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Ismaël Soudères

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# A RELATIVE BASIS FOR MIXED TATE MOTIVES OVER THE PROJECTIVE LINE MINUS THREE POINTS

ISMAEL SOUDÈRES

Abstract. In a previous work, the author have built two families of distinguished algebraic cycles in Bloch-Kriz cubical cycle complex over the projective line minus three points.

The goal of this paper is to show how these cycles induce well-defined elements in the $H^0$ of the bar construction of the cycle complex and thus generated comodules over this $H^0$, that is a mixed Tate motives as in Bloch and Kriz construction.

In addition, it is shown that out of the two families only ones is needed at the bar construction level. As a consequence, the author obtains that one of the family gives a basis of the tannakian coLie coalgebra of mixed Tate motives over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ relatively to the tannakian coLie coalgebra of mixed Tate motives over $\mathrm{Spec}(\mathbb{Q})$. This in turns provides a new formula for Goncharov motivic coproduct, which really should be think as a coaction.

Note : Preliminary version. Missing :

- introduction;
- typos, English and writing corrections.

## Contents



## 1. Introduction

## 2. Bar and cobar adjunctions

In this section we recall how the bar/cobar constructions gives a pair of adjoint functors in the two following cases:

$$B : \left\{ \begin{array}{c} \text{diff. gr. ass.} \\ \text{algebras} \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{diff. gr. coass.} \\ \text{coalgebras} \end{array} \right\} : \Omega$$

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None of this would have been possible without M. Levine patience, his explanations and deep inputs. The author is grateful to the MPIM in Bonn for providing ideal working conditions and support during my stay there as a guest where this work has been accomplished.

and

$$B_c : \left\{ \begin{array}{c} \text{diff. gr. com.} \\ \text{ass. algebras} \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{diff. gr. coLie.} \\ \text{coalgebras} \end{array} \right\} : \Omega_L.$$

As a differential graded commutative algebras $A$ is also and associative algebra, we will recall how the two construction are related in this case and how suitable degree 1 differential systems in $A$ naturally induce families of 0 cocycle in the bar construction $B(A)$.

The material developed here is well known and can be found in Ginzburg-Kapranov [GK94] work even if their use of graded duals replace coalgebra structure by algebra structures. The presentation used here is closer to the Kozul duality as developed by Jones and Getzler in [GJ94]. We follow here the signs conventions and the formalism presented by Loday and Vallette in [LV12]. The associative case is directly taken from [LV12, Chap. 2] in a cohomological version.

## 2.1. **Notation and convention.**

2.1.1. *Koszul sign rule.* The objects are all object of the category of (sign) *graded $\mathbb{Q}$-vector spaces* with the tensor product $\otimes$. The degree of an homogeneous element $v$ in $V$ is denoted by $|v|$ or $|v|_V$ if we want to emphasis where $v$ is. The symmetric structure is given by the switching map

$$\tau : V \otimes W \longrightarrow W \otimes V\,, \qquad \tau(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

For any maps $f : V \to V'$ and $g : W \to W'$ of graded spaces, the tensor product

$$f \otimes g : V \otimes W \longrightarrow V' \otimes W'$$

is defined by

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

A *differential graded vector space* abbreviated into dg vector space is a graded vector space equipped with a *differential* $d_V$ (or when the context is clear enough simply $d$) ; that is a degree 1 linear map satisfying $d_V^2 = 0$.

**Definition 2.1.** Let $V = \oplus_n V^n$ and $W = \oplus_n W^n$ be two graded vector spaces. A *morphism of degree $r$*, say $f : V \longrightarrow W$, of graded vector spaces is a collection of morphisms $f_n : V^n \longrightarrow W^{n+r}$. Let $\mathrm{Hom}(V, W)_r$ be the vector space of morphisms of degree $r$ from $V$ to $W$.

The vector space $\mathrm{Hom}(V, W) = \oplus \mathrm{Hom}(V, W)_r$ is graded by the degree of the morphisms. When $V$ and $W$ are morphisms of dg vector spaces, $\mathrm{Hom}(V, W)$ turns into a a dg vector space with differential $d_{\mathrm{Hom}}(f)$ given by :

$$d_{\mathrm{Hom}}(f) = d_W \circ f - (-1)^r f \circ d_V$$

for any homogeneous element $f$ of degree $r$.

A degree $r$ *dg morphism* $f : V \longrightarrow W$ of dg vector space is a degree $r$ morphism of graded vector space $f : V^\bullet \longrightarrow W^{\bullet+r}$ satisfying $d_w \circ f = (-1)^r f \circ d_V$; that is such that $d_{\mathrm{Hom}}(f) = 0$.

The dual of a graded vector space $V = \oplus_n V^n$ is defined by

$$V^* = \oplus_n \mathrm{Hom}_{Vect}(V^{-n}, \mathbb{Q}) = \mathrm{Hom}(V, N)$$

where the dg vector space $N$ is defined by $N = \mathbb{Q}$ concentrated in degree 0 with 0 differential.

One has an obvious notion of cohomology on dg vector space.

2.1.2. *Symmetric and antisymmetric tensor products.* We review here the definition in the dg context of the tensor product and its symmetric and anti-symmetric version

Let $(V, d_V)$ and $(W, d_W)$ be two dg vector space, their tensor product $(V \otimes W)$ is defined by

$$(V \otimes W)^r = \oplus_{p+q=r} V^p \otimes W^q \qquad \text{and} \qquad d_{V\otimes W} \operatorname{id}_V \otimes d_W + d_V \otimes \operatorname{id}_W .$$

It is the total complex of the obvious double complex.

The *symmetric product* $V \odot V$ is the quotient of the dg vector space $V \otimes V$ modulo the relations

$$a \otimes b - (-1)^{|a||b||} b \otimes a = 0.$$

The *anti-symmetric product* $V \oslash V$ is the quotient of the dg vector space $V \otimes V$ modulo the relations

$$a \otimes b + (-1)^{|a||b||} b \otimes a = 0.$$

**Definition 2.2** ((de)suspension)**.** Let $S$ be the 1 dimensional dg vector space concentrated in degree 1 (that is $d_S = 0$) generated by $s$ : that is $S = s\mathbb{Q}$

The dual of $S$ is a one dimensional dg vector space denoted by $S^1$ and generated by a degree one element $s^{-1}$ dual to $s$.

Let $V, d_V$ be a dg vector space. Its *suspension* $(sV, d_{sV})$, is the dg vector space $S \otimes V$. Its desuspension $(s^{-1}V, d_{s^{-1}V})$ is $S^{-1} \otimes V$.

**Lemma 2.3.** *There is a canonical identification* $V^{n-1} \simeq (sV)^n$ *given by the dg morphism of degree* 1

$$i_s : V \longrightarrow sV \qquad v \longmapsto (-1)^{|v|_V} s \otimes v.$$

*Under this identification* $d_{sV} = -d_V$.

2.1.3. *Associative dg algebra / coassociative dg coalgebra.* A *differential graded associative* algebra $(A, d_A)$ abbreviated into dga algebra is a dg vector space equipped with a unital associative product $\mu_A$ of degree 0 commuting with the differential :

$$d_A \circ \mu_A = \mu_A \circ d_{A\otimes A}$$

and satisfying the usual commutative diagrams for an associative algebra, all the maps involved being maps of dg vector spaces.

This last equality is nothing but Leibniz rules making $d_A$ into a derivation for the product. Observe that the unit $1_A$ belongs to $A^0$. On elements, one writes $a \cdot_A b$ or simply $a \cdot b$ instead of $\mu_A(a \otimes b)$.

**Definition 2.4.** The dga $A$ is *connected* if $A^0 = 1_A\mathbb{Q}$.

A dga algebra $A$ is *quasi-free* if, as a graded algebra, it is *free* over some graded vector space $V$.

**Definition 2.5** (Tensor algebra)**.** The tensor algebra over a dg vector space $V$ is defined by

$$T(V) = \bigoplus_{n\geqslant 0} V^{\otimes n}$$

and equipped with the differential induced on each $V^{\otimes n}$ by $d_V$ and with the concatenation product given by

$$[a_1|\cdots|a_n] \otimes [a_{n+1}|\cdots a_{n+m}] \longmapsto [a_1|\cdots|a_n|a_{n+1}|\cdots a_{n+m}]$$

where the "bar" notation $[a_1|\cdots|a_n]$ stand for $a_1 \otimes \cdots \otimes a_n$ in $V^{\otimes n}$.

Note that the degree of $[a_1|\cdots|a_n]$ is $|a_1|_V + \cdots |a_n|_V$ and that $T(V)$ admits a natural augmentation given by $\varepsilon([a_1|\cdots|a_n]) = 0$ for $n > 0$ and the convention $V^{\otimes 0} = \mathbb{Q}$.

The concatenation product is clearly associative and the natural inclusion $i_V : V \longrightarrow T(V)$ into the tensor degree 1 part makes $T(V)$ into the quasi-free unital dga algebra over $V$.

2.1.4. *Commutative and anti commutative algebras.* A commutative dga algebra $(A, d_A, \mu_A)$ or cdga algebra is a dga algebra such that the multiplication commutes with the switching map

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\quad\tau\quad} & A\otimes A \\ & {\scriptstyle\mu_A}\searrow \quad \swarrow{\scriptstyle\mu_A} & \\ & A & \end{array}$$

On homogeneous elements, this reads as

$$a \cdot b = (-1)^{|a||b|} b \cdot a.$$

Let $V$ be a dg vector space and $n \geqslant 1$ an integer. The symmetric group $\mathbb{S}_n$ acts on $V^{\otimes n}$ in two ways : the symmetric action $\rho_S$ and the antisymmetric action $\rho_\Lambda$ (both graded).

For $i$ in $\{1, \dots n-1\}$, let $\tau_i$ be the permutation exchanging $i$ and $i+1$. As any permutation can be decomposed as a product of transposition $\tau_i$, it is enough to define the action for the $\tau_i$:

$$\rho_S(\tau_i) = \underbrace{\mathrm{id} \otimes \cdots \mathrm{id}}_{i-1 \text{ factors}} \otimes \tau \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}$$

and

$$\rho_\Lambda(\tau_i) = \underbrace{\mathrm{id} \otimes \cdots \mathrm{id}}_{i-1 \text{ factors}} \otimes (-\tau) \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}\ .$$

Note that both action involved signs. The graded signature $\varepsilon^{gr}(\sigma) \in \{\pm 1\}$ of a permutation $\sigma$ is defined by

$$\rho_S(\sigma)(v_1 \otimes \cdots \otimes v_n) = \varepsilon^{gr}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

Then, one has

$$\rho_\Lambda(\sigma)(v_1 \otimes \cdots \otimes v_n) = \varepsilon(\sigma)\varepsilon^{gr}(\sigma) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

where $\varepsilon(\sigma)$ is the usual signature. When we want to keep track of where the action takes place, we will write $\rho_{S,V}$, $\rho_{\Lambda,V}$ and $\varepsilon^{gr,V}(\sigma)$.

**Definition 2.6.** The (graded) symmetric algebra $S^{gr}(V)$ over a dg vector space $V$ is defined as the quotient of $T(V)$ by the two side ideal generated by $(\mathrm{id} - \tau)(a \otimes b)$.

On can write

$$S^{gr}(V) = \bigoplus_{n \geqslant 0} V^{\odot n}$$

where $V^{\odot n} = S^{gr,n}(V)$ is the quotient of $V^{\otimes n}$ by the symmetric action of $\mathbb{S}_n$.

Let $p_{S,n}$ be the projector defined on $V^{\otimes n}$ by

$$p_{S,n} = \frac{1}{n!}\left(\sum_{\sigma \in \mathbb{S}_n} \rho_S(\sigma)\right).$$

On also has the isomorphism $S^{gr,n}(V) = p_{S,n}(V^{\otimes n})$. $S^{gr}(V)$ is the free commutative algebra over $V$.

**Definition 2.7.** The (graded) antisymmetric algebra $\Lambda^{gr}(V)$ over a dg vector space $V$ is defined as the quotient of $T(V)$ by the two side ideal generated by $(\mathrm{id} + \tau)(a \otimes b)$.

On can write

$$\Lambda^{gr}(V) = \bigoplus_{n \geqslant 0} V^{\oslash n}$$

where $V^{\oslash n} = \Lambda^{gr,n}(V)$ is the quotient of $V^{\otimes n}$ by the antisymmetric action of $\mathbb{S}_n$.

As in the symmetric case, $\Lambda^{gr}(V)$ is also the image of $V^{\otimes n}$ by the projector $p_{\Lambda,n} = 1/(n!)(\sum_\sigma \rho_\Lambda(\sigma))$ and $\Lambda(V)$ is the free antisymmetric algebra over $V$.

Considering a dg morphism of degree $r$ $f : V \to V'$, we have the following relations on the antisymmetric and symmetric representations.

**Lemma 2.8.** *Let $V$ and $V'$ be two dg vector space and $f$ be dg morphism of degree $r$. Let $f^{\otimes n} : V^{\otimes n} \longrightarrow (V')^{\otimes n}$ be the induced morphism. Then for any permutation $\sigma$ in $\mathbb{S}_n$, one has*

$$f^{\otimes n} \circ \rho_{\Lambda,V}(\sigma) = \begin{cases} \rho_{\Lambda,sV}(\sigma) \circ f^{\otimes n} & \text{if } r \text{ even} \\ \rho_{S,sV}(\sigma) \circ f^{\otimes n} & \text{if } r \text{ odd} \end{cases}$$

*Thus, when $f$ is of degree 1, $f^{\otimes}$ induces a degree $n$ morphism of dg vector space*

$$\Lambda^{gr,n}(V) \xrightarrow{f^{\otimes n}} S^{gr,n}(sV).$$

*This applies in particular to the dg isomorphism of degree 1, $i_s : V \longrightarrow sV$.*

*Proof.* Let $f : V \longrightarrow V'$ a dg morphism of degree $r$. First, it is enough to check the relation when $\sigma$ is any transposition $\tau_i$ exchanging $i$ and $i+1$ (for $1 \leqslant i \leqslant n-1$. This case follows from the case $n = 2$ and $\tau_i = \tau$ the switching morphism. As $\varepsilon(\tau_i) = -1$, it is enough to compute $(f \otimes f) \circ (-\tau)$

A direct computation shows that for any $v_1 \otimes v_2$ in $V \otimes V$ one has:

$$\begin{aligned}(f \otimes f) \circ (-\tau)(v_1 \otimes v_2) &= (-1)^{|v_1||v_2|+1+r|a_2|} f(v_2) \otimes f(v_1) \\ &= (-1)^{r|a_1|}(-1)^{-r^2+1}\tau(f(v_1) \otimes f(v_2)).\end{aligned}$$

This last equality concludes the proof. □

2.1.5. *Associative coalgebra.* A *differential graded associative* coalgebra $(C, d_C)$ abbreviated into dga coalgebra is a dg vector space equipped with a counital associative coproduct $\Delta_C$ of degree 0 commuting with the differential :

$$d_{C\otimes C} \circ \Delta_C = \Delta_C \circ d_C$$

and satisfying the usual commutative diagrams for an associative coalgebra, all the maps involved being maps of dg vector spaces.

One defines an iterated coproduct $\Delta^n : C \longrightarrow C^{\otimes(n+1)}$ by

$$\Delta^n = (\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}\,)\Delta^{n-1} \qquad \text{and} \qquad \Delta^1 = \Delta.$$

Note that, in the above formula$\Delta^n$ is independent of the place of the $\Delta$ factor (here in first position) because of the associativity of the coproduct. We may use Sweedler's notation:

$$\Delta(x) \sum x_{(1)} \otimes x_{(2)}\,, \qquad (\Delta) \otimes \mathrm{id}\,)\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$$

and

$$\Delta^n(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(n+1)}.$$

A coaugmentation on $C$ is a morphism of dga coalgebra $u : \mathbb{Q} \to C$. In this case, $C$ is canonically isomorphic to $\ker(\varepsilon) \oplus \mathbb{Q}u(1)$. Let $\bar{C} = \ker(\varepsilon)$ be the kernel of the counit.

If $C$ is coaugmented, its admits a decomposition $C = \mathbb{Q} \oplus \bar{C}$. The *reduced coproduct* is $\bar{\Delta} = \Delta - 1 \otimes \mathrm{id} - \mathrm{id} \otimes 1$. It is associative and there is an iterated reduced coproduct $\bar{\Delta}^n$.

We may also use Sweedler's notation for the reduced coproduct $\bar{\Delta}^n(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(n+1)}$.

**Definition 2.9.** $C$ is *conilpotent* if it is coaugmented and if for any $x$ in $C$ $\bar{\Delta}^n(x)$ vanishes for $n$ large enough.

A *cofree* associative dga coalgebra over the dg vector space is by definition a *conilpotent* dga coalgebra $F^c(V$ equipped with a linear map of degree 0 $p : F^c(C) \longrightarrow V$ commuting with the differential such that $p(1) = 0$ which factor any morphism of dg vector space $\phi : C \longrightarrow V$ where $C$ is a conilpotent dga coalgebra with $\phi(1) = 0$.

**Definition 2.10** (Tensor coalgebra)**.** The tensor coalgebra over a dg vector space $V$ is defined by

$$T^c(V) = \bigoplus_{n \geqslant 0} V^{\otimes n}$$

and equipped with the differential induced on each $V^{\otimes n}$ by $d_V$ and with the deconcatenation coproduct given by

$$[a_1|\cdots|a_n] \longmapsto \sum_{i=0}^{n+1} [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots a_n].$$

The deconcatenation coproduct is clearly associative. The natural projection $\pi_V : T^c(V) \longrightarrow \mathbb{Q} = V^{\otimes 0}$ onto the tensor degree 0 part is a counit for $T^c(V)$ while the inclusion $\mathbb{Q} = V^{\otimes 0} \longrightarrow T^c(V)$ gives the coaugmentation. The tensor coalgebra $T^c(V)$ is the cofree counital dga coalgebra over $V$ ; the conilpotency condition which is part of the definition of cofree is clearly satisfied.

Let $C$ be a conilpotent counital dg coalgebra and $\phi : C \longrightarrow V$ be a dg morphism of degree 0 with $\phi(1) = 0$. As essentially shown in [LV12, Proposition 1.2.7], there exist a unique morphism of (conilpotent counital) dga coalgebra $\tilde{\phi} : C \longrightarrow T^c(V)$ extending $\phi$.

2.1.6. *dg Lie algebra.* We review here the definition of Lie algebra and coLie coalgebra in the dg formalism.

For any dg vector space $V$, let $\xi$ be the cyclic permutation of $V \otimes V \otimes V$ defined by

$$\xi = (\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id}).$$

It corresponds to the cycle sending 1 to 3, 3 to 2 and 2 to 1.

**Definition 2.11.** A dg Lie algebra $L$ is a dg vector space equipped with a degree 0 map of dg vector spaces $c : L \otimes L \longrightarrow L$ ($c$ stands for "crochet") satisfying

$$c \circ \tau = -c \qquad \text{and} \qquad c \circ (c \otimes \mathrm{id}) \circ (\mathrm{id} + \xi + \xi^2) = 0$$

where $\tau$ is the switching morphism on $L \otimes L$ and $xi$ is the above permutation of the factor on $L \otimes L \otimes L$.

On elements, we will use a bracket notation $[x, y]$ instead of $c(x \otimes y)$.

In the above definition, the first relation is the usual antisymmetry of the bracket which gives in the dg context:

$$[x, y] = (-1)^{|x||y|}[y, x].$$

The second relation is the Jacobi relation:

$$[[x, y], z] + (-1)^{|x|(|y|+|z|)}[[y, z], x] + (-1)^{|z|(|y|+|x|)}[[z, x], y] = 0.$$

One remarks that $(c \otimes \mathrm{id}\,) \circ \xi = \tau \circ (\mathrm{id} \otimes c)$ and that $(c \otimes \mathrm{id}\,) \circ \xi^2 = ((c \circ \tau) \otimes \mathrm{id}\,) \circ (\mathrm{id} \otimes \tau)$. Using this and the antisymmetry relation, one can rewrite the Jacobi relation as a Leibniz relation:

$$c \circ (c \otimes \mathrm{id}\,) = c \circ (\mathrm{id} \otimes c) + c \circ (c \otimes \mathrm{id}\,) \circ (\mathrm{id} \otimes \tau).$$

2.1.7. *dg coLie coalgebra.* The definition of a dg coLie coalgebra is dual to the definition of a Lie algebra.

**Definition 2.12.** A dg coLie coalgebra $\mathcal{L}^c$ is a dg vector space equipped with a degree 0 map of dg vector spaces $\delta : \mathcal{L}^c \longrightarrow \mathcal{L}^c \otimes \mathcal{L}^c$ satisfying

$$\tau \circ \delta = -\delta \qquad \text{and} \qquad (\mathrm{id} + \xi + \xi^2) \circ (\delta \otimes \mathrm{id}\,) \circ \delta = 0$$

where $\tau$ is the switching morphism on $\mathcal{L}^c \otimes \mathcal{L}^c$ and $xi$ is the above permutation of the factor on $\mathcal{L}^c \otimes \mathcal{L}^c \otimes \mathcal{L}^c$.

The first condition shows that $\delta$ induces a map (again denoted by $\delta$)

$$\delta : \mathcal{L}^c \longrightarrow \mathcal{L}^c \oslash \mathcal{L}^c.$$

Then, the second condition show the following.

**Lemma 2.13.** *Let $\tau_{12} : \mathcal{L}^{c\otimes 3} \longrightarrow \mathcal{L}^{c\otimes 3}$ be the permutation exchanging the two first factors. The following diagram is commutative*

$$\begin{array}{ccccc} & & \mathcal{L}^c \otimes \mathcal{L}^c & \xrightarrow{\delta \otimes \mathrm{id} - \mathrm{id} \otimes \delta} & \mathcal{L}^c \otimes \mathcal{L}^c \otimes \mathcal{L}^c \\ & \overset{\delta}{\nearrow} & \uparrow & & \downarrow{\scriptstyle 1/6(\mathrm{id} - \tau_{12})(\mathrm{id} + \xi + \xi^2)} \\ \mathcal{L}^c & & & & \\ & \underset{\delta}{\searrow} & \Lambda^{gr,2}(\mathcal{L}^c) & \xrightarrow{\delta \oslash \mathrm{id} - \mathrm{id} \oslash \delta} & \Lambda^{gr,3}(\mathcal{L}^c) \end{array}$$

*and the composition going through the bottom line is* 0

*Proof.* We remark that

$$(\mathrm{id} \otimes \delta) = \xi^2 \circ (\delta \otimes \mathrm{id}\,) \circ \tau$$

which show that

$$((-1)\,\mathrm{id} \otimes \delta) \circ \delta = \xi^2 \circ (\delta \otimes \mathrm{id}\,) \circ \delta.$$

Thus, using the fact that $(\mathrm{id} + \xi + \xi^2) \circ (\delta \otimes \mathrm{id}\,) \circ \delta = 0$, one gets

$$(\mathrm{id} - \tau_{12}) \circ (\mathrm{id} + \xi + \xi^2) \circ (\delta \otimes \mathrm{id} - \mathrm{id} \otimes \delta) \circ \delta = 2(\mathrm{id} - \tau_{12}) \circ (\mathrm{id} + \xi + \xi^2) \circ (\delta \otimes \mathrm{id}\,) \circ \delta = 0.$$

□

2.1.8. *Twisting morphisms.* Let $(C, d_C)$ be a differential graded coaugmented coalgebra and let $(A, d_A)$ be a differential graded augmented algebra.

**Definition 2.14.** Let $f$, $g : C \longrightarrow A$ be two linear maps. The composition

$$f \star g = \mu \circ (f \otimes g) \circ \Delta : C \longrightarrow C \otimes C \longrightarrow A \otimes A \longrightarrow A \tag{1}$$

defines a internal composition law on $\mathrm{Hom}(C, V)$ called the *convolution* of $f$ and $g$.

Note that $f \star g$ is defined as soon as there are dg morphisms $C :\xrightarrow{\Delta} C \otimes C$ and $A \otimes A \xrightarrow{\mu} A$.

**Proposition 2.15** (Proposition 2.1.2 in [LV12])**.** *The composition law $\star$ on the dg vector space* $\mathrm{Hom}(C, A)$ *satisfies the Leibniz rule:*

$$d_{\mathrm{Hom}}(f \star g) = d_{\mathrm{Hom}}(f) \star g + (-1)^p f \star d_{\mathrm{Hom}}(g)$$

*for any morphism $f$ and $g$ of degree $p$ and $q$ respectively.*

*When $A$ is associative with unit $u$ and $C$ is coassociative with counit $\epsilon$ then* $(\mathrm{Hom}(C, A), d_{\mathrm{Hom}}, \star)$ *is a dga algebra with unit given by $u \circ \epsilon$.*

**Definition 2.16** (Twisting morphisms-associative case)**.** Let $(C, d_C)$ be a differential graded coaugmented coalgebra and let $(A, d_A)$ be a differential graded augmented algebra.

A *twisting morphism* $f : C \longrightarrow A$ is a degree 1 morphisms satisfying the Mauer-Cartan equation

$$d_{\mathrm{Hom}}(f) + f \star f = 0 \tag{2}$$

and null when composed with either the augmentation of $A$ or the coaugmentation of $C$.

The set of twisting morphisms is denoted by $\mathrm{Tw}(C, A)$.

**Definition 2.17** (Twisting morphisms-coLie case)**.** Let $(\mathcal{L}^c, d_{\mathcal{L}^c})$ be a differential graded coLie coalgebra and let $(A, d_A)$ be a differential graded augmented commutative algebra.

Loday and Vallette explained in [LV12, Proposition 6.4.7], that the natural structure on $\mathrm{Hom}(\mathcal{L}^c, A)$ is a dg Lie algebra structure given by

$$[f, g] = f \star g - g \star f.$$

A *twisting morphism* $f : C \longrightarrow A$ is a degree 1 morphisms satisfying the Mauer-Cartan equation which in this case is written in the equivalent form (coefficients are rational):

$$d_{\mathrm{Hom}}(f) + \frac{1}{2}[f, f] = 0. \tag{3}$$

and null when composed with the augmentation of $A$.

## 2.2. **Bar/cobar adjunction: associative case.**

2.2.1. *Bar construction.* In this subsection, we recall briefly the bar/cobar construction and how they give a pair of adjoint functor in the associative case.

$$B : \left\{ \begin{array}{c} \text{diff. gr. ass.} \\ \text{algebras} \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{diff. gr. coass.} \\ \text{coalgebras} \end{array} \right\} : \Omega.$$

Let $(A, d_A, \mu_A, \varepsilon_A)$ be an augmented dga algebra and $\bar{A} = \ker(\varepsilon_A)$ its augmentation ideal. The bar construction of $A$ is obtained by twisting the differential of the dga free coalgebra $T^c(s^{-1}\bar{A})$.

The differential $d_A$ makes $\bar{A}$ and thus $s^{-1}\bar{A}$ into a dg vector vector space. Let $D_1$ denote the induced differential on $T^c(s^{-1}\bar{A})$ which in tensor degree $n$ is:

$$\sum_{i=1}^{n} \mathrm{id}^{\,i-1} \otimes d_{s^{-1}\bar{A}} \otimes \mathrm{id}^{\,n-i}.$$

$S^{-1} = s^{-1}\mathbb{Q}$ admits an associative product-like map of degree $+1$ defined by:

$$\Pi_s : s^{-1}\mathbb{Q} \otimes s^{-1}\mathbb{Q} \longrightarrow s^{-1}\mathbb{Q} \qquad \Pi_s(s^{-1} \otimes s^{-1}) = s^{-1}.$$

The map $\Pi_s$ and the restriction $\mu\bar{A}$ of the multiplication $\mu_A$ to $\bar{A}$ induce the following map:

$$f : s^{-1}\mathbb{Q} \otimes \bar{A} \otimes s^{-1}\mathbb{Q} \otimes \bar{A} \xrightarrow{\mathrm{id} \otimes \tau \otimes \mathrm{id}} s^{-1}\mathbb{Q} \otimes s^{-1}\mathbb{Q} \otimes \bar{A} \otimes \bar{A} \xrightarrow{\Pi_s \otimes \mu_{\bar{A}}} s^{-1}\mathbb{Q} \otimes \bar{A}.$$

This map induces a degree 1 map $D_2 : T^c(s^{-1}\bar{A}) \longrightarrow T^c(s^{-1}\bar{A})$ which satisfies $D_2^2 = 0$ because of the associativity of $\mu_A$

One check that the degree 1 morphism $D_1$ and $D_2$ commutes (in the graded sense):

$$D_1 \circ D_2 + D_2 \circ D_1 = 0$$

The coproduct on $T^c(s^{-1}\bar{A})$ is given by the deconcatenation coproduct. From these definitions, one obtains (see [LV12][Section 2.2.1]) the following.

**Lemma 2.18.** *The complex $B(A) = (T^c(s^{-1}\bar{A}), d_B)$ with $d_B = D_1 + D_2$ and endowed with the deconcatenation coproduct $\Delta$ is a conilpotent dga coalgebra.*

We recall below the explicit formulas related to the bar construction $B(A)$:

- An homogeneous element $\mathbf{a}$ of tensor degree $n$ is denoted by
$$[s^{-1}a_1|\cdots|s^{-1}a_n]$$
or when the context is clear enough not to forget the shifting simply by $[a_1|\cdots|a_n]$. Its degree is given by:
$$\deg_B(\mathbf{a}) = \sum_{i=1}^{n} \deg_{s^{-1}\bar{A}}(sa_i) = \sum_{i=1}^{n} (\deg_A(a_i) - 1)$$
- the coproduct is given by:
$$\Delta(\mathbf{a}) = \sum_{i=1}^{n} [s^{-1}a_1|\cdots|s^{-1}a_i] \otimes [s^{-1}a_{i+1}|\cdots|s^{-1}a_n].$$
- Let $\eta_{\mathbf{a}}(i)$ or simply $\eta(i)$ denote the "partial degree" of $\mathbf{a}$:
$$\eta_{\mathbf{a}}(i) = \sum_{k=1}^{i} \deg_{s^{-1}\bar{A}}(s^{-1}a_k) = \sum_{k=1}^{i} (\deg_A(a_k) - 1).$$
- The differential $D_1$ and $D_2$ are explicitly given by the formulas:
$$D_1(\mathbf{a}) = -\sum_{i=1}^{n} (-1)^{\eta(i-1)} [s^{-1}a_1|\cdots|s^{-1}d_A(a_i)|\cdots|s^{-1}a_n]$$
and
$$D_2(\mathbf{a}) = -\sum_{i=1}^{n} (-1)^{\eta(i)} [s^{-1}a_1|\cdots|s^{-1}\mu_A(a_i, a_{i+1})|\cdots|s^{-1}a_n].$$
The global minus sign in $D_1$ appears because the differential of the dg vector space $s^{-1}\bar{A}$ is given by $d_{s^{-1}\bar{A}}(s^{-1}a) = -s^{-1}d_A(a)$. The other signs are due to the Kozul sign rules taking care of the shifting.

*Remark* 2.19. This construction can be seen as a simplicial total complex associated to the complex $A$ (as in [BK94]). Here, the augmentation makes it possible to use directly $\bar{A}$ without referring to the tensor coalgebra over $A$ and without the need of killing the degeneracies. However this simplicial presentation usually masks the need of working with the shifted complex.

The bar construction $B(A)$ also admits a product ш which shuffles the tensor factors. However, this extra structure becomes more interesting when $A$ is graded commutative and we will present it in the next section.

The bar construction is a quasi-isomorphism invariant as shown in [LV12] (Proposition 2.2.4) and the construction provides a functor:

$$B : \{\text{aug. dga algebra}\} \longrightarrow \{\text{coaug. dga coalgebra}\}.$$

2.2.2. *Cobar construction.* Analogously, one constructs the cobar functor. Let $(C, d_C, \Delta_C, \varepsilon_C$ be a coaugmented dga coalgebra decomposed as $C = \bar{C} \oplus \mathbb{Q}$ with . Consider $T(s\bar{C})$ the free algebra over $s\bar{C}$ (with concatenation product). The differential on $C$ induces a differential $d_1$ on $T(s\bar{C})$. $S = s\mathbb{Q}$ comes with a coproduct-like degree 1 map:

$$\Delta_s : s\mathbb{Q} \longrightarrow s\mathbb{Q} \otimes s\mathbb{Q}\,, \qquad \Delta_s(s) = -s \otimes s.$$

The map $\Delta_s$ and the restriction of the reduced coproduct $\bar{\Delta}_c$ to $\bar{C}$ induce the following map:

$$g : s\bar{C} \xrightarrow{\Delta_s \otimes \bar{\Delta}_C} s\mathbb{Q} \otimes s\mathbb{Q} \otimes \bar{C} \otimes \bar{C} \xrightarrow{\mathrm{id}\ \otimes \tau \otimes \mathrm{id}} s\mathbb{Q} \otimes \bar{C} \otimes s\mathbb{Q} \otimes \bar{C}$$

which induces a degree 1 map $d_2$ on $T(s\bar{C})$ satisfying $d_2^2 = 0$ because of the coassociativity of $\Delta_c$. The two degree 1 maps $d_1$ and $d_2$ commutes (in the graded sense):

$$d_1 \circ d_2 + d_2 \circ d_1 = 0.$$

**Lemma 2.20.** *The complex $\Omega(C) = (T(s\bar{C}), d_\Omega)$ with $d_B = D_1 + D_2$ and endowed with the concatenation product is a augmented dga algebra called the cobar construction of $C$.*

Note that the cobar construction is not in general a quasi-isomorphisms invariant. The reader may look at [LV12, Section 2.4] for more details.

2.2.3. *Adjunction.* The two functors bar and cobar induces an adjunction described as follows:

**Theorem 2.21** ([LV12, Theorem 2.2.9 and Corollary 2.3.4])**.** *For every augmented dga algebra $A$ and every conilpotent dga coalgebra $C$ there exist natural bijections*

$$\mathrm{Hom}_{\mathrm{dga\ alg}}(\Omega(C), A) \simeq \mathrm{Tw}(C, A) \simeq \mathrm{Hom}_{\mathrm{dga\ coalg}}(C, B(A)).$$

*This adjunction, gives two universal twisting morphisms : $\pi : B(A) \longrightarrow A$ induced by the projection onto the tensor degree 1 onto $s^{-1}\bar{A}$; $\iota : C \longrightarrow \Omega(C)$ induced by the inclusion of $s\bar{C}$ into the tensor degree 1 part.*

*The unit $\upsilon : C \longrightarrow B \circ \Omega(C)$ and the counit $\epsilon : \Omega \circ B(A) \longrightarrow A$ are quasi-isomorphisms of dga coalgebras and algebras respectively.*

2.3. **Bar/cobar adjunction: commutative algebras/coLie Coalgebras.** In this section we recall the bar/cobar adjunction in the commutative/coLie case giving a pair of functors:

$$B_c : \left\{ \begin{array}{c} \text{diff. gr. com.} \\ \text{ass. algebras} \end{array} \right\} \rightleftharpoons \left\{ \begin{array}{c} \text{diff. gr. coLie.} \\ \text{coalgebras} \end{array} \right\} : \Omega_L.$$

The cobar construction in the coLie case is a little more delicate. We will concentrate on this construction. The bar construction in the commutative case, will be presented as the set of indecomposable elements of the associative bar construction. A direct construction can be found in [SW11a]. Other description were given in [GJ94, GK94].

2.3.1. *Cobar construction for coLie coalgebras.* The construction essentially follows the one given for commutative dga coalgebra, however the lack of associativity and the use of the symmetric algebra makes the construction a little more delicate.

First we need a notion of conilpotency for a coLie algebra $(\mathcal{L}^c, \delta, d)$. As $\delta$ is not associative, one can not directly use the reduced iterated coproduct and one needs to introduce trivalent trees.

A rooted trivalent tree, or simply a tree when the context is clear, is a planar tree (at each internal vertex a cyclic ordering of the incident edges is given) where

vertices have valency 1 (external vertices) or 3 (internal vertices) together with a distinguished external vertex (the root); other external vertices are called leaves.

Trees will be drawn with the convention that the cyclic ordering of the edges around an internal vertex is displayed in counterclockwise direction. The root vertex in the case of a rooted tree is displayed at the top. A tree comes with a canonical numbering of its edge, starting from the root edge and induced by the cyclic ordering at each vertex. This numbering of the edge, induce a canonical numbering of the leaves (the root has number 0).

Let $(\mathcal{L}^c, \delta, d)$ be a dg coLie coalgebra. Recall that $\delta$ is a dg morphism $\delta : \mathcal{L}^c \longrightarrow \mathcal{L}^c \otimes \mathcal{L}^c$. In order to emphasize the target space, we may write $\delta_\otimes$ for $\delta$ and $\delta_\Lambda$ for the induced morphism $\delta_\Lambda : \mathcal{L}^c \longrightarrow \mathcal{L}^c \Lambda \mathcal{L}^c$.

**Definition 2.22.** Let $T$ be a tree with $n$ as above and let $\{e_1, \dots, e_n\}$ be the set of its leaves ($e_i$ is the $i$-th leaf). $T$ induces a morphism

$$\delta_T : \mathcal{L}^c \longrightarrow \mathcal{L}^{c \otimes n}$$

as follows:

- if $T$ has $n = 1$ leaf, $\delta_T = \mathrm{id}_{\mathcal{L}^c}$;
- if $T$ has $n = 2$ leaves, then $T =$  and $\delta_T = \delta_\otimes$;
- if $T$ has $n \geqslant 3$ leaves, then there exists at least one leaf $e_i$ in a strict subtree of the form

  $T_0 =$ 

  where $v$ is an internal vertex of $T$. Let $T'$ be the tree $T \setminus T_0$, where the internal vertex $v$ of $T$ is the $i$-th leaf (in $T'$). The morphism $\delta_T$ is defined by

  $$\delta_T = (\mathrm{id}^{\otimes(i-1)} \otimes \delta \otimes \mathrm{id}^{\otimes(n-i)}) \circ \delta_{T'}.$$

This definition does not depend on the choice of the leaf $e_i$; it is as if each internal vertex of $T$ contributes for a $\delta$ keeping track of the positions of the factors in $\mathcal{L}^c$ in the tensor products.

We can now define a conilpotent dg coLie coalgebra by analogy with the associative case.

**Definition 2.23.** We say that $(\mathcal{L}^c, \delta, d)$ is *conilpotent* if for any $x \in \mathcal{L}^c$ there exist $n$ big enough such that for any tree $T$ with $k$ leaves, $k \geqslant n$, $\delta_T(x) = 0$.

We now fix a conilpotent dg coLie coalgebra $(\mathcal{L}^c, \delta, d_{\mathcal{L}^c})$. Its cobar construction is given by twisting the differential of the free commutative dga $S^{gr}(s\mathcal{L}^c)$.

The differential $d_{\mathcal{L}^c}$ induces a differential $d_{s\mathcal{L}^c}$ on $s\mathcal{L}^c$ and thus on $(s\mathcal{L}^c)^{\otimes n}$ given by

$$\sum_{i=1}^{n} id^{\otimes(i-1)} \otimes d_{s\mathcal{L}^c} \otimes \mathrm{id}^{\otimes(n-i)} : (s\mathcal{L}^c)^{\otimes n} \longrightarrow (s\mathcal{L}^c)^{\otimes n}.$$

This differential goes down to a differential on $n$-th symmetric power of $s\mathcal{L}^c$:

$$D_1 : S^{gr,n}(s\mathcal{L}^c) \longrightarrow S^{gr,n}(s\mathcal{L}^c).$$

Using the map $\Delta_s$ and the cobracket $\delta_{\mathcal{L}^c}$, one has a morphism:

$$g_L : s\mathcal{L}^c \xrightarrow{\Delta_s \otimes \mathcal{L}^c} s\mathbb{Q} \otimes s\mathbb{Q} \otimes \mathcal{L}^c \otimes \mathcal{L}^c \xrightarrow{\mathrm{id} \otimes \tau \otimes id} s\mathbb{Q} \otimes \mathcal{L}^c \otimes s\mathbb{Q} \otimes \mathcal{L}^c.$$

which induces a degree 1 map $\delta^s : s\mathcal{L}^c \longrightarrow S^{2,gr}(s\mathcal{L}^c)$ because

$$\begin{aligned}-\tau_{s\mathcal{L}^c\otimes s\mathcal{L}^c}\circ g_L &= (\mathrm{id}\otimes\tau\,\mathrm{id})\circ(-\tau_{S\otimes S}\otimes\tau_{\mathcal{L}^c\otimes\mathcal{L}^c})\circ(\Delta_s\otimes\delta_{\mathcal{L}^c})\\ &= (\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ((\mathrm{id}_{S\otimes S}\circ\Delta_s)\otimes(\tau_{\mathcal{L}^c\otimes\mathcal{L}^c}\circ\delta_{\mathcal{L}^c}))\end{aligned}$$

which shows that

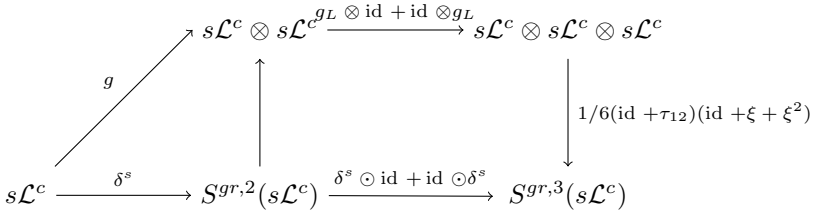
$$(\mathrm{id}-\tau)_{s\mathcal{L}^c\otimes s\mathcal{L}^c}\circ g_L = (\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\Delta_s\otimes((\mathrm{id}+\tau)_{\mathcal{L}^c\otimes\mathcal{L}^c}\circ\delta_{\mathcal{L}^c})) = 0$$

Now, we want to show that $\delta_s$ induce a differential $D_2$ on $S^{gr}(s\mathcal{L}^c)$ given on $S^{gr,n}(s\mathcal{L}^c)$ by the formula:

$$D_2|_{S^{gr,n}(s\mathcal{L}^c)} = \sum_{i=1}^{n}\mathrm{id}^{\otimes(i-1)}\otimes g_L\otimes\mathrm{id}^{\otimes(n-i)}\,.$$

In order to do so, it is enough to prove the following lemma.

**Lemma 2.24.** *The composition given by the bottom line of commutative diagram below is zero:*




*where $\tau_{12}$ exchange the two first factors of $s\mathcal{L}^c$ and $\xi$ is the cyclic permutation of the factors $s\mathcal{L}^c$ previously describe $(1\to3\to2\to1)$.*

*Proof.* In order to check that the bottom line is 0, we will work with the following spaces :

$$s\mathcal{L}^c = s\mathbb{Q}\otimes\mathcal{L}^c\,,\quad (s\mathbb{Q})^{\otimes2}\otimes(\mathcal{L}^c)^{\otimes2}\,,\quad (s\mathcal{L}^c)^{\otimes2}\quad (s\mathbb{Q})^{\otimes3}\otimes(\mathcal{L}^c)^{\otimes3}\quad (s\mathcal{L}^c)^{\otimes3}.$$

Let $\phi$ be an element of $\mathbb{Q}[\mathbb{S}_n]$, we shall denote by $(\phi)_V$ the induce morphism on $V^{\otimes n}$. Let $f : (s\mathbb{Q})^{\otimes3}\otimes(\mathcal{L}^c)^{\otimes3}\longrightarrow(s\mathcal{L}^c)^{\otimes3}$ be the morphism defined by (note that $S = s\mathbb{Q}$)

$$\begin{aligned}f =&(\mathrm{id}_{S\otimes\mathcal{L}^c\otimes S}\otimes\tau\otimes\mathrm{id}_{\mathcal{L}^c})\circ(\mathrm{id}_S\otimes\tau\otimes\mathrm{id}_{S\otimes\mathcal{L}^c\otimes\mathcal{L}^c})\circ(\mathrm{id}_{S\otimes S}\otimes\tau\otimes\mathrm{id}_{\mathcal{L}^c\otimes\mathcal{L}^c})\\ =&(\mathrm{id}_S\otimes\tau\otimes\mathrm{id}_{\mathcal{L}^c\otimes S\otimes\mathcal{L}^c})\circ(\mathrm{id}_{S\otimes S\otimes\mathcal{L}^c}\otimes\tau\otimes\mathrm{id}_{\mathcal{L}^c})\circ(\mathrm{id}_{S\otimes S}\otimes\tau\otimes\mathrm{id}_{\mathcal{L}^c\otimes\mathcal{L}^c}).\end{aligned}$$

Then for any permutation $\sigma$ in $\mathbb{S}_3$, one has

$$(\sigma)_{s\mathcal{L}^c} = f\circ((\sigma)_S\otimes(\sigma)_{\mathcal{L}^c})\circ f^{-1}.$$

Thus, as $\tau_{12}$ and $\xi$ act on $S^3$ by $-\mathrm{id}$ and $\mathrm{id}$ respectively, one gets

$$(4)\quad (\mathrm{id}+\tau_{12})_{s\mathcal{L}^c}(\mathrm{id}+\xi+\xi^2)_{s\mathcal{L}^c} = f\circ(\mathrm{id}_{S^{\otimes3}}\otimes(\mathrm{id}-\tau_{12})_{\mathcal{L}^c}(\mathrm{id}+\xi+\xi^2)_{\mathcal{L}^c})\circ f^{-1}.$$

Now, we remark that

$$(g_L\otimes\mathrm{id}_{s\mathcal{L}^c})\circ g_L = f\circ((\Delta_s\otimes\mathrm{id}_S)\circ\Delta_S)\otimes((\delta_{\mathcal{L}^c}\otimes\mathrm{id}_{\mathcal{L}^c})\circ\delta_{\mathcal{L}^c})$$

and

$$(\mathrm{id}_{s\mathcal{L}^c}\otimes g_L)\circ g_L = f\circ((\mathrm{id}_S\otimes\Delta_s)\circ\Delta_S)\otimes((\mathrm{id}_{\mathcal{L}^c}\otimes\delta_{\mathcal{L}^c})\circ\delta_{\mathcal{L}^c})\,.$$

Because $(\mathrm{id}_S\otimes\Delta_s)\circ\Delta_S = -(\Delta_s\otimes\mathrm{id}_S)\circ\Delta_S$, the above relations show that

$$\begin{aligned}(5)\quad &(g_L\otimes\mathrm{id}_{s\mathcal{L}^c}+\mathrm{id}_{s\mathcal{L}^c}\otimes g_L)\circ g_L =\\ &f\circ((\Delta_s\otimes\mathrm{id}_S)\circ\Delta_S\otimes)\otimes(\delta_{\mathcal{L}^c}\otimes\mathrm{id}_{\mathcal{L}^c}-\mathrm{id}_{\mathcal{L}^c}\otimes\delta_{\mathcal{L}^c})\,\delta_{\mathcal{L}^c}.\end{aligned}$$

Putting equations (4) and (5) together, Lemma 2.13 concludes the proof.

□

The differential $D_1$ and $D_2$ commute (in the graded sens), that is $D_1 \circ D_2 + D_2 \circ D_1 = 0$ and one obtains the following.

**Lemma 2.25.** *The complex $\Omega_{coL}(\mathcal{L}^c) = (S^{gr}(s\mathcal{L}^c), d_{\Omega,coL})$ with the symmetric concatenation product is an augmented commutative dga algebra called the cobar-coLie construction of $\mathcal{L}^c$.*

2.3.2. *Bar construction for commutative dga algebras.* Let $(A, d_A, \mu_A, \varepsilon_A)$ be an augmented commutative dga algebra and $\bar{A} = \ker(\varepsilon_A)$. One can consider its bar construction $B(A)$ as associative algebra. One defines on the coalgebra $B(A)$ an associative product ш by the formula

$$\begin{aligned}[x_1|\cdots|x_n] \text{ ш } [x_{n+1}|\cdots|x_{n+m}] &= \sum_{\sigma\in \mathrm{sh}(n,m)} \rho_S(\sigma)([x_1|\cdots|x_n|x_{n+1}|\cdots|x_{n+m}]) \\ &= \sum_{\sigma\in \mathrm{sh}(n,m)} \varepsilon^{gr}(\sigma)([x_{\sigma^{-1}(1)}|\cdots|x_{\sigma^{-1}(n+m)}])\end{aligned}$$

where $\mathrm{sh}(n,m)$ denotes the subset $\mathbb{S}_{n+m}$ preserving the order of the ordered sets $\{1,\dots,n\}$ and $\{n+1,\dots,n+m\}$. For grading reasons, and thus for signs issues, it is important to note that, in the above formula, the $x_i$'s are elements of $s^{-1}A$.

A direct computation shows that ш turns $B(A)$ into an augmented commutative dga Hopf algebra; that is it respects the expected diagrams for a Hopf algebra, all the morphisms involved being dg-morphisms.

The augmentation of $B(A)$ is the projection onto the tensor degree 0 part. Let $B(A)^+$ be the kernel of the augmentation of $B(A)$ and

$$Q_B(A) = B(A)^+/(B(A)^+)^2$$

be the set of its indecomposable elements. Ree's theorem [LV12, Theorem 1.3.9] (originally in [Ree58]) shows the following

**Lemma 2.26.** *The reduced coproduct $\Delta'$ induce a cobracket $\delta_Q = \Delta' - \tau\Delta'$ on $Q_B(A)$ making it a conilpotent coLie dg coalgebra. The differential $d_B$ induces a differential $d_Q$ on $Q_B(A)$.*

*The complex $(Q_B(A), d_Q)$ endowed with the cobracket $\delta_Q$ is the* commutative bar construction *of $A$ and denoted by $B_{com}(A)$.*

Working with the indecomposable elements is not always easy as one has to work modulo products. In particular, obtaining explicit formula may be problematic as such a formula would be defined up to shuffle. Moreover, some structure, say for example extra filtrations, may not behave well by taking a quotient. For this purpose, R. Hain, dealing with Hodge structure problems, gave in [Hai86] a splitting

$$i_Q : Q_B(A) \longrightarrow B(A)^+$$

of the projection $p_Q : B(A)^+ \longrightarrow Q_B(A)$ commuting with the differential. The projector of $B(A)^+$ given by the composition $p_{\text{ш}} = i_Q \circ p_Q$ can be express using the following explicit formula

$$p_{\text{ш}}([a_1|\cdots|a_n]) = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i} \text{ ш} \circ (\bar{\Delta})^{i-1}([a_1|\cdots|a_n])$$

where the associative product ш has been extend to $B(A)^{\otimes n}$ for all $n \geqslant 2$ and where $\bar{\Delta}^{(0)} = \mathrm{id}$.

2.3.3. *Adjunction.* As in the case of associative algebras and coalgebras, the functors $\Omega_{coL}$ and $B_{com}$ are adjoint.

**Theorem 2.27.** *For any augmented commutative dga algebra $A$ and any conilpotent dg coLie coalgebra $\mathcal{L}^c$ there exist natural bijections*

$$\mathrm{Hom}_{\text{com dga alg}}(\Omega_{coL}(\mathcal{L}^c), A) \simeq \mathrm{Tw}_{coL;com}(\mathcal{L}^c, A) \simeq \mathrm{Hom}_{\text{conil dg coLie}}(\mathcal{L}^c, B_{com}(A)).$$

*This adjunction, gives two universal twisting morphisms : $\pi : B_{com}(A) \longrightarrow A$ induced by the projection onto the tensor degree 1 onto $s^{-1}\bar{A}$; $\iota : \mathcal{L}^c \longrightarrow \Omega_{coL}(\mathcal{L}^c)$ induced by the inclusion of $s\bar{\mathcal{L}}^c$ into the tensor degree 1 part of $S^{gr}(s\mathcal{L}^c)$.*

*The unit $\upsilon : \mathcal{L}^c \longrightarrow B_{com} \circ \Omega_{coL}(\mathcal{L}^c)$ and the counit $\epsilon : \Omega_{coL} \circ B_{com}(A) \longrightarrow A$ are quasi-isomorphisms of conilpotent dg co Lie algebras and commutative dga algebras respectively.*

Note that the following diagram of vector space is commutative:

$$\begin{array}{ccc} \mathcal{L}^c & \longrightarrow & \Omega_{coL}(\mathcal{L}^c) = (S^{gr}(s\mathcal{L}^c), d_{\Omega,coL}) \\ & \overset{\upsilon}{\searrow} & \uparrow{\scriptstyle \pi_1} \\ & & B_{com}(\Omega_{coL}(\mathcal{L}^c)) = Q_B\left(S^{gr}(s\mathcal{L}^c), d_{\Omega,coL}\right) \end{array}$$

where $\pi_1$ is the projection onto the tensor degree 1 part $\pi_1 : \oplus_{n\geqslant 0}\Omega(\mathcal{L}^c)^{\otimes n}$ restricted to the set of indecomposable elements, that is restricted to the image of $p_{\text{ш}}$.

2.4. **An explicit map.** In this last section about the bar/cobar adjunction, we present an explicit description for the unit $\upsilon : \mathcal{L}^c \longrightarrow B_{com} \circ \Omega_{coL}(\mathcal{L}^c)$ with image in $T^c(S^{gr}(s\mathcal{L}^c))$ using the projector $p_{\text{ш}}$ onto the indecomposable elements.

In order to make explicit the morphism $\upsilon$, we need to introduce the following definition.

**Definition 2.28.** Let $n$ be a positive integer.

- The Catalan number $C(n-1)$ gives the number of rooted trivalent trees with $n$ leaves.
- Let $T$ be such a rooted trivalent tree with $n$ leaves. Define $\tilde{\Delta}_T$ by
$$\tilde{\Delta}_T = \frac{1}{2^n}\Delta_T$$
where the map $\phi_T : \mathcal{L}^c \longrightarrow \mathcal{L}^{c\otimes n}$ was introduced at Definition 2.22.
- The morphism $\tilde{\Delta}_n$ is up to a normalizing coefficient the sums for all trees $T$ with $n$ Leaves of the morphism $\tilde{\Delta}_T$:
$$\tilde{\Delta}_n = \sum_{T \text{ s.t. } |T|=n} \frac{1}{nC(n-1)}\tilde{\Delta}_T.$$
where $|T|$ denotes the number of leaves of the tree $T$.

Note that using the identification $\mathcal{L}^c \simeq s^{-1} \otimes s \otimes \mathcal{L}^c$ given by $x \mapsto s^{-1} \otimes s \otimes x$, the morphism $\tilde{\Delta_n}$ induces a morphism

$$\mathcal{L}^c \longrightarrow \left(s^{-1} \otimes s \otimes \mathcal{L}^c\right)^{\otimes n}$$

again denoted by $\tilde{\Delta_n}$.

**Claim.** *The composition*

$$\phi_{\mathcal{L}^c} = p_{\text{ш}} \circ \left(\sum_{n\geqslant 1} \tilde{\Delta_n}\right)$$

*which lands into the indecomposable elements of the bar construction over $\Omega_{coL}(\mathcal{L}^c)$ (which is as algebra isomorphic to $S^{gr}(s\mathcal{L}^c)$ but not with the dg structure):*

$$\phi_{cL} : \mathcal{L}^c \longrightarrow B_{com} \circ \Omega_{coL}(\mathcal{L}^c) = \left(Q_B\left(S^{gr}(s\mathcal{L}^c), d_{\Omega,coL}\right), d_B\right)$$

*is equal to the unit of the adjunction; that is*

$$\phi_{\mathcal{L}^c} = \upsilon.$$

Note that by definition $\sum_{n\geqslant 1} De\tilde{l}ta_n$ lands into $\left(s^{-1}\otimes s\otimes \mathcal{L}^c\right)^{\otimes n}$ which is a subspace of

$$\left(s^{-1}S^{gr}(s\mathcal{L}^c)\right)^{\otimes n} \subset B(S^{gr}(s\mathcal{L}^c))^+.$$

Then, the projection $p_{\text{ш}}$ makes sure that the morphism $\phi_{\mathcal{L}^c}$ lands into the set of indecomposable elements.

The idea for this formula comes from mimicking the associative case where the unit morphism being a coalgebra morphisms has to be compatible with the iterated reduced coproduct. Similarly, in the coLie case, the unit have to be compatible with the map $\Delta_{T,\mathcal{L}^c}$ and $\Delta_{T,Q_B(S^{gr}(s\mathcal{L}^c))}$. In order to obtain a direct expression in $\sum_{n\geqslant 1}\left(s^{-1}\otimes s\otimes \mathcal{L}^c\right)^{\otimes n}$ one needs to relate the morphisms $\Delta_T$ with the iterated reduced deconcatenation coproduct on $B(S^{gr}(s\mathcal{L}^c))$. This leads us to the normalizing coefficients. This formula can be derived as a consequence of the work of Getzler and Jones [GJ94] or of the work of Sinha and Walter [SW11b].

## 3. Families of bar elements

3.1. **From differential systems to bar elements.** Let $(A,\mu_A,d_A,\varepsilon_A)$ be an augmented cdga algebra. As previously, the product $\mu_A(x,y)$ can be denoted simply by $x\cdot y$. Under some additional conditions on $A$, the indecomposable elements of the $\mathrm{H}^0$ of the bar-construction $B(A)$ provides generating the 1-minimal model of $A$ in the sens of Sullivan [Sul77]. In [DGMS75], the authors gave a concrete construction for such a model. This construction induces (and is essentially encoded) by a family of elements of degree 1 $\{v_{i,k}\}_{i\in I_k,\, k\geqslant 1}$ where $I_k$ is a finite set and a differential system in $A$ relating the elements $v_{i,k}$:

$$d_A(v_{i,k}) = \sum_{\substack{l+l'=k\\ j\in I_l,\, j'\in I_{l'}}} a^{i,k}_{j,l\;\;j',l'}\, v_{j,l}\cdot v_{j',l'}.$$

A general question is to relate such a construction of the 1-minimal model to the bar construction setting, in particular to relate such a differential system to specific elements in $B(A)$ providing the expected classes in $Q_B(\mathrm{H}^0(B(A)))$.

The goal of this section is to show how, some differential systems in $A$ (recovering partly the above question) induce naturally elements in $B(A)$ lying into the image of $p_{\text{ш}}$ and having 0 differential; these elements which can written directly in terms of $A^{\otimes n}$ provide classes in $Q_B(\mathrm{H}^0(B(A)))$.

**Definition 3.1.** (Adams or weight graded cdga Algebra) An augmented cdga $(A,\mu_A,d_A,\varepsilon_A)$ is weight (or Adams) graded if the dg vector space $A = A^\bullet$ admits a decomposition into sub-dg vector space $A^\bullet = \oplus_{n\geqslant 0} A^\bullet(p)$ such that:

- the multiplication restricts as $\mu_A : A^k(p)\otimes A^l(q) \longrightarrow A^{k+l}(p+q)$ and does not see the weight grading; as if the weight grading was only even.
- $A(0) = \mathbb{Q}$ is the image of the unit morphism $\mathbb{Q} \longrightarrow A$.

We now assume that $(A,\mu_A,d_A,\varepsilon_A)$ is weight graded. Consider a family of elements $\{v_{i,p}\}_{i\in I_p,\, p\geqslant 1}$ in $A^1$ such that

(1) $I_p$ is a finite set of indices depending on $p$;

(2) for any $p \geqslant 1$ and any $i \in I_p$, the element $v_{i,p}$ is in $A^1(p)$; that is of degree 1 and weight $p$;
(3) elements of weight 1 are mapped to 0 by the differential:
$$\forall i \in I_1 \qquad d_A(v_{i,1}) = 0;$$
(4) the differential of any element $v_{i,p}$ is decomposable inside the family:
$$\forall p \geqslant 2, \quad \forall i \in I_p \qquad d_A(v_{i,p}) = \sum_{\substack{p_1+p_2=p \\ i_1 \in I_{p_1},\, i_2 \in I_{p_2}}} a^{i,p}_{\substack{i_1,p_1 \\ i_2,p_2}} v_{i_1,p_1} \cdot v_{i_2,p_2}$$
where the coefficients $a^{i,p}_{\substack{i_1,p_1 \\ i_2,p_2}}$ are rationals;
(5) the coefficients $a^{i,p}_{\substack{i_1,p_1 \\ i_2,p_2}}$ alone show that $d_A^2(v_{i,p}) = 0$ for any $p$ and $i \in I_p$; that is no relation between the $v_{i,p}$ is used in the relation $d_A^2(v_{i,p}) = 0$.

**Proposition 3.2.** *Let $A$ and a differential system of elements $v_{i,p}$ be as above. Then for any $p \geqslant 1$ and any $i$ in $I_p$ there exist an element $v^B_{i,p}$ in $B(A)$ the bar construction over $A$ satisfying:*

(1) *Let $\pi_1 : B(A) \longrightarrow A$ be the projection onto the tensor degree 1 part, then $\pi_1(v^B_{i,p}) = v_{i,p}$;*
(2) $v^B_{i,p} = p_{\text{ш}}(v^B_{i,p})$;
(3) $v^B_{i,p}$ *is of degree 0 in $B(A)$ and*
$$d_{B(A)}(v^B_{i,p}) = 0;$$
(4) *Writing $\overline{v^B_{i,p}}$ for the class of $v^B_{i,p}$ in the $\mathrm{H}^0(Q_B(A))$ which is the set of indecomposable elements of the $\mathrm{H}^0(B(A))$, one has for all $p \geqslant 2$ and all $i \in I_p$*
$$\delta_Q(\overline{v^B_{i,p}}) = \sum_{\substack{p_1+p_2=p \\ i_1 \in I_{p_1},\, i_2 \in I_{p_2}}} -a^{i,p}_{\substack{i_1,p_1 \\ i_2,p_2}} \overline{v^B_{i_1,p_1}} \wedge \overline{v^B_{i_2,p_2}}$$
*where $\delta_Q : \mathrm{H}^0(Q_B(A)) \longrightarrow \mathrm{H}^0(Q_B(A)) \wedge \mathrm{H}^0(Q_B(A))$ is the cobracket induced on $\mathrm{H}^0(Q_B(A))$ by the coproduct on $B(A)$; with a slight abuse of notation $\delta_Q = 1/2(\bar{\Delta} - \tau \circ \bar{\Delta})$.*

*In weight 1 one has $\delta_Q(v_{i,1}) = 0$ for any $i \in I_1$.*

The general idea is to relate the above differential system to a coLie algebra in order to use the unit of the adjunction bar/cobar in the commutative/coLie case. The condition (5) on the coefficients of the differential system insures that one has an "underlying" coLie algebra while condition (3) and the weight grading insures the conilpotency of this coLie algebra.

*Proof.* Let $A$ and a family of elements $\{v_{i,p}\}$ be as above. Let $\mathcal{L}^c$ be the dg-vector space concentrated in degree 0 generated by formal elements $f_{i,p}$:
$$\mathcal{L}^c = \oplus_{p \geqslant 1} \oplus_{i \in I_p} \mathbb{Q} f_{i,p}.$$
The differential system satisfied by the $v_{i,p}$'s induces a cobracket $\delta : \mathcal{L}^c \longrightarrow \mathcal{L}^c \otimes \mathcal{L}^c$:
$$\forall i \in I_1, \qquad \delta(f_{i,1}) = 0$$
and
$$\forall p \geqslant 2\, \forall i \in I_p \qquad \delta(v_{i,p}) = \sum_{\substack{p_1+p_2=p \\ i_1 \in I_{p_1},\, i_2 \in I_{p_2}}} -\frac{1}{2}\, a^{i,p}_{\substack{i_1,p_1 \\ i_2,p_2}} (f_{i_1,p_1} \otimes f_{i_2,p_2} - f_{i_2,p_2} \otimes f_{i_1,p_1}).$$

Note that, as the $f_{i,p}$ are of degree 0, $\delta$ satisfies
$$(\mathrm{id}_{\mathcal{L}^c \otimes \mathcal{L}^c} - \tau) \circ \delta = 0$$

by definition.

We now check the co-Jacobi identity; that is

$$(\mathrm{id} + \xi + \xi^2) \circ (\delta \otimes \mathrm{id}\,) \circ \delta = 0.$$

The anti-symmetry of $\delta$ implies that

$$(\mathrm{id} \otimes \delta) \circ \delta = -\xi^2 \circ (\delta \otimes \mathrm{id}\,) \circ \delta$$

and

$$(\mathrm{id} - \tau_{12})(\mathrm{id} + \xi + \xi^2) \circ (\delta \otimes \mathrm{id}\,) = 2(\mathrm{id} + \xi + \xi^2) \circ (\delta \otimes \mathrm{id}\,)$$

where $\tau_{12}$ exchanges the two first factors of $\mathcal{L}^{c\otimes 3}$. Hence, in order to prove co-Jacobi, it is enough to prove that

$$(\mathrm{id} - \tau_{12})(\mathrm{id} + \xi + \xi^2) \circ (\delta \otimes \mathrm{id} - \mathrm{id} \otimes \delta) \circ \delta = 0.$$

That is, one can compute directly in the exterior algebra $\Lambda^{gr,3}(\mathcal{L}^c)$. As the $v_{i,p}$'s are of the degree 1 and the $f_{i,p}$ of degree 0, the computation above gives the same coefficients as $d_A \circ d_A(v_{i,p})$. Thus, condition (5) on the coefficients $a^{i,p}_{\substack{i_1,p_1\\ i_2,p_2}}$'s induces the co-Jacobi identity.

Now, the natural morphism $\psi : \Omega_{coL}(\mathcal{L}^c) = S^{gr}(s\mathcal{L}^c) \longrightarrow A$ sending $s \otimes f_{i,p}$ to $v_{i,p}$ for any $p \geqslant 1$ and any $i \in I_p$ is a morphism of cdga algebra; its commutes with the differential on $\Omega_{coL}(\mathcal{L}^c)$ because $\mathcal{L}^c$ is concentrated in degree 0 and thus has 0 differential.

It induces a morphism on the bar construction (for the associative case)

$$\psi_B : B(\Omega_{coL}(\mathcal{L}^c)) \longrightarrow B(A)$$

compatible with projection on tensor degree $n$ (for any $n$) and with the projector $p_{\mathrm{III}}$ onto the indecomposable elements.

Hence we obtain the following commutative diagram (of vector space)

$$\begin{array}{ccccc} & & \Omega_{coL}(\mathcal{L}^c) & \xrightarrow{\quad\psi\quad} & A \\ & \nearrow & \uparrow{\scriptstyle \pi_1} & & \uparrow{\scriptstyle \pi_1} \\ \mathcal{L}^c & \xrightarrow{\quad\upsilon\quad} & B_{com}(\Omega_{coL}(\mathcal{L}^c)) & \xrightarrow{\quad\psi_{\mathbb{Q}}\quad} & Q_B(A) \end{array}$$

where the morphisms in the bottom line are morphisms of dg coLie algebras and where $\upsilon$ is the unit of the bar/cobar adjunction.

For any $p \geqslant 1$ and any $i \in I_p$, one defines $v^B_{i,p}$ as the image:

$$v^B_{i,p} = \psi_Q \circ \upsilon(f_{i,p}).$$

Recall that $\mathcal{L}^c$ is concentrated in degree 0 and that the composition $\pi_1 \circ \upsilon$ sends $f_{i,p}$ to $sf_{i,p}$. The properties of the $v^B_{i,p}$ now follow because $\psi_Q \circ \upsilon$ is a morphism of dg coLie algebra. □

3.2. **A differential system in Bloch's cycle algebra over $\mathbb{P}^1 \setminus \{0,1,\infty\}$.** In [Sou12a], the author defined a family of algebraic cycles $\mathcal{L}^{\varepsilon}_W$ indexed by couple $(W,\varepsilon)$ where $W$ is a Lyndon word and $\varepsilon$ in $\{\emptyset, 1\}$. Note that, all words considered in this work are words in the two letters 0 and 1.

One of the idea underling the construction of the cycles was to follow explicitly a 1-minimal model construction as in [DGMS75] with the hope to use the relation between 1-minimal model and bar construction in order to obtain motives over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ in the sense of Bloch and Kriz [BK94].

The construction of the family of cycles provides in fact a differential system for these cycles together with an underlying combinatorial differential system for a

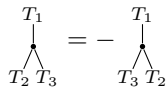
family of rooted trivalent trees. In this subsection we will associated bar elements to the previously defined algebraic cycles.

Before dealing with the algebraic cycles situations, we need to recall the combinatorial situation from [Sou12a]. This is need for the condition (5) on the differential system and will be used later on.

As previously, a *tree* is a finite tree whose internal vertices have valency $\geqslant 3$ and where at each vertex a cyclic ordering of the incident edges is given. A *rooted* tree have a distinguished external vertex called the *root* and a *forest* is a disjoint union of trees.

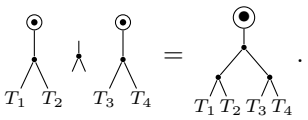
Note that even if the we have already used rooted trees before, the trees below are unrelated to the one previously used.

3.2.1. *Trees, Lie algebras and Lyndon words.* Let $\mathcal{T}^{tri}$ be the $\mathbb{Q}$-vector space generated by rooted trivalent trees with leaves decorated by 0 and 1 modulo the relation




where the $T_i$'s are subtrees (and $T_1$ contains the root of the global tree). Note that in the above definition, the root is not decorated.

Define on $\mathcal{T}^{tri}$ the internal law $\curlywedge$ by




and extend it by bilinearity. One remarks that by definition $\curlywedge$ is antisymmetric. Identifying $\{0,1\}$ with $\{X_0, X_1\}$ by the obvious morphism and using the correspondence $\curlywedge \leftrightarrow [,]$, this internal law allows us to identity the free Lie algebra $\mathrm{Lie}(X_0, X_1)$ with $\mathcal{T}^{tri}$ modulo the Jacobi identity. Thus, one can identify the (graded) dual of $\mathrm{Lie}(X_0, X_1)$ as a subspace of $\mathcal{T}^{tri}$

$$\mathrm{Lie}(X_0, X_1)^* \subset \mathcal{T}^{tri}.$$

A Lyndon word in 0 and 1 is a word in 0 and 1 strictly smaller than any of its nonempty proper right factors for the lexicographic order with $0 < 1$ (for more details, see [Reu93]). The standard factorization $[W]$ of a Lyndon word $W$ is defined inductively by $[0] = X_0$, $[1] = X_1$ and otherwise by $[W] = [[U], [V]]$ with $W = UV$, $U$ and $V$ nontrivial and such that $V$ is minimal. The sets of Lyndon brackets $\{[W]\}$, that is Lyndon words in standard factorization, form a basis of $\mathrm{Lie}(X_0, X_1)$ which can then be used to write the Lie bracket

$$[[U], [V]] = \sum_{\substack{W \text{ Lyndon} \\ \text{words}}} \alpha_{U,V}^{W} [W].$$

with $U < V$ Lyndon words.

**Example 3.3.** Lyndon words in letters $0 < 1$ in lexicographic order are up to weight 5:

$$0 < 00001 < 0001 < 00011 < 001 < 00101 < 0011 < 00111 < 01 < 01011 < 011 < 0111 < 01111 < 1$$

The above identification between $\mathrm{Lie}(X_0, X_1)$ as a quotient of $\mathcal{T}^{tri}$ and the basis of Lyndon brackets allows us to define a family of trees dual to the Lyndon bracket

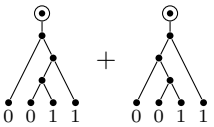
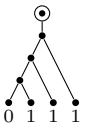
basis beginning with $T_{0^*} =$  and $T_{1^*} =$  and then setting

$$T_{W^*} = \sum_{U<V} \alpha_{U,V}^{W} T_{U^*} \curlywedge T_{V^*}. \tag{6}$$

**Example 3.4.** We give below the corresponding dual trees in weight 1, 2 and 3

$T_{0^*} =$ , $T_{1^*} =$ , $T_{01^*} =$  , $T_{001^*} =$  , $T_{011^*} =$  .

In weight 4 appears the first linear combination

$T_{0001^*} =$  , $T_{0011^*} =$  , $T_{0111^*} =$  .

due to the fact that both $[0] \wedge [011]$ and $[001] \wedge [1]$ are mapped onto $[0011] = [X_0, [[X_0, X_1], X_1]]$ under the bracket map.

3.2.2. *The combinatorial context.* Now, we introduced rooted oriented decorated trees which give the combinatorial context.

**Definition 3.5.** A rooted oriented decorated tree $(T, \omega)$ (not necessarily trivalent) is a rooted tree $T$ such that:

- the root vertex is decorated by 0, 1, or a formal parameter $t$;
- the leaves are decorated by 0 or 1;
- $\omega$ is a numbering of the edge of $T$, that is a map from $\omega : E(T) \longrightarrow \{1, \ldots, n\}$ where $E(T)$ is the set of edge of $T$ and $n = e(T)$ the number of edges.

Note that there is an obvious direction on the edges of a rooted tree $T$: away from the root. A rooted tree $T$ comes with a canonical numbering, starting from the root edge and induced by the cyclic ordering at each vertex.

The above definition extends naturally to forest, that is to disjoint union of rooted oriented decorated trees.

**Example 3.6.** With our convention, an example of this canonical ordering is shown at Figure 1; we recall that by convention we draw trees with the root at the top and the cyclic order at internal vertices counterclockwise.

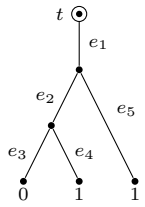



FIGURE 1. A tree with its canonical orientation, that is the canonical numbering of its edges.

**Definition 3.7.**

- Let $V^t$ be the $\mathbb{Q}$-vector space generated by a unit 1 and oriented forests of rooted oriented decorated trees. The degree of such a tree $T$ is $2w(T) - e(T)$ where $w(T)$ denotes the number of leaves in $T$.

- Let $\cdot$ denote the product induced by the disjoint union of the trees and shift of the numbering for the orientation of the second factor. That is the product of $(F_1, \omega_1)$ and $(F_2, \omega_2)$ is the forest $F = F_1 \sqcup F_2$ together with the numbering $\omega$ satisfying $\omega|_{E(F_1)} = \omega_1$ and $\omega|_{E(F_2)} = \omega_2 + n_1$ where $n_1 = e(F_1)$ is the number of edges in $F_1$. Note that here, by convention, the empty tree is 0 and the unit for $\cdot$ is the extra generator 1.
- Define $\mathcal{T}^{dec,or}$ to be the algebra $V^t$ modulo the relations:

$$(T, \sigma(\omega)) = \varepsilon(\sigma)(T, \omega), \qquad \overset{0}{\underset{T_1 T_2}{\bullet}} = 0 \qquad \text{and} \qquad \overset{1}{\underset{0}{\bullet}} = 0.$$

  for any permutation $\sigma$ and where $\varepsilon(\sigma)$ denotes the usual signature of the permutation $\sigma$.

The algebra $\mathcal{T}^{dec,or}$ endowed with the product $\cdot$ is graded commutative. Note that for any forest $F$ one has $(-1)^{\deg(F)} = (-1)^{e(F)}$.

The algebra $\mathcal{T}^{dec,or}$ is endowed with a differential $d_{cy}$ given essentially by contraction of internal edges and contraction and splitting of external edges, see [Sou12a] or [GGL09].

Note that there are two morphisms $\varphi_t, \varphi_1 : \mathcal{T}^{tri} \longrightarrow \mathcal{T}^{dec,or}$, sending a rooted trivalent tree in $\mathcal{T}^{tri}$ to the same tree (with same decoration of leaves), canonical ordering of the edge and with the root vertex decoration equal to $t$ and 1 respectively. The differential $d_{cy}$ on $\mathcal{T}^{dec,or}$ is in particular defined in order to mimic the differential on algebraic cycle (see [GGL09]). The main result of the combinatorial aspects shows that the image of the linear combination $T_{U^*}$ in $\mathcal{T}^{dec,or}$ have decomposable differential under $d_{cy}$.

**Theorem 3.8.** *By an abuse of notation, for any Lyndon word $U$ the image of $T_{U^*}$ in $\mathcal{T}^{dec,or}$ with root vertex decorated by $t$ and canonical orientation is also denoted by $T_{U^*}(t)$. The image of $T_{U^*}$ in $\mathcal{T}^{dec,or}$ with root vertex decorated by 1 and canonical orientation is denoted by $T_{U^*}(1)$.*

*Let $W$ be a Lyndon word. Then, the following equality holds in $\mathcal{T}^{dec,or}$:*

$$\text{(ED-T)} \qquad d_{cy}(T_{W^*}(t)) = \sum_{U<V} \alpha^W_{U,V} T_{U^*}(t) \cdot T_{V^*}(t) + \sum_{U,V} \beta^W_{U,V} T_{U^*}(t) \cdot T_{V^*}(1)$$

*where the $\alpha^W_{U,V}$ are the ones from Equation* (6)*. In the above equation, coefficients $\alpha$'s and $\beta$'s are in $\mathbb{Z}$.*

*Remark* 3.9. They are no relations between the $T_{U^*}(t)$ and $T_{U^*}(1)$, thus computing $d^2_{cy}(T_{U^*}(t)) = 0$ provides relations among the coefficients $\alpha$'s and $\beta$'s (explicitly given in [Sou12a]).

*Remark* 3.10. One remarks that contracting the root edge and then splitting in two new rooted trees in all the trees appearing in $T_{U^*}$ is exactly the cobracket operation dual to the Lie bracket in $\mathrm{Lie}(X_0, X_1)$ viewed as quotient of $\mathcal{T}^{tri}$. This explains why the coefficients $\alpha$'s from (6) appear in the above equation.

We give below examples in weight 1, 2 and 3 of the differential $d_{cy}$ which will be useful later on.

**Example 3.11.** The trees are endowed with their canonical numbering. First, trees with only one edge are mapped to 0 so

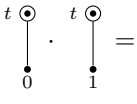
$$d_{cy}(T_{0^*}) = d_{cy}\Big(\overset{t}{\underset{0}{\bullet}}\Big) = 0 \qquad \text{and} \qquad d_{cy}(T_{1^*}) = d_{cy}\Big(\overset{t}{\underset{1}{\bullet}}\Big) = 0.$$

We recall that a tree with root decorated by 0 is 0 in $\mathcal{T}^{dec,or}$. As applying an odd permutation to the numbering changes the sign of the tree, the trivalency of

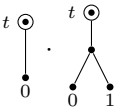
the tree $T_{W^*}$ shows that some trees arising from the computation of $d_{cy}$ are 0 in $\mathcal{T}^{dec,or}$ because they contain a symmetric subtree; that is they contain a subtree of the form


where $T$ is a trivalent tree.

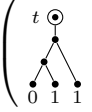
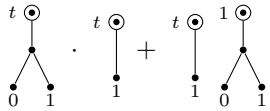
Using the fact that the tree  is 0 in $\mathcal{F}^{\bullet}_{\mathbb{Q}}$, one computes in weight 2

$$d_{cy}(T_{01^*}) = d_{cy}\Big($$  $$\Big) =$$  $$= T_{0^*}\, T_{1^*}.$$

In weight 3, one has

$$d_{cy}(T_{001^*}) = d_{cy}\Big($$  $$\Big) =$$  $$= T_{0^*}\, T_{01^*}$$

and

$$d_{cy}(T_{011^*}) = d_{cy}\left($$  $$\right) =$$ 

$$= T_{01^*}\, T_{1^*} + T_{1^*}\, T_{01^*}(1).$$

3.2.3. *A differential system for cycles.* From this differential system, the author defined in [Sou12a] a twisted version defining the following coefficients.

**Definition 3.12.** Let $W$ be a Lyndon word and $U$, $V$ two Lyndon words. We set:

$$\begin{aligned} a^W_{U,V} &= \alpha^W_{U,V} + \beta^W_{U,V} - \beta^W_{V,U} && \text{for } U < V \\ b^W_{U,V} &= -\beta^W_{U,V} && \text{for any } U,\, V \end{aligned} \tag{7}$$

and

$$\begin{aligned} {a'}^W_{U,V} &= -a^W_{U,V} && \text{for } 0 < U < V, \\ {b'}^W_{U,V} &= a^W_{U,V} + b^W_{U,V} && \text{for } 0 < U < V, \\ {b'}^W_{V,U} &= -a^W_{U,V} + b^W_{V,U} && \text{for } 0 < U < V, \\ {a'}^W_{0,V} &= a_{0,V} && \text{for any } V, \\ {b'}^W_{U,U} &= b^W_{U,U} && \text{for any } U. \end{aligned} \tag{8}$$

It is also convenient to define ${b'}^W_{0,V} = {b'}^W_{V,0} = 0$.

The introduction of coefficient $a$'s and $b$'s corresponds to changing the basis in which one writes the trees differential system (ED-T). More precisely, instead of writing (ED-T) with trees $T_{U^*}(t)$ and $T_{U^*}(1)$, one writes two differential systems using the trees $T_{U^*}(t)$ and $T^1_{U^*} = T_{U^*}(t) - T_{U^*}(1)$.

As explained in [Sou12a], this change in the coefficient is required by the geometric situation in Bloch cycle algebra $\mathcal{N}^{\bullet}_{\mathbb{A}^1}$ and the need to work with equidimensional cycles.

The projective line minus three points $\mathbb{P}^1 \setminus \{0,1,\infty\}$ will be simply denoted by $X$. Recall that in section **??** were defined two weight 1 degree 1 equidimensional

cycles in $\mathcal{N}_X^1$:

$$\mathcal{L}_0 = \mathcal{A}lt([t;t]) \subset X \times \square^1, \quad \text{and} \quad \mathcal{L}_1 = \mathcal{A}lt([t;1-t]) \subset X \times \square^1,$$

using the graph of the identity

$$\mathbb{A}^1 \xrightarrow{t \mapsto t} \mathbb{A}^1$$

and the graph of the involution

$$\mathbb{A}^1 \xrightarrow{t \mapsto 1-t} \mathbb{A}^1.$$

This allows to obtain in the geometric situation:

**Theorem 3.13** ([Sou12a])**.** *Let $j$ be the inclusion $X \hookrightarrow \mathbb{A}^1$. For any Lyndon word of length $p$ greater than or equal to 2, there exist two cycles $\mathcal{L}_W$ and $\mathcal{L}^1_W$ in $\mathcal{N}^1_X(p)$ such that:*

- *$\mathcal{L}_W$, $\mathcal{L}^1_W$ are elements in $\mathcal{N}_X^{eq,\,1}(p)$.*
- *There exist cycles $\overline{\mathcal{L}_W}$, $\overline{\mathcal{L}^1_W}$ in $\mathcal{N}_{\mathbb{A}^1}^{eq,\,1}(p)$ such that*

$$\mathcal{L}_W = j^*(\overline{\mathcal{L}_W}) \qquad \text{and} \qquad \mathcal{L}^1_W = j^*(\overline{\mathcal{L}^1_W}).$$

- *The restriction of $\overline{\mathcal{L}_W}$ (resp. $\overline{\mathcal{L}^1}$) to the fiber $t = 0$ (resp. $t = 1$) is empty.*
- *The cycle $\mathcal{L}_W$ and $\mathcal{L}^1_W$ satisfy the following differential equations in $\mathcal{N}_X^\bullet$:*

$$\text{(ED-}\mathcal{L}\text{)} \qquad \partial(\mathcal{L}_W) = \sum_{U<V} a^W_{U,V} \mathcal{L}_U \mathcal{L}_V + \sum_{U,V} b^W_{U,V} \mathcal{L}_U \mathcal{L}^1_V$$

*and*

$$\text{(ED-}\mathcal{L}^1\text{)} \qquad \partial(\mathcal{L}^1_W) = \sum_{0<U<V} a'^W_{U,V} \mathcal{L}^1_U \mathcal{L}^1_V + \sum_{U,V} b'^W_{U,V} \mathcal{L}_U \mathcal{L}^1_V + \sum_V a'_{0,V} \mathcal{L}_0 \mathcal{L}_V$$

*The same equations hold for their extension to $\mathcal{N}_{\mathbb{A}^1}^{eq,\bullet}$ $\overline{\mathcal{L}_W}$ and $\overline{\mathcal{L}^1_W}$.*

One will write generically a cycle in the above families as $\mathcal{L}^\varepsilon_W$ with $\varepsilon$ in $\{\emptyset, 1\}$ when working over $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $\overline{\mathcal{L}^\varepsilon_W}$ with $\varepsilon$ in $\{\emptyset, 1\}$ when working over $\mathbb{A}^1$.

*Remark* 3.14. The above theorem gives us two differential systems : one in $\mathcal{N}_X^{eq,\bullet}$ given by the $\mathcal{L}^\varepsilon_W$'s and one in $\mathcal{N}_X^{eq,\bullet}$ given by the $\overline{\mathcal{L}^\varepsilon_W}$'s. Over $X$ or over $\mathbb{A}^1$, which will be the interesting case for us, the two different differential systems are related by the following relation

$$\text{(ED-}\overline{\mathcal{L}} - \overline{\mathcal{L}^1}\text{)} \qquad \partial\left(\overline{\mathcal{L}_W} - \overline{\mathcal{L}^1_W}\right) = \sum_{0<U<V<1} a^W_{U,V} \left(\overline{\mathcal{L}_U} - \overline{\mathcal{L}^1_U}\right)\left(\overline{\mathcal{L}_V} - \overline{\mathcal{L}^1_V}\right).$$

This relation is a direct consequence of the relation between the coefficient of Definition 3.12.

When $W$ is a Lyndon word of length greater or equal to 2, the equidimensionality of the cycle $\overline{\mathcal{L}^\varepsilon_W}$ allows us to consider its fiber at a point $x \in A^1$; in particular its fiber at 1. This fiber can now be extended as a constant equidimensional cycle to $\mathbb{A}^1$ and then restricted (when needed) to $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. All these operations are compatible with the differential providing us with three other differential systems (one over the point 1, one over $\mathbb{A}^1$ and one over $X$). Following the previous section each of these systems of equations induced a family of elements in the corresponding bar construction (over the point, over $\mathbb{A}^1$, and over $X$). The rest of this section makes the above idea precise and the next subsection shows how the mentioned differential systems are related.

The geometric situation relates $\mathbb{P}^1 \setminus \{0,1,\infty\}$, $\mathbb{A}^1$ and the point $\{1\}$ as follows:

$$\begin{array}{ccc} X = \mathbb{P}^1 \setminus \{0,1,\infty\} & \xrightarrow{\quad j \quad} & \mathbb{A}^1 \\ & & {\scriptstyle p_1}\downarrow \uparrow {\scriptstyle i_1} \\ & & \{1\} \end{array}$$

where $j$ is the open inclusion, $p_1$ is the projection onto $\{1\}$ and $i_1$ the closed inclusion (or the 1-section). On the cycle algebra level one gets

$$\mathcal{N}_X^\bullet \qquad\qquad \mathcal{N}_{\mathbb{A}^1}^\bullet$$

$$\mathcal{N}_X^{eq,\bullet} \qquad\qquad \mathcal{N}_{\mathbb{A}^1}^{eq,\bullet}$$

$$\mathcal{N}_{\mathrm{Spec}(\mathbb{Q})}^{eq,\bullet}$$

Then $\overline{\mathcal{L}_W}$ is in $\mathcal{N}_{\mathbb{A}^1}^{eq,\,1}(p)$ and is equidimensional over $\mathbb{A}^1$. Its pull-back $i_1^*(\overline{\mathcal{L}_W})$ is simply its fiber at $\{1\}$ and one can pull this fiber back again to $\mathbb{A}^1$ as a constant equidimensional cycle in $\mathcal{N}_{A^1}^{eq,\,1}(p)$.

**Definition 3.15.** Let $W$ be a Lyndon word of length $p \geqslant 2$. One defines the cycle $\overline{\mathcal{L}_W(1)}$ as:

$$\overline{\mathcal{L}_W(1)} = p_1^* \circ i_1^*(\overline{\mathcal{L}_W}) = p^1(\overline{\mathcal{L}_W}|_{t=1})$$

together with its restriction to $X$: $\mathcal{L}_W(1) = j^*(\overline{\mathcal{L}_W(1)})$.

Morphisms $i_1^*$, $p_1^*$ and $j^*$ commute with the differential on $\mathcal{N}_{\mathrm{Spec}(\mathbb{Q})}^{eq,\bullet}$, $\mathcal{N}_{\mathbb{A}^1}^{eq,\bullet}$ and $\mathcal{N}_X^{eq,\bullet}$ respectively. Note that from [Sou12a, Remark 4.2], the coefficients $a_{U,1}^W$'s are 0 and that $\overline{\mathcal{L}_0}$ has empty fiber at 1. As cycles $\overline{\mathcal{L}_V^1}$ have empty fiber at 1, one obtains the following differential system for the $\mathcal{L}_W(1)$.

$$\text{(ED-}\mathcal{L}(1)\text{)} \qquad\qquad \partial(\mathcal{L}_W(1)) = \sum_{0<U<V<1} a_{U,V}^W \mathcal{L}_U(1)\mathcal{L}_V(1)$$

for any Lyndon word $W$ of length greater or equal to 2. The same holds for the family over $\mathbb{A}^1$.

Let $B_X$, $B_{\mathbb{A}^1}$ and $B_{\{1\}}$ denote the bar construction over $\mathcal{N}_X^{eq,\bullet}$, $\mathcal{N}_{\mathbb{A}^1}^{eq,\bullet}$ and $\mathcal{N}_{\mathrm{Spec}(\mathbb{Q})}^{eq,\bullet}$ respectively. Let $Q_X$, $Q_{\mathbb{A}^1}$ and $Q_{\{1\}}$ be the corresponding set of indecomposable elements. By an abuse of notation, we will write $d_B$, $\Delta$, $\text{ш}$ and $\delta_Q$ the natural operation in the corresponding spaces. When required, the "base" space will be precise by the subscript $X$, $\mathbb{A}^1$ and $\{1\}$ respectively.

The bar elements $\mathcal{L}_0^B$ and $\mathcal{L}_1^B$ in $B_X$ defined by

$$\mathcal{L}_0^B = [\mathcal{L}_0] \qquad \text{and} \mathcal{L}_1^B = [\mathcal{L}_1]$$

are of bar degree 0 with bar differential equals to 0. They give non-trivial class in $Q_X$ and map to 0 under the cobracket $\delta_Q$ (because their image under the reduced coproduct $\bar{\Delta}$ is 0).

**Theorem 3.16** (bar elements)**.** *For any Lyndon word $W$ of length $p \geqslant 2$ there exist elements $\mathcal{L}_W^B$, $\mathcal{L}_W^{1,B}$ and $\mathcal{L}_W^B(1)$ in the bar construction $B_X$ satisfying:*

- *Their images under the projection onto the tensor degree* 1 *part* $\pi_1 : B_X \longrightarrow \mathcal{N}_X^\bullet$ *are the expected cycles:*

$$\pi_1(\mathcal{L}_W^B) = \mathcal{L}_W, \quad \pi_1(\mathcal{L}_W^{1,B}) = \mathcal{L}_W^1 \quad \textit{and} \quad \pi_1(\mathcal{L}_W^B(1)) = \mathcal{L}_W(1).$$

- *They are invariant under* $p_{\text{ш}}$*; that is they are in the image of the projector* $p_{\text{ш}}$.
- *They are of bar degree* 0 *and their image under* $d_B$ *is* 0*. That is they induced classes in* $\mathrm{H}^0(B_X)$ *and in* $\mathrm{H}^0(Q_X)$.
- *Their image under* $\delta_Q$ *is given up to a global minus sign by the differential equations* (ED-$\mathcal{L}$), (ED-$\mathcal{L}^1$) *and* (ED-$\mathcal{L}(1)$).

*Proof.* The main point is to apply Proposition 3.2. In order to do so, we need to describe the family $v_{i,p}$ and to check that it fulfill the condition of Proposition 3.2.

In weight 1, the set of indices $I_1$ is the set $I_1 = \{0, 1\}$. In weight $p$, the set of indices is

$$I_p = \{\emptyset, 1\} \times \{W \text{ s. t. } W \text{is a Lyndon word of length } p\}.$$

The family of elements $v_{i,p}$ is then given by

$$v_{0,1} = \mathcal{L}_0, \quad v_{1,1} = \mathcal{L}_1$$

and

$$v_{(\varepsilon,W),p} = \begin{cases} \mathcal{L}_W & \text{if } \varepsilon = \emptyset \\ \mathcal{L}_W^1 & \text{if } \varepsilon = 1 \end{cases}$$

Then, Remark 3.9 and the definition of coefficients $a$'s, $a'$'s, $b$'s and $b'$'s in terms of the coefficients of (ED-T) (Definition 3.12) insure that the condition (5) about the coefficients $a^{i,p}_{\substack{i_1,p_1\\ i_2,p_2}}$ is satisfied (for explicit computations one can see [Sou12a, Proposition 4.3]). The theorem is then a consequence of Proposition 3.2. Similar arguments works for the elements $\mathcal{L}_W^B(1)$. □

**Proposition 3.17.** *The family of elements in* $\mathcal{N}_{\mathbb{A}^1}^{eq,\,1}$ *given by* $\overline{\mathcal{L}_W} - \overline{\mathcal{L}_W^1}$ *for any Lyndon word* $W$ *of length* $p \geqslant 2$ *gives a family of bar elements* $\overline{\mathcal{L}_W^{\emptyset-1,B}}$ *in* $B(\mathcal{N}_{\mathbb{A}^1}^{eq,\bullet})$ *satisfying*

- *Their image under the projection onto the tensor degree* 1 *part* $\pi_1 : B_{\mathbb{A}^1} \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{eq,\bullet}$ *is the expected cycle*

$$\pi_1(\overline{\mathcal{L}_W^{\emptyset-1,B}}) = \overline{\mathcal{L}_W} - \overline{\mathcal{L}_W^1}$$

- *They are in the image of the projector* $p_{\text{ш}}$.
- *They are of bar degree* 0 *and their image under* $d_B$ *is* 0*. That is they induce classes in* $\mathrm{H}^0(B_X)$ *and in* $\mathrm{H}^0(Q_X)$.
- *Their image under* $\delta_Q$ *is given up to a global minus sign by the differential equations* (ED-$\overline{\mathcal{L}} - \overline{\mathcal{L}^1}$).
- *Their restrictions* $j^*(\overline{\mathcal{L}_W^{\emptyset-1,B}})$ *to* $B(\mathcal{N}_X^{eq,\bullet})$ *equal*

$$j^*(\overline{\mathcal{L}_W^{\emptyset-1,B}}) = \mathcal{L}_W^B - \mathcal{L}_W^{1,B}.$$

*The family* $i_1^*(\overline{\mathcal{L}_W})$ *for Lyndon word* $W$ *of length* $p \geqslant 2$ *gives a family of bar elements* $\mathcal{L}_{W,t=1}^B$ *in* $B_{\{1\}}$ *satisfying*

*the following properties :*

- *Their image under* $\pi_1 : B_1 \longrightarrow \mathcal{N}_{\mathrm{Spec}(\mathbb{Q})}^{eq,\bullet}$ *is* $\pi_1(\overline{\mathcal{L}_W^{\emptyset-1,B}}) = i_1^*(\overline{\mathcal{L}_W})$
- *They are invariant under* $p_{\text{ш}}$*;*
- *They induce classes in* $\mathrm{H}^0(B_X)$ *and in* $\mathrm{H}^0(Q_X)$.

- *Their images under $\delta_Q$ is given by*

$$\delta_Q(\mathcal{L}^B_{W,t=1}) = -\sum_{0<U<V<1} \mathcal{L}^B_{U,t=1} \wedge \mathcal{L}^B_{V,t=1}$$

  *which is corresponding to* (ED-$\mathcal{L}(1)$).

*They induce (by pull-back on $\mathbb{A}^1$) bar elements $p_1^*(\mathcal{L}^B_{W,t=1})$ and are related to the elements $\mathcal{L}^B_W(1)$ by*

$$j^* \circ p_1^*(\mathcal{L}^B_{W,t=1}) = \mathcal{L}^B_W(1)$$

*where*

$$B(\mathcal{N}^\bullet_X) \qquad\qquad B(\mathcal{N}^\bullet_{\mathbb{A}^1})$$

$$B(\mathcal{N}^{eq,\bullet}_X) \qquad\qquad B(\mathcal{N}^{eq,\bullet}_{\mathbb{A}^1})$$

$$B(\mathcal{N}^{eq,\bullet}_{\mathrm{Spec}(\mathbb{Q})})$$

*We will write $\overline{\mathcal{L}^B_W(1)} = p_1^*(\mathcal{L}^B_{W,t=1})$ as these pull-back give also the bar elements corresponding to the family $\overline{\mathcal{L}_W(1)}$ from Definition 3.15.*

*Proof.* The existence of the bar elements goes as in Theorem 3.16 above and rely on Proposition 3.2 and on [Sou12a, Proposition 4.3]) for the explicit relations on the coefficient.

The relations between constructions over $\{1\}$, $\mathbb{A}^1$ and $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$ follow because $i_1^*$, $p_1^*$ and $j^*$ are morphisms of cdga algebra. □

The cycle $\overline{\mathcal{L}_0}$ and $\overline{\mathcal{L}_1}$ are not equidimensional over $\mathbb{A}^1$, even if they are well-defined element in $\mathcal{N}^1_{\mathbb{A}^1}$. Thus differentials $\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_W})$ and $\partial_{\mathbb{A}^1}(\overline{\mathcal{L}^1_W})$ are not decomposable in $\mathcal{N}^{eq,\,2}_{\mathbb{A}^1}$. In particular, one can not consider bar elements corresponding to $\overline{\mathcal{L}_W}$ and $\mathcal{L}^1_W$ in $B_{\mathbb{A}^1}$. However the differential $\partial_{A^1}(\overline{\mathcal{L}_W} - \overline{\mathcal{L}^1_W})$ is a linear combination of products in $\mathcal{N}^{eq,\,2}_{\mathbb{A}^1}$ which allows us to define the family of bar elements $\overline{\mathcal{L}^{\emptyset-1,B}_W}$.

These elements $\overline{\mathcal{L}^{\emptyset-1,B}_W}$ correspond to the difference $\overline{\mathcal{L}_W} - \overline{\mathcal{L}^1_W}$ and working over $\mathbb{A}^1$ makes it possible to take the fiber at 1 of relations coming from the bar construction level. This is a key point in the next section.

3.3. **Relations between bar elements.** This subsection proves that the correspondence

$$\mathcal{L}_W - \mathcal{L}_W(1) \leftrightarrow \mathcal{L}^1_W$$

for $W$ a Lyndon word of length greater or equal to 2 is an equality in the $\mathrm{H}^0$ of the bar construction modulo shuffle products; that is

$$\mathcal{L}^B_W - \mathcal{L}^B_W(1) = \mathcal{L}^{1,B}_W \qquad \in \mathrm{H}^0(Q_X).$$

This relation is coming from the geometric situation and its expectation was one of the reasons to introduce the cycles $\mathcal{L}^1_W$ with empty fiber at 1 instead of the difference $\mathcal{L}_W - \mathcal{L}_W(1)$.

A key point in order to build cycles $\mathcal{L}_W$ and $\mathcal{L}^1_W$ in [Sou12a] was a pull-back by the multiplication. More precisely, the usual multiplication $\mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ composed with the isomorphism $\mathbb{A}^1 \times \square^1 \simeq \mathbb{A}^1 \times \mathbb{A}^1$ gives a multiplication

$$m_0 : \mathbb{A}^1 \times \square^1 \longrightarrow \mathbb{A}^1$$

sending $\{0\} \times \square^1$ to 0. Twisting $m_0$ by $\theta : t \mapsto t$ gives a "twisted multiplication"

$$m_1 = (\theta \times \mathrm{id}\,) \circ m_0 \circ \theta : \mathbb{A}^1 \times \square^1 \longrightarrow \mathbb{A}^1$$

sending $\{1\} \times \square^1$ to 1.

**Proposition 3.18** ([Sou12a, Propositions 4.9 and 4.11])**.** *For $\varepsilon = 0, 1$ the morphism $m_\varepsilon$ induces a linear morphism*

$$m_\varepsilon^* : \mathcal{N}_{\mathbb{A}^1}^{eq,\,k} \longrightarrow \mathcal{N}_{\mathbb{A}^1}^{eq,\,k-1}$$

*giving a homotopy*

$$\partial_{\mathbb{A}^1} \circ m_\varepsilon^* + m_\varepsilon^* \circ \partial_{\mathbb{A}^1} = \mathrm{id} - p_\varepsilon^* \circ i_\varepsilon^*$$

*where $p_\varepsilon : \mathbb{A}^1 \longrightarrow \{\varepsilon\}$ is the projection onto the point $\{\varepsilon\}$ and $i_\varepsilon$ its inclusion is $\mathbb{A}^1$.*

From this homotopy property, one derives the following relation between $m_0^*$ and $m_1^*$.

**Lemma 3.19.** *One has:*

$$m_1^* = m_0^* - p_1^* \circ i_1^* \circ m_0^* - \partial_{\mathbb{A}^1} \circ m_1^* \circ m_0^* + m_1^* \circ m_0^* \circ \partial_{\mathbb{A}^1} + m_1^* \circ p_0^* \circ i_0^* \tag{9}$$

*and*

$$m_0^* = m_1^* - p_0^* \circ i_0^* \circ m_1^* - \partial_{\mathbb{A}^1} \circ m_0^* \circ m_1^* + m_0^* \circ m_1^* \circ \partial_{\mathbb{A}^1} + m_0^* \circ p_1^* \circ i_1^* \tag{10}$$

*In particular, when $b \in \mathcal{N}_{\mathbb{A}^1}^{eq,\,k}$ satisfies $\partial_{\mathbb{A}^1} = (b)0$ and $i_0(b) = 0$, one has:*

$$m_1^*(b) = m_0^*(b) - p_1^* \circ i_1^*(m_0^*(b)) + \partial_{\mathbb{A}^1}(m_0^* \circ m_1^*(b))$$

*Proof.* Let $b$ be in $\mathcal{N}_{\mathbb{A}^1}^{eq,\,k}$. We treat only Equation 9. Using the homotopy property for $m_0^*$, one writes

$$b = \partial_{\mathbb{A}^1} \circ m_0^*(b) + m_0^* \circ \partial_{\mathbb{A}^1} + p_0^* \circ i_0^*(b).$$

Computing $m_1^*$, the homotopy property

$$m_1^* \circ \partial_{\mathbb{A}^1}(m_0(b)) = m_0^*(b) - p_1^* \circ i_1^*(m_0(b)) - \partial_{\mathbb{A}^1} \circ m_1^*(m_0^*(b))$$

gives the desired formula. □

Writing $\overline{A_W}$ (resp. $\overline{A_W^1}$) the $A^1$-extension of the right hand side of Equation (ED-$\mathcal{L}$) (resp. Equation (ED-$\mathcal{L}^1$)), cycles $\mathcal{L}_W = j^*(\overline{\mathcal{L}_W})$ (resp $\mathcal{L}_W^1 = j^*(\overline{\mathcal{L}_W^1})$ are obtained in [Sou12a] as

$$\overline{\mathcal{L}_W} = m_0^*(\overline{A_W}) \quad \left(\text{resp} \quad \overline{\mathcal{L}_W^1} = m_1^*(\overline{A_W^1})\right).$$

The main difficulties are to show that $\overline{A_W}$ (resp. $\overline{A_W^1}$) has 0 differential and that it has empty fiber at 0 (resp. 1).

The tree weight 2 example 3.4 together with Definition 3.12 show that

$$\overline{\mathcal{L}_{01}} = m_0^*(\overline{\mathcal{L}_0}\,\overline{\mathcal{L}_1}) \quad \text{and} \quad \overline{\mathcal{L}_{01}^1} = m_1^*(\overline{\mathcal{L}_0}\,\overline{\mathcal{L}_1})$$

Using Lemma 3.19, one gets

$$\overline{\mathcal{L}_{01}^1} = \overline{\mathcal{L}_{01}} - \overline{\mathcal{L}_{01}(1)} + \partial_{\mathbb{A}^1}(m_1^* \circ m_0^*(\overline{\mathcal{L}_0}\,\overline{\mathcal{L}_1})). \tag{11}$$

Note that as a parametrized cycle,

$$m_1^* \circ m_0^*(\overline{\mathcal{L}_0}\,\overline{\mathcal{L}_1}) = m_1^*(\overline{\mathcal{L}_{01}})$$

can be written (omitting the projector $\mathcal{A}lt$) as

$$m_1^* \circ m_0^*(\overline{\mathcal{L}_0}\,\overline{\mathcal{L}_1}) = \left[t; \frac{y-t}{y-1}, 1 - \frac{y}{x}, x, 1-x\right] \subset \mathbb{A}^1 \times \square^4.$$

This expression coincides, up to reparametrization, with the expression of $C_{01}$ given in [Sou12b, Example 5.5] relating $\mathcal{L}_{01}$ and $\mathcal{L}_{01}^1$.

Thus, $\overline{\mathcal{L}_{01}} - \overline{\mathcal{L}_{01}(1)}$ and $\overline{\mathcal{L}^1_{01}}$ differs only by a boundary. The differential of $\overline{\mathcal{L}_{01}(1)}$ is zero and one can compute explicitly the corresponding bar elements:

$$\overline{\mathcal{L}_{01}^{\emptyset-1,B}} = [\mathcal{L}_{01}] - [\mathcal{L}^1_{01}], \quad \mathcal{L}^B_{01}(1) = [\mathcal{L}_{01}(1)]$$

**Lemma 3.20.** *In $B(\mathcal{N}^{eq,\bullet}_{A^1})$, one has the following relation*

$$\overline{\mathcal{L}_{01}^{\emptyset-1,B}} - \mathcal{L}^B_{01}(1) = d_B([m_1^*(\overline{\mathcal{L}_{01}})]). \tag{12}$$

*Thus in $\mathrm{H}^0(B_{\mathbb{A}^1})$ as in $\mathrm{H}^0(Q_{\mathbb{A}^1})$ one has the equality between the respective classes*

$$\overline{\mathcal{L}_{01}^{\emptyset-1,B}} - \mathcal{L}^B_{01}(1) = 0.$$

*Taking the restriction to $\mathbb{P}^1 \setminus \{0,1,\infty\}$, one obtains in $\mathrm{H}^0(Q_X)$*

$$\mathcal{L}^B_{01} - \mathcal{L}^{1,B}_{01} = j^*(\overline{\mathcal{L}_{01}^{\emptyset-1,B}}) = \mathcal{L}^B_{01}(1). \tag{13}$$

For $W$ a Lyndon word of length $p \geqslant 2$, the explicit comparison between $\mathcal{L}^B_W$ and $\mathcal{L}^{1,B}_W$ is in general much more complicated as

$$\partial_{\mathbb{A}^1}(\overline{\mathcal{L}_W} - \overline{\mathcal{L}^1_W}) = \sum_{0<U<V<1} a^W_{U,V}(\overline{\mathcal{L}_U} - \overline{\mathcal{L}^1_U})(\overline{\mathcal{L}_V} - \overline{\mathcal{L}^1_V}) \neq 0.$$

However, working at the bar construction level in $\mathrm{H}^0(Q_{\mathbb{A}^1})$ allows to use an induction argument.

**Theorem 3.21.** *For any Lyndon word $W$ of length $p \geqslant 2$ the following relation holds*

$$\overline{\mathcal{L}_W^{\emptyset-1,B}} = \overline{\mathcal{L}^B_W(1)} \qquad \text{in} \quad \mathrm{H}^0(Q_{A^1}) = Q_{\mathrm{H}^0(B_{\mathbb{A}^1})}. \tag{14}$$

*Taking the restriction to $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$, one obtains in $\mathrm{H}^0(Q_X) = Q_{\mathrm{H}^0(B_X)}$*

$$\mathcal{L}^B_W - \mathcal{L}^{1,B}_W = j^*(\overline{\mathcal{L}_W^{\emptyset-1,B}}) = \mathcal{L}^B_W(1). \tag{15}$$

*Proof.* From Lemma 3.20 above it is true for $p = 2$ as there is then only one Lyndon word to consider $W = 01$.

Now we assume that the theorem is true for all Lyndon words of length $k$ with $2 \leqslant k \leqslant p-1$. Let $W$ be a Lyndon word of length $p$.

From Proposition 3.17, one has in $Q_{\mathbb{A}^1}$ and in particular in $\mathrm{H}^0(Q_{\mathbb{A}^1})$

$$\delta_Q(\overline{\mathcal{L}_W^{\emptyset-1,B}}) = -\sum_{0<U<V<1} a^W_{U,V}(\overline{\mathcal{L}_U^{\emptyset-1,B}}) \wedge (\overline{\mathcal{L}_V^{\emptyset-1,B}})$$

and

$$\delta_Q(\overline{\mathcal{L}^B_W(1)}) = -\sum_{0<U<V<1} a^W_{U,V}(\overline{\mathcal{L}^B_U(1)}) \wedge (\overline{\mathcal{L}^B_V(1)}).$$

Using the induction hypothesis, one has in $\mathrm{H}^0(Q_{A^1})$

$$\delta_Q(\overline{\mathcal{L}_W^{\emptyset-1,B}}) = -\sum_{0<U<V<1} a^W_{U,V}(\overline{\mathcal{L}^B_U(1)}) \wedge (\overline{\mathcal{L}^B_V(1)})$$

and thus

$$\delta_Q\left(\overline{\mathcal{L}_W^{\emptyset-1,B}} - \overline{\mathcal{L}^B_W(1)}\right) = 0 \qquad \text{in} \quad \mathrm{H}^0(Q_{A^1}).$$

Let $C_W$ be the class of $\overline{\mathcal{L}_W^{\emptyset-1,B}} - \overline{\mathcal{L}^B_W(1)}$ in $\mathrm{H}^0(Q_{A^1})$ and $sC_W$ its image in $\Omega_{coL}(\mathrm{H}^0(Q_{A^1})) = S^{gr}(s\mathbb{Q} \otimes \mathrm{H}^0(Q_{A^1}))$.

As $\delta_Q(C_W) = 0$, $d_{\Omega,coL}(sC_W) = 0$ and $sC_W$ gives a class in

$$\mathrm{H}^1(\Omega_{coL}(\mathrm{H}^0(Q_{A^1}))) \simeq \mathrm{H}^1(\mathcal{N}^{eq,\bullet}_{\mathbb{A}^1});$$

where the was isomorphism is given by Bloch and Kriz in [BK94, Corollary 2.31] after a choice of a 1-minimal model in the sens of Sullivan. Using the comparison

with $\mathrm{H}^1(\mathcal{N}^\bullet_{\mathbb{A}^1})$ and the $\mathbb{A}^1$-homotopy invariance for higher Chow groups, this class can be represented by $p_1^*(C)$ in $\mathcal{N}^{eq,\,1}_{\mathbb{A}^1}$ with $C$ a cycle in $\mathcal{N}^1_X$.

The cycle $p_1^*(C)$ satisfies $\partial_{\mathbb{A}^1}(p_1^*(C)) = 0$ and $[p_1^*(C)]$ gives a degree 0 bar element $C^B$ in $B_{\mathbb{A}^1}$ with bar differential and reduced coproduct equal to 0.

From this, one gets a class $\tilde{C}_W = C_W - C^B$ in $\mathrm{H}^0(\mathbb{Q}_{\mathbb{A}^1})$. Its image $s\tilde{C}_W$ in $\Omega_{coL}(\mathrm{H}^0(Q_{A^1}))$ also gives a class in

$$\mathrm{H}^1(\Omega_{coL}(\mathrm{H}^0(Q_{A^1})))$$

which is 0 by construction.

As, on the degree 0 part of $\Omega_{coL}(\mathrm{H}^0(Q_{A^1}))$, the differential $d_{Omega,coL}$ is zero, one obtains that $s\tilde{C}_W = 0$ in $s\mathbb{Q}\otimes\mathrm{H}^0(Q_{A^1})$ and thus $\tilde{C}_W$ is zero in $\mathrm{H}^0(Q_{A^1}) = Q_{\mathrm{H}^0(B_{\mathbb{A}^1})}$.

So far one has obtained that in $B_{\mathbb{A}^1}$

$$\overline{\mathcal{L}_W^{\emptyset-1,B}} - \overline{\mathcal{L}_W^B(1)} - [p_1^*(C)] = d_B(b) \qquad \text{modulo } \text{ш} \text{ products} \tag{16}$$

with $b$ in the degree $-1$ part of $B_{\mathbb{A}^1} = B(\mathcal{N}^{eq,\,\bullet}_{\mathbb{A}^1})$.

In the above equation, all elements involved are in $\oplus_n s\mathcal{N}^{eq,\,\bullet\otimes n}_{A^1}$ that is each tensor component is equidimensional over $\mathbb{A}^1$. Because taking the fiber at 1 commutes with products and differential, one gets modulo shuffles

$$i_1^*(\overline{\mathcal{L}_W^{\emptyset-1,B}}) - \mathcal{L}^B_{W,t=1} - [C] = d_B(i_1^*(b)).$$

The bar element $i_1 * (\overline{\mathcal{L}_W^{\emptyset-1,B}})$ is the bar element corresponding to the differential system

$$\partial_{\{1\}}(i_1^*(\mathcal{L}_W - \mathcal{L}^1_W)) = \sum_{0<U<V<1} a^W_{U,V} i_1^*(\mathcal{L}_U - \mathcal{L}^1_U) i_1^*(\mathcal{L}_V - \mathcal{L}^1_V)$$

which is nothing but the differential system for $i_1^*(\mathcal{L}_W)$ because $i_1^*(\mathcal{L}^1_{W'}) = 0$ for any Lyndon words. Moreover the bar elements $i_1^*(\overline{\mathcal{L}_W^{\emptyset-1,B}})$ and $\mathcal{L}^B_{W,t=1}$ agree on the tensor degree 1 part. Thus, one has

$$-[C] = d_B(i_1^*(b)) + \text{ shuffle products}$$

which shows that $[p^*(C)]$ is zero in $\mathrm{H}^0(B_{\mathbb{A}^1})$ modulo shuffles. Hence Equation (16) can be written has

$$\overline{\mathcal{L}_W^{\emptyset-1,B}} - \overline{\mathcal{L}_W^B(1)} = 0 \qquad \text{in} \quad Q_{\mathrm{H}^0(B_{\mathbb{A}^1})} = \mathrm{H}^0(Q_{\mathbb{A}^1})$$

Finally, taking the restriction to $\mathbb{P}^1 \setminus \{0,1,\infty\}$, one has $\mathcal{L}^B_W - \mathcal{L}^{1,B}_W = j^*(\overline{\mathcal{L}_W^{\emptyset-1,B}})$ because the construction of the bar elements is linear. □

The main consequence of Equation (15) in the previous theorem is that in $Q_{\mathrm{H}^0(B_X)}$ one can replace the bar avatar of the geometric differential system (ED-$\mathcal{L}$) by a bar avatar of the differential system (ED-T) coming from trees.

**Corollary 3.22.** *In $Q_{\mathrm{H}^0(B_X)}$, the set of indecomposable elements of $\mathrm{H}^0(B(\mathcal{N}^{eq,\,\bullet}_X))$, the following holds for any (non-empty) Lyndon word $W$:*

$$\text{(ED-}Q_X\text{)} \qquad \delta_Q(\mathcal{L}^B_W) = -\sum_{U<V} \alpha^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^B_V - \sum_{U,V} \beta^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^B_V(1).$$

*Proof.* Let $W$ be a Lyndon word. The statement holds when $W$ has length equal 1 and one can assume that $W$ has length greater or equal to 2. One begins with the formula giving $\delta_Q(\mathcal{L}^B_W)$ from Theorem 3.16:

$$\delta_Q(\mathcal{L}^B_W) = -\sum_{U<V} a^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^B_V - \sum_{U,V} b^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^{1,B}_V.$$

Then using the relations given by Equation (15), one has

$$\delta_Q(\mathcal{L}^B_W) = -\sum_{U<V} a^W_{U,V}\mathcal{L}^B_U \wedge \mathcal{L}^B_V - \sum_{U,V} b^W_{U,V}\mathcal{L}^B_U \wedge \left(\mathcal{L}^B_V - \mathcal{L}^B_V(1)\right).$$

Expanding terms as $\mathcal{L}^B_U\left(\mathcal{L}^B_V - \mathcal{L}^B_V(1)\right)$ and using the Definition 3.12 of coefficients $a$'s and $b$' in terms of $\alpha$'s and $\beta$'s concludes the proof. □

## 4. A relative basis

This section shows that the image of the family of bar elements $\mathcal{L}^B_W$ in Deligne-Goncharov motivic fundamental coLie coalgebra is a basis of this coLie coalgebra. Hence the family $\mathcal{L}^B_W$ induced a basis of the tannakian coLie coalgebra of mixed Tate motives over $\mathbb{P}^1 \setminus \{0,1,\infty\}$ relative to the one for mixed Tate motives over $\mathbb{Q}$.

More precisely, Together with defining an avatar of $\mathcal{N}_X$ in $\mathrm{DM}(X)$, Levine [Lev11][Theorem 5.3.2 and beginning of the section 6.6] shows that when the motive of $X$ is in $\mathrm{DMT}(\mathbb{Q})$ and satisfies the Beilinson-Soulé conjecture, one can identify the Tannakian group associated with $\mathrm{MTM}(X)$ with the spectrum of the $\mathrm{H}^0$ of the bar construction over the cdga $\mathcal{N}_X$:

$$G_{\mathrm{MTM}(X)} \simeq \mathrm{Spec}(\mathrm{H}^0(B(\mathcal{N}_X))).$$

Then, he uses a relative bar-construction in order to relate the natural morphisms

$$p^* : \mathrm{DMT}(\mathrm{Spec}(\mathbb{Q})) \longrightarrow \mathrm{DMT}(X) \qquad x^* : \mathrm{DMT}(X) \longrightarrow \mathrm{DMT}(\mathrm{Spec}(\mathbb{Q})),$$

induced by the structural morphism $p : X \to \mathrm{Spec}(\mathbb{Q})$ and a choice of a $\mathbb{Q}$-point $x$, to the motivic fundamental group of $X$ at the base point $x$ defined by Goncharov and Deligne, $\pi^1_{mot}(X,x)$ (see [Del89] and [DG05]).

In particular, applying this to the the case $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$, we have the following result.

**Theorem 4.1** ([Lev11][Corollary 6.6.2])**.** *Let $x$ be a $\mathbb{Q}$-point of $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$. Then there is a split exact sequence:*

$$1 \longrightarrow \pi^{1\,DG}_{mot}(X,x) \longrightarrow \mathrm{Spec}(\mathrm{H}^0(B(\mathcal{N}_X))) \underset{x^*}{\overset{p^*}{\rightleftarrows}} \mathrm{Spec}(\mathrm{H}^0(B(\mathcal{N}_{\mathbb{Q}}))) \longrightarrow 1$$

*where $p$ is the structural morphism $p : \mathbb{P}^1 \setminus \{0,1,\infty\} \longrightarrow \mathrm{Spec}(\mathbb{Q})$ and where $\pi^{1\,DG}_{mot}(X,x)$ is the Deligne-Goncharov motivic fundamental group of $X$ described in* [DG05].

Theorem 4.1 can be reformulate in terms of coLie algebras, looking at indecomposable elements of the Hopf algebras of the functions on the pro-unipotent radical of above pro-algebraic pro-groups.

**Proposition 4.2.** *There is a split exact sequence of coLie algebras:*

$$0 \longrightarrow Q_{\mathrm{H}^0(B_{\mathbb{Q}})} \underset{\tilde{x}}{\overset{\tilde{p}}{\rightleftarrows}} Q_{\mathrm{H}^0(B_X)} \xrightarrow{\phi} Q_{geom} \longrightarrow 0$$

*where $Q_{geom}$ is the set of indecomposable elements of $\mathcal{O}(\pi^{1\,DG}_{mot}X,x)$ and is isomorphic as coLie algebra to the graded dual of the Lie algebra associated to $\pi^1_{mot}(X,x)$. Hence, $Q_{geom}$ is isomorphic as coLie coalgebra to the graded dual of the free Lie algebra on two generators* $\mathrm{Lie}(X_0,X_1)$.

Considering the family of bar elements $\mathcal{L}^B_W$ for all Lyndon words $W$ in this short exact sequence of coLie algebra, ones gets

**Theorem 4.3.** *The family $\phi(\mathcal{L}^B_W)$ for any Lyndon words $W$ is a basis of the coLie algebra $Q_{geom}$. Hence the family $\mathcal{L}^B_W$ is a basis of $Q_{\mathrm{H}^0(B_X)}$ relatively to $Q_{\mathrm{H}^0(B_{\mathbb{Q}})}$.*

*Proof.* The above short exact sequence being a sequence of coLie coalgebra, one has

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q_{\mathrm{H}^0(B_{\mathbb{Q}})} \wedge Q_{\mathrm{H}^0(B_{\mathbb{Q}})} & \xrightarrow{\tilde{p}\wedge\tilde{p}} & Q_{\mathrm{H}^0(B_X)} \wedge Q_{\mathrm{H}^0(B_X)} & \xrightarrow{\phi\wedge\phi} & Q_{geom} \wedge Q_{geom} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \delta_{Q,\mathbb{Q}}} & & \uparrow{\scriptstyle \delta_{Q,X}} & & \uparrow{\scriptstyle \delta_{geom}} & & \\
0 & \longrightarrow & Q_{\mathrm{H}^0(B_{\mathbb{Q}})} & \xrightarrow{\tilde{p}} & Q_{\mathrm{H}^0(B_X)} & \xrightarrow{\phi} & Q_{geom} & \longrightarrow & 0
\end{array}$$

As $\delta_{Q,X}(\mathcal{L}^B_0) = \delta_{Q,X}(\mathcal{L}^B_1) = 0$, weight reasons show that $\phi(\mathcal{L}^B_0)$ and $\phi(\mathcal{L}^B_1)$ are dual to the weight 1 generators ($X_0$ and $X_1$) of $\mathrm{Lie}(X_0, X_1)$.

In order to show that the family $\phi(\mathcal{L}^B_W)$ is a basis of $Q_{geom}$, it is enough to show that the elements $\phi(\mathcal{L}^B_W)$ satisfy:

$$\delta_{geom}(\phi(\mathcal{L}^B_W)) = -\sum_{U<V} \alpha^W_{U,V} \phi(\mathcal{L}^B_U) \wedge \phi(\mathcal{L}^B_V)$$

because $\delta_{geom}$ is dual to the bracket $[\,,\,]$ of $\mathrm{Lie}(X_0, X_1)$ and using the beginning of Subsection 3.2.1, in particular equation (6) ; for more details one could see [Sou12a, Proposition 3.19].

As $\phi$ commutes with the cobracket, it is enough to compute $(\phi \wedge \phi) \circ \delta_X(\mathcal{L}^B_W)$ :

$$\begin{aligned}
\delta_{geom}(\phi(\mathcal{L}^B_W)) =& (\phi \wedge \phi) \circ \delta_X(\mathcal{L}^B_W) \\
=& (\phi \wedge \phi) \left( -\sum_{U<V} \alpha^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^B_V - \sum_{U,V} \beta^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^B_V(1) \right) \\
=& -\sum_{U<V} \alpha^W_{U,V} \phi(\mathcal{L}^B_U) \wedge \phi(\mathcal{L}^B_V) - \sum_{U,V} \beta^W_{U,V} \phi(\mathcal{L}^B_U) \wedge \phi(\mathcal{L}^B_V(1))
\end{aligned}$$

By construction $\phi(\mathcal{L}^B_V(1))$ is zero which gives the desired formula for $\delta_{geom}(\phi(\mathcal{L}^B_W))$. □

Note that $\delta_X$ gives the coaction of $Q_{\mathrm{H}^0(B_{\mathbb{Q}})}$ on $Q_{geom}$ described in [Bro12] in relation with Goncharov motivic coproduct $\Delta^{mot}$. Thus, Equation (ED-$Q_X$)

$$\delta_Q(\mathcal{L}^B_W) = -\sum_{U<V} \alpha^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^B_V - \sum_{U,V} \beta^W_{U,V} \mathcal{L}^B_U \wedge \mathcal{L}^B_V(1)$$

is nothing but another expression for the cobracket $1/2(\Delta^{mot} - \tau\Delta^{mot})$. This new expression has the advantage that it is stable under the generating family $\mathcal{L}^B_W$. The price to pay is that there is so fare no combinatorial closed formula computing the coefficients $\alpha$'s and $\beta$'s.

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Max-Planck-Insititut für Mathematik, Bonn, Vivatsgasse 7,, 53111 Bonn,, Germany, souderes@mpim-bonn.mpg.de