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# Scalable and Quasi-Contractive Markov Coupling of Maxwell Collisions

Mathias Rousset\*, †

December 8, 2013

## Abstract

This paper considers space homogenous Boltzmann kinetic equations in dimension  $d$  with Maxwell molecules (without Grad's cut-off). An explicit Markov coupling of the associated conservative (Nanbu)  $N$ -particle system is constructed, using plain parallel coupling of isotropic random walks on the sphere of two-body collisional directions. The resulting coupling distance is almost surely decreasing, and the  $L_2$ -coupling creation is computed explicitly. Some quasi-contractive and uniform in  $N$  coupling / coupling creation inequalities are then proved, relying on  $> 2$ -moments of velocity distributions. The latter are based on an original sharp inequality, which bound from above the coupling distance of two centered and normalized random variables  $(U, V)$  in  $\mathbb{R}^d$ , with the average square parallelogram area spanned by  $(U - U_*, V - V_*)$ ,  $(U_*, V_*)$  denoting an independent copy. Two counter-examples proving the necessity of the dependence on  $> 2$ -moments and the impossibility of strict contractivity are provided. The paper is self-contained and uses only elementary tools.

## Contents

<b>1</b>	<b>Summary and contents</b>	<b>2</b>
1.1	Summary . . . . .	2
1.2	Contents . . . . .	5
<b>2</b>	<b>Context and results</b>	<b>6</b>
2.1	Context . . . . .	6
2.2	Markov coupling and particle systems . . . . .	7
2.3	Coupled collisions . . . . .	9
2.4	Kinetic equations . . . . .	11
2.5	Results . . . . .	12
2.6	Two counter-examples . . . . .	14

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<b>3</b>	<b>Standard notation and facts</b>	<b>15</b>
3.1	Scattering . . . . .	15
3.2	Isotropic random walk on sphere . . . . .	16
3.3	Two-body random collisions . . . . .	16
3.4	Two-body mean-field (Nanbu) particle systems . . . . .	17
<b>4</b>	<b>Coupled collisions</b>	<b>18</b>
4.1	Spherical coupling . . . . .	18
4.2	Contractivity of spherical couplings . . . . .	19
4.3	Coupled Nanbu particle systems . . . . .	21
4.4	Proof of Corollary (2.9) and (2.10) . . . . .	22
<b>5</b>	<b>Coupling / coupling creation inequalities</b>	<b>22</b>
5.1	The special inequality . . . . .	23
5.2	Proof of coupling creation estimates . . . . .	26

# 1 Summary and contents

## 1.1 Summary

This paper considers space homogenous Boltzmann kinetic equations in dimension  $d$ , with conservative (two-body) collisions of Maxwell type. Emphasis is made on the associated conservative (or Nanbu)  $N$ -particle system; a Markov process denoted (up to particle permutation)

$$t \mapsto V_t^N = (V_{t,(1)}, \dots, V_{t,(N)}) \in (\mathbb{R}^d)^N, \quad (1)$$

and satisfying the following conservation laws (for any  $t \geq 0$ ):

$$\begin{cases} \langle V_t^N \rangle_N = 0 & a.s., \\ \langle |V_t^N|^2 \rangle_N = 1 & a.s. \end{cases} \quad (2)$$

In the above,  $\langle \cdot \rangle_N$  denotes the averaging over particles. The latter is moreover reversible with respect to the invariant probability distribution

$$\text{Unif}_{0,1}^N \stackrel{\text{def}}{=} \text{Unif} \left\{ v \in (\mathbb{R}^d)^N \mid (2) \text{ holds} \right\}$$

The Markov dynamics of (1) is specified by two-body random collisions of (Levy) jump type, satisfying conservation of momentum and kinetic energy (particles have identical masses). By definition of Maxwell collisions, the rate of the two-body collisions is constant; and by energy and momentum conservation, a random two-body collision is necessarily an isotropic random step on the euclidean sphere of possible collisional directions. The *collisional direction* is the direction of the velocity difference of a particle pair. The angular size ( $\theta \in [0, \pi]$ ) of the step is the so-called *scattering (or deviation) angle*. The *angular collision kernel*  $b(d\theta)$  is a positive Levy measure on  $[0, \pi]$  generating the random steps of the scattering angle, with Levy condition

$$\lambda \stackrel{\text{def}}{=} \int_{[0,\pi]} \sin^2 \theta b(d\theta) < +\infty. \quad (3)$$

It is thus possible to construct an explicit Markov coupling (*i.e.* a probabilistic coupling of Markov processes which is itself again Markov) denoted

$$t \mapsto (U_t^N, V_t^N) \equiv (U_{t,(1)}, V_{t,(1)}, \dots, U_{t,(N)}, V_{t,(N)}) \in (\mathbb{R}^d \times \mathbb{R}^d)^N, \quad (4)$$

such that both  $U_t^N \in (\mathbb{R}^d)^N$  and  $V_t^N \in (\mathbb{R}^d)^N$  satisfy the conservation laws (2), and such that collisions are coupled using the following set of rules:

- (i) Collision times and collisional particles are the same.
- (ii) Scattering angles are the same.
- (iii) The isotropic random step on the collisional direction is coupled using plain parallel transport (in the geometric sense), with no reflexion. The sphere being a strictly positively curved manifold, the latter coupling is bound to be almost surely decreasing, in the sense that for any initial condition and  $0 \leq t \leq t + h$

$$\left\langle |U_{t+h}^N - V_{t+h}^N|^2 \right\rangle_N \leq \left\langle |U_t^N - V_t^N|^2 \right\rangle_N \quad a.s.. \quad (5)$$

The goal of this paper is to study the quasi-contractivity of the latter coupling, uniformly in the number of particles  $N$ . We first compute the coupling creation by computing the time derivative of the average coupling distance

$$\frac{d}{dt} \mathbb{E} \left\langle |U_t^N - V_t^N|^2 \right\rangle_N = -\mathbb{E} \langle \mathcal{C}(U_t^N, V_t^N, U_{*,t}^N, V_{*,t}^N) \rangle_N \leq 0, \quad (6)$$

where in the above we use the following notation

$$\langle o(U^N, V^N; U_*^N, V_*^N) \rangle_N \stackrel{\text{def}}{=} \frac{1}{N^2} \sum_{n_1, n_2=1}^N o(U_{(n_1)}^N, V_{(n_1)}^N, U_{(n_2)}^N, V_{(n_2)}^N)$$

to account for averages over particles of a two-body observable  $o : (\mathbb{R}^d \times \mathbb{R}^d)^2 \rightarrow \mathbb{R}$ . It turns out that the two-body ‘‘coupling creation’’ functional in the right hand side of (6) is given by the following alignment between the velocity difference  $v - v_* \in \mathbb{R}^d$ , and its coupled counterpart  $u - u_* \in \mathbb{R}^d$ :

$$\mathcal{C}(u, v, u_*, v_*) = \lambda \frac{c_{d-1}}{c_{d-3}} |u - u_*| |v - v_*| - (u - u_*) \cdot (v - v_*) \geq 0. \quad (7)$$

In the above  $c_d = \int_0^{\pi/2} \sin^d(\varphi) d\varphi$  denotes the  $d$ 'th Wallis integral, and  $\lambda > 0$  is the variance of the scattering angle kernel, as defined in (3).

In order to relate the coupling and the coupling creation, we have introduced an original general sharp inequality (see (28) below) that holds for any couple of centered and normalized random variables in  $\mathbb{R}^d$ . In the present context, it takes the following form

$$f \left( \left\langle |u^N - v^N|^2 \right\rangle_N \right) \leq \kappa \left( \langle u^N \otimes u^N \rangle_N \right) \left\langle |u^N - u_*^N|^2 |v^N - v_*^N|^2 - ((u^N - u_*^N) \cdot (v^N - v_*^N))^2 \right\rangle_N, \quad (8)$$

for any vectors  $u^N \in (\mathbb{R}^d)^N$  and  $v^N \in (\mathbb{R}^d)^N$  both satisfying the conservation laws (2). In the above, the condition number

$$\kappa(S) \stackrel{\text{def}}{=} (1 - \|S\|)^{-1} \quad (9)$$

is a function of the spectral radius  $\|S\| \leq 1$  of a positive trace 1 symmetric matrix, and

$$\begin{aligned} f &: [0, 4] \rightarrow [0, 1] \\ x &\mapsto x - \frac{x^2}{4}, \end{aligned} \quad (10)$$

is a positive concave function satisfying  $f(x) \underset{x \rightarrow 0}{\sim} x$  that will be used throughout the paper. The equality case in (8) is satisfied under the following sufficient two conditions:

1. Co-linearity of  $\frac{u_{(n)}}{|u_{(n)}|}$  and  $\frac{v_{(n)}}{|v_{(n)}|}$  for all  $1 \leq n \leq N$ ;
2. Equality of co-variances  $\langle u^N \otimes u^N \rangle_N = \langle v^N \otimes v^N \rangle_N$ .

It is then of interest to compare that the alignment functional in the right hand side of (8) (which is a sharp upper bound of the square coupling distance), and the coupling creation functional (7). They differ by a weight of the form  $|u - u_*| |v - v_*|$  which forbids any strong ‘‘coupling/coupling creation’’ inequality of the form

$$\frac{\mathbb{E} f \left( \left\langle |U^N - V^N|^2 \right\rangle_N \right)}{\mathbb{E} \langle \mathcal{C}(U^N, V^N, U_*^N, V_*^N) \rangle_N} \leq r < +\infty$$

for some universal constant  $r > 0$  independant of the distribution of the pair  $(U^N, V^N) \in (\mathbb{R}^d \times \mathbb{R}^d)^N$  both satisfying the conservation laws (2). However, a direct Hölder inequality yields some weak ‘‘coupling/coupling creation inequalities’’ (see Section 2.5) of the form

$$\frac{\mathbb{E} f \left( \left\langle |U^N - V^N|^2 \right\rangle_N \right)}{\left( \mathbb{E} \langle \mathcal{C}(U^N, V^N, U_*^N, V_*^N) \rangle_N \right)^{\frac{\alpha}{1+\alpha}}} \leq r_{\alpha, \text{Law}(U^N), \text{Law}(V^N)} < +\infty, \quad (11)$$

for some  $\alpha \in ]0, +\infty[$ . The inequality (11) can be interpreted as a power law estimates of the trend to equilibrium of order  $\alpha$  (*i.e.* with speed  $\underset{t \rightarrow +\infty}{\sim} t^{-\alpha}$ ); and as expected,  $r_{\alpha, \text{Law}(U^N), \text{Law}(V^N)}$  depends only on finite moments of order  $2 + \alpha$  of the velocity distributions.

Such results are similar to the classical results ([13, 8, 4, 9, 31]) that are obtained with entropy methods using ‘‘entropy/entropy creation’’ inequalities. Yet, the coupling method has some noticeable specificities:

**Maxwell restriction** The analysis is restricted to Maxwell collisions.

**Angular condition** The analysis is independent of the scattering angular distribution of collisions.

**Particle system size** The analysis is independent of the particle system size  $N$ . It works similarly for the kinetic equation.

**A priori estimates** The analysis depends on higher  $> 2$  moments of velocity distributions, and not on regularity estimates. Such estimates are available for the kinetic equation, but unfortunately, up to our knowledge, not directly for the  $N$ -particle system in the spirit of [21]. However, they can be obtained (indirectly and at least in principle) by pull-back, by using uniform in time large  $N$  propagation of chaos, as proven in [24].

**Constants** Constants are simple and explicit.

**Sharpness** The weak “coupling/coupling creation inequalities” are derived via Hölder inequality from the key sharp inequality (8).

The latter method then yields some lower bound estimates on the contraction rate (of the particle system distribution, or of the kinetic equation) with respect to the  $L^2$ -Wasserstein metric. In the case of the kinetic equation with Maxwell collisions, the latter contraction has been shown to be strictly positive (thus implying uniqueness) in the classical paper by Tanaka [29]. In a sense, the analysis in the present paper makes Tanaka’s argument quantitative.

We also suggest some negative results in the form of two counterexamples to stronger versions of “coupling/coupling creation inequalities”. These are the counterpart of counterexamples to Cercignani’s conjecture ([5, 4, 33]) in the entropy context.

- (i) Velocity distributions with sufficiently heavy tails can make the coupling creation vanish. This first counterexample shows that any “coupling/coupling contraction inequality” must involve some higher order (say,  $> 2$ ) velocity distribution moments.
- (ii) There exists a continuous perturbation of the identity coupling at equilibrium for which however the coupling creation is sub-linearly smaller than the coupling itself. This second type of counterexample shows that even with some reasonable moment or coupling domain restrictions, a sub-exponential trend is unavoidable.

## 1.2 Contents

In Section 2.1, we summarize the literature related to the present work. In Section 2.2, we recall some notation and basic concepts related to probabilistic couplings for Markov particle systems. In Section 2.3, we define the random collisions coupling of interest. In Section 2.4, we detail the associated kinetic equations. In Section 2.5, the main results of the present paper are stated. In Section 2.6, the two counter-examples showing the necessity of a sub-exponential trend and of finite higher moments are presented and proved.

In 3 and sub-sections there in, standard facts on conservative random collisions and associated particle systems are recalled.

In 4.1, an explicit (*i.e.* coordinate) formulation of spherical couplings is detailed. In 4.2, the contractivity of spherical couplings is computed. In (4.3), the coupling creation of the coupled particle system is computed. In 4.4, the proof of the quasi-contractivity in Wasserstein distance of the kinetic equation and its associated conservative particle system is detailed.

In 5.1, the special inequality between coupling distance and colinearity of coupled pairs is proven. In (5.2), the quasi-contractive coupling / coupling creation estimate is deduced.

## 2 Context and results

### 2.1 Context

The mathematical literature on the space homogenous Boltzmann kinetic equations (and related models) is extremely vast, especially in the case of Maxwell collisions, and we refer to the classical reviews [14, 32]. In the same way, the use of explicit coupling methods to study the trend to equilibrium of Markov processes (or Markov chains) is now a classical topic on its own, especially for discrete models (see *e.g.* [22]). It is also a well-established topic for continuous models, as well as for non-linear partial differential equations that have an interpretation in terms of Markovian particles. Let us mention some classical papers more closely related to the present study, with a sample of more recent references.

**Well-posedness** For Maxwell molecules, well-posedness of the kinetic equation as a probability flow using a probabilistic coupling was initiated in [29], with a sophisticated probabilistic technology. In [27, 30], an ad hoc spectral theory (Fourier) enabled to simplify the argument, and to push forward the theory. A recent example of the use of the coupling method to non-Maxwell molecules is available in [20].

**Moments** Moments production and moments propagation for the Boltzmann kinetic equation is a well established phenomenon. For Maxwell molecules, some explicit calculations are possible ([21]), and finite vs. infinite moments strictly propagates. In general, higher moments estimates typically rely on the so-called Pozner inequality ([34, 3, 9]). It is still a field of study ([23, 1]), with emphasis on moments production without any regularity theory. However, we are not aware of such results directly carried out on the associated  $N$ -particle systems.

**Trend to equilibrium (i)** As said before, most of the results on trend to equilibrium for Boltzmann kinetic equations, rely on “entropy / entropy creation” analysis ([13, 8, 4, 9, 31]), which try to circumvent the breakdown of Cercignani’s conjecture (counterexamples: [5, 4, 33]). It is of interest to interpret the latter counterexamples with the Markovian viewpoint: they amount of the impossibility of obtaining a uniform in  $N$  log-Sobolev (in the sense of entropy/entropy creation) inequality of the associated  $N$ -particle systems. For Maxwell collisions, or Kac’s caricature, special simplifications enables to push forward the trend estimations, for instance using Wild’s expansion ([12]), or central limit theorem to obtain sharp exponential rates ([16, 17]).

**Trend to equilibrium (ii)** For other models of non-linear partial differential equations, a Markov coupling can give exponential trend to equilibrium, by using a “strong coupling/coupling creation inequality” (see for instance [6, 7], for non-linear Fokker-Planck models with convexity assumption on potentials). However, the latter cases differ from the case of Boltzmann collisions by their exponential behavior. Indeed, and loosely speaking, in the following classical chain of implications

$$\begin{aligned} \text{Contractive Markov coupling} &\Rightarrow \text{Wasserstein contractivity} \\ &\Rightarrow \text{Bakry-Emery criterion} \Rightarrow \text{Log-Sobolev inequality} , \end{aligned}$$

the underlying constants are in fact uniform in  $N$ . From the breakdown of Cercignani’s conjecture, this cannot be true for Boltzmann collision processes.

**Trend to equilibrium (iii)** Direct probabilistic methods studying the trend of equilibrium of the Boltzmann-Maxwell  $N$ -particle system (or Kac’s caricature) have been undertaken [15, 10, 11, 26]. The main striking feature of the latter list is the difficulty to achieve the so-called “Kac’s program” ([24]), by obtaining a scalable (in  $N$ ) analysis of the trend to equilibrium of the particle system. In [10, 11], the analysis computes exact spectral gaps, which are not  $N$ -scalable measures of trends to equilibrium. In [26], an explicit coupling method is used but the estimate focus on the contraction constant which is not  $N$ -scalable.

**Propagation of chaos** In [24] have reversed the latter point of view, and proved the trend to equilibrium of the  $N$ -particle system by pulling-back the long time stability of the kinetic equation with uniform in time propagation of chaos. Moreover, the use of coupling methods in this context have been undertaken in [19] to obtain sharper chaos propagation rates, at the price of time stability. These are nonetheless fairly indirect and impressively technical viewpoints.

In the present work, the whole analysis itself is independent of  $N$ , and the study is fully elementary, the only advanced tool being the final Hölder inequality. This generality and simplicity is one of the main motivation of the present work.

## 2.2 Markov coupling and particle systems

Let us now make a brief summary of some basic concepts related to coupling methods for Markov particle systems. Consider an exchangeable (particle permutation symmetric) and coupled  $N$ -particle system of the form (4). Strictly speaking, the state space is obtained by quotienting  $(\mathbb{R}^d \times \mathbb{R}^d)^N / \text{Sym}_N$  with particle permutations.

We also assume that the latter is a *two-body interaction* system, so that its Markov generator has the usual structure:

$$\mathcal{L}_c^N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n,m=1}^N L_c^{(n,m)} (= N \times \langle L_c \rangle_N), \quad (12)$$

where  $L_c$  is a Markov generator with state space  $(\mathbb{R}^d \times \mathbb{R}^d)^2$ , the superscript  $(n, m)$  denotes the action on the corresponding pair of particles, and the particle averaging  $\langle \cdot \rangle_N$  have been obviously extended to operators.

We say that the latter process is a (time homogenous) *Markov coupling* if the following two conditions hold:

- (i) The marginal distribution of the two processes  $t \mapsto U_t^N \in (\mathbb{R}^d)^N$  and  $t \mapsto V_t^N \in (\mathbb{R}^d)^N$  are Markov with the same generator of the form

$$\mathcal{L}^N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{n,m=1}^N L^{(n,m)} (= N \times \langle L \rangle_N). \quad (13)$$

- (ii) If  $U_0^N = V_0^N$  *a.s.* then  $U_t^N = V_t^N$  *a.s.* for any  $t \geq 0$ .



Formally, the first point is direct consequence of the fact that

$$L_c(\varphi \otimes \mathbf{1}) = L_c(\mathbf{1} \otimes \varphi) = L(\varphi), \quad (14)$$

for any test function  $\varphi$  ranging in a domain of the generator  $\mathcal{L}$ .

Assuming that the coupling is almost surely increasing ((5) holds), the two-body interaction structure implies that  $L^2$  coupling creation as defined by (6) is given by a two-body positive coupling creation functional

$$\mathcal{C} : (\mathbb{R}^d \times \mathbb{R}^d)^2 \rightarrow \mathbb{R}^+$$

given by the general formula

$$\mathcal{C}(u, v, u_*, v_*) \stackrel{\text{def}}{=} -L^c \circ \left( (u, v, u_*, v_*) \mapsto |u - v|^2 + |u_* - v_*|^2 \right). \quad (15)$$

Next, for any coupled random variables  $(U^N, V^N) \in (\mathbb{R}^d \times \mathbb{R}^d)^N$ , and any  $\alpha > 0$ , we will consider weak “coupling / coupling creation” inequalities of the form (11). If the particle system has an invariant probability distribution, and if the constant in the right hand side of (11) is uniformly bounded in time and particle size  $N$ , a computable  $N$ -uniform trend to equilibrium of power law type  $\underset{t \rightarrow +\infty}{\sim} t^{-\alpha}$  can be obtained. If the latter holds for any large  $\alpha > 0$ , the coupling may be called  $N$ -uniform quasi-contractive.

Weak forms of contractivity in terms of Wasserstein metric can be obtained as a corollary. We can first introduce the “two-step” Wasserstein distance on exchangeable particle system.

**Definition 2.1.** Let  $U^N \in (\mathbb{R}^d)^N$  and  $V^N \in (\mathbb{R}^d)^N$  two exchangeable random vectors satisfying the conservation laws (2). Denote by

$$\eta_{U^N} \stackrel{\text{def}}{=} \frac{1}{N} \sum_n^N \delta_{U_{(n)}} \in \mathcal{P}(\mathbb{R}^d)$$

the associated empirical distributions. Then the two-step  $L^2$ -Wasserstein distance denoted

$$d_{\mathcal{W}_2^N}(\text{Law}(U^N), \text{Law}(V^N)) \stackrel{\text{def}}{=} d_{\mathcal{W}_2, (\mathcal{P}(\mathbb{R}^d), d_{\mathcal{W}_2})}(\text{Law}(\eta_{U^N}), \text{Law}(\eta_{V^N}))$$

between  $\text{Law}(U^N)$  and  $\text{Law}(V^N)$  is induced by the  $L^2$ -Wasserstein distance between  $\text{Law}(\eta_{U^N})$  and  $\text{Law}(\eta_{V^N})$  on the space of random probability distributions in  $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ , where  $\mathcal{P}(\mathbb{R}^d)$  is itself endowed with the euclidean  $L_2$ -Wasserstein distance. It is equivalent to the  $L_2$ -Wassertsein in the quotient space  $(\mathbb{R}^d)^N / \text{Sym}_N$  endowed with the quotient euclidean distance.

This yields the following general upper bound for any  $t_0 \geq 0$

$$\frac{d^+}{dt} d_{\mathcal{W}_2^N}^2(\text{Law}(U_t^N), \text{Law}(V_t^N)) \Big|_{t=t_0} \leq -\mathbb{E} \left\langle \mathcal{C} \left( \tilde{U}_{t_0}^N, \tilde{V}_{t_0}^N, \tilde{U}_{t_0,*}^N, \tilde{V}_{t_0,*}^N \right) \right\rangle_N. \quad (16)$$

In the above,  $(\tilde{U}^N, \tilde{V}^N) \in (\mathbb{R}^d \times \mathbb{R}^d)^N$  is a random variable representation of the optimal coupling and  $\frac{d^+}{dt}$  is the right derivative which exists in  $[0, +\infty]$  by monotony.

## 2.3 Coupled collisions

As usual, the post-collisional velocities of a particle pair are given by the collision mapping

$$\begin{cases} v' = \frac{1}{2}(v + v_*) + \frac{1}{2}|v - v_*|n'_v, \\ v'_* = \frac{1}{2}(v + v_*) - \frac{1}{2}|v - v_*|n'_v, \end{cases} \quad (17)$$

where  $(n_v, n'_v) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$  denote the collisional/post-collisional directions. A Maxwell collision process for two particles in  $\mathbb{R}^d \times \mathbb{R}^d$  is the Markov process constructed from the following Levy generator on  $\mathbb{R}^d \times \mathbb{R}^d$ :

$$L(\varphi)(v, v_*) \stackrel{\text{def}}{=} \int_{\mathbb{S}^{d-1} \times [0, \pi]} (\varphi(v', v'_*) - \varphi(v, v_*)) c_\theta(n_v, dn'_v) b(d\theta), \quad (18)$$

where in the above  $\varphi$  is a test function,  $b$  is an angular collisional kernel satisfying the Levy condition (3), and  $c_\theta$  is the isotropic probability transition on the collisional direction (the sphere  $\mathbb{S}^{d-1}$ ) with prescribed scattering (or deviation) angle  $\theta$ .

The 2-body coupled process is then obtained by (i) coupling the scattering angles  $\theta$ ; (ii) coupling the post-collisional directions  $n'_u$  and  $n'_v$ , using *parallel transport* between  $n_u$  and  $n_v$  on the sphere  $\mathbb{S}^{d-1}$  (no reflexion).

We state without proof (the reader may resort to a drawing here) two equivalent elementary descriptions of the parallel transport coupling on the sphere. We will call the latter *spherical coupling*.

**Definition 2.2.** *Let  $(n_u, n_v) \in (\mathbb{S}^{d-1})^2$  be given with condition  $n_u \neq -n_v$ . There is a unique rotation of  $\mathbb{S}^{d-1}$  denoted*

$$n'_u \mapsto n'_v = \text{Coupl}_{n_u, n_v}(n'_u) \in \mathbb{S}^{d-1},$$

*called spherical coupling, satisfying  $n'_u = n'_v$  if  $n_u = n_v$ , and equivalently defined as follows for  $n_u \neq n_v$ .*

- (i)  $n'_v$  is obtained from  $n'_u$  by performing the elementary rotation in  $\text{span}(n_u, n_v)$  bringing  $n_u$  to  $n_v$ .
- (ii) Denote by  $t_u$  a tangent vector of  $\mathbb{S}^{d-1}$  at base point  $n_u$  of a geodesic of length  $\theta$  bringing  $n_u$  to  $n'_u$ . Generate  $t_v$  from  $t_u$  by using parallel transport in  $\text{span}(n_u, n_v)$  from base point  $n_u$  to base point  $n_v$ . Generate  $n'_v$  as the endpoint of the geodesic of length  $\theta$  and tangent to  $t_v$  at base point  $n_v$ .

Moreover, it satisfies by construction the symmetry condition

$$\text{Coupl}_{n_v, n_u} = \text{Coupl}_{n_u, n_v}^{-1}. \quad (19)$$

It is necessary to keep in mind that the full mapping  $(n_u, n_v) \mapsto \text{Coupl}_{n_u, n_v}$  is smooth, but has a singularity on the extremity set  $\{n_u, n_v \in \mathbb{S}^{d-1} | n_u = -n_v\}$ . This fact has already been pointed out ([29, 20, 19]) in slightly different contexts, and causes difficulty in order to define uniquely regular Levy generators and kinetic equations with such couplings. However, we will avoid such technical issues by using Grad's cut-off (see (24), and we will consider coupled Levy generators only on the formal level.

Anyway, the resulting probability transition is then a symmetric Markov coupling (by the symmetry condition (19)).

**Definition 2.3.** A spherically coupled random collision with deviation angle  $\theta$  is defined by the following probability transition on coupled collisional directions (in  $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$ ):

$$c_{c,\theta}(n_u, n_v, dn'_u dn'_v) \stackrel{\text{def}}{=} \left( \mathbf{1}_{n_u \neq -n_v} \delta_{\text{Coupl}_{n_u, n_v}(n'_u)}(dn'_v) + \mathbf{1}_{n_u = -n_v} \delta_{\text{Coupl}_{n_u, \sigma}(n'_u)}(dn'_v) \text{Unif}_{\mathbb{S}^{d-1}}(d\sigma) \right) c_\theta(n_u, dn'_u), \quad (20)$$

**Lemma 2.4.** The spherically coupled probability transition (20) verifies the symmetry condition

$$c_{c,\theta}(n_u, n_v, dn'_u dn'_v) = c_{c,\theta}(n_v, n_u, dn'_v dn'_u). \quad (21)$$

It is thus a symmetric Markov coupling of  $c_\theta$ .

*Proof.* By construction,  $\text{Coupl}_{n_u, n_v}$  and  $\text{Coupl}_{n_u, \sigma}$  are isometries. On the other hand, by isotropy, for any isometry  $R$  and vector  $n_v \in \mathbb{S}^{d-1}$  we have  $R^{-1}c_\theta(Rn_v, \cdot) = c_\theta(n_v, \cdot)$ . Finally, the symmetry condition (19) yields (21).  $\square$

In Section REF, the following basic formula will be proved.

**Lemma 2.5.** The Markov coupling defined by  $c_{c,\theta}$  is almost surely contractive. Moreover, we have the average quadratic contraction estimate

$$\int_{\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}} |n'_u - n'_v|^2 c_{c,\theta}(n_u, n_v, dn'_u dn'_v) - |n_u - n_v|^2 = -\sin^2 \theta \frac{c_{d-1}}{c_{d-3}} |n_u - n_v|^2, \quad (22)$$

where in the above  $c_d = \int_0^{\pi/2} \sin^d(\varphi) d\varphi$  denotes the  $d$ 'th Wallis integral.

We can now consider a coupled Nanbu (two-body)  $N$ -particle system (see [25]), each particle being endowed with *two* velocities. It is a Markov process in state space  $(\mathbb{R}^d \times \mathbb{R}^d)^N$ , with generator of the form (12). The two-body Levy generator  $L_c$  is a coupled collision operator defined when acting on two particles by:

$$L_c(\psi)(u, v) \stackrel{\text{def}}{=} \int_{\mathbb{S}^{d-1} \times [0, \pi]} (\psi(u', v') - \psi(u, v)) c_{c,\theta}(n_u, n_v, dn'_u dn'_v) b(d\theta), \quad (23)$$

for any test function  $\psi$  on  $(\mathbb{R}^d \times \mathbb{R}^d)^2$ . The resulting particle system is by construction a formal Markov coupling of the usual Nanbu  $N$ -particle system process with two-body Levy generator  $L$  given in (18). We recall that such particle systems conserve total momentum and energy, and have the uniform probability on the sphere defined by these conservation laws as an invariant probability distribution.

In order to be rigorous (at least for uniqueness), let us recall that the latter process can be constructed using Grad's cut-off:

$$b_\varepsilon(d\theta) \stackrel{\text{def}}{=} \mathbf{1}_{\theta \geq \varepsilon} b(d\theta) \quad \bar{b}_\varepsilon \stackrel{\text{def}}{=} \int_{[0, \pi]} b_\varepsilon(d\theta) < +\infty. \quad (24)$$

- (i) Each particle perform a collision with a fixed rate  $\bar{b}_\varepsilon$ , and with a uniformly randomly chosen other particle.

- (ii) The scattering angle of each collision is independently sampled according to the probability  $\frac{b_\varepsilon(d\theta)}{\bar{b}_\varepsilon}$ .
- (iii) The coupled random post-collisional directions  $(n'_u, n'_v)$  (with scattering angle prescribed by (ii)) are sampled using the coupled isotropic probability transition on sphere  $c_{c,\theta}$ .

The general case of Levy grazing collisions can then be considered as a the formal  $\varepsilon \rightarrow 0$  limit of the latter.

## 2.4 Kinetic equations

Consider a  $N$ -particle system. We say that propagation of chaos holds if the marginal distribution of  $k$ -particles ( $k$  being fixed) is converging (in law) to a product measure when  $N \rightarrow +\infty$ . Under this assumption, the limit of the *one body* distribution  $\mu_t \in \mathcal{P}(\mathbb{R}^d)$  of the particle system satisfies formally an evolution equation in closed form with a quadratic non-linearity given by:

$$\frac{d}{dt} \int_{\mathbb{R}^d} \varphi d\mu_t = \int_{\mathbb{R}^d \times \mathbb{R}^d} L(\varphi \otimes \mathbf{1}) d\mu_t \otimes \mu_t, \quad (25)$$

where in the above  $\varphi$  is a test function of  $\mathbb{R}^d$ . When  $L$  is the collision operator (18) with given kernel  $b$ , then the non-linear equation (25) is the Boltzmann equation in  $\mathbb{R}^d$  with Maxwell collision kernel  $b$ . The usual expression on the particle velocity density, denoted  $f_t(v)dv$ , is then:

$$\frac{d}{dt} f_t(v) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1} \times [0, \pi]} (f_t(v')f_t(v'_*) - f_t(v)f_t(v_*)) dv_* c_\theta(n_v, dn'_{v_*}) b(d\theta). \quad (26)$$

In the above, the collision mapping (17) is used implicitly, and detailed balance has been used to remove test functions.

In the same way for the case coupled case, one obtains an evolution equation on the one body coupled particle distribution in the form

$$\frac{d}{dt} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi \mu_{c,t}(dudv) = \int_{(\mathbb{R}^d \times \mathbb{R}^d)^2} L_c(\psi \otimes \mathbf{1}) \mu_{c,t}(dudv) \mu_{c,t}(du_*dv_*).$$

When the underlying generator is  $L_c$ , the coupled collision operator (23), the associated non-linear kinetic equation in measure form is then

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{2d}} \psi d\mu_{c,t} &= \int_{(\mathbb{R}^d \times \mathbb{R}^d)^2} \int_{[0, \pi] \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}} \\ &(\psi(u', v') - \psi(u, v)) c_{c,\theta}(n_u, n_v, dn'_u dn'_v) b(d\theta) \mu_{c,t}(dudv) \mu_{c,t}(du_*dv_*). \end{aligned} \quad (27)$$

wher  $\mu_{c,t} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$  is the one body distribution of the coupled velocities. In the case of Boltzmann collisions, there is no hope to obtain on the coupled density (the density of  $\mu_{c,t}$ ) an explicit simple expression similar to (26). Indeed, this would require some kind of detailed balance in product space  $\mathbb{R}^d \times \mathbb{R}^d$ , which is broken for contractive coupling.

Here again, the lack of smoothness of the mapping  $(n_u, n_v) \mapsto c_{c,\theta}(n_u, n_v; dn'_u, dn'_v)$  at the set  $\{n_u, n_v \in \mathbb{S}^{d-1} | n_u = -n_v\}$  is causing difficulties to obtain an appropriate Cauchy theory for (27). This difficulty is removed under Grad's cut-off (24) (using, say, total variation distance).

## 2.5 Results

We can now detail the results of the present paper.

We first compute the coupling creation functional (7) of the coupled Nanbu particle system as defined by (15)-(23). The fact that the average of this functional can be bounded from above in some way by the coupling distance itself is not obvious. Fortunately, we have found a remarkable general inequality which enables to do so.

**Proposition 2.6.** *Let  $(U, V) \in \mathbb{R}^d \times \mathbb{R}^d$  a couple of centered and normalized ( $\mathbb{E}|U|^2 = \mathbb{E}|V|^2 = 1$ ) random variables in euclidean space. Let  $(U_*, V_*) \in \mathbb{R}^d \times \mathbb{R}^d$  be an i.i.d. copy. We have:*

$$f\left(\mathbb{E}|U - V|^2\right) \leq \kappa(\mathbb{E}(U \otimes U))\mathbb{E}\left(|U - U_*|^2|V - V_*|^2 - ((U - U_*) \cdot (V - V_*))^2\right), \quad (28)$$

where in the above we have used (10)-(9) (i.e.  $f(x) = x - x^2/4$ , and  $\kappa(S) = (1 - \|S\|)^{-1} \geq 0$  is a condition number of a symmetric positive matrix  $S$  of trace 1 and maximal eigenvalue  $\|S\|$ ). Moreover, a sufficient condition for the equality case in (28) is given by the following isotropy and co-linear coupling conditions

(i)  $\frac{U}{|U|} = \frac{V}{|V|}$  a.s..

(ii)  $\mathbb{E}(U \otimes V) = \mathbb{E}(U \otimes U)$ .

It is now possible to apply the inequality (28) to coupled particle velocities  $(u, v) \in (\mathbb{R}^d \times \mathbb{R}^d)^N$  with null momentum  $\langle u \rangle_N = \langle v \rangle_N = 0$  and normalized energy  $\langle |u|^2 \rangle_N = \langle |v|^2 \rangle_N = 1$  on the probability space  $([1, N], \text{Unif})$  generated by averaging over particles. Using Hölder inequality, we obtain the following quasi-contractive estimate:

**Proposition 2.7.** *Let  $(U^N, V^N) \in (\mathbb{R}^d \times \mathbb{R}^d)^N$  be random variables satisfying the centering and normalization condition (conservation laws) (2). Let  $\alpha > 0$  and  $p_1, p_2 > 1$  with  $1/p_1 + 1/p_2 = 1$  be given. Then, we have the inequality:*

$$\frac{\mathbb{E}f\left(\langle |U^N - V^N|^2 \rangle_N\right)}{\left(\frac{1}{\lambda} \frac{c_{d-3}}{c_{d-1}} \mathbb{E} \langle \mathcal{C}(U^N, V^N, U_*^N, V_*^N) \rangle_N\right)^{\frac{\alpha}{1+\alpha}}} \leq \frac{d}{d-1} k_\alpha \tilde{m}_{U^N, p_1, \alpha}^{\frac{2+\alpha}{1+\alpha}} m_{V^N, p_2(2+\alpha)}^{\frac{2+\alpha}{1+\alpha}},$$

where in the above the right-hand side is defined by:

- The bounded above and below constant

$$k_\alpha \stackrel{\text{def}}{=} (1 + \alpha) \left(\frac{2}{2 + \alpha}\right)^{\frac{2+\alpha}{1+\alpha}} \left(\xrightarrow[\alpha \rightarrow 0 \text{ or } +\infty]{} 2 \text{ or } 1\right).$$

- The modified  $p_1(2 + \alpha)$ -moment

$$\tilde{m}_{U^N, p_1, \alpha} \stackrel{\text{def}}{=} \left(\frac{d-1}{d}\right)^{\frac{1+\alpha}{2+\alpha}} \mathbb{E}\left(\kappa^{p_1(1+\alpha)}\left(\mathbb{E}\langle U^N \otimes U^N \rangle_N\right) \left\langle |U^N - U_*^N|^{p_1(2+\alpha)} \right\rangle_N\right)^{1/p_1(2+\alpha)}.$$

- The  $p_1(2 + \alpha)$ -moment

$$m_{V^N, p_2(2+\alpha)} \stackrel{\text{def}}{=} \mathbb{E} \left( \left\langle |V^N - V_*^N|^{p_2(2+\alpha)} \right\rangle_N \right)^{1/p_2(2+\alpha)}.$$

Moreover, if  $U^N$  is uniformly distributed ( $\sim \text{Unif}_{0,1}^N$ ), one has  $\tilde{m}_{U^N, p_1, \alpha} < +\infty$  as soon as  $p_1(1 + \alpha) \leq (d - 1)(N - d - 1)/2$ , and moreover

$$\lim_{N \rightarrow +\infty} \tilde{m}_{U^N, p_1, \alpha} = \left( \mathbb{E} |N - N_*|^{p_1(2+\alpha)} \right)^{1/p_1(2+\alpha)} < +\infty.$$

where  $(N, N_*) \in \mathbb{R}^d \times \mathbb{R}^d \sim \mathcal{N}_{0,1} \times \mathcal{N}_{0,1}$  is a couple of i.i.d. centered and normalized ( $\mathbb{E} |N|^2 = 1$ ) normal random variables.

Proposition (2.7) is the main result of this paper, and immediately shows that the trend to equilibrium of a Nanbu  $N$ -particle system is controlled by the velocity moments of order  $2 + \alpha > 2$ . It is useful to state the counterpart of Proposition 2.7 for the formal limit  $N = +\infty$ .

**Proposition 2.8.** *Let  $(U, V) \in \mathbb{R}^d \times \mathbb{R}^d$  be two centered and normalized random vectors, and assume the isotropy condition:*

$$\mathbb{E}(U \otimes U) = \frac{1}{d} \text{Id}.$$

Let  $(U_*, V_*)$  be an i.i.d. copy. Denote  $f(x) = x - x^2/4$ . Then we have the inequality:

$$\frac{f \left( \mathbb{E} |U - V|^2 \right)}{\left( \mathbb{E} \langle |U - U_*| |V - V_*| - (U - U_*) \cdot (V - V_*) \rangle_N \right)^{\frac{\alpha}{1+\alpha}}} \leq k_\alpha \frac{d}{d-1} m_{U, p_1(2+\alpha)}^{\frac{2+\alpha}{1+\alpha}} m_{V, p_2(2+\alpha)}^{\frac{2+\alpha}{1+\alpha}},$$

where in the above the definitions (2.7)-(2.7) are used.

We then obtain consequences on the contraction in Wasserstein distance.

**Corollary 2.9.** *Any Nanbu particle system with Maxwell collisions  $t \mapsto V_t^N$  satisfies the following weak coupling - coupling creation inequality.*

$$\frac{d_{\mathcal{W}_2^N}^2(\text{Law}(V_t^N), \text{Unif}_{0,1}^N)}{\left( -\frac{1}{\lambda} \frac{c_{d-3}}{c_{d-1}} \frac{d^+}{dt} d_{\mathcal{W}_2^N}^2(\text{Law}(V_t^N), \text{Unif}_{0,1}^N) \right)^{\frac{\alpha}{1+\alpha}}} \leq 2 \frac{d}{d-1} k_\alpha \tilde{m}_{\text{Unif}_{0,1}^N, p_1, \alpha}^{\frac{2+\alpha}{1+\alpha}} m_{\text{Law}(V_t^N), p_2, \alpha}^{\frac{2+\alpha}{1+\alpha}}.$$

A similar result holds for the associated kinetic equation

**Corollary 2.10.** *The centered and normalized measure solution  $(\mu_t)_{t \geq 0} \in \mathcal{P}(\mathbb{R}^d)$  of the space homogenous kinetic equation with Maxwell collisions (26) satisfies the following weak coupling - coupling creation inequality. Denote  $\mathcal{N}_{0,1}$  the associate reduced normal (Maxwellian) distribution in  $\mathbb{R}^d$ . Then we have*

$$\frac{f \left( d_{\mathcal{W}_2}^2(\mu_t, \mathcal{N}_{0,1}) \right)}{\left( -\frac{1}{\lambda} \frac{c_{d-3}}{c_{d-1}} \frac{d^+}{dt} d_{\mathcal{W}_2}^2(\mu_t, \mathcal{N}_{0,1}) \right)^{\frac{\alpha}{1+\alpha}}} \leq \frac{d}{d-1} k_\alpha m_{\mathcal{N}(0,1), p_1(2+\alpha)}^{\frac{2+\alpha}{1+\alpha}} m_{\mu_t, p_2(2+\alpha)}^{\frac{2+\alpha}{1+\alpha}}.$$

## 2.6 Two counter-examples

Let us finally present two negative results, that demonstrates, in coupling- coupling creation inequalities, the necessity of sub-exponential power law estimates on the one hand, and the necessity to resort on higher moments of velocity distribution on the other hand. We give the counter-examples in the form lemmas, with proofs. In both cases, we consider a coupled distribution in the form of random variables

$$(U, V) \in \mathbb{R}^d \times \mathbb{R}^d, \quad U \sim \mathcal{N}(0, \frac{1}{d}\text{Id}). \quad (29)$$

which are centered and normalized, with normal distribution in the first variable.  $(U_*, V_*)$  is an i.i.d. copy.

**Lemma 2.11** (The necessity of  $> 2$ -moments). *Let (29) holds. Denote the order  $< 2$  moment*

$$m_q \stackrel{\text{def}}{=} \mathbb{E}(|V|^q)^{1/q}, \quad (30)$$

for some  $0 < q < 1$ . Then we have the following degeneracy of the coupling creation

$$\lim_{m_q \rightarrow 0} \mathbb{E}(|U - U_*| |V - V_*| - (U - U_*) \cdot (V - V_*)) = 0.$$

*Proof.* Condition (30) obviously implies via Hölder inequality that  $\lim_{m_q \rightarrow 0} \mathbb{E}(|V - U|^2) = 2 \neq 0$ . Moreover, we have

$$\begin{aligned} \mathbb{E}(|U - U_*| |V - V_*| - (U - U_*) \cdot (V - V_*)) &\leq 2\mathbb{E}(|U - U_*| |V - V_*|) \\ &\leq \mathbb{E}^{1/p}(|U - U_*|^p) \mathbb{E}^{1/q}(|V - V_*|^q) \xrightarrow{i \rightarrow +\infty} 0. \end{aligned}$$

□

**Lemma 2.12** (The necessity of sub-exponential rates). *Let (29) holds. Consider the isotropic and co-linear coupling*

$$\frac{V}{|V|} \stackrel{\text{def}}{=} \frac{U}{|U|} \quad \text{a.s.},$$

with moreover the following radial coupling perturbation on some interval  $0 < r_- < r_+ < +\infty$ :

$$|V| \stackrel{\text{def}}{=} |U| \mathbf{1}_{|U| \notin [r_-, r_+]} + \mathbb{E}^{1/2}(|U|^2 | |U| \in [r_-, r_+]) \mathbf{1}_{|U| \in [r_-, r_+]}$$

Then we have the following degeneracy of the coupling - coupling creation estimate

$$\lim_{r_- \rightarrow +\infty} \lim_{r_+ \rightarrow r_-} \frac{f\left(\mathbb{E}\left(|V - U|^2\right)\right)}{\mathbb{E}(|U - U_*| |V - V_*| - (U - U_*) \cdot (V - V_*))} = +\infty.$$

*Proof.* First, for such isotropic and co-linear couplings, the key inequality (28) is in fact an equality. Denoting:

$$A \stackrel{\text{def}}{=} 1 - \frac{((U - U_*) \cdot (V - V_*))}{|U - U_*| |V - V_*|},$$

we obtain easily (since  $U \cdot V \geq 0$  *a.s.*)

$$\mathcal{R}(r_-, r_+) \stackrel{\text{def}}{=} \frac{f\left(\mathbb{E}\left(|V - U|^2\right)\right)}{\mathbb{E}\left(|U - U_*||V - V_*| - (U - U_*) \cdot (V - V_*)\right)} \geq \frac{d}{d-1} \frac{\mathbb{E}\left((U - U_*)^2 (V - V_*)^2 A\right)}{\mathbb{E}\left((U - U_*) (V - V_*) A\right)}.$$

Since  $A = 0$  when both  $|U| \notin [r_-, r_+]$  and  $|U_*| \notin [r_-, r_+]$ , we can replace  $A$  with

$$A\left(\mathbb{1}_{|U| \in [r_-, r_+]} + \mathbb{1}_{|U_*| \in [r_-, r_+]}\right),$$

and the smoothness of Gaussian density yields

$$\lim_{r_+ \rightarrow r_-} \mathcal{R}(r_-, r_+) = \frac{d}{d-1} \frac{\mathbb{E}\left(|R_U - U_*|^2 |R_U - U_*|^2\right)}{\mathbb{E}\left(|R_U - U_*| |R_U - U_*|\right)},$$

where  $R_U$  is distributed uniformly on the sphere with radius  $r_-$ . Taking the limit  $r_- \rightarrow +\infty$  yields the result.  $\square$

### 3 Standard notation and facts

#### 3.1 Scattering

As usual, the velocities of a pair of collisional particles are denoted

$$(v, v_*) \in \mathbb{R}^d \times \mathbb{R}^d,$$

and the post-collisional quantities are denoted by adding the superscript  $'$ . All particles are assumed to have the same mass so that the conservation of momentum imposes

$$v' + v_*' = v + v_*,$$

and conservation of energy

$$|v'|^2 + |v_*'|^2 = |v|^2 + |v_*|^2.$$

As a consequence, the relative speed is also conserved

$$|v' - v_*'| = |v - v_*|,$$

and a collision can be fully described by only using the normalized velocity difference, called the *collisional direction*. It is denoted

$$n_v \stackrel{\text{def}}{=} \frac{v - v_*}{|v - v_*|} \in \mathbb{S}^{d-1},$$

and in the same way  $n_v'$  is called the *post-collisional direction*. The collisional and post-collisional directions being given, velocities are then determined by the standard involutive collision mapping (17) with inverse

$$\begin{cases} v = \frac{1}{2}(v' + v_*') + \frac{1}{2}|v' - v_*'| n_v, \\ v_* = \frac{1}{2}(v' + v_*') - \frac{1}{2}|v' - v_*'| n_v. \end{cases}$$



The scattering or deviation angle

$$\theta \in [0, \pi]$$

of the collision is then uniquely defined as the half-line angle between the collisional and the post-collisional direction:

$$\cos \theta \stackrel{\text{def}}{=} \frac{v' - v'_*}{|v' - v'_*|} \cdot \frac{v - v_*}{|v - v_*|} = n'_v \cdot n_v.$$

### 3.2 Isotropic random walk on sphere

Isotropy (or invariance by rotation, which is equivalent to the physical Galilean invariance in the case of velocity differences), does not impose any condition on the scattering angle  $\theta$ ; however, it requires other degrees of freedom to be uniformly distributed. This is made precise in the following definition.

**Definition 3.1.** *Let  $\theta \in [0, \pi]$  be given. The isotropic probability transition on the sphere  $\mathbb{S}^{d-1}$  with scattering angle  $\theta \in [0, \pi]$  is the unique probability transition  $c_\theta : \mathbb{S}^{d-1} \rightarrow \mathcal{P}(\mathbb{S}^{d-1})$  invariant under isometries, and generating states at a prescribed angular distance  $\theta$ . Formally*

$$c_\theta(n_v, dn'_v) \stackrel{\text{def}}{=} \text{Unif}_{\{n'_v \in \mathbb{S}^{d-1} \mid n_v \cdot n'_v = \cos \theta\}}(dn'_v), \quad (31)$$

or equivalently

$$c_\theta(n_v, \cdot) \stackrel{\text{Law}}{=} \text{rot}(\theta, n_v, \Sigma_1, \Sigma_2) n_v \quad \Sigma_1, \Sigma_2 \sim \text{Unif}_{\mathbb{S}^{d-1}} \otimes \text{Unif}_{\mathbb{S}^{d-1}}, \quad (32)$$

where in (32),  $\text{rot}(\theta, n_v, \sigma_1, \sigma_2)$  is the elementary<sup>1</sup> rotation of  $\mathbb{R}^d$  with rotation angle  $\theta$ , rotation plane  $\Pi_{n_v, \sigma_1} \stackrel{\text{def}}{=} \text{span}(n_v, \sigma_1)$ , and orientation prescribed by  $P_{\Pi_{n_v, \sigma_1}}(\sigma_1, \sigma_2)$  where  $P$  stands for orthogonal projection.

Clearly, invariance under isometries implies that the uniform distribution is an invariant distribution of  $c_\theta$ . The expression (32) is in fact useful to see why detailed balance (reversibility) hold; using the geometric inversion formula

$$n'_v = \text{rot}(\theta, n_v, \sigma_1, \sigma_2) n_v \Leftrightarrow n_v = \text{rot}(\theta, n_v, -\sigma_1, -\sigma_2) n'_v,$$

and the invariance of the uniform distribution under the parity transformation  $\sigma \mapsto -\sigma$ . This yields:

**Lemma 3.2.**  *$c_\theta$  is reversible with invariant probability the uniform distribution. Formally,*

$$dn_v c_\theta(n_v, dn'_v) = dn'_v c_\theta(n'_v, dn_v).$$

### 3.3 Two-body random collisions

The following definition of a random collision will be useful (its properties are directly inherited from those of the isotropic probability transition on the sphere):

---

<sup>1</sup>*i.e.* fixing the orthogonal of a plane

**Definition 3.3.** Consider two collisional particles with velocity pair in euclidean  $\mathbb{R}^d \times \mathbb{R}^d$ . We call a (two body) random collision with scattering angle  $\theta$ , the probability transition in  $\mathbb{R}^d \times \mathbb{R}^d$  induced by the collision mapping (17) and the isotropic probability transition on sphere  $c_\theta(n_v, dn'_v)$ . It satisfies:

- (i) Invariance (in law) under isometries of  $\mathbb{R}^d$  (applied simultaneously to each particle velocity).
- (ii) Almost surely conservation of the total energy and momentum (conservation laws).
- (iii) Reversibility with respect to uniform distributions. Formally,

$$dv dv_* c_\theta(n_v, dn'_v) = dv' dv'_* c_\theta(n'_v, dn_v),$$

where in the above the collision mappings (17)-(3.1) are implicitly used.

In the same way, we can consider continuous time collision (Levy) processes on  $\mathbb{R}^d \times \mathbb{R}^d$ , generated by a ( non-negative ) Levy measure:

$$b : \mathbb{R}_*^+ \rightarrow \mathcal{M}_+([0, \pi]),$$

usually called an *angular kernel*. Recall that Levy measures are measures with finite diffusive intensity  $\lambda$  as defined in (3). The latter generates a Markov (isotropic Levy) process on the sphere  $\mathbb{S}^{d-1}$  through the generator:

$$\int_{[0, \pi]} (\varphi(n'_v) - \varphi(n_v)) c_\theta(n_v, dn'_v) b(d\theta).$$

where in the above  $\varphi$  is a test function on  $\mathbb{S}^{d-1}$ . By extension using the collision mapping, we can define a *two-body collision process* on velocity pairs ( $\mathbb{R}^d \times \mathbb{R}^d$ ) with generator (18).

**Lemma 3.4.** The two-body collision Levy process with generator (18) satisfies the following properties:

- (i) Invariant (in law) under isometries of  $\mathbb{R}^d$  (applied simultaneously to velocity particles),
- (ii) Almost surely conservation of the total energy and momentum (conservation laws),
- (iii) Reversibility with respect to the uniform distribution.

**Remark 3.5.** In all the definitions above, Galilean invariance (rotation and translation for velocities) allow that  $b$  may depend on the system state  $(v, v_*)$  through the absolute collision speed  $|v - v_*|$  (a conserved quantity). Note that the fact that  $|v - v_*|$  is conserved is necessary to keep the reversibility properties. The case where  $b$  is in fact independent of  $|v - v_*|$  is exactly what is called a *Maxwell collision*.

### 3.4 Two-body mean-field (Nanbu) particle systems

When  $L$  is the collision process (18), the  $N$ -particle system generated by (13) is called the Nanbu particle system.

**Definition 3.6.** The particle system in  $(\mathbb{R}^d)^N$  generated by the generator (13), with Maxwell collision generator (18) is called a Nanbu particle system ([25]).

The tensorial structure of (13) implies:

**Lemma 3.7.** *If  $L$  is stationary (resp. reversible) with product invariant probability  $\mu \otimes \mu \in E \times E$ , then so is  $\mathcal{L}^N$  with product invariant probability  $\mu^{\otimes N} \in E^N$ .*

As a consequence, the Nanbu particle system satisfies the following properties, directly inherited from the two body case.

**Lemma 3.8.** *A Nanbu particle system satisfies*

- (i) *Invariance under isometries of  $\mathbb{R}^d$  (applied to each particle velocity).*
- (ii) *Almost surely conservation of total kinetic energy and momentum (conservation laws).*
- (iii) *Reversibility (detailed balance) with uniform invariant distribution.*

## 4 Coupled collisions

In this section we will consider a pair of collisional particles with coupled velocities

$$(u, v, u_*, v_*) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}.$$

A coupled collision can then be described by expressing the post-collisional velocities

$$(u', u'_*, v', v'_*) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$$

using coupled collision parameters. According to Section 3, it is sufficient in order to obtain the above coupling to express, using the *same* collision random parameters, the collision and post-collisional directions  $(n_u, n'_u, n_v, n'_v) \in (\mathbb{S}^{d-1})^2 \times (\mathbb{S}^{d-1})^2$ . This is *naturally* done by what we have called in the present work a *spherical coupling*.

### 4.1 Spherical coupling

We give a special description of the isotropic probability transition with scattering angle  $\theta$ .

**Lemma 4.1.** *Let  $\theta \in [0, \pi]$  be given, as well as  $(n_v, m_v)$  two orthonormal vectors in  $\mathbb{S}^{d-1}$ . Consider the spherical change of variable*

$$n'_v = \cos \theta n_v + \sin \theta \cos \varphi m_v + \sin \theta \sin \varphi l \in \mathbb{S}^{d-1} \tag{33}$$

where  $\varphi \in [0, \pi]$  is an azimuthal angle and  $l \in \mathbb{S}^{d-1}$  is such that  $(n_v, m_v, l)$  is an orthonormal triplet. Then the image by the transformation (33) of the probability distribution

$$\sin^{d-3} \varphi \frac{d\varphi}{c_{d-3}} \text{Unif}_{(n_v, m_v)^\perp \cap \mathbb{S}^{d-1}}(dl), \tag{34}$$

is the isotropic probability transition  $c_\theta(n_v, dn'_v)$  with initial state  $n_v$  and scattering angle  $\theta$  ( $c_{d-3}$  denotes the Wallis integral normalization). In particular, the latter does not depend on the choice of  $m_v$ .

*Proof.*  $c_\theta(n_v, dn'_v)$  is defined as the uniform distribution induced by the euclidean structure on the submanifold of  $\mathbb{S}^{d-1}$  defined by  $n'_v \cdot n_v = \cos \theta$ . Moreover the expression of volume elements in (hyper)spherical coordinates implies that for any  $m_v \in \mathbb{S}^{d-1}$ , the vector  $\cos \varphi m_v + \sin \varphi l \in \mathbb{S}^{d-1}$  is distributed (under (34)) uniformly in the  $d - 2$ -dimensional sphere  $n_v^\perp \cap \mathbb{S}^{d-1}$ . The result follows.  $\square$

Of course in the above, only the scattering angle  $\theta$  has an intrinsic physical meaning, the azimuthal angle  $\varphi$  being dependent of the arbitrary choice of the pair  $(m_v, l)$ . This leads to the core analysis of a spherical coupling.

**Lemma 4.2.** *Let  $(n_u, n_v) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$  be given. A pair  $(n'_u, n'_v) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$  is spherically coupled, in the sense that  $n'_v = \text{Coupl}(\sigma, n_u, n_v; n'_u)$  for some uniformly distributed  $\sigma \in \mathbb{S}^{d-1}$  (the spherical coupling mapping is defined in Definition 2.2) if and only if*

$$\begin{cases} n'_u = \cos \theta n_u + \sin \theta \cos \varphi m_u + \sin \theta \sin \varphi l, \\ n'_v = \cos \theta n_v + \sin \theta \cos \varphi m_v + \sin \theta \sin \varphi l, \end{cases} \quad (35)$$

where in the above  $(n_u, m_u, l)$  and  $(n_v, m_v, l)$  are both orthonormal sets of vectors such that  $(n_u, m_u)$  and  $(n_v, m_v)$  belong to the same plane  $\text{span}(n_u, n_v)$  (or  $\text{span}(n_u, \sigma)$  if  $|n_u \cdot n_v| = 1$ ) and have the same orientation with respect to  $l$ . Note that the pair  $(m_u, m_v)$  and the angle  $\varphi$  are defined uniquely up to a common involution (a change of sign of the vectors and the reflexion  $\varphi \rightarrow \pi - \varphi$ ).

*Proof.* Assume  $n_u \neq n_v$ . Denote by  $R_\theta$  the unique elementary rotation bringing  $n_u$  to  $n_v$ . By construction  $R_\theta m_u = m_v$ , and  $R_\theta l = l$   $\square$

This immediately implies that the coupled probability transition  $c_{c,\theta}(n_u, n_v; dn_u dn_v)$  is the image using the mapping (35) above of the uniform probability described in  $(\varphi, l)$ -variables in (34).

## 4.2 Contractivity of spherical couplings

We can now state the contractivity ("coupling creation") equation satisfied by spherical couplings.

**Lemma 4.3.** *Let  $(n_u, n_v) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$  be given, with a spherically coupled pair  $(n'_u, n'_v) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$  (Definition 2.2). Then the contractivity formula (22) holds.*

*Proof.* Next, we expand  $n'_u \cdot n'_v$  using (35) and obtain:

$$n'_u \cdot n'_v = (\cos \theta n_u + \sin \theta \cos \varphi m_u) \cdot (\cos \theta n_v + \sin \theta \cos \varphi m_v) + \sin^2 \theta \sin^2 \varphi.$$

Next by construction,  $(m_u, m_v)$  is obtained from a  $\frac{\pi}{2}$ -rotation of  $(n_u, n_v)$ , so that  $n_u \cdot n_v = m_u \cdot m_v$  and  $n_u \cdot m_v = -m_u \cdot n_v$  and

$$n'_u \cdot n'_v = (\cos^2 \theta + \sin^2 \theta \cos^2 \varphi) n_u \cdot n_v + \sin^2 \theta \sin^2 \varphi.$$

Using  $1 = \cos^2 \theta + \sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi$  we obtain

$$n'_u \cdot n'_v - n_u \cdot n_v = -\sin^2 \theta \sin^2 \varphi (n_u \cdot n_v - 1),$$

and the result follows.  $\square$

We can then compute the consequence on velocities.

**Lemma 4.4.** *Consider coupled collisional and post-collisional velocities  $(u, u_*, v, v_*) \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}$ , and a spherical coupling as defined by Definition 4.2. Then we have:*

$$\begin{aligned} & |u' - v'|^2 + |u'_* - v'_*|^2 - |u - v|^2 - |u_* - v_*|^2 = \\ & - \sin^2 \theta \sin^2 \varphi (|u - u_*| |v - v_*| - (u - u_*) \cdot (v - v_*)) \leq 0. \end{aligned} \quad (36)$$

The above quantity vanishes if and only if the coupled collision directions are aligned with the same orientation ( $n_u \cdot n_v = 1$ ). If moreover the coupled velocities  $(u, u_*)$  and  $(v, v_*)$  have the same total momentum and energy, then we have the contractivity equality:

$$|u' - v'|^2 + |u'_* - v'_*|^2 - |u - v|^2 - |u_* - v_*|^2 = -\sin^2 \theta \sin^2 \varphi \frac{1}{4} (|u - v|^2 + |u_* - v_*|^2). \quad (37)$$

*Proof.* We use the following change of variable:

$$\begin{cases} s_v \stackrel{\text{def}}{=} \frac{1}{2} (v + v_*) \\ d_v \stackrel{\text{def}}{=} \frac{1}{2} (v - v_*) \end{cases} \Leftrightarrow \begin{cases} v = s_v + d_v \\ v_* = s_v - d_v \end{cases}.$$

First remark that

$$\begin{aligned} |u - v|^2 + |u_* - v_*|^2 &= |s_u - s_v + d_u - d_v|^2 + |s_u - s_v - d_u + d_v|^2 \\ &= 2|s_u - s_v|^2 + 2|d_u - d_v|^2 \end{aligned} \quad (38)$$

Developing the left hand side of (36), and using the conservation laws ( $s' = s$  and  $|d'| = |d|$ ), we obtain

$$\begin{aligned} & |u' - v'|^2 + |u'_* - v'_*|^2 - |u - v|^2 - |u_* - v_*|^2 \\ &= 2|d'_u - d'_v|^2 - 2|d_u - d_v|^2 \\ &= -(u' - u'_*) \cdot (v' - v'_*) + (u - u_*) \cdot (v - v_*) \\ &= -|u - u_*| |v - v_*| (n'_u \cdot n'_v - n_u \cdot n_v) \\ &= \frac{1}{2} |u - u_*| |v - v_*| (|n'_u - n'_v|^2 - |n_u - n_v|^2). \end{aligned}$$

Then the contractivity formula (22) yields the first result. If  $(u, u_*)$  and  $(v, v_*)$  have the same momentum and energy, then (38) implies

$$\begin{aligned} |u - v|^2 + |u_* - v_*|^2 &= 0 + 2|d_u|^2 + 2|d_v|^2 - 4d_u \cdot d_v \\ &= 4|u - u_*| |v - v_*| (1 - n_u \cdot n_v), \end{aligned}$$

□

### 4.3 Coupled Nanbu particle systems

In this section, we detail properties of the coupled Nanbu particle system.

**Definition 4.5.** *A particle system in  $(\mathbb{R}^{2d} \times \mathbb{R}^{2d})^N$  with generator (13) associated to the two-body coupled Levy collision generator  $L_c$  in (23) is called a (spherically) coupled Nanbu particle system.*

Note that the possibility of constructing coupled Nanbu system relies on the use of Maxwell collisions. We indeed need a process where collision parameters are independent on the specific invariants (energy and momentum) of a particle pair. Again, uniqueness of such processes requires some further analysis, unless Grad's cut-off (24) is used.

**Lemma 4.6.** *Consider a Nanbu particle system with two-body generator defined in (23). Then, the resulting process is a coupling of the Nanbu particle system of Definition 3.6.*

*Proof.* (i) The fact that the marginal processes have the good distribution is a consequence of (21). (ii) The fact that an identity coupling ( $U_0^N = V_0^N$  a.s.) remains as such ( $U_t^N = V_t^N$  a.s.  $\forall t$ ) is implied by the decrease of spherical couplings.  $\square$

By construction of the spherical coupling, a generated coupled Nanbu system then satisfies the following expected properties.

**Lemma 4.7.** *A coupled Nanbu particle system denoted  $t \mapsto (U_t^N, V_t^N, U_{*,t}^N, V_{*,t}^N) \in (\mathbb{R}^d \times \mathbb{R}^d)^N$  satisfy the following properties.*

(i) *It is invariant under isometries of  $\mathbb{R}^d$  (applied simultaneously to each of the  $2 \times N$  velocities).*

(ii) *The coupling is almost surely decreasing (for all  $0 \leq t \leq t+h$ ):*

$$\left\langle |U_{t+h}^N - V_{t+h}^N|^2 \right\rangle_N \leq \left\langle |U_t^N - V_t^N|^2 \right\rangle_N \quad a.s..$$

**Remark 4.8.** In the special case  $N = 2$ , and if the coupled velocities have the same total momentum and kinetic energy, the system is exponentially contractive (from (37)) with explicit rate:

$$\frac{d}{dt} \ln \mathbb{E} \left( |U_t^N - V_t^N|^2 + |U_{*,t}^N - V_{*,t}^N|^2 \right) = -\frac{1}{4} \lambda \frac{c_{d-1}}{c_{d-3}} < 0.$$

It is of interest to remark that the exponential rate is in fact uniform in the dimension, since:

$$\lim_{d \rightarrow +\infty} \frac{c_{d-1}}{c_{d-3}} = 1.$$

We can finally compute the coupling creation functional for the coupled Nanbu particle system.

**Proposition 4.9.** *Consider the coupled Nanbu particle system in state space  $(\mathbb{R}^{2d})^N$ . Then the coupling creation functional is given by (7).*

*Proof.* The calculation uses the definition of the Nanbu particle generator, the definition of the coupling creation functional  $\mathcal{C}$ , and the two-body spherical coupling contractivity in Lemma (4.4). We have

$$\begin{aligned} \mathcal{C}(u, v, u_*, v_*) &= L_c \left( (u, v, u_*, v_*) \mapsto |u - v|^2 + |u_* - v_*|^2 \right) \\ &= - \int_{[0, \pi]^2} \sin^2 \theta \sin^2 \varphi b(d\theta) \frac{\sin^{d-3} \varphi d\varphi}{c_{d-3}} (|u - u_*| |v - v_*| - (u - u_*) \cdot (v - v_*)). \end{aligned}$$

□

#### 4.4 Proof of Corollary (2.9) and (2.10)

The two corollaries (2.9) and (2.10) can now be deduced from Proposition 2.7 and 2.8 respectively.

*Proof of Corollary (2.9).* Let  $t_0 \geq 0$  be given, and consider as an initial condition  $(\tilde{U}_t^N, \tilde{V}_t^N) \in (\mathbb{R}^d \times \mathbb{R}^d)^N$ , a random variable representation of the  $d_{\mathcal{W}_2^N}$ -optimal coupling between  $\text{Unif}_{0,1}^N$  and  $\text{Law}(V_t^N)$ . Consider next the solution  $h \mapsto (U_{\varepsilon, t+h}^N, V_{\varepsilon, t+h}^N)$  of the coupled Nanbu collision process with the latter initial condition, and subject to Grad's cut-off (24) with  $\varepsilon > 0$ . By construction, the latter satisfies

$$\begin{aligned} d_{\mathcal{W}_2}^2(\text{Law}(V_{\varepsilon, t+h}^N), \text{Unif}_{0,1}^N) - d_{\mathcal{W}_2}^2(\text{Law}(V_{\varepsilon, t}^N), \text{Unif}_{0,1}^N) \\ \leq - \int_t^{t+h} \mathbb{E} \langle \mathcal{C}(U_{\varepsilon, t+h'}^N, V_{\varepsilon, t+h'}^N, U_{*, \varepsilon, t+h'}^N, V_{*, \varepsilon, t+h'}^N) \rangle_N dh'. \end{aligned}$$

We can remark that for any pair  $(\mu, \nu) \in \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d)$  satisfying the conservation laws,  $d_{\mathcal{W}_2}^2(\mu, \nu) \leq 2$  (try the trivial product coupling). As a consequence, we obtain the general upper bound for random exchangeable particle systems satisfying the conservation laws:

$$d_{\mathcal{W}_2^N}^2(\text{Law}(V^N), \text{Law}(U^N)) \leq 2 \mathbb{E} f \left( \left\langle |V^N - U^N|^2 \right\rangle_N \right).$$

We can then combine the moment inequality in Proposition 2.7, with the continuity with respect to weak convergence of  $\varepsilon \rightarrow \text{Law}(V_{\varepsilon, t+h}^N)$  (this is a standard result of approximation of Levy processes by jump processes with bounded intensity, see [18]). Since the Wasserstein distance metrizes weak convergence and moments are continuous observables, the result follows. □

*Proof of Corollary (2.10).* The proof is similar to the one of Corollary (2.9), except that we need to justify the continuity of  $\varepsilon \rightarrow \mu_{\varepsilon, t}$  with respect to weak convergence, as well as uniform control of higher moments, where  $\mu_{\varepsilon, t}$  the solution of the kinetic equation with Grad's cut-off. The continuity can be found in Section 5 of [30]. The uniform control on higher moments is standard for Maxwell molecules, where explicit computations of the latter can be carried out (see the classical paper [21]). □

## 5 Coupling / coupling creation inequalities

In this section, the key inequalities between coupling and coupling creation are proven. The key point is to compare the *coupling  $L^2$ -distance*, with the *average parallelogram square area* spanned by the difference of two independent copies. This is the content of the first section.

## 5.1 The special inequality

Let  $(U, V)$  be two random vectors in a Euclidean space, and  $(U_*, V_*)$  an i.i.d. copy. We assume the latter are *centered*

$$\mathbb{E}(U) = \mathbb{E}(V) = 0. \quad (39)$$

If  $(Z_1, Z_2)$  are two centered random vectors, we will use the notation

$$C_{Z_1, Z_2} \stackrel{\text{def}}{=} \mathbb{E}(Z_1 \otimes Z_2).$$

The goal is to bound from above the *coupling distance*  $\mathbb{E}(|U - V|^2)$  with the following alignment average

$$\mathbb{E}\left(|U - U_*|^2 |V - V_*|^2 - ((U - U_*) \cdot (V - V_*))^2\right),$$

which is the average parallelogram area spanned by the two vector differences  $U - U_*$  and  $V - V_*$ .

The main computation is based on an expansion of the parallelogram area formula, and will involve four positive quantities. The first one is the average parallelogram area spanned by  $U$  and  $V$  alone, and quantify the degree of alignment of the coupling between the two vectors (*a* stands for *alignement*):

$$a_{U,V} \stackrel{\text{def}}{=} \mathbb{E}\left(|U|^2 |V|^2 - (U \cdot V)^2\right) \geq 0.$$

The second one quantify the asymmetry of the covariance between  $U$  and  $V$  (*s* stands for *symmetry*):

$$s_{U,V} \stackrel{\text{def}}{=} \text{Tr}((C_{U,V} - C_{V,U})(C_{V,U} - C_{U,V})) \geq 0.$$

The third and most important one is given by

$$d_{U,V} \stackrel{\text{def}}{=} \text{Tr}(C_{U,U}) \text{Tr}(C_{V,V}) - \text{Tr}(C_{U,V})^2 \geq 0,$$

and quantify the coupling distance (*d* stands for *distance*) if the two vectors have the same variance. Finally, the fourth one is defined by

$$\tilde{d}_{U,V} \stackrel{\text{def}}{=} \text{Tr}(C_{U,U}C_{V,V}) - \text{Tr}(C_{U,V}C_{V,U}) \geq 0,$$

and is related to  $d_{U,V}$  through a general trace inequality for positive definite symmetric matrices in even dimension. In the present context, the latter matrix is given by the full co-variance operator

$$C_{(U,V),(U,V)} = \begin{pmatrix} C_{U,U} & C_{U,V} \\ C_{V,U} & C_{V,V} \end{pmatrix},$$

and the trace inequality is detailed in the following lemma.

**Lemma 5.1.** *Let  $U$  and  $V$  be two centered random vectors. Then we have*

$$\tilde{d}_{U,V} \leq \frac{\|C_{U,U}\|}{\text{Tr}(C_{U,U})} d_{U,V},$$

where  $\|\cdot\|$  denotes the spectral radius (maximal eigenvalue) of a symmetric non-negative operator. Moreover, the equality case holds if (sufficient condition)  $C_{U,U}$  and  $C_{U,V}$  are co-linear.



*Proof.* In an orthonormal basis where  $C_{U,U}$  is diagonal, we have the expression, for  $i, j, k \in \llbracket 1, d \rrbracket$ :

$$\begin{aligned} \text{Tr}(C_{U,U}C_{V,V} - C_{U,V}C_{V,U}) &= \sum_i C_{U,U}^{i,i} C_{V,V}^{i,i} - \sum_{j,k} C_{U,V}^{j,k} C_{V,U}^{k,j}, \\ &= \sum_i \left( C_{U,U}^{i,i} C_{V,V}^{i,i} - (C_{U,V}^{i,i})^2 \right) - \underbrace{\sum_{j \neq k} (C_{U,V}^{j,k})^2}_{\leq 0}. \end{aligned}$$

Then, by definition of the maximal eigenvalue of  $C_{U,U}$ :

$$\sum_i C_{U,U}^{i,i} C_{V,V}^{i,i} - (C_{U,V}^{i,i})^2 \leq \|C_{U,U}\| \left( \text{Tr}(C_{V,V}) - \sum_i \frac{(C_{U,V}^{i,i})^2}{C_{U,U}^{i,i}} \right),$$

so that by Cauchy-Schwarz inequality

$$\left( \sum_i C_{U,V}^{i,i} \right)^2 \leq \left( \sum_i \frac{(C_{U,V}^{i,i})^2}{C_{U,U}^{i,i}} \right) \times \left( \sum_i C_{U,U}^{i,i} \right),$$

and we eventually get

$$\text{Tr}(C_{U,U}C_{V,V} - C_{U,V}C_{V,U}) \leq \frac{\|C_{U,U}\|}{\text{Tr}(C_{U,U})} \left( \text{Tr}(C_{V,V}) \text{Tr}(C_{U,U}) - \text{Tr}(C_{U,V})^2 \right).$$

Moreover the equality holds if and only if

- (i)  $(C_{U,V}^{j,k})^2 = 0 \quad \forall j \neq k$ ,
- (ii)  $\frac{C_{U,V}^{i,i}}{C_{U,U}^{i,i}}$  is independent of  $i$ ,

that is to say if and only if  $C_{U,U}$  and  $C_{U,V}$  are co-linear. The result follows.  $\square$

**Proposition 5.2.** *Let  $U$  and  $V$  be two centered and normalized (i.e. verifying (39)-(42)) random vector in a Euclidean space with scalar product  $|\cdot|$ . Let  $(U_*, V_*)$  be a i.i.d. copy. Then we have the following decomposition:*

$$\mathbb{E} \left( |U - U_*|^2 |V - V_*|^2 - ((U - U_*) \cdot (V - V_*))^2 \right) = 2a_{U,V} + s_{U,V} + 2 \left( d_{U,V} - \tilde{d}_{U,V} \right). \quad (40)$$

Moreover, we have the inequality

$$\begin{aligned} &\mathbb{E} \left( |U - U_*|^2 |V - V_*|^2 - ((U - U_*) \cdot (V - V_*))^2 \right) \\ &\geq 2 \left( 1 - \frac{\|C_{U,U}\|}{\text{Tr}(C_{U,U})} \right) d_{U,V} \geq 0, \end{aligned} \quad (41)$$

and the equality case holds if (sufficient condition)

(i) The coupling between  $U$  and  $V$  is co-linear

$$\frac{U}{|U|} = \frac{V}{|V|} \quad a.s.,$$

(ii)  $C_{U,V}$  and  $C_{U,U}$  are co-linear symmetric matrices.

Note that if  $C_{U,U} = \frac{1}{d}\text{Id}$ , then the latter condition is necessary.

*Proof.* Before computing terms, recall that if  $M, N$  are two square matrices, then

$$\text{Tr}(MN) = \text{Tr}(NM) = \text{Tr}(M^T N^T) = \text{Tr}(N^T M^T).$$

Let us expand the alignment functional (the left hand side of (40)). We have first,

$$\begin{aligned} \mathbb{E}\left(|U - U_*|^2 |V - V_*|^2\right) &= 2\mathbb{E}\left(|U|^2 |V|^2\right) + 2\mathbb{E}\left(|U|^2\right) \mathbb{E}\left(|V|^2\right) + 4\mathbb{E}(U \cdot U_* V \cdot V_*) + 8 \times 0 \\ &= 2\mathbb{E}\left(|U|^2 |V|^2\right) + 2\text{Tr}(C_{U,U}) \text{Tr}(C_{V,V}) + 4\text{Tr}(C_{U,V} C_{V,U}), \end{aligned}$$

and second,

$$\begin{aligned} \mathbb{E}\left(\left((U - U_*) \cdot (V - V_*)\right)^2\right) &= 2\mathbb{E}(U \cdot V^2) + 2\mathbb{E}(U \cdot V)^2 + 2\mathbb{E}(U \cdot V_*^2) + 2\mathbb{E}(U_* \cdot V U \cdot V_*) + 8 \times 0 \\ &= 2\mathbb{E}(U \cdot V^2) + 2\text{Tr}(C_{U,V})^2 + 2\text{Tr}(C_{U,U} C_{V,V}) + 2\text{Tr}(C_{U,V}^2). \end{aligned}$$

On the other hand,

$$\text{Tr}\left((C_{U,V} - C_{V,U})(C_{V,U} - C_{U,V})\right) = -2\text{Tr}(C_{U,V}^2) + 2\text{Tr}(C_{U,V} C_{V,U}),$$

and the result then follows.  $\square$

Two remarks.

**Remark 5.3.** In the following, the random vectors will be *normalized*

$$\mathbb{E}(|U|^2) = \mathbb{E}(|V|^2) = 1, \quad (42)$$

so that we have  $\text{Tr}(C_{U,U}) = \text{Tr}(C_{V,V}) = 1$ , and

$$d_{U,V} = \frac{1}{2} f\left(\mathbb{E}\left(|U - V|^2\right)\right),$$

with the function  $f(x) = x - \frac{x^2}{4} \underset{x \rightarrow 0}{\sim} x$  defined in (10).

**Remark 5.4.** Assuming that the normalization (42) above holds, and that  $C_{U,U} = \frac{1}{d}\text{Id}$  is isotropic, then (41) becomes

$$\begin{aligned} \mathbb{E}\left(|U - U_*|^2 |V - V_*|^2 - \left((U - U_*) \cdot (V - V_*)\right)^2\right) &\geq \frac{d-1}{d} f\left(\mathbb{E}\left(|U - V|^2\right)\right). \end{aligned} \quad (43)$$

Moreover, a sufficient condition for equality in (43) is given by strongly isotropic distributions with co-linear coupling, defined by the fact that the lengths  $(|U|, |V|) \in \mathbb{R}_+^2$  are independent of the perfectly coupled directions  $\frac{U}{|U|} = \frac{V}{|V|}$ , which are moreover uniformly distributed.

## 5.2 Proof of coupling creation estimates

It is now possible to use Hölder inequality to relate the coupling distance with the coupling contraction functional (7), by using the special inequality (41). This yields Proposition 2.7 and Proposition 2.8. Here are the proofs of the latter.

*Proof of Corollary 2.7.* It is a consequence of Hölder inequality applied to the right hand side of (41). We first apply (41) on the pair  $(U^N, V^N)$  with respect to the probability space generated by the particle averaging operator  $\langle \cdot \rangle_N$ . We obtain using the normalization assumption:

$$\begin{aligned} I &\stackrel{\text{def}}{=} \left\langle |U^N - U_*^N|^2 |V^N - V_*^N|^2 - ((U^N - U_*^N) \cdot (V^N - V_*^N))^2 \right\rangle_N \\ &\geq (1 - \|\langle U^N \otimes U_*^N \rangle_N\|) f(\langle |U^N - V^N|^2 \rangle_N) \quad a.s.. \end{aligned}$$

Next, let us denote

$$\mathcal{A}_{+/-} = |U^N - U_*^N| |V^N - V_*^N| + / - (U^N - U_*^N) \cdot (V^N - V_*^N),$$

and introduce  $q = 1 + \alpha$ ,  $p = 1 + 1/\alpha$  so that  $1/p + 1/q = 1$ . Using Hölder inequality two times yields

$$\begin{aligned} I &= \langle \mathcal{A}_+ \mathcal{A}_- \rangle = \left\langle \mathcal{A}_+ \mathcal{A}_-^{1/q} \mathcal{A}_-^{1/p} \right\rangle \\ &\leq q \left( \frac{2}{q+1} \right)^{1/q+1} \left\langle |U^N - U_*^N|^{1+1/q} |V^N - V_*^N|^{1+1/q} \mathcal{A}_-^{1/p} \right\rangle \\ &\leq q \left( \frac{2}{q+1} \right)^{1/q+1} \left\langle |U^N - U_*^N|^{1+q} |V^N - V_*^N|^{1+q} \right\rangle_N^{1/q} \langle \mathcal{A}_- \rangle_N^{1/p} \quad a.s., \end{aligned}$$

where in the line before last line, we have use the sharp inequality  $(1 - \theta)(1 + \theta)^{1/q} \leq q \left( \frac{2}{q+1} \right)^{1/q+1}$  that holds for any  $\theta \in [-1, 1]$ . Taking the expectation, and using Hölder inequality on the expectation with  $1/qp_1 + 1/qp_2 + 1/p = 1$ , as well as Jensen, yields

$$\begin{aligned} &\mathbb{E} \left( (1 - \|\langle U^N \otimes U_*^N \rangle_N\|) I \right) \\ &\leq q \left( \frac{2}{q+1} \right)^{1/q} \mathbb{E}^{1/qp_1} \left( (1 - \|\langle U^N \otimes U_*^N \rangle_N\|)^{-qp_1} \left\langle |U^N - U_*^N|^{p_1(1+q)} \right\rangle_N \right) \\ &\quad \times \mathbb{E}^{1/qp_2} \left\langle |V^N - V_*^N|^{p_2(1+q)} \right\rangle_N \mathbb{E}^{1/p} \langle \mathcal{A}_- \rangle_N. \end{aligned}$$

Then, remarking that  $k_\alpha = q \left( \frac{2}{q+1} \right)^{1/q}$  and  $p_1(1+q)/qp_1 = \frac{\alpha+2}{\alpha+1}$  yields the first part of the result.

Finally, the integrability condition and the limit in (2.7) can be justified using the explicit expression for the probability distribution of eigenvalues of Wishart matrices (see *e.g* [2, 28]). To prove the limit, consider some  $M > 0$  and higher finite  $l$ -moments of the following random variable

$$\kappa \left( \langle U^N \otimes U^N \rangle_N \right) \mathbf{1}_{\kappa(\langle U^N \otimes U^N \rangle_N) \leq M}.$$

My Markov inequality, the result will follow from dominated convergence and the fact that  $\kappa \left( \langle U^N \otimes U^N \rangle_N \right)$  has finite  $l$ -moments uniformly bounded in  $N$ . Using the explicit expression

(see Section 1 in [28]), such moments can be bounded above by ( $\Gamma$  is the usual so-called function)

$$\frac{\Gamma(Nd/2)}{\Gamma((N-l)d/2)} \frac{\prod_{i=0}^{d-1} \Gamma((N-i-l)/2)}{\prod_{i=0}^{d-1} \Gamma((N-i)/2)} \underset{N \rightarrow +\infty}{\sim} N^{ld/2} \times (N^{-l/2})^d \rightarrow 1.$$

□

*Proof of Proposition 2.8.* Strictly similar to Proposition 2.7. □

## References

- [1] Ricardo Alonso, Jose A Canizo, Irene Gamba, and Clément Mouhot. A new approach to the creation and propagation of exponential moments in the boltzmann equation. *Communications in Partial Differential Equations*, 38(1):155–169, 2013.
- [2] Theodore Wilbur Anderson, Theodore Wilbur Anderson, Theodore Wilbur Anderson, and Theodore Wilbur Anderson. *An introduction to multivariate statistical analysis*, volume 2. Wiley New York, 1958.
- [3] Aleksandr V Bobylev. Moment inequalities for the boltzmann equation and applications to spatially homogeneous problems. *Journal of statistical physics*, 88(5-6):1183–1214, 1997.
- [4] Alexander V. Bobylev and Carlo Cercignani. On the rate of entropy production for the boltzmann equation. *Journal of Statistical Physics*, 94(3-4):603–618, 1999.
- [5] AV Bobylev. The theory of the nonlinear boltzmann equation for maxwell molecules. *Mathematical physics reviews*, 7:111, 1988.
- [6] François Bolley, Ivan Gentil, and Arnaud Guillin. Convergence to equilibrium in wasserstein distance for fokker–planck equations. *Journal of Functional Analysis*, 2012.
- [7] François Bolley, Ivan Gentil, and Arnaud Guillin. Uniform convergence to equilibrium for granular media. *Archive for Rational Mechanics and Analysis*, pages 1–17, 2012.
- [8] EA Carlen and MC Carvalho. Strict entropy production bounds and stability of the rate of convergence to equilibrium for the boltzmann equation. *Journal of statistical physics*, 67(3-4):575–608, 1992.
- [9] EA Carlen, E Gabetta, and G Toscani. Propagation of smoothness and the rate of exponential convergence to equilibrium for a spatially homogeneous maxwellian gas. *Communications in mathematical physics*, 199(3):521–546, 1999.
- [10] Eric A Carlen, Maria C Carvalho, and Michael Loss. Determination of the spectral gap for kac’s master equation and related stochastic evolution. *Acta mathematica*, 191(1):1–54, 2003.
- [11] Eric A Carlen, Jeffrey S Geronimo, and Michael Loss. Determination of the spectral gap in the kac’s model for physical momentum and energy-conserving collisions. *SIAM Journal on Mathematical Analysis*, 40(1):327–364, 2008.

- [12] Eric A Carlen and Xuguang Lu. Fast and slow convergence to equilibrium for maxwellian molecules via wild sums. *Journal of statistical physics*, 112(1-2):59–134, 2003.
- [13] C Cercignani. H-theorem and trend to equilibrium in the kinetic theory of gases. *Archiv of Mechanics, Archiwum Mechaniki Stosowanej*, 34:231–241, 1982.
- [14] Carlo Cercignani. *Mathematical methods in kinetic theory*. Plenum Press New York, 1969.
- [15] Persi Diaconis and Laurent Saloff-Coste. Bounds for kac’s master equation. *Communications in Mathematical Physics*, 209(3):729–755, 2000.
- [16] Emanuele Dolera, Ester Gabetta, and Eugenio Regazzini. Reaching the best possible rate of convergence to equilibrium for solutions of kac equation via central limit theorem. *The Annals of Applied Probability*, 19(1):186–209, 2009.
- [17] Emanuele Dolera and Eugenio Regazzini. The role of the central limit theorem in discovering sharp rates of convergence to equilibrium for the solution of the kac equation. *The Annals of Applied Probability*, 20(2):430–461, 2010.
- [18] S. N. Ethier and T. G. Kurtz. *Markov Processes. Characterization and Convergence*. Wiley Series in Probability and Mathematical Statistics, 1985.
- [19] Nicolas Fournier and Stéphane Mischler. Rate of convergence of the nanbu particle system for hard potentials. *arXiv preprint arXiv:1302.5810*, 2013.
- [20] Nicolas Fournier and Clément Mouhot. On the well-posedness of the spatially homogeneous boltzmann equation with a moderate angular singularity. *Communications in Mathematical Physics*, 289(3):803–824, 2009.
- [21] E Ikenberry and C Truesdell. On the pressures and flux of energy in a gas according to {M} axwell’s kinetic theory. *J. Rat. Mech. Anal.*, 5, 1956.
- [22] David Asher Levin, Yuval Peres, and Elizabeth Lee Wilmer. *Markov chains and mixing times*. AMS Bookstore, 2009.
- [23] Xuguang Lu and Clément Mouhot. On measure solutions of the boltzmann equation, part i: moment production and stability estimates. *Journal of Differential Equations*, 252(4):3305–3363, 2012.
- [24] Stéphane Mischler and Clément Mouhot. About kac’s program in kinetic theory. *Comptes Rendus Mathématique*, 349(23):1245–1250, 2011.
- [25] Kenichi Nanbu. Direct simulation scheme derived from the boltzmann equation. i. mono-component gases. *J. Phys. Soc. Japan*, 49(5):2042–2049, 1980.
- [26] Roberto Imbuzeiro Oliveira. On the convergence to equilibrium of kac’s random walk on matrices. *The Annals of Applied Probability*, 19(3):1200–1231, 2009.
- [27] A Pulvirenti and G Toscani. The theory of the nonlinear boltzmann equation for maxwell molecules in fourier representation. *Annali di Matematica Pura ed Applicata*, 171(1):181–204, 1996.

- [28] FJ Schuurmann, PR Krishnaiah, and AK Chattopadhyay. On the distributions of the ratios of the extreme roots to the trace of the wishart matrix. *Journal of Multivariate Analysis*, 3(4):445–453, 1973.
- [29] Hiroshi Tanaka. Probabilistic treatment of the boltzmann equation of maxwellian molecules. *Probability Theory and Related Fields*, 46(1):67–105, 1978.
- [30] G Toscani and C Villani. Probability metrics and uniqueness of the solution to the boltzmann equation for a maxwell gas. *Journal of statistical physics*, 94(3-4):619–637, 1999.
- [31] G Toscani and C Villani. Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous boltzmann equation. *Communications in mathematical physics*, 203(3):667–706, 1999.
- [32] Cédric Villani. A review of mathematical topics in collisional kinetic theory. *Handbook of mathematical fluid dynamics*, 1:71–74, 2002.
- [33] Cédric Villani. Cercignani’s conjecture is sometimes true and always almost true. *Communications in mathematical physics*, 234(3):455–490, 2003.
- [34] Bernt Wennberg. Entropy dissipation and moment production for the boltzmann equation. *Journal of Statistical Physics*, 86(5-6):1053–1066, 1997.