The s-cobordism theorem seen as a particular case of Latour's theorem

Carlos Moraga Ferrandiz

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Abstract

We show how Latour's theorem ([Lat94]) can be understood as a natural generalization of the s-cobordism theorem for cohomology classes $u \in H^1(M;\mathbb{R})$. The s-cobordism theorem becomes a special degenerate case when $u = 0$.

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1 The s-cobordism theorem: the exact case

Two connected, closed and oriented manifolds N_0^n, N_1^n are cobordant if there exists a compact oriented manifold W^{n+1} such that $\partial W^{n+1} = (-N_0^n) \sqcup (N_1^n)$. Superscripts denote dimension while $(-N)$ represents the manifold N with reversed orientation. Such a triad $(W; N_0, N_1)$ is said to be an h-cobordism if both inclusions $N_0 \xhookrightarrow{i_0} W \xhookleftarrow{i_1} N_1$ are homotopy equivalences.
Let π be the fundamental group of W; we denote by $\Lambda := \mathbb{Z}[\pi]$ its group ring. To each h-cobordism we can associate its torsion $\tau(W, N_0)$ which lives in the Whitehead group $\mathrm{Wh}(\pi) := \frac{K_1(\Lambda)}{\pm\pi}$ (see [Coh73] for a definition).

The s-cobordism theorem, which can be found in [Ker65], states that $\tau(W, N_0) = 0$ is a sufficient condition[1] for W being diffeomorphic to $N_0 \times [0,1]$, provided $n \geq 5$.

We can reformulate this theorem into a statement about non triviality of some functional space: consider $\mathcal{F}$ the space of $\mathcal{C}^\infty$-functions $f : W \to [0,1]$ such that $f^{-1}(i) = N_i, i = 0, 1$ with the $\mathcal{C}^\infty-$topology. Its subspace $\mathcal{E}$ consisting of functions without critical points is non-empty if and only if $W \cong N_0 \times [0,1]$, as it suffices to pick some $f \in \mathcal{E}$ and to integrate the vector field $\frac{\nabla f}{\|\nabla f\|}$ relative to some Riemannian metric on W in order to find a diffeomorphism from $N_0 \times [0,1]$ to W. We obtain so:

Theorem 1 (*Functional formulation of the s-cobordism theorem*)**.** *Let $n \geq 5$,*

$$\mathcal{E} \neq \varnothing \quad \Leftrightarrow \quad \tau(W, N_0) = 0$$

□

[1] Trivially, the condition $\tau(W, N_0) = 0$ is also necessary for $W \cong N_0 \times [0,1]$.

Remark 2. The relative homology $H_*(W, N_0)$ vanishes since i_0 is a homotopy equivalence. This is indeed a necessary condition for $\mathcal{E} \neq \varnothing$, since the Morse complex $C_*(f)$ of a Morse function $f \in \mathcal{E}$ is zero in every degree, and the homology of $C_*(f)$ is isomorphic to $H_*(W, N_0)$ (see [Mil63]).
By Lefschetz duality, we deduce that $H^1(W, N_1) \approx H_n(W, N_0)$ vanishes. The same holds for $H^1(W, N_0) \approx H_n(W, N_1)$ by using the fact that i_1 is also a homotopy equivalence.

We are going to consider the s-cobordism theorem and the one from Latour as statements about the relative cohomologies $H^1(X; Y)$ and $H^1(X; Z)$ of a triad $(X; Y, Z)$. Since the only relative cohomology class of degree 1 to consider in the case of an h-cobordism is $u = 0$, we will talk about *the exact case* to refer to the context of this section.

2 The theorem of Latour

Consider now a closed manifold M^{n+1}. We ask M to fiber over the circle $\mathbb{S}^1$, which is equivalent by Tischler's theorem [Tis70] to the existence of a non-singular closed 1-form on M.

We say that a cohomology class is non-singular if it is representable by a non-singular closed 1-form. It is clear that there is no chance for $u = 0 \in H^1(M; \mathbb{R})$ to be non-singular since M is closed. Latour's theorem characterizes degree one de Rham cohomology classes $0 \neq u$ that are non-singular. Within the context of this section, here is the statement:

Theorem 3 ([Lat94]). *Let $n \geq 5$, and let Ω^u_{NS} denote the space of non-singular closed 1-forms representing u. We have:*

$$\Omega^u_{NS} \neq \varnothing \quad \Leftrightarrow \quad \begin{cases} H_*(M, -u) = 0, \\ \tau(-u) = 0, \\ u \text{ and } -u \text{ are stable.} \end{cases}$$

□

Notice that a $\alpha \in \Omega^u_{NS}$ determines a whole ray $ru = [r\alpha], r \in \mathbb{R}^*$ of non-singular cohomology classes. These form so a cone into $H^1(M; \mathbb{R})$. In particular $\Omega^u_{NS} \neq \varnothing \Leftrightarrow \Omega^{-u}_{NS} \neq \varnothing$.

A degree one cohomology class can be seen as a morphism $u : \pi \to \mathbb{R}$ just by integrating representatives of loops in M. The Novikov ring associated to u, denoted by Λ_u, is a completion of the group ring Λ. Elements of Λ_u are formal sums $\lambda := \sum n_i g_i, n_i \in \mathbb{Z}$ such that, for every fixed $C \in \mathbb{R}$, there are only finitely many terms g_i verifying $u(g_i) < C$. The homology $H_*(M; -u)$ which appears in Latour's theorem is the Novikov homology, which was first constructed in [Nov81]. The Novikov complex is the free finite Λ_{-u}-module $\left(\mathcal{N}^{-u}_* := \Lambda_{-u} \otimes_\Lambda S_*(\widetilde{M}), \partial_*\right)$, where $S_*(\widetilde{M})$ denotes the simplicial/cellular chain complex of the universal cover of M associated to a given triangulation/cell structure on M.

Remark 4. Is important to notice that Latour's theorem, which is a property of the cohomology class u, is stated in terms related to Λ_{-u}-modules.

The second right-side condition of theorem 3 contains indeed the first: in order to define the torsion $\tau(-u)$, we need the Novikov complex to be acyclic. In this case, $\tau(-u)$ is defined as follows: by setting a base of $\mathcal{N}^{-u}_*$, we obtain a contraction $\delta_* : \mathcal{N}^{-u}_* \to \mathcal{N}^{-u}_{*+1}$ as in [Mau67, §4].

The map $(\partial+\delta)_* : \mathcal{N}_{\mathrm{ev}}^{-u} \to \mathcal{N}_{\mathrm{odd}}^{-u}$ is then an isomorphism and we can consider S, the class in $K_1(\Lambda_{-u})$ of its associated matrix in the fixed basis. This class may depend on the choice of the basis (compare to [Mil66, §7]); in order to remove this indeterminacy, Latour defined the Whitehead group associated to $-u$ as $\mathrm{Wh}(-u) := \frac{K_1(\Lambda_{-u})}{T_{-u}}$, where the class $[S]$ depends only on $-u$. Here, $T_{-u} := \pm\pi \cdot \big(1 + \big(u < 0\big)\big) \subset \Lambda_{-u}^{\times}$ is the subgroup of the so-called trivial units. The torsion $\tau(-u)$ is defined by $[S] \in \mathrm{Wh}(-u)$.

An explanation about the stability condition of $\pm u$ is postponed to subsection 4.1.

As he pointed out in his introduction, Latour's strategy to prove theorem 3 is similar to that of the s-cobordism theorem; the goal of the present paper is to show that Latour's theorem is indeed a natural generalization of s-cobordism theorem for relative cohomology classes.

3 A generalization framework

In Latour's theorem, the notion of u-stability is related with *unbounded* primitives of $p^*(u)$ where $p : \widehat{M} \to M$ is the abelian cover of u having $\pi_1(\widehat{M})$ equal to $\ker(u)$. If we try to extend this notion to a null class $u = 0$, the cover coincides with $\mathrm{Id} : M \to M$ and we have no unbounded primitives of 0.
However, we only want to extend the notion of u-stability for null classes of the relative 1-cohomology of an h-cobordism. We replace so the notion of h-cobordism in the most trivially possible way in order to have unbounded primitives in the exact context when $u = 0 \in H^1(W, N_0) \cup H^1(W, N_1)$:

Definition 5. *From any h-cobordism $(W; N_0, N_1)$, we construct the triad $(W_\pm; N_-, N_+)$ by setting:*

- $N_- := N_0 \times (-\infty, 0], N_+ := N_1 \times [1, \infty)$ *and*
- $W_\pm := N_- \underset{\mathrm{Id}_{N_0}}{\coprod} W \underset{\mathrm{Id}_{N_1}}{\coprod} N_+$.

We call $(W_\pm; N_-, N_+)$ the extended triad *of $(W; N_0, N_1)$.*

In particular the cohomologies of an h-cobordism and of its extended triad are the same and W is trivial if and only if $W_\pm$ is diffeomorphic to $N_0 \times \mathbb{R}$. We can so state the s-cobordism theorem in terms of extended triads.

Remark 6. Of course, the extended triad is not strictly an h-cobordism since $W_\pm$ has no boundary, but the inclusion $i : (W; N_0, N_1) \hookrightarrow (W_\pm; N_-, N_+)$ is nevertheless a simple homotopy equivalence: any cell of, say N_-, is of the form $\Delta \times \mathbb{R}^-$ where Δ is a cell of N_0 and we have a natural collapse $c : N_- \to N_0$.

4 Comparison of the two theorems

Let us study how Latour's conditions relative to $u \in H^1(M; \mathbb{R}) \setminus \{0\}$ of closed manifolds M degenerate to the s-cobordism theorem condition for extended triads of h-cobordisms $(W; N_0, N_1)$

as in section 3.

Firstly, regard the closed manifold M as the triad $(M;\varnothing_-,\varnothing_+)$ and the cohomology class as living in $u \in H^1(M;\mathbb{R}) = H^1(M,\varnothing_-;\mathbb{R})$. Latour's conditions applied to $-u$ should be regarded as a statement about $-u \in H^1(W,\varnothing_+;\mathbb{R})$ since in this case, the associated Novikov complex is constructed using Λ_u-modules instead of Λ_{-u}-modules.

Secondly, consider the h-cobordism replaced by its extended triad $(W_\pm;N_-,N_+)$ as in definition 5. We distinguish the null-elements of the relative cohomologies by setting $H^1(W_\pm,N_-) = \{+0\}$ and $H^1(W_\pm,N_+) = \{-0\}$.

Now we study what happens to Latour's conditions when they are interpreted relatively to the extended triad $(W_\pm;N_-,N_+)$ for $u = +0 \in H^1(W_\pm,N_-)$:

- The Novikov homology $H_*\big((W_\pm,N_-),-0\big)$ is computed from the complex $\mathcal{N}_*^{-0}$. This complex is $\Lambda_{-0}\otimes_\Lambda \mathcal{S}_*(\widetilde{W_\pm},\widetilde{N_-})$ by definition, but the ring Λ_{-0} trivially coincides with the group ring Λ and hence the Novikov complex $\mathcal{N}_*^{-0}$ is nothing but $\mathcal{S}_*(\widetilde{W_\pm},\widetilde{N_-})$. So $H_*\big((W_\pm,N_-),-0\big) = H_*(\widetilde{W_\pm},\widetilde{N_-})$ which is isomorphic to $H_*(\widetilde{W},\widetilde{N_0}) = 0$ since both pairs are homotopy equivalent.
  The first condition of Latour is so trivially true for h-cobordisms as we have noticed on remark 2.

- Since the set of trivial units $T_{-0} = \pm\pi$, the group $\mathrm{Wh}(-0)$ defined by Latour reduces to the usual Whitehead group $\mathrm{Wh}(\pi)$. The torsion $\tau(-0)$ is $\tau(W_\pm,N_-)$, since $\mathcal{N}_*^{-0} = \mathcal{S}_*(\widetilde{W_\pm},\widetilde{N_-})$. But the latter torsion coincides with the Milnor torsion $\tau(W,N_0)$ since the pairs $(W_\pm,N_-)$ and (W,N_0) are simply homotopy equivalent by remark 6.
  The condition $\tau(-0) = 0$ of Latour is so the equivalent condition of theorem 1 for an h-cobordism to be trivial.

Remark 7. The corresponding statements about $u = -0 \in H^1(W_\pm,N_+)$ yield the vanishing of the relative homology $H_*(\widetilde{W},\widetilde{N_1})$ and associated torsion $\tau(W,N_1)$, which is an equivalent formulation of the s-cobordism theorem.

Note that the previous observations do not need the notion of extended triad and can be applied to the h-cobordism $(W;N_0,N_1)$ directly. We have established so far that the first two conditions of Latour's theorem reduce to theorem 1 when applied to an h-cobordism or to its extended triad. We need so to prove that the third condition relative to stability holds trivially when reducing to $u = \pm 0$. This will be proved below in proposition 9, where the convenience of the concept of extended triad will become more apparent.

4.1 The stability condition

To prove his theorem, Latour showed that every Morse closed 1-form α representing u gives raise to a complex $C_*(\alpha)$ of Λ_{-u}-modules which is simply equivalent to the Novikov complex $\mathcal{N}_*^{-u}$. The two first conditions that we have analized allow one to proceed as in the s-cobordism theorem in order to recurrently eliminate zeros of index/coindex i by eventually adding zeroes of index/coindex $i+2$, apart from the case $i = 2$ which is special. Adding $\pm u$-stability, Latour

obtained a sufficient condition to handle with this special case (compare with [Dam00]). Since critical points of index/coindex 2 do not represent a natural obstruction in the exact case, ± 0-stability should hold trivially. Let us recall what u-stability means, as in [Lat94, §5.4]:

Consider $p : \widehat{M} \to M$ the covering whose fundamental group is $\ker u$. Its transformation group is $\frac{\pi}{\ker u} \approx \mathbb{Z}^{\mathrm{irr}(u)}$. Since the class $p^*(u)$ vanishes, any closed 1-form α representing u admits a *primitive*: a function $\widehat{f} : \widehat{M} \to \mathbb{R}$ verifying $d\widehat{f} = p^*(\alpha)$ and $\widehat{f}(g \cdot x) = u(g) + \widehat{f}(x)$ for every pair (g, x) in $\mathbb{Z}^{\mathrm{irr}(u)} \times \widehat{M}$. It is easy to see that for every $t \in \mathbb{R}$, $\widehat{f}^{-1}\big([t, \infty)\big)$ has only one connected component where $\widehat{f}$ is not bounded; denote it by $\widehat{M}_t$. The inclusions $\big(\widehat{M}_s \overset{i_t^s}{\hookrightarrow} \widehat{M}_t\big)_{s>t}$ induce a projective system $\mathcal{P}(u) := \Big(\pi_1(\widehat{M}_t)\Big)_{t\in\mathbb{R}}$.
Latour showed that this system does not depend on the choice of $\widehat{f}$ but only on u, up to projective isomorphism (see [Lat94, Lemme 5.7]). The u-stability is a condition about $\mathcal{P}(u)$.

Definition 8. *A cohomology class $u \in H^1(M; \mathbb{R})$ is* stable *if there exists an increasing sequence $(t_n)_{n\in\mathbb{N}} \to \infty$ where the restrictions to the images of $\mathcal{P}(u)$ are isomorphisms. More precisely, if we set $I_n := \mathrm{Im}\big((\pi_1)_*(i_{t_n}^{t_{n+1}})\big)$ and $j_n := (\pi_1)_*(i_{t_n}^{t_{n+1}})|_{I_{n+1}}$, then $j_n : I_{n+1} \to I_n$ are isomorphisms for every $n \in \mathbb{N}$.*

The next proposition shows how u-stability reduces to a condition which holds trivially for extended triads of h-cobordisms.

Proposition 9. *The extended triad $(W_\pm; N_-, N_+)$ of any h-cobordism is ± 0-stable.*

Proof. Let us deal with (-0)-stability. Here $\{-0\} = H^1(W_\pm, N_+)$. In this situation $\ker(-0)$ is identified with the whole $\pi_1(W_\pm, N_+)$, and the covering pair $(\widehat{W_\pm}, \widehat{N_+})$ to consider coincides with the pair $(W_\pm, N_+)$ itself. By relative de Rham theory (see [BT82, Ch.1,§6] for example), the class -0 is represented by the pair $(df, f|_{N^+})$ with $f : W_\pm \to \mathbb{R}$. We are free to choose f verifying $f(x, t) = t$ for every $(x, t) \in N_+ \cup N_-$; since $W \hookrightarrow W_\pm$ is compact, there exists some $1 \leq t_0 \in \mathbb{R}$ such that for every $t \geq t_0$, the unique unbounded component W_t of $f^{-1}\big([t, \infty)\big)$ equals $N_1 \times [t, \infty)$. The projective system $\pi_1(W_t)$ is constantly $\pi_1(N_1)$ with inclusions inducing the identity if $t \geq t_0$. By choosing any increasing sequence (t_n) starting at t_0, stability for the class -0 holds. □

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GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO
3-8-1 KOMABA MEGURO-KU, TOKYO 153-8914, JAPAN.
Telephone: (+81) (0)3-5465-8292
E-mail address: carlos@ms.u-tokyo.ac.jp