T(u)|^2 \leq \frac{1}{1 + \eta \xi^2 u^2}.$$

Consequently, we find from (6.11) that for  $T$  large enough,  $\Phi_T$  belongs to  $L^2(\mathbb{R})$ . Therefore, we obtain from Parseval's formula that  $B_T$ , given by (6.5), can be rewritten as

$$(6.12) \quad B_T = \frac{1}{2\pi a_c \sigma_c \sqrt{T}} \int_{\mathbb{R}} \left( 1 + \frac{i u}{a_c \sigma_c \sqrt{T}} \right)^{-1} \Phi_T(u) du.$$

However, we deduce from Lemma 2.2 that, for all  $u \in \mathbb{R}$ ,

$$\begin{aligned}
\lim_{T \rightarrow \infty} \Phi_T(u) &= \lim_{T \rightarrow \infty} \exp \left( T L_T \left( a_c + \frac{i u}{\sigma_c \sqrt{T}} \right) - T L_T(a_c) \right), \\
&= \exp \left( -\frac{u^2}{2\sigma_c^2} L''(a_c) \right), \\
(6.13) \quad &= \exp \left( -\frac{u^2}{2} \right)
\end{aligned}$$

as  $L''(a_c) = \sigma_c^2$ , which means that the distribution of  $U_T$  under  $\mathbb{P}_T$  converges to the standard  $\mathcal{N}(0, 1)$  distribution. Finally, it follows from (6.12), (6.13) and the Lebesgue dominated convergence theorem that

$$B_T = \frac{1}{a_c \sigma_c \sqrt{2\pi T}} (1 + o(1))$$

which completes the proof of Lemma 6.1.  $\square$

**Proof of Theorem 4.1, first part.** The expansion (4.3) immediately follows from the conjunction of (6.4) and (6.6). The proof of (4.4) follows exactly the same lines, the only notable point to mention being a change of sign in Parseval's formula.  $\square$

**6.2. Proof of Theorem 4.1, second part.** We shall now proceed to the proof of the SLDP in the more complex case  $c > \theta/3$  with  $c \neq 0$ . One can easily check that the function  $L$ , given by (4.1), is decreasing and reaches its minimum at the right boundary point  $a_c = 2(c - \theta)$  of the domain  $\mathcal{D}_{\mathcal{L}}$ . Therefore, as in [2] or [1], it is necessary to make use of a slight modification of the usual strategy of change of probability given in (6.3). There exists a unique  $a_T$ , which belongs to the interior of  $\mathcal{D}_{\mathcal{L}}$  and converges to its border  $a_c$ , solution of the implicit equation

$$(6.14) \quad L'(a_T) + \frac{1}{T}H'(a_T) = 0.$$

It leads to the decomposition  $\mathbb{P}(\widehat{\theta}_T \geq c) = A_T B_T$  with

$$(6.15) \quad A_T = \exp(TL_T(a_T)),$$

$$(6.16) \quad B_T = \mathbb{E}_T \left[ \exp(-Z_T(a_T)) \mathbf{I}_{Z_T(a_T) \geq 0} \right],$$

where  $\mathbb{E}_T$  stands for the expectation after the time-varying change of probability

$$(6.17) \quad \frac{d\mathbb{P}_T}{d\mathbb{P}} = \exp(Z_T(a_T) - TL_T(a_T)).$$

We deduce from (4.1), (4.2) together with (6.14) that

$$(\theta^2 T + \gamma^2) \tau(a_T) = \frac{\theta^2 (c(2\varphi(a_T) - a_T) + \theta(\theta - 3c))}{\varphi(a_T)(\varphi(a_T) + c)}$$

where  $\varphi(a) = \sqrt{\theta^2 + 2ac}$  and  $\tau(a) = \varphi(a) - a - \theta$ . Consequently, it follows from straightforward calculations that

$$(6.18) \quad \lim_{T \rightarrow \infty} T(\varphi(a_T) + 2c - \theta) = -\frac{c}{3c - \theta},$$

$$(6.19) \quad \lim_{T \rightarrow \infty} T(a_T - a_c) = -\frac{(2c - \theta)}{3c - \theta},$$

$$(6.20) \quad \lim_{T \rightarrow \infty} T\tau(a_T) = \frac{c - \theta}{3c - \theta}.$$

Moreover, we can show via Lemma B.4 and B.5 that  $R_T(a_T)$  remains bounded when  $T$  goes to infinity. Hence, Lemma 2.2 together with convergences (6.18), (6.19), and (6.20) imply that

$$(6.21) \quad \begin{aligned} A_T &= \exp(TL(a_T) + H(a_T)) \left(1 + o(1)\right), \\ &= \exp\left(-TI_{\theta}(c) - \frac{\gamma^2}{\theta^2}(2c - \theta)\right) \left(\frac{2eT(2c - \theta)^3(3c - \theta)}{\theta^2(c - \theta)}\right)^{1/2} \left(1 + o(1)\right). \end{aligned}$$

Moreover, the second term  $B_T$  can be rewritten as

$$(6.22) \quad B_T = \mathbb{E}_T \left[ \exp(-a_T T U_T) \mathbf{I}_{U_T \geq 0} \right]$$

where

$$U_T = \frac{Z_T(1)}{T}.$$

**Lemma 6.2.** *For  $c > \theta/3$  with  $c \neq 0$ , we have*

$$(6.23) \quad B_T = \frac{1}{a_c b_c T \sqrt{2\pi e}} \left(1 + o(1)\right)$$

where

$$(6.24) \quad b_c = -L'(a_c) = \frac{3c - \theta}{2(2c - \theta)}.$$

*Proof.* Denote by  $\Phi_T$  the characteristic function of  $U_T$  under  $\mathbb{P}_T$ . We infer from (6.17) that for all  $u \in \mathbb{R}$ ,

$$(6.25) \quad \Phi_T(u) = \exp \left( TL_T \left( a_T + \frac{iu}{T} \right) - TL_T(a_T) \right).$$

Moreover, we obtain from (6.18) and (6.19) that for  $T$  large enough and for all  $u \in \mathbb{R}$  such that  $|u| = o(T)$ ,

$$\exp \left( TL \left( a_T + \frac{iu}{T} \right) - TL(a_T) \right) = \exp \left( -ib_c u - \frac{\sigma_c^2 u^2}{2T} \right) \left( 1 + O \left( \frac{|u|^3}{T^2} \right) \right)$$

where  $\sigma_c^2$  and  $b_c$  are given by (4.8) and (6.24). Consequently, as soon as  $|u| = o(T^{2/3})$ ,

$$(6.26) \quad \exp \left( TL \left( a_T + \frac{iu}{T} \right) - TL(a_T) \right) = \exp \left( -ib_c u - \frac{\sigma_c^2 u^2}{2T} \right) \left( 1 + o(1) \right)$$

and the remainder  $o(1)$  is uniform. By the same token,

$$(6.27) \quad \lim_{T \rightarrow \infty} \exp \left( H \left( a_T + \frac{iu}{T} \right) - H(a_T) \right) = \frac{1}{\sqrt{1 - 2ib_c u}}.$$

Therefore, we deduce from Lemma 2.2 together with (6.26), (6.27) and the boundedness of  $R_T(a_T)$ , that for all  $u \in \mathbb{R}$  such that  $|u| = o(T^{2/3})$ ,

$$(6.28) \quad \Phi_T(u) = \Phi(u) \exp \left( -\frac{\sigma_c^2 u^2}{2T} \right) \left( 1 + o(1) \right)$$

where

$$\Phi(u) = \frac{1}{\sqrt{1 - 2ib_c u}} \exp(-ib_c u).$$

It means that the distribution of  $U_T$  under  $\mathbb{P}_T$  converges to  $b_c(\xi^2 - 1)$ , where  $\xi$  stands for an  $\mathcal{N}(0, 1)$  random variable. It also implies that, for  $T$  large enough,  $\Phi_T$  belongs to  $L^2(\mathbb{R})$ . Hereafter, we deduce from Parseval's formula that  $B_T$ , given by (6.22), can be rewritten as

$$(6.29) \quad B_T = \frac{1}{2\pi T a_T} \int_{\mathbb{R}} \left( 1 + \frac{iu}{a_T T} \right)^{-1} \Phi_T(u) du.$$

We split  $B_T$  into two terms,  $B_T = C_T + D_T$ , where

$$(6.30) \quad C_T = \frac{1}{2\pi T a_T} \int_{|u| \leq s_T} \left(1 + \frac{i u}{T a_T}\right)^{-1} \Phi_T(u) du,$$

$$(6.31) \quad D_T = \frac{1}{2\pi T a_T} \int_{|u| > s_T} \left(1 + \frac{i u}{T a_T}\right)^{-1} \Phi_T(u) du$$

with  $s_T = T^{2/3}$ . On the one hand, it follows from (6.31) that  $D_T$  is negligible, as

$$(6.32) \quad D_T = o\left(\exp\left(-\frac{\sigma_c^2 s_T^2}{2T}\right)\right).$$

On the other hand, we find from (6.28) that for  $T$  large enough

$$2\pi T a_T C_T = \int_{|u| \leq s_T} \Phi(u) \exp\left(-\frac{\sigma_c^2 u^2}{2T}\right) (1 + o(1)) du,$$

which leads, thanks to Lemma 7.3 in [2], to

$$(6.33) \quad \lim_{T \rightarrow \infty} 2\pi T a_T C_T = \frac{\sqrt{2\pi}}{b_c \sqrt{e}}.$$

Finally, (6.32) together with (6.33) clearly imply (6.23).  $\square$

**Proof of Theorem 4.1, second part.** We immediately deduce (4.7) from (6.21) and (6.23).  $\square$

**6.3. Proof of Theorem 4.1, third part.** Assume now that  $c = \theta/3$  which means that  $a_c = a_\theta$  with  $a_\theta = -4\theta/3$ . There exists a unique  $a_T$ , which belongs to the interior of  $\mathcal{D}_{\mathcal{L}}$  and converges to its border  $a_\theta$ , solution of the implicit equation

$$(6.34) \quad L'(a_T) + \frac{1}{T} H'(a_T) = 0.$$

We deduce from (4.1), (4.2) together with (6.34) that

$$(\theta^2 T + \gamma^2) \tau(a_T) = \frac{\theta^2 c (2\varphi(a_T) - a_T)}{\varphi(a_T) (\varphi(a_T) + c)}$$

where  $\varphi(a) = \sqrt{\theta^2 + 2ac}$  and  $\tau(a) = \varphi(a) - a - \theta$ . We obviously have

$$\tau(a) = -\frac{(\varphi(a) + c)(\varphi(a) - \theta)}{2c}$$

which leads to

$$(\theta^2 T + \gamma^2) (\varphi(a_T) + c)^2 = \frac{2\theta^2 c^2 (a_T - 2\varphi(a_T))}{\varphi(a_T) (\varphi(a_T) - \theta)}.$$



It implies after some elementary calculations that

$$(6.35) \quad \lim_{T \rightarrow \infty} T(\varphi(a_T) + c)^2 = -\frac{\theta}{3},$$

$$(6.36) \quad \lim_{T \rightarrow \infty} T(a_T - a_\theta)^2 = -\frac{\theta}{3},$$

$$(6.37) \quad \lim_{T \rightarrow \infty} \sqrt{T}\tau(a_T) = 2\sqrt{-\frac{\theta}{3}}.$$

Hereafter, we shall make use of the decomposition  $\mathbb{P}(\widehat{\theta}_T \geq c) = A_T B_T$  given by

$$(6.38) \quad A_T = \exp(TL_T(a_T)),$$

$$(6.39) \quad B_T = \mathbb{E}_T \left[ \exp(-Z_T(a_T)) \mathbf{I}_{Z_T(a_T) \geq 0} \right],$$

where  $\mathbb{E}_T$  stands for the expectation after the time-varying change of probability

$$(6.40) \quad \frac{d\mathbb{P}_T}{d\mathbb{P}} = \exp(Z_T(a_T) - TL_T(a_T)).$$

We obtain from Lemma B.4 and B.5 that  $R_T(a_T)$  remains bounded when  $T$  goes to infinity. Hence, it follows from Lemma 2.2 together with convergences (6.35), (6.36), and (6.37) that

$$(6.41) \quad A_T = \frac{\sqrt{2}}{3} \exp\left(-TI_\theta(c) + \frac{\gamma^2}{3\theta}\right) \left(\frac{-\theta eT}{3}\right)^{1/4} (1 + o(1)).$$

On the other hand,  $B_T$  can be rewritten as

$$(6.42) \quad B_T = \mathbb{E}_T \left[ \exp(-a_T \sqrt{T} U_T) \mathbf{I}_{U_T \geq 0} \right]$$

where

$$U_T = \frac{Z_T(1)}{\sqrt{T}}.$$

**Lemma 6.3.** *For  $c = \theta/3$ , we have*

$$(6.43) \quad B_T = \frac{1}{4\pi\sqrt{a_\theta T}} \exp(-1/4)\Gamma(1/4) (1 + o(1)).$$

*Proof.* Via the same lines as in the proof of Lemma 6.1, we find that the characteristic function  $\Phi_T$  of  $U_T$ , under  $\mathbb{P}_T$ , belongs to  $L^2(\mathbb{R})$ . Hence, it follows from Parseval's formula that

$$(6.44) \quad B_T = \frac{1}{2\pi a_T \sqrt{T}} \int_{\mathbb{R}} \left(1 + \frac{i u}{a_T \sqrt{T}}\right)^{-1} \Phi_T(u) du.$$

However, we obtain from (6.35) and (6.36) that

$$(6.45) \quad \lim_{T \rightarrow \infty} T \left( L\left(a_T + \frac{i u}{\sqrt{T}}\right) - L(a_T) \right) = -i d_\theta u - \frac{\sigma_\theta^2 u^2}{2}$$

where  $\sigma_\theta^2$  is given by (4.11) and  $d_\theta = \sigma_\theta/\sqrt{2}$ . By the same token,

$$(6.46) \quad \lim_{T \rightarrow \infty} \exp \left( H\left(a_T + \frac{i u}{\sqrt{T}}\right) - H(a_T) \right) = \frac{1}{\sqrt{1 - 2i d_\theta u}}.$$

Therefore, we deduce from Lemma 2.2 together with (6.45), (6.46) and the boundedness of  $R_T(a_T)$ , the pointwise convergence

$$(6.47) \quad \lim_{T \rightarrow \infty} \Phi_T(u) = \Phi(u)$$

where

$$\Phi(u) = \frac{1}{\sqrt{1 - 2id_\theta u}} \exp\left(-id_\theta u - \frac{\sigma_\theta^2 u^2}{2}\right).$$

It shows that the distribution of  $U_T$  under  $\mathbb{P}_T$  converges to  $\sigma_\theta \zeta + d_\theta(\xi^2 - 1)$ , where  $\zeta$  and  $\xi$  are two independent random variables sharing the same  $\mathcal{N}(0, 1)$  distribution. Finally, we obtain from (6.44) together with (6.47) and the Lebesgue dominated convergence theorem that

$$\begin{aligned} B_T &= \frac{1}{2\pi a_T \sqrt{T}} \int_{\mathbb{R}} \Phi(u) du \left(1 + o(1)\right), \\ &= \frac{1}{4\pi \sqrt{a_\theta T}} \exp(-1/4) \Gamma(1/4) \left(1 + o(1)\right) \end{aligned}$$

which achieves the proof of Lemma 6.3.  $\square$

**Proof of Theorem 4.1, third part.** The proof of (4.10) immediately follows from the conjunction of (6.41) and (6.43).  $\square$

**6.4. Proof of Theorem 4.1, fourth part.** Assume now that  $c = 0$ . We want to obtain the leading asymptotic behavior of the probability

$$\mathbb{P}(\widehat{\theta}_T \geq 0) = \mathbb{P}(X_T^2 - 2X_T \bar{X}_T \geq T).$$

For all  $\alpha > 0$ , we have the decomposition  $\mathbb{P}(X_T^2 - 2X_T \bar{X}_T \geq T) = A_T + B_T$  where

$$\begin{aligned} A_T &= \mathbb{P}(X_T^2 - 2X_T \bar{X}_T \geq T, |\bar{X}_T| \leq \alpha), \\ B_T &= \mathbb{P}(X_T^2 - 2X_T \bar{X}_T \geq T, |\bar{X}_T| > \alpha). \end{aligned}$$

First of all, if

$$\alpha = \left| \frac{\gamma}{\theta} \right| + \frac{2}{\sqrt{-\theta}},$$

it is not hard to see that  $B_T$  is negligible. As a matter of fact, we deduce from the simple upper bound

$$B_T \leq \mathbb{P}(|\bar{X}_T| > \alpha)$$

together with (5.7) that

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \log B_T \leq 2\theta,$$

leading to

$$B_T = o\left(\exp\left(\frac{3\theta T}{2}\right)\right).$$

Next, we recall that the sequence  $(\widehat{\theta}_T)$  satisfies an LDP with good rate function  $I_\theta$  given by (3.2). Consequently,

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}(\widehat{\theta}_T \geq 0) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}(X_T^2 - 2X_T \overline{X}_T \geq T) = \theta$$

which clearly implies that

$$\mathbb{P}(X_T^2 - 2X_T \overline{X}_T \geq T) = A_T (1 + o(1)).$$

From now on, it only remains to establish the leading asymptotic behavior of  $A_T$ . We shall show in Appendix A that the random vector

$$\begin{pmatrix} X_T \\ \overline{X}_T \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} m_T \\ \mu_T \end{pmatrix}, \Gamma_T(\theta) \right)$$

where  $m_T$  and  $\mu_T$  are defined in (5.5), (5.6) and the covariance matrix

$$\Gamma_T(\theta) = \begin{pmatrix} a_T & b_T \\ b_T & c_T \end{pmatrix}$$

is given by (A.2). Therefore,

$$A_T = \int_{-\alpha}^{\alpha} h_T(x) f_T(x) dx$$

where

$$h_T(x) = \mathbb{P}(X_T^2 - 2X_T \overline{X}_T \geq T | \overline{X}_T = x)$$

and  $f_T$  is the Gaussian probability density function of  $\overline{X}_T$ . Moreover, as  $c_T > 0$ , it is well-known that the conditional distribution of  $X_T$  given  $\overline{X}_T = x$  is  $\mathcal{N}(\nu_T, s_T^2)$  with

$$\nu_T = m_T + \frac{b_T}{c_T}(x - \mu_T) \quad \text{and} \quad s_T^2 = a_T - \frac{b_T^2}{c_T}.$$

Furthermore, for all  $x \in \mathbb{R}$ ,  $h_T(x)$  can be rewritten as

$$h_T(x) = \mathbb{P} \left( \frac{X_T - \nu_T}{s_T} \leq -y_T | \overline{X}_T = x \right) + \mathbb{P} \left( \frac{X_T - \nu_T}{s_T} \geq z_T | \overline{X}_T = x \right)$$

where

$$y_T = \frac{-x + \sqrt{x^2 + T} + \nu_T}{s_T} \quad \text{and} \quad z_T = \frac{x + \sqrt{c^2 + T} - \nu_T}{s_T}.$$

One can easily check that

$$\liminf_{T \rightarrow +\infty} \{y_T, z_T\} \geq \liminf_{T \rightarrow +\infty} \frac{-\alpha + \sqrt{T} - m_T - b_T c_T^{-1}(\alpha - \mu_T)}{s_T} = +\infty.$$

Consequently, it follows from standard asymptotic analysis of Gaussian distribution tails that

$$h_T(x) = \frac{1}{y_T \sqrt{2\pi}} \exp \left( -\frac{y_T^2}{2} \right) (1 + o(1)) + \frac{1}{z_T \sqrt{2\pi}} \exp \left( -\frac{z_T^2}{2} \right) (1 + o(1))$$

where  $o(1)$  is uniform with respect to  $x$ . We split  $A_T$  into two terms,  $A_T = C_T + D_T$ , where

$$\begin{aligned} C_T &= \int_{-\alpha}^{\alpha} \frac{1}{y_T \sqrt{2\pi}} \exp\left(-\frac{y_T^2}{2}\right) \frac{1}{\sqrt{2\pi c_T}} \exp\left(-\frac{(x - \mu_T)^2}{2c_T}\right) dx \left(1 + o(1)\right), \\ D_T &= \int_{-\alpha}^{\alpha} \frac{1}{z_T \sqrt{2\pi}} \exp\left(-\frac{z_T^2}{2}\right) \frac{1}{\sqrt{2\pi c_T}} \exp\left(-\frac{(x - \mu_T)^2}{2c_T}\right) dx \left(1 + o(1)\right). \end{aligned}$$

We find from a careful asymptotic expansion inside the integral  $C_T$  together with the change of variables  $y = -\theta(x + \gamma/\theta)\sqrt{T}$  and Lebesgue's dominated convergence theorem, that

$$\begin{aligned} \lim_{T \rightarrow +\infty} \sqrt{2\pi} \sqrt{-2\theta T} e^{-\theta T} C_T &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(2y + \frac{\gamma^2}{\theta} - \frac{y^2}{2}\right) dy, \\ &= \exp\left(\frac{\gamma^2}{\theta} + 2\right). \end{aligned}$$

By the same token, we also obtain that

$$\lim_{T \rightarrow +\infty} \sqrt{2\pi} \sqrt{-2\theta T} e^{-\theta T} D_T = \exp\left(\frac{\gamma^2}{\theta} + 2\right)$$

which is exactly what we wanted to prove.  $\square$

#### APPENDIX A: PROOF OF LEMMA 2.1.

For all  $(a, b, c) \in \mathbb{R}^3$ , let

$$\mathcal{Z}_T(a, b, c) = a\sqrt{T}X_T + b \int_0^T X_t^2 dt + c \int_0^T X_t dt.$$

We shall calculate the limit  $\Lambda$  of the normalized cumulant generating function  $\Lambda_T$  of the random variable  $\mathcal{Z}_T(a, b, c)$ . First of all, it follows from Girsanov's formula associated with (1.1) that

$$\begin{aligned} \Lambda_T(a, b, c) &= \frac{1}{T} \log \mathbb{E} \left[ \exp(\mathcal{Z}_T(a, b, c)) \right], \\ &= \frac{1}{T} \log \mathbb{E}_{\varphi, \delta} \left[ \exp\left((\theta - \varphi) \int_0^T X_t dX_t + \frac{1}{2}(2b - \theta^2 + \varphi^2) \int_0^T X_t^2 dt - \xi_T\right) \right] \end{aligned}$$

where  $\mathbb{E}_{\varphi, \delta}$  stands for the expectation after the change of probability

$$\frac{d\mathbb{P}_{\varphi, \delta}}{d\mathbb{P}_{\theta, \gamma}} = \exp\left((\varphi - \theta) \int_0^T X_t dX_t - \frac{1}{2}(\varphi^2 - \theta^2) \int_0^T X_t^2 dt + \zeta_T\right)$$

with

$$\begin{aligned} \xi_T &= \zeta_T - a\sqrt{T}X_T - cT\bar{X}_T, \\ \zeta_T &= (\theta\gamma - \varphi\delta)T\bar{X}_T - (\gamma - \delta)X_T + \frac{1}{2}(\gamma^2 - \delta^2)T. \end{aligned}$$

Consequently, if we assume that  $\theta^2 - 2b > 0$  and if we choose  $\varphi = \sqrt{\theta^2 - 2b}$  and  $\delta = 0$ ,  $\Lambda_T(a, b, c)$  reduces to

$$(A.1) \quad \Lambda_T(a, b, c) = \frac{\varphi - \theta - \gamma^2}{2} + \frac{1}{T} \log \mathbb{E}_{\varphi,0} \left[ \exp \left( -\frac{1}{2} V_T' J V_T + U_T' V_T \right) \right]$$

where the vectors  $U_T$  and  $V_T$  are given by

$$U_T = \begin{pmatrix} a\sqrt{T} + \gamma \\ T(c - \theta\gamma) \end{pmatrix} \quad \text{and} \quad V_T = \begin{pmatrix} X_T \\ \bar{X}_T \end{pmatrix}$$

and  $J$  is the diagonal matrix of order two

$$J = \begin{pmatrix} \varphi - \theta & 0 \\ 0 & 0 \end{pmatrix}.$$

Under the new probability  $\mathbb{P}_{\varphi,0}$ ,  $V_T$  is Gaussian random vector with zero mean and covariance matrix  $\Gamma_T(\varphi)$  given by

$$(A.2) \quad \Gamma_T(\varphi) = \begin{pmatrix} a_T(\varphi) & b_T(\varphi) \\ b_T(\varphi) & c_T(\varphi) \end{pmatrix}$$

where

$$\begin{aligned} a_T(\varphi) &= \frac{1}{2\varphi} (e^{2\varphi T} - 1), \\ b_T(\varphi) &= \frac{1}{2\varphi^2 T} (e^{\varphi T} - 1)^2, \\ c_T(\varphi) &= \frac{1}{\varphi^2 T^2} \left( \frac{1}{2\varphi} (e^{2\varphi T} - 1) - \frac{2}{\varphi} (e^{\varphi T} - 1) + T \right). \end{aligned}$$

As a matter of fact,

$$a_T(\varphi) = \mathbb{E}_{\varphi,0}[X_T^2] = e^{2\varphi T} \int_0^T e^{-2\varphi t} dt = \frac{1}{2\varphi} (e^{2\varphi T} - 1),$$

while

$$\begin{aligned} b_T(\varphi) &= \mathbb{E}_{\varphi,0}[X_T \bar{X}_T] = \frac{1}{T} \int_0^T \mathbb{E}_{\varphi,0}[X_t X_T] dt, \\ &= \frac{1}{2\varphi T} \int_0^T e^{\varphi(T+t)} (1 - e^{-2\varphi t}) dt, \\ &= \frac{1}{2\varphi^2 T} (e^{\varphi T} (e^{\varphi T} - 1) + e^{\varphi T} (e^{-\varphi T} - 1)), \\ &= \frac{1}{2\varphi^2 T} (e^{\varphi T} - 1)^2, \end{aligned}$$

and

$$\begin{aligned}
c_T(\varphi) &= \mathbb{E}_{\varphi,0}[(\bar{X}_T)^2] = \frac{1}{T^2} \int_0^T \int_0^T \mathbb{E}_{\varphi,0}[X_s X_t] ds dt, \\
&= \frac{1}{\varphi T^2} \int_0^T \int_0^t e^{\varphi(s+t)} (1 - e^{-2\varphi s}) ds dt, \\
&= \frac{1}{\varphi^2 T^2} \int_0^T (e^{\varphi t} (e^{\varphi t} - 1) + e^{\varphi t} (e^{-\varphi t} - 1)) dt, \\
&= \frac{1}{\varphi^2 T^2} \left( \frac{1}{2\varphi} (e^{2\varphi T} - 1) - \frac{2}{\varphi} (e^{\varphi T} - 1) + T \right).
\end{aligned}$$

Denote by  $M_T(a, b, c)$  the square matrix of order two

$$M_T(a, b, c) = \mathbf{I}_2 + J\Gamma_T(\varphi) = \begin{pmatrix} 1 + (\varphi - \theta)a_T(\varphi) & (\varphi - \theta)b_T(\varphi) \\ 0 & 1 \end{pmatrix}$$

where  $\mathbf{I}_2$  stands for the identity matrix of order two. We clearly have

$$\det M_T(a, b, c) = 1 + (\varphi - \theta)a_T(\varphi) = 1 + \frac{\varphi - \theta}{2\varphi} (e^{2\varphi T} - 1)$$

which leads to

$$(A.3) \quad \lim_{T \rightarrow \infty} \frac{\det M_T(a, b, c)}{e^{2\varphi T}} = \frac{\varphi - \theta}{2\varphi}.$$

Hence, as  $\theta < 0 < \varphi$ , it follows from (A.3) that for  $T$  large enough, the matrix  $M_T(a, b, c)$  is positive definite. It is also not hard to see from (A.2) that

$$(A.4) \quad \lim_{T \rightarrow \infty} \frac{T \det \Gamma_T(\varphi)}{e^{2\varphi T}} = \frac{1}{2\varphi^3}.$$

Therefore, we obtain from standard Gaussian calculations that

$$(A.5) \quad \Lambda_T(a, b, c) = \frac{\varphi - \theta - \gamma^2}{2} - \frac{1}{2T} \log(\det M_T(a, b, c)) + \frac{1}{T} H_T(a, b, c)$$

where

$$H_T(a, b, c) = \frac{1}{2} U_T' \Gamma_T(\varphi) M_T^{-1}(a, b, c) U_T.$$

On the one hand, we immediately obtain from (A.3) that

$$(A.6) \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \log(\det M_T(a, b, c)) = \varphi.$$

On the other hand, we clearly have

$$H_T(a, b, c) = \frac{1}{2 \det M_T(a, b, c)} \left( (a\sqrt{T} + \gamma)^2 a_T(\varphi) + 2T d_T(\varphi) + T^2 (c - \theta\gamma)^2 e_T(\varphi) \right)$$

where  $d_T(\varphi) = (a\sqrt{T} + \gamma)(c - \theta\gamma)b_T(\varphi)$  and  $e_T(\varphi) = c_T(\varphi) + (\varphi - \theta) \det \Gamma_T(\varphi)$ . Consequently, we obtain from (A.3) and (A.4) that

$$(A.7) \quad \lim_{T \rightarrow \infty} \frac{1}{T} H_T(a, b, c) = \frac{1}{2} \left( \frac{a^2}{\varphi - \theta} + \frac{(c - \theta\gamma)^2}{\varphi^2} \right).$$

Finally, we deduce from (A.5) together with (A.6) and (A.7) that

$$(A.8) \quad \lim_{T \rightarrow \infty} \Lambda_T(a, b, c) = -\frac{1}{2}(\theta + \varphi + \gamma^2) + \frac{1}{2} \left( \frac{a^2}{\varphi - \theta} \right) + \frac{1}{2} \left( \frac{c - \theta\gamma}{\varphi} \right)^2,$$

which is exactly what we wanted to prove.  $\square$

## APPENDIX B: PROOF OF LEMMA 2.2.

Our goal is to establish the full asymptotic expansion for the normalized cumulant generating function  $\mathcal{L}_T$  of the random variable  $\mathcal{Z}_T(a, b)$ . First of all, as in the proof of Lemma 2.1, it follows from Girsanov's formula associated with (1.1) that

$$\begin{aligned} \mathcal{L}_T(a, b) &= \frac{1}{T} \log \mathbb{E} \left[ \exp(\mathcal{Z}_T(a, b)) \right], \\ &= \frac{1}{T} \log \mathbb{E}_{\varphi, \delta} \left[ \exp \left( (a + \theta - \varphi) \int_0^T X_t dX_t + \frac{1}{2} (2b - \theta^2 + \varphi^2) \int_0^T X_t^2 dt - \xi_T \right) \right] \end{aligned}$$

where

$$\begin{aligned} \xi_T &= aX_T \bar{X}_T + \zeta_T + bT(\bar{X}_T)^2, \\ \zeta_T &= (\theta\gamma - \varphi\delta)T\bar{X}_T - (\gamma - \delta)X_T + \frac{1}{2}(\gamma^2 - \delta^2)T. \end{aligned}$$

Consequently, if we assume that  $\theta^2 - 2b > 0$  and if we choose  $\varphi = \sqrt{\theta^2 - 2b}$  and  $\delta = 0$ ,  $\mathcal{L}_T(a, b)$  reduces to

$$\mathcal{L}_T(a, b) = \frac{\tau - \gamma^2}{2} + \frac{1}{T} \log \mathbb{E}_{\varphi, 0} \left[ \exp \left( -\frac{1}{2} V_T' J_T V_T + \gamma U_T' V_T \right) \right]$$

with  $\tau = \varphi - (a + \theta)$ , where the vectors  $U_T$  and  $V_T$  are given by

$$U_T = \begin{pmatrix} 1 \\ -\theta T \end{pmatrix} \quad \text{and} \quad V_T = \begin{pmatrix} X_T \\ \bar{X}_T \end{pmatrix}$$

and  $J_T$  is the diagonal matrix of order two

$$J_T = \begin{pmatrix} \tau & a \\ a & 2bT \end{pmatrix}.$$

We already saw in Appendix A that under the new probability  $\mathbb{P}_{\varphi, 0}$ ,  $V_T$  is Gaussian random vector with zero mean and covariance matrix  $\Gamma_T(\varphi)$  given by (A.2). Let  $M_T(a, b)$  be the square matrix of order two

$$M_T(a, b) = \mathbf{I}_2 + J_T \Gamma_T(\varphi) = \begin{pmatrix} 1 + \tau a_T(\varphi) + ab_T(\varphi) & \tau b_T(\varphi) + ac_T(\varphi) \\ 2bTb_T(\varphi) + aa_T(\varphi) & 1 + 2bTc_T(\varphi) + ab_T(\varphi) \end{pmatrix}.$$

It is not hard to see that

$$(B.1) \quad \det(M_T(a, b)) = 1 + 2ab_T(\varphi) + 2bTc_T(\varphi) + \tau a_T(\varphi) + (2\tau T b - a^2) \det \Gamma_T(\varphi).$$

Hence, we deduce from (A.4) that

$$(B.2) \quad \lim_{T \rightarrow \infty} \frac{\det M_T(a, b)}{e^{2\varphi T}} = \frac{\tau\theta^2}{2\varphi^3}.$$

Consequently, as soon as  $\tau > 0$ , we find from (B.2) that for  $T$  large enough, the matrix  $M_T(a, b)$  is positive definite. Therefore, it follows from standard Gaussian calculations that

$$(B.3) \quad \mathcal{L}_T(a, b) = \frac{\tau - \gamma^2}{2} - \frac{1}{2T} \log(\det M_T(a, b)) + \frac{\gamma^2}{2T} U_T' \Gamma_T(\varphi) M_T^{-1}(a, b) U_T.$$

We are now in position to improve convergence (B.2) as follows.

**Lemma B.4.** *We have for  $T$  large enough*

$$(B.4) \quad \frac{1}{2T} \log(\det M_T(a, b)) = \varphi + \frac{1}{2T} \log\left(\frac{\tau\theta^2}{2\varphi^3}\right) + \frac{1}{2T} \log(1 + \mathcal{K}_T(a, b)) + r_T(a, b)$$

where  $\mathcal{K}_T(a, b)$  is given by

$$\mathcal{K}_T(a, b) = \frac{1}{\tau\varphi\theta^2} \left( \frac{1}{T} (2b\varphi + 2a\varphi^2 - a^2\varphi - 4\tau b) + \frac{2a^2}{T^2} \right),$$

and  $r_T(a, b)$  is such that

$$|r_T(a, b)| \leq P(a, b, \varphi, T) e^{-\varphi T}$$

with  $P$  a rational function.

**Proof of Lemma B.4.** We already saw from (B.1) that

$$\det(M_T(a, b)) = 1 + \Delta_T(a, b) + (2\tau T b - a^2) \det \Gamma_T(\varphi).$$

where  $\Delta_T(a, b) = 2ab_T(\varphi) + 2bT(\varphi)c_T(\varphi) + \tau a_T(\varphi)$ . It follows from straightforward calculations that

$$\det \Gamma_T(\varphi) = \frac{(T\varphi - 2)e^{2\varphi T} + 4e^{\varphi T} - (T\varphi + 2)}{2\varphi^4 T^2}.$$

In addition, we deduce from the definition of  $a_T(\varphi)$ ,  $b_T(\varphi)$  and  $c_T(\varphi)$  that

$$\Delta_T(a, b) = \left( \frac{\tau}{2\varphi} + \frac{a\varphi + b}{\varphi^3 T} \right) e^{2\varphi T} - \frac{2a\varphi + 4b}{\varphi^3 T} e^{\varphi T} + \frac{2a\varphi + 6b + 4bT\varphi - \tau\varphi^2 T}{2\varphi^3 T}.$$

Consequently,

$$\begin{aligned} \det(M_T(a, b)) &= \left( \frac{\tau(\varphi^2 + 2b)}{2\varphi^3} + \frac{2b\varphi + 2a\varphi^2 - 4\tau b - a^2\varphi}{2\varphi^4 T} + \frac{2a^2}{2\varphi^4 T^2} \right) e^{2\varphi T} \\ &\quad + P_1(a, b, \varphi, T) e^{\varphi T} + P_2(a, b, \varphi, T) \end{aligned}$$

where  $P_1$  and  $P_2$  are rational functions, which achieves the proof of Lemma B.4.  $\square$

Finally, the following lemma gives us an expansion for the last term in (B.3).



**Lemma B.5.** *We have for  $T$  large enough*

$$(B.5) \quad \frac{\gamma^2}{2T} U_T' \Gamma_T(\varphi) M_T^{-1}(a, b) U_T = \frac{\gamma^2}{2} - \frac{1}{2T} \frac{\gamma^2(\varphi + a + \theta)}{\theta^2} + \frac{1}{T} \mathcal{J}_T(a, b) + \rho_T(a, b),$$

where  $\mathcal{J}_T(a, b)$  is given by

$$\mathcal{J}_T(a, b) = T \frac{\gamma^2}{2} \left( \frac{1 + \frac{f}{T} + \frac{g}{T^2}}{1 + \frac{h}{T} + \frac{\ell}{T^2}} - 1 + \frac{\varphi + a + \theta}{\theta^2 T} \right),$$

with

$$f = \frac{2\theta a\varphi + 2a\theta^2 + 4\theta b}{\theta^2 \varphi T}, \quad g = \frac{-4(b + \theta a)}{\theta^2 \varphi T},$$

$$h = \frac{2a\theta^2 + 4b\theta - a^2\varphi - 2b\varphi}{\theta^2 \varphi T}, \quad \ell = \frac{2a^2}{\theta^2 \varphi T},$$

and  $\rho_T(a, b)$  is such that

$$|\rho_T(a, b)| \leq Q(a, b, \varphi, T) e^{-\varphi T}$$

with  $Q$  a rational function.

**Proof of Lemma B.5.** First of all, we have previously remark that for  $T$  large enough,  $\det(M_T(a, b)) > 0$ . Consequently,  $M_T^{-1}(a, b)$  is well defined. With very tedious but straightforward calculations, we obtain that

$$\frac{\gamma^2}{2T} U_T' \Gamma_T(\varphi) M_T^{-1}(a, b) U_T = \frac{\gamma^2}{2} \frac{N_T(a, b)}{D_T(a, b)}$$

with

$$N_T(a, b) = (\theta^2 \varphi T^2 + (2\theta a\varphi + 2a\theta^2 + 4\theta b)T - 4(b + \theta a)) e^{2\varphi T}$$

$$+ (-2\theta(\theta a + 2b)T + 8(b + \theta a)) e^{\varphi T} + Q_1(a, b, \varphi, T)$$

$$D_T(a, b) = (\theta^2 \varphi T^2 + (2a\theta^2 + 4b\theta - a^2\varphi - 2b\varphi)T + 2a^2) e^{2\varphi T}$$

$$+ (-4\theta(\theta a + 2b)T - 4a^2) e^{\varphi T} + Q_2(a, b, \varphi, T)$$

where  $Q_1$  and  $Q_2$  are polynomial functions, which clearly leads to (B.5).  $\square$

**Proof of Lemma 2.2.** Lemma 2.2 follows from the conjunction of (B.3), (B.4) and (B.5).  $\square$

### APPENDIX C: PROOF OF LEMMA 2.3.

First of all, one can observe that the Ornstein-Uhlenbeck process  $(X_T)$  with linear shift  $\gamma$  can be rewritten as  $X_T = Y_T + m_T$  where

$$m_T = \mathbb{E}[X_T] = -\frac{\gamma}{\theta}(1 - e^{\theta T})$$

and  $(Y_T)$  is the Ornstein-Uhlenbeck process without shift

$$Y_T = e^{\theta T} \int_0^T e^{-\theta t} dB_t.$$

By the same token,  $\bar{X}_T = \bar{Y}_T + \mu_T$  where

$$\mu_T = \mathbb{E}[\bar{X}_T] = -\frac{\gamma}{\theta} \left( 1 + \frac{1}{\theta T} (1 - e^{\theta T}) \right)$$

with

$$\bar{Y}_T = \frac{1}{T} \int_0^T Y_t dt.$$

It follows from (2.3) together with Itô's formula (5.12) that

$$(C.1) \quad \mathcal{Z}_T(a, b) = \frac{a}{2} X_T^2 - \frac{aT}{2} - a \bar{X}_T X_T + b \int_0^T X_t^2 dt - bT (\bar{X}_T)^2.$$

Consequently, we have the decomposition

$$\mathcal{Z}_T(a, b) = \mathcal{Z}_T^0(a, b) + \mathcal{Z}_T^1(a, b) + \mathcal{Z}_T^2(a, b),$$

where

$$\begin{aligned} \mathcal{Z}_T^0(a, b) &= \mathbb{E}[\mathcal{Z}_T(a, b)], \\ \mathcal{Z}_T^1(a, b) &= a(m_T - \mu_T)Y_T - am_T \bar{Y}_T + 2b \int_0^T (m_t - \mu_T)Y_t dt, \\ \mathcal{Z}_T^2(a, b) &= \frac{a}{2} \left( Y_T^2 - \mathbb{E}[Y_T^2] \right) - a \left( \bar{Y}_T Y_T - \mathbb{E}[\bar{Y}_T Y_T] \right) - bT \left( (\bar{Y}_T)^2 - \mathbb{E}[(\bar{Y}_T)^2] \right) \\ &\quad + b \left( \int_0^T Y_t^2 dt - \mathbb{E} \left[ \int_0^T Y_t^2 dt \right] \right). \end{aligned}$$

By using the same notations as in Chapters 2 and 6 of Janson [6], we clearly have

$$\mathcal{Z}_T^0(a, b) \in H^{:0:}, \quad \mathcal{Z}_T^1(a, b) \in H^{:1:}, \quad \mathcal{Z}_T^2(a, b) \in H^{:2:},$$

where  $H^{:n:}$  stands for the homogeneous chaos of order  $n$ . Hence, we deduce from Theorem 6.2 of [6] that

$$(C.2) \quad \mathcal{Z}_T(a, b) = \mathbb{E}[\mathcal{Z}_T(a, b)] + \sum_{k=1}^{\infty} \alpha_k^T (\varepsilon_k^2 - 1) + \beta_k^T \varepsilon_k$$

where  $(\varepsilon_k)$  are independent standard  $\mathcal{N}(0, 1)$  random variables. We also obtain from Theorem 6.2 of [6] that

$$(C.3) \quad \sum_{k=1}^{+\infty} (\beta_k^T)^2 = \mathbb{E} \left[ (\mathcal{Z}_T^1(a, b))^2 \right].$$

In addition, some rough estimates give us that the right-hand side of (C.3) is uniformly bounded by some constant  $B > 0$ , depending only on  $a$  and  $b$ . As a matter of fact, it exists some constant  $\zeta(a, b) > 0$  such that

$$(C.4) \quad \mathbb{E} \left[ (\mathcal{Z}_T^1(a, b))^2 \right] \leq \zeta(a, b) \left( \mathbb{E}[Y_T^2] + \mathbb{E}[(\bar{Y}_T)^2] + \mathbb{E}[\Delta_T^2] \right),$$

where

$$\begin{aligned}\mathbb{E}[Y_T^2] &= \int_0^T e^{2\theta(T-t)} dt \leq -\frac{1}{2\theta}, \\ \mathbb{E}[(\bar{Y}_T)^2] &\leq \frac{1}{T} \int_0^T \mathbb{E}[Y_t^2] ds \leq -\frac{1}{2\theta},\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[\Delta_T^2] &= \mathbb{E}\left[\left(\int_0^T (m_t - \mu_T) Y_t dt\right)^2\right] = \frac{\gamma^2}{\theta^2} \mathbb{E}\left[\left(\int_0^T \left(\frac{1 - e^{\theta T}}{\theta T} + e^{\theta t}\right) Y_t dt\right)^2\right], \\ &\leq \frac{2\gamma^2}{\theta^4} \mathbb{E}\left[\left(\frac{1}{T} \int_0^T |Y_t| dt\right)^2\right] + \frac{2\gamma^2}{\theta^2} \mathbb{E}\left[\left(\int_0^T e^{\theta t} |Y_t| dt\right)^2\right], \\ &\leq \frac{2\gamma^2}{\theta^4} \frac{1}{T} \int_0^T \mathbb{E}[Y_t^2] dt + \frac{2\gamma^2}{\theta^2} \int_0^T e^{\theta t} dt \int_0^T e^{\theta t} \mathbb{E}[Y_t^2] dt \leq \frac{-2\gamma^2}{\theta^5}.\end{aligned}$$

Therefore, we clearly obtain (2.9) from (C.3) and (C.4). It now remains to show that it exists some constant  $A > 0$  that do not depend on  $T$ , such that  $|\alpha_k^T| \leq A$  for all  $k \geq 1$ . Since  $\mathcal{D}_{\mathcal{L}}$  is an open set and the origin belongs to the interior of  $\mathcal{D}_{\mathcal{L}}$ , it exists  $\varepsilon > 0$  such that

$$\{(xa, xb) \in \mathcal{D}_{\mathcal{L}} / |x| < \varepsilon\} \subset \mathcal{D}_{\mathcal{L}}.$$

However, for all  $(a, b) \in \mathcal{D}_{\mathcal{L}}$  and for  $T$  large enough, we deduce from Lemma 2.2 that  $\exp(T\mathcal{L}_T(xa, xb)) = \mathbb{E}[\exp(x\mathcal{Z}_T(a, b))]$  is finite. It means that the Laplace transform of  $\mathcal{Z}_T(a, b)$  is well defined on  $[-\varepsilon, \varepsilon]$ . Hence, Theorem 6.2 of [6] ensures that the characteristic function of  $\mathcal{Z}_T(a, b)$  is analytic in the strip

$$\left\{z \in \mathbb{C} / |\operatorname{Im}z| < \frac{1}{2}(\max_{k \geq 1} |\alpha_k^T|)^{-1}\right\}.$$

So, we necessarily obtain that for  $T$  large enough,

$$\max_{k \geq 1} |\alpha_k^T| < A$$

with  $A = 1/2\varepsilon$ . Hereafter, the decomposition of  $\mathcal{L}_T(xa, xb)$  given in Lemma 2.3, directly follows from equation (6.7) in Theorem 6.2 of Janson [6]. Our goal is now to pass through the limit in  $\mathcal{L}_T(xa, xb)$ . If we choose  $x \in \mathbb{R}$  such that  $|x| \leq 1/4A$ , we have

$$(C.5) \quad \lim_{T \rightarrow \infty} \frac{1}{2T} \sum_{k=1}^{\infty} \frac{(x\beta_k^T)^2}{1 - 2x\alpha_k^T} \leq \lim_{T \rightarrow \infty} \frac{B}{16A^2T} = 0.$$

Moreover, we deduce from Lemma 2.2 that

$$(C.6) \quad \lim_{T \rightarrow \infty} \mathcal{L}_T(xa, xb) = \mathcal{L}(xa, xb) = -\frac{1}{2} \left( xa + \theta + \sqrt{\theta^2 - 2xb} \right).$$

Furthermore, we already saw in Section 5 that

$$\lim_{T \rightarrow \infty} \mathbb{E}[X_T^2] = \frac{\gamma^2}{\theta^2} - \frac{1}{2\theta} \quad \text{and} \quad \lim_{T \rightarrow \infty} \mathbb{E}[(\bar{X}_T)^2] = \frac{\gamma^2}{\theta^2},$$

which clearly implies

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[X_T^2] = 0, \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{E}[X_t^2] dt = \frac{\gamma^2}{\theta^2} - \frac{1}{2\theta}, \quad \lim_{T \rightarrow \infty} \mathbb{E}[X_T \bar{X}_T] = 0.$$

Then, we find from (C.1) that

$$(C.7) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E}[\mathcal{Z}_T(xa, xb)] = -\frac{x}{2} \left( a + \frac{b}{\theta} \right).$$

Finally, we obtain from the decomposition of  $\mathcal{L}_T(xa, xb)$  together with (C.5), (C.6), and (C.7) that

$$(C.8) \quad \begin{aligned} \lim_{T \rightarrow +\infty} \frac{1}{2T} \sum_{k=1}^{\infty} \left( \log(1 - 2x\alpha_k) + 2x\alpha_k^T \right) &= -\frac{xb}{2\theta} + \frac{1}{2} \left( \theta + \sqrt{\theta^2 - 2xb} \right), \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} f_x(bg(y)) dy, \end{aligned}$$

where the spectral density  $g$  is given by (2.7) and for all  $x \in \mathbb{R}$  such that  $|x| \leq 1/4A$ ,

$$f_x(y) = \frac{1}{2} \left( \log(1 - 2xy) + 2xy \right).$$

Hence, it follows from (C.8) together with the elementary Taylor expansion

$$\log(1 - z) + z = - \sum_{p=2}^{\infty} \frac{z^p}{p}$$

and classical complex analysis results that, for any integer  $p \geq 2$ ,

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{k=1}^{\infty} (\alpha_k^T)^p = \frac{1}{2\pi} \int_{\mathbb{R}} (bg(y))^p dy,$$

Therefore, we obtain the weak convergence (2.11) on the class of functions  $\mathcal{F}$  from the Stone-Weierstrass theorem, which completes the proof of Lemma 2.3.  $\square$

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