Serre duality and Hörmander's solution of the $\bar{\partial}$ -equation.

Eric Amar

SERRE DUALITY AND HÖRMANDER'S SOLUTION OF THE $\bar\partial$ -EQUATION.

ERIC AMAR

ABSTRACT. We use duality in the manner of Serre to generalize a theorem of Hedenmalm on solution of the $\bar\partial$ equation with inverse of the weight in Hörmander L^2 estimates.

1. Introduction

Let φ be a $\mathcal{C}^2$ strictly sub harmonic function in the complex plane $\mathbb{C}$, i.e. with Δ the Laplacian, $\Delta\varphi > 0$ in $\mathbb{C}$. Let $A^2(\mathbb{C}, e^{-2\varphi})$ be the set of all holomorphic functions g in $\mathbb{C}$ such that

$$\|g\|_{L^2(\mathbb{C},e^{-2\varphi})} := \int_{\mathbb{C}} |g|^2 e^{-2\varphi} dA < \infty,$$

with dA the Lebesgue measure in $\mathbb{C}$. Suppose that $f \in L^2(\mathbb{C}, e^{2\varphi})$ verifies

$$\forall g \in A^2(\mathbb{C}, e^{-2\varphi}),\ \int_{\mathbb{C}} fg\,dA = 0 \tag{1.1}$$

then in a recent paper H. Hedenmalm [2] proved

Theorem 1.1. *Suppose that $f \in L^2(\mathbb{C}, e^{2\varphi})$ verifies condition (1.1) then there exists a solution to the $\bar\partial$ -equation $\bar\partial u = f$ with*

$$\int_{\mathbb{C}} |u|^2 e^{2\varphi}\Delta\varphi\, dA \leq \frac{1}{2}\int_{\mathbb{C}} |f|^2 e^{2\varphi} dA.$$

He adds in remark 1.3. that this theorem should generalize to the setting of several complex variables. The aim of this note is to show that he was right and in fact we have to make slight modifications in our paper [1], inspired by Serre's duality theorem [4] to get it. The paper [1] contains more material and here we extract just the part we need to make this work self contained.

Let φ be a strictly plurisubharmonic function of class $\mathcal{C}^2$ in the Stein manifold Ω. Let $c_\varphi(z)$ be the smallest eigenvalue of $\partial\bar\partial\varphi(z)$, then $\forall z \in \Omega,\ c_\varphi(z) > 0$. We denote by $L^2_{p,q}(\Omega, e^\varphi)$ the set of (p, q) currents ω whose coefficients are in the space $L^2(\Omega, e^\varphi)$, i.e. there is a constant $C > 0$ such that in a coordinates patch (U, ψ) , we have, with dm the Lebesgue measure in $\mathbb{C}^n$,

$$\psi^*\omega = \sum_{I,J} \omega_{I,J} dz^I d\bar z^J,\ \int_{\psi(U)} |\omega_{I,J}|^2 e^{\varphi\circ\psi^{-1}dm} \leq C.$$

We denote by $L^{2,c}_{p,q}(\Omega, e^\varphi)$ the currents in $L^2_{p,q}(\Omega, e^\varphi)$ with *compact support* in Ω and $\mathcal{H}_p(\Omega)$ the set of all $(p,\ 0)$, $\bar\partial$ closed forms in Ω. If $p = 0$, $\mathcal{H}_0(\Omega) = \mathcal{H}(\Omega)$ is the set of holomorphic functions in Ω.

We shall prove

Theorem 1.2. *Let Ω be a pseudo convex domain in $\mathbb{C}^n$; if $\omega \in L^{2,c}_{p,q}(\Omega,\ e^\varphi)$ with $\bar\partial\omega = 0$ if $q < n$ and $\omega \in L^2_{p,q}(\Omega, e^\varphi)$ with $\omega \perp \mathcal{H}^p(\Omega)$ if $q = n$, then there is $u \in L^2_{p,q-1}(\Omega, c_\varphi e^\varphi)$ such that $\bar\partial u = \omega$, and*

$$\|u\|_{L^2(\Omega,c_\varphi e^\varphi)} \leq C\|\omega\|_{L^2(\Omega,e^\varphi)}.$$

In the case Ω is a Stein manifold, the result is more restrictive :

Theorem 1.3. *Let Ω be a Stein manifold ; there is a convex increasing function χ such that, with $\psi:=\chi(\varphi)$, if $\omega\in L^{2,c}_{p,q}(\Omega,\ e^{\psi})$ with $\bar\partial\omega=0$ if $q<n$ and $\omega\in L^2_{p,q}(\Omega,e^{\psi})$ with $\omega\perp\mathcal{H}^p(\Omega)$ if $q=n$, then there is $u\in L^2_{p,q-1}(\Omega,c_\psi e^{\psi})$ such that $\bar\partial u=\omega$, and*
$$\|u\|_{L^2(\Omega,c_\psi e^{\psi})}\leq C\|\omega\|_{L^2(\Omega,e^{\psi})}.$$

Clearly theorem 1.2 generalizes Hedenmalm's theorem, because in one variable, we have $q=n=1$ and no compactness assumption is required.

2. The proof.

Set a weight $\eta:=e^{-\varphi}$ and $\mu:=c_\varphi^{-1}\eta$ such that if $\alpha\in L^2_{(p,q)}(\Omega,\mu)$ is such that $\bar\partial\alpha=0$ in Ω then there is $(p,q-1)$ current $\varphi\in L^2(\Omega,\eta)$ with $\bar\partial\varphi=\alpha$. Moreover we want that if $\beta\in L^2_{(p,q)}(\Omega,\eta)$, $\bar\partial\beta=0$, there is a $\gamma\in L^2_{(p,q-1)}(\Omega,loc)$ such that $\bar\partial\gamma=\beta$.

Now we suppose that, if $q<n$ then ω has a compact support, $\omega\in L^{2,c}_{(p,q)}(\Omega,\eta^{-1})$ and $\bar\partial\omega=0$ and if $q=n$, then $\omega\perp\mathcal{H}_{n-p}(\Omega)$ and $\omega\in L^2_{(p,q)}(\Omega,\eta^{-1})$, but ω needs not have compact support. We copy lemma 3.5 from [1].

Lemma 2.1. *For $\Omega,\ \omega,\ \eta$ as above, the function $\mathcal{L}=\mathcal{L}_\omega$, defined on $(n-p,n-q+1)$ form $\alpha\in L^2(\Omega,\beta)$, $\bar\partial$ closed in Ω, as follows :*
$$\mathcal{L}(\alpha):=(-1)^{p+q-1}\langle\omega,\varphi\rangle,\ \textit{where}\ \varphi\in L^2(\Omega,\eta)\ \textit{is such that}\ \bar\partial\varphi=\alpha\ \textit{in}\ \Omega$$
is well defined and linear.

Proof.
We have $\alpha\in L^2(\Omega,\mu),\bar\partial\alpha=0$, then such a $\varphi\in L^2(\Omega,\eta)$ exists by hypothesis and the pairing $\langle\omega,\varphi\rangle$ is well defined because $\omega\in L^2_{(p,q)}(\Omega,\eta^{-1})$.
Let us see that $\mathcal{L}$ is well defined.

Suppose first that $q<n$.
In order for $\mathcal{L}$ to be well defined we need
$$\forall\varphi,\psi\in L^2_{(n-p,n-q)}(\Omega,\eta),\ \bar\partial\varphi=\bar\partial\psi\Rightarrow\langle\omega,\varphi\rangle=\langle\omega,\psi\rangle.$$
Then we have $\bar\partial(\varphi-\psi)=0$ hence we can solve $\bar\partial$ in $L^2(\Omega,loc)$:
$$\exists\gamma\in L^2_{(n-p,n-q-1)}(\Omega,loc)::\bar\partial\gamma=(\varphi-\psi).$$
So $\langle\omega,\varphi-\psi\rangle=\left\langle\omega,\bar\partial\gamma\right\rangle=(-1)^{p+q-1}\left\langle\bar\partial\omega,\gamma\right\rangle=0$ because ω is compactly supported in Ω.
Hence $\mathcal{L}$ is well defined in that case.

Suppose now that $q=n$.
Of course $\bar\partial\omega=0$ and we have that $\varphi,\ \psi$ are $(n-p,\ 0)$ currents hence $\bar\partial(\varphi-\psi)=0$ means that $h:=\varphi-\psi$ is a $\bar\partial$ closed $(n-p,\ 0)$ current hence $h\in\mathcal{H}_{n-p}(\Omega)$. Hence the hypothesis, $\omega\perp\mathcal{H}_{n-p}(\Omega)$ gives $\langle\omega,h\rangle=0$, and $\mathcal{L}$ is also well defined in that case and no compactness on the support of ω is required here.

It remains to see that $\mathcal{L}$ is linear, so let $\alpha=\alpha_1+\alpha_2$, with $\alpha_j\in L^2(\Omega,\mu),\ \bar\partial\alpha_j=0,\ j=1,2$; we have $\alpha=\bar\partial\varphi,\ \alpha_1=\bar\partial\varphi_1$ and $\alpha_2=\bar\partial\varphi_2$, with $\varphi,\ \varphi_1,\ \varphi_2$ in $L^2(\Omega,\eta)$ so, because $\bar\partial(\varphi-\varphi_1-\varphi_2)=0$, we have

- for $q<n$,
$$\varphi=\varphi_1+\varphi_2+\bar\partial\psi,\ \text{with}\ \psi\ \text{in}\ L^2(\Omega,loc),$$

so
$$\mathcal{L}(\alpha)=(-1)^{p+q-1}\langle\omega,\varphi\rangle=(-1)^{p+q-1}\left\langle\omega,\varphi_1+\varphi_2+\bar\partial\psi\right\rangle=\mathcal{L}(\alpha_1)+\mathcal{L}(\alpha_2)+(-1)^{p+q-1}\left\langle\omega,\bar\partial\psi\right\rangle,$$

but $(-1)^{p+q-1}\langle\omega,\bar\partial\psi\rangle=\langle\bar\partial\omega,\psi\rangle=0$, because $\operatorname{Supp}\omega\Subset\Omega$ implies there is no boundary term so $\mathcal{L}(\alpha)=\mathcal{L}(\alpha_1)+\mathcal{L}(\alpha_2)$.

• for $q=n$,

$\varphi=\varphi_1+\varphi_2+h$, with $h\in\mathcal{H}_{n-p}(\Omega)$ hence

$$\mathcal{L}(\alpha)=(-1)^{p+q-1}\langle\omega,\varphi\rangle=(-1)^{p+q-1}\langle\omega,\varphi_1+\varphi_2+h\rangle=\mathcal{L}(\alpha_1)+\mathcal{L}(\alpha_2)+(-1)^{p+q-1}\langle\omega,h\rangle,$$

so, because $\langle\omega,h\rangle=0$, we still have $\mathcal{L}(\alpha)=\mathcal{L}(\alpha_1)+\mathcal{L}(\alpha_2)$ without compactness assumption on the support of ω.

The same for $\alpha=\lambda\alpha_1$ and the linearity. ■

Lemma 2.2. *Still with the same hypotheses as above there is a $(p,q-1)$ current u such that*

$$\forall\alpha\in L^2_{(n-p,n-q+1)}(\Omega,\mu),\ \langle u,\alpha\rangle=\mathcal{L}(\alpha)=(-1)^{p+q-1}\langle\omega,\varphi\rangle,$$

and

$$\sup_{\alpha\in L^2(\Omega,\mu),\ \|\alpha\|_{L^2(\Omega,\mu)}\leq 1}|\langle u,\alpha\rangle|\leq C\|\omega\|_{L^2(\Omega,\eta)}.$$

Proof.

By lemma 2.1 we have that $\mathcal{L}$ is a linear form on $(n-p,n-q+1)$ forms $\alpha\in L(\Omega,\mu)$, $\bar\partial$ closed in Ω.

We have

$$\exists\varphi\in L^2_{(n-p,n-q)}(\Omega,\eta)::\bar\partial\varphi=\alpha,\ \|\varphi\|_{L^2(\Omega,\eta)}\leq C\|\alpha\|_{L^2(\Omega,\mu)}.$$

Hence $\mathcal{L}(\alpha)=(-1)^{p+q-1}\langle\omega,\varphi\rangle$ and by Cauchy Schwarz inequality

$$|\mathcal{L}(\alpha)|\leq\|\omega\|_{L^2(\Omega,\eta^{-1})}\|\varphi\|_{L^2(\Omega,\eta)}\leq C\|\omega\|_{L^2(\Omega,\eta^{-1})}\|\alpha\|_{L^2(\Omega,\mu)},$$

hence

$$|\mathcal{L}(\alpha)|\leq C\|\omega\|_{L^2(\Omega,\eta^{-1})}\|\alpha\|_{L^2(\Omega,\ \mu)}.$$

So we have that the norm of $\mathcal{L}$ is bounded on the subspace of $\bar\partial$ closed forms in $L^2(\Omega,\mu)$ by $C\|\omega\|_{L^2(\Omega,\eta^{-1})}$.

We apply the Hahn-Banach theorem to extend $\mathcal{L}$ with the *same* norm to *all* $(n-p,n-q+1)$ forms in $L^2(\Omega,\ \mu)$. As in Serre's duality theorem ([4], p. 20) this is one of the main ingredient in the proof.

This means, by the definition of currents, that there is a $(p,q-1)$ current u which represents the extended form $\mathcal{L}$, i.e.

$$\forall\alpha\in L^2_{(n-p,n-q+1)}(\Omega,\mu),\ \exists\varphi\in L^2(\Omega,\eta)::\langle u,\alpha\rangle=\mathcal{L}(\alpha)=(-1)^{p+q-1}\langle\omega,\varphi\rangle,$$

and such that

$$\sup_{\alpha\in L^2(\Omega,\mu),\ \|\alpha\|_{L^2(\Omega,\mu)}\leq 1}|\langle u,\alpha\rangle|\leq C\|\omega\|_{L^2(\Omega,\eta^{-1})}.$$

■

Proof of the theorem 1.2 and theorem 1.3.

Let φ be a strictly plurisubharmonic function in the Stein manifold Ω. Let $c_\varphi(z)$ be the smallest eigenvalue of $\partial\bar\partial\varphi(z)$, then $\forall z\in\Omega,\ c_\varphi(z)>0$.

If Ω is a pseudo convex domain in $\mathbb{C}^n$, by lemma 4.4.1. of Hörmander [3], p. 92, we have that, with $\eta=e^{-\varphi},\ \mu=c_\varphi^{-1}e^{-\varphi}$,

$$\forall\alpha\in L^2_{(n-p,n-q+1)}(\Omega,\mu),\ \bar\partial\alpha=0,\ \exists\varphi\in L^2_{(n-p,n-q)}(\Omega,\eta)::\bar\partial\varphi=\alpha,$$

and by theorem 4.2.2. still from Hörmander [3], p. 84, because if $\beta\in L^2_{(n-p,n-q)}(\Omega,\eta)$ then $\beta\in L^2_{(n-p,n-q)}(\Omega,loc)$, we have

$$\forall\beta\in L^2_{(n-p,n-q)}(\Omega,\eta),\ \bar\partial\beta=0,\ \exists\gamma\in L^2_{(n-p,n-q)}(\Omega,loc)::\bar\partial\gamma=\beta.$$

Hence it remains to apply lemma 2.3, proved in the next section, with $r=2$ and $\eta=c_{\varphi}^{-1}e^{-\varphi}$ to get the theorem 1.2.

If Ω is a Stein manifold, we need c_{φ} big enough to verify inequality (5.2.13), p. 125 in [3]. To have this we can replace φ by $\psi:=\chi(\varphi)$ with χ a convex increasing function, as in [3], p. 125. Then we still can apply what precedes to have the theorem 1.3. ■

2.1. **A duality lemma with weight.**

This is still in [1] lemma 3.1., but I repeat it completely for the reader's convenience.

Let $\mathcal{I}_p$ be the set of multi-indices of length p in $(1,...,n)$. We shall use the measure defined on $\Gamma:=\Omega\times\mathcal{I}_p\times\mathcal{I}_q$ the following way :

$$d\mu(z,k,l)=d\mu_{\eta,p,q}(z,k,l):=\eta(z)dm(z)\otimes\sum_{|I|=p,\ |J|=q}\delta_I(k)\otimes\delta_J(l),$$

where $\delta_I(k)=1$ if the multi-index k is equal to I and $\delta_I(k)=0$ if not.
This means, if $f(z,I,J)$ is a function defined on Γ, that

$$\int f(z,k,l)d\mu_{\eta,p,q}(z,k,l):=\sum_{|I|=p,\ |J|=q}\int_{\Omega}f(z,I,J)\eta(z)dm(z).$$

If I is a multi-index of length p, let I^c be the unique multi-index, ordered increasingly, such that $I\cup I^c=(1,\ 2,...,\ n)$; then I^c is of length $n-p$.

To $t=\ \sum_{|I|=p,\ |J|=q}t_{I,J}(z)dz^I\wedge d\bar{z}^J$ a (p,q) form, we associate the function on Γ :

$$T(z,I,J):=(-1)^{s(I,J)}t_{I,J}(z),$$

where

$$s(I,J)=0 \text{ if } dz^I\wedge d\bar{z}^J\wedge dz^{I^c}\wedge d\bar{z}^{J^c}=dz_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n \text{ as a } (n,n) \text{ form}$$

and

$$s(I,J)=1 \text{ if not.}$$

If $\varphi=\ \sum_{|I|=p,\ |J|=q}\varphi_{I^c,J^c}(z)dz^I\wedge d\bar{z}^J$ is of complementary bi-degree, associate in the same manner :

$\Phi^*(z,\ I,\ J):=\varphi_{I^c,J^c}(z)$. This is still a function on Γ.

Now we have, for $1<r<\infty$, if $T(z,I,J)$ is a function in Ω with $L^r(\Omega)$ coefficients and with $\mu=\mu_{\eta,p,q}$,

$$\|T\|_{L^r(d\mu)}^r:=\int|T(z,I,J)|^r\,d\mu_{\eta,p,q}(x,\ I,\ J)=\sum_{|I|=p,\ |J|=q}\|T(\cdot,I,J)\|_{L^r(\Omega,\eta)}^r.$$

For $1\leq r<\infty$ the dual of $L^r(\mu)$ is $L^{r'}(\mu)$ where r' is the conjugate of r, $\dfrac{1}{r}+\dfrac{1}{r'}=1$, and the norm is defined analogously with r' replacing r.

We also know that, for $p,\ q$ fixed,

$$\|T\|_{L^r(\mu)}=\sup_{\Phi\in L^{r'}(\mu)}\frac{\left|\int T\Phi d\mu\right|}{\|\Phi\|_{L^{r'}(\mu)}}. \tag{2.2}$$

For a $(p,\ q)$ form $t=\sum_{|J|=p,\ |K|=q}t_{J,K}dz^J\wedge d\bar{z}^K$, and a weight $\eta>0$ we define its norm by :

$$\|t\|_{L^r(\Omega,\eta)}^r:=\sum_{|J|=p,\ |K|=q}\|t_{J,K}\|_{L^r(\Omega,\eta)}^r=\|T\|_{L^r(\mu)}^r. \tag{2.3}$$

Now we can state

Lemma 2.3. *Let $\eta>0$ be a weight. If u is a (p,q) current defined on $(n-p,n-q)$ forms in $L^{r'}(\Omega,\eta)$ and such that*

$$\forall\alpha\in L^{r'}_{(n-p,n-q)}(\Omega,\eta),\ \ \lvert\langle u,\ \alpha\rangle\rvert\leq C\lVert\alpha\rVert_{L^{r'}(\Omega,\eta)},$$

then $\lVert u\rVert_{L^r(\Omega,\eta^{1-r})}\leq C.$

Proof.

Let us take the measure $\mu=\mu_{\eta,p,q}$ as above. Let Φ^* be the function on Γ associated to α and T the one associated to u. We have, by definition of the measure μ applied to the function

$$f(z,I,J):=T(z,I,J)\eta^{-1}\Phi^*(z,I,J),$$

$$\int T\eta^{-1}\Phi^*d\mu=\int f(z,\ k,\ l)d\mu(z,k,l):=\sum_{|I|=p,\ |J|=q}\int_\Omega f(z,\ I,\ J)\eta(z)dm(z)=$$

$$=\sum_{|I|=p,\ |J|=q}\int_\Omega T(z,I,J)\eta^{-1}(z)\Phi^*(z,I,J)\eta(z)dm(z)=\langle u,\ \alpha\rangle,$$

by definition of T and Φ^*.

Hence we have, by (2.2)

$$\lVert T\eta^{-1}\rVert_{L^r(\mu)}=\sup_{\Psi\in L^{r'}(\mu)}\frac{\lvert\langle u,\ \alpha\rangle\rvert}{\lVert\Psi\rVert_{L^{r'}(\mu)}}.$$

But $\lVert T\eta^{-1}\rVert_{L^r(\mu)}=\lVert u\eta^{-1}\rVert_{L^r(\Omega,\ \eta)}$ by (2.3), and

$$\lVert f\eta^{-1}\rVert^r_{L^r(\Omega,\eta)}=\int_\Omega\lvert f\eta^{-1}\rvert^r\eta dm=\int_\Omega\lvert f\rvert^r\eta^{1-r}dm=\lVert f\rVert_{L^r(\Omega,\eta^{1-r})},$$

so we get

$$\lVert u\rVert_{L^r(\Omega,\eta^{1-r})}=\sup_{\Psi\in L^{r'}(\mu)}\frac{\lvert\langle u,\ \alpha\rangle\rvert}{\lVert\Psi\rVert_{L^{r'}(\mu)}},$$

which implies the lemma because, still by (2.2), we can take $\Psi=\Phi^*$ and $\lVert\Psi\rVert_{L^{r'}(\mu)}=\lVert\alpha\rVert_{L^{r'}(\Omega,\eta)}$. ■

It may seem strange that we have such an estimate when the dual of $L^{r'}(\Omega,\eta)$ is $L^r(\Omega,\eta)$, but the reason is, of course, that in the duality forms-currents there is no weights.

References

[1] Eric Amar. An Andreotti-Grauert theorem with L^r estimates. *arXiv:1203.0759v5*, 2012.
[2] H. Hedenmalm. On Hörmander's solution of the $\bar\partial$-equation. *arXiv:1311.2020v2*, 2013.
[3] L. Hörmander. *An introduction to complex analysis in several variables.* North-Holland/American Elsevier, 1994.
[4] J-P. Serre. Un théorème de dualité. *Comment. Math. Helv.*, 29:9–26, 1955.