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Abstract

We mainly prove that most d-dimensional convex surfaces Σ have a set of endpoints of Hausdorff dimension at least $d/3$.

An *endpoint* means a point not lying in the interior of any shorter path in Σ. "Most" means that the exceptions constitute a meager set, relatively to the usual Hausdorff-Pompeiu distance.

The proof employs some of the ideas used in [9] about a similar question. However, our result here is just an estimation about a still unsolved question, as much as we know.

Keywords: cut locus, Hausdorff dimension, convex body.

Mathematical Subject Classifications (2010): 28A78, 28A80, 53C22, 54E52, 52A20.

1 Notation, Introduction

Throughout this paper, d is an integer ≥ 2. $\mathcal{B}$ denotes the set of all *convexe bodies* of the $d+1$-dimensional Eudlidean space $\mathbb{E}^{d+1}$, i. e. the compact convex subsets with non empty interior.

For such a convex body $C \in \mathcal{B}$, we are interested in the corresponding *convex surface*, that is the boundary $\Sigma = \partial C$, endowed with its inner geodesic distance. More precisely $\mathcal{E}_C$ or $\mathcal{E}_\Sigma$ denotes the set of all *endpoints* of Σ, that is the points who are not in the interior of some shorter path in Σ. We focus on the Hausdorff dimension of $\mathcal{E}_C$ for typical $C \in \mathcal{B}$, when one endows $\mathcal{B}$ with the usual *Pompeiu-Hausdorff metric*:

$$d_{\mathrm{H}}(A, B) = \sup(\sup_A \operatorname{dist}(., B), \sup_B \operatorname{dist}(., A)).$$

DEFINITION 1 *By a* function dimension *we mean a continuous nondecreasing map* $\mathbb{R}_+ \to \mathbb{R}_+$ *such that* $f(t) = 0 \Leftrightarrow t = 0$.

If A is some metric space, its measure relatively to h is defined by $\mathcal{H}^h(A) = \sup_{\varepsilon>0} \mathcal{H}^h_\varepsilon(A)$*, where* $\mathcal{H}^h_\varepsilon(A)$ *is the infimum of the sums* $\sum h(\operatorname{diam} A_n)$ *associated to some countable covering of A by sets with diameters* $\operatorname{diam} A_i \leq \varepsilon$.

When $h(t) = t^s$, we also write $\mathcal{H}^s$ and $\mathcal{H}^s_\varepsilon$ instead of $\mathcal{H}^h$ and $\mathcal{H}^h_\varepsilon$. Here $\mathcal{H}^s$ is the s-dimensional measure.

The Hausdorff dimension $\dim_{\mathrm{H}}(A)$ *of A is defined by the fact that* $\mathcal{H}^s(A) = 0$ *if* $s > \dim_{\mathrm{H}}(A)$ *and that* $\mathcal{H}^s(A) = \infty$ *if* $0 < s < \dim_{\mathrm{H}}(A)$.

h is said doubling *when $h(2t)/h(t)$ is bounded on* $\mathbb{R}^*_+$.

We said that most *elements, or* typicall *elements, of A share a property when the set of exceptions is* meager*, that is included in a countable union of closed sets with empty interiors.*

When h is doubling, the properties $\mathcal{H}^h(A) > 0$ and $\mathcal{H}^h(A) < \infty$ are invariant under bi-Lipschitz maps, because of this we will be sometime imprecise on the distance involved.

We state now our main result:

THEOREM 1 *Most* $C \in \mathcal{B}$ *satisfy* $\dim_{\mathrm{H}} \mathcal{E}_C \geq d/3$.

It will be proved in Section 3. Section 2 contains preliminary estimations.

The endpoints belong to the *cut locus* $\mathcal{C}_a$ of every point $a \in \Sigma$ (that is the set of points who are never interior points of shorter paths to a) and thus for most Σ and any $a \in \Sigma$ we also have $\dim_{\mathrm{H}} \mathcal{C}_a \geq d/3$.

Otsu and Shioya have proved in [8], in the more general framework of d-dimensional spaces of curvature bounded below, that the cut locus $\mathcal{C}_a$, and thus the set of endpoints, is always $\mathcal{H}^d$-negligible. But we address the

QUESTION 1 *Does it exist a convex body $C \in \mathcal{B}$ satisfying $\mathcal{H}^{d/3}(\mathcal{E}_C) > 0$?*

In particular we dont know if we can have $\dim_{\mathrm{H}} \mathcal{E}_C > d/3$, although it would not be surprising that most $C \in \mathcal{B}$ satisfy $\dim_{\mathrm{H}} \mathcal{E}_C = d$.

Existence of endpoints a certainly involves a kind of concentration of the curvature of Σ around a, however Σ *could* be quite flat in an endpoint. Adiprasito [1] recently solved an old standing question from Zamifirescu in proving that for most $C \in \mathcal{B}$, $\Sigma = \partial C$ have some point a with infinite lower curvature (that means that Σ admits in a “locally supporting spheres” with radius arbitrarily small). But conversely, such a point a is not necessarily an endpoint.

Zamfirescu proved many curvature properties about typical convex surfaces Σ: most of their points are endpoints [14] and for most $a \in \Sigma$ and every tangent direction of Σ in a, the lower curvature is zero or the upper curvature is infinite [13]. Using an Alexandrov theorem [3], he also got that Σ is flat almost everywhere [12] (this suggests that high concentration of curvature should exist), however it is shown in [2] that Σ has no pair of opposite flat points, that is with parallel tangent space. We also recall that a Klee result [7] states that most $C \in \mathcal{B}$ are smooth and are strictly convex.

Let us observe now that for any $C \in \mathcal{B}$, we have

$$\mathcal{E}_C = \bigcap_{\varepsilon > 0} \mathcal{E}_{C,\varepsilon}$$

where $\mathcal{E}_{C,\varepsilon} = \mathcal{E}_{\partial C,\varepsilon}$ is the set of point of ∂C which are not the middle of some shorter path of ∂C with length 2ε. The sets $\mathcal{E}_{C,\varepsilon}$ are open subsets of ∂C because of classical closeness properties of shorter paths that we will often use implicitly. They mainly are described in the following lemma, of which we have just used before the case where $C_n = C$ (and often used with Blaschke th.):

LEMMA 1 ([4] Th1 p.91 and Ascoli th.)
Consider a converging sequence $C_n \to C$ in $\mathcal{B}$, and let $\gamma_n : [0,1] \to \partial C_n$ be a shorter path parametrized proportionally to the length. Then theree exists a subsequence (γ_{N_n}) uniformly convergent to a shorter path $\gamma : [0,1] \to \partial C$, moreover the length of γ is then the limit of the length of γ_n.

Thus one sees that $\mathcal{E}_C$ is a G_δ set of ∂C (and of $\mathbb{E}^{d+1}$).

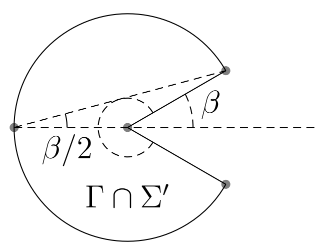
It makes also natural for our purpose to estimate the bigness of the sets $\mathcal{E}_{C,\varepsilon}$. In the following section we will make such an estimation around a conical point created by slightly modifying a sphere. Because the modifications in view are small, we focus on cases where the cone is almost flat, that means that some angle α vill be very small.

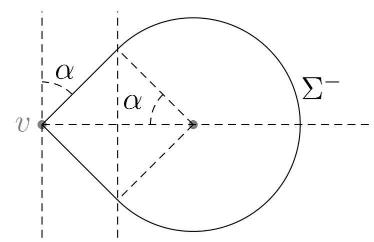
2 Around the vertices of a modified sphere

We consider here a sphere Σ of radius 1 (other cases will follow by homogeneity), and an oriented line Δ containing the center of Σ. We choose some (small) $0 < \alpha < \pi/2$ and denote by Σ^- the skullcap of all $x \in \Sigma$ making with Δ an angle $\geq \alpha$.

Our modified sphere is then the boundary Σ' of the largest convex set among all those whose boundaries contain Σ^-. Σ' has a conical point v, vertex of a cone Γ of revolution around Δ and such that $\Sigma' \subset \Gamma \cup \Sigma$.

Then $S = \Gamma \cap \Sigma$ is a $d-1$ sphere of radius $\sin\alpha$ in $\mathbb{E}^{d+1}$ and $\tan\alpha$ in Σ'.





We want to estimate, for a small α, the smallest possible distance r_α from the vertex v to a shorter path γ, in Σ' and between points of S. Such a γ lies in $\Gamma \cap \Sigma'$ (in other words $\Gamma \cap \Sigma'$ is convex in Σ') and we note that

the open ball in Σ' of center v and of radius r_α is included in $\mathcal{E}_{\Sigma',\varepsilon_\alpha}$

with $\varepsilon_\alpha = 4\tan\alpha$. Actuallly this remains true if we substitute to Σ' any convex surface Σ'' containing $\Gamma \cap \Sigma'$ as a convex subset, so that we can also consider a sphere modified by finitely many spaced enough conical point creations.

Because of the revolution symmetry of Σ', it is easy to check that γ belongs to some tridimensional affine space, so that we can suppose that $d = 2$ for the estimation of r_α.

The curvature 2β concentrated at the vertices v is then given by

$$2\pi\cos\alpha = 2\pi - 2\beta$$

($\Gamma \cap \Sigma'$ can be usually obtained by cutting from a disk with radius $\tan\alpha$ a sector with angle 2β and by pasting the part remaining). Thus we have

$$\beta = \pi(1-\cos\alpha) \sim \frac{\pi}{2}\alpha^2$$

(when α tends to zero) and

$$r_\alpha = \tan\alpha \sin\frac{\beta}{2} \sim \frac{\pi}{4}\alpha^3.$$

We can remember the estimations (the precise constants does not matter):
$r_\alpha \sim \frac{\pi}{4}\alpha^3$ and $\varepsilon_a \sim 4\alpha$.

Because of them, we will consider, for a function dimension h such that $h(t) = \mathrm{o}(t^3)$, small compact sets K in the following meaning:

DEFINITION 2 *Let h be some function dimension and K a metric space. Then K is* strongly radially porous*, respectively* h-radially porous*, if for all $x \in K$, there exists a sequence of balls such that for each n we have* $\mathrm{B}_K(x, r_n) \subset \mathrm{B}_K(x, r_n/n)$*, respectively* $\mathrm{B}_K(x, r_n) \subset \mathrm{B}_K(x, h(r_n))$*, and with the radius sequence (r_n) decreasing of null limit.*

One can equivalently use open or closed ball. h-radial porosity implies strong radial porosity. When K is compact, strong radial porosity means that given any $n, r > 0$, there exists a finite covering $(\mathrm{B}(x_i, r_i])$ of K by pairwise disjointed closed balls satisfying $K \cap \mathrm{B}(x, r_i] \subset \mathrm{B}(x, r_i/n]$, $r_i < r$ (it also implies the h-radial porosity of K for an h depending on K); and h-radial porosity means similarly that given any $r > 0$, there exists a finite covering of K by pairwise disjointed closed balls satisfying $K \cap \mathrm{B}(x_i, r_i] \subset \mathrm{B}(x_i, h(r_i)]$ and $r_i < r$.

3 Most $C \in \mathcal{B}$ satisfy $\dim_{\mathrm{H}} \mathcal{E}_C \geq d/3$

Roughly speaking, our strategy for proving bigness of typical $\mathcal{E}_\Sigma$ is to find some big fixed compact set K who is included in most $\mathcal{E}_\Sigma$, up to some parametrization of Σ by the unit sphere S^d. They are two canonical such parameterizations for $\Sigma = \partial C$:

$\Phi_C(x) = \mathrm{argmax}_C(. \mid x)$ where $(. \mid .)$ denotes the inner product of $\mathbb{E}^{d+1}$, and when C is strictly convex,

$\{\Phi_C(x)\} = \Sigma \cap (c_C + \mathbb{R}_+ x)$ where $c_C \in \operatorname{Int} C$ is the usual isobarycenter of C (or for instance the center of the smallest ball containing C).

The first idea is not good because the map will not be bi-Lipschitz at all for typical C. The second idea works well but induces some technical complication: when modifying slightly C you must also care about the modification of the center. Because of this we will choose a fixed center in our proof.

Let $\mathcal{B}_c = \{C \in \mathcal{B} \mid c \in \operatorname{Int} C\}$, it is an open subset of $\mathcal{B}$, isometric (both for the Euclidean and for the Hausdorff metric!) to $\mathcal{B}_0$ and $\mathcal{B}$ admits a (countable) covering by such $\mathcal{B}_c$. Because of this it will be very simple to check that when some property (invariant under Euclidean isometries) is shared by most $C \in \mathcal{B}_0$, then it is also shared by most $C \in \mathcal{B}$.

Thus, for $C \in \mathcal{B}_0$ we will consider the bi-Lipchitzian map $\Phi_C : S^d \to \partial C$ defined by

$$\{\Phi_C(x)\} = \Sigma \cap (\mathbb{R}_+ x).$$

We then define for any compact subset K of S^d :

$$G_K = \{C \in \mathcal{B}_0 \mid \Phi_C(K) \subset \mathcal{E}_C\} = \bigcap_{\varepsilon>0} G_{K,\varepsilon}$$

where $G_{K,\varepsilon} = \{C \in \mathcal{B}_0 \mid \Phi_C(K) \subset \mathcal{E}_{C,\varepsilon}\}$ is an open subset of $\mathcal{B}_0$ (because of Lemma 1). So by the Baire theorem, to prove that most $C \in \mathcal{B}_0$ belong to G_K, it is enough to check that every $G_{K,\varepsilon}$ is dense in $\mathcal{B}_0$:

PROPOSITION 1 *If K is an h-radially porous compact set of S^d, with $h(t) = \mathrm{o}(t^3)$, then $G_{K,\varepsilon}$ is dense in $\mathcal{B}_0$ for all $\varepsilon > 0$.*

We will use the following Lemma (whose corresponding would be more intricate if we had choosed a center c_C depending on C, to define the parametrization Φ_C):

LEMMA 2 *Let K be a compact subset of the unit sphere S^d and let $\mathcal{D}_K$ be the set of all $C \in \mathcal{B}_0$ such that there exists a finite family (B_i) of balls such that*

$$\Phi_C(K) \subset \bigcup \operatorname{int}_{\partial C}(\partial C \cap \partial B_i).$$

Then a sufficient condition for $\mathcal{D}_K$ to be dense in $\mathcal{B}_0$ is that K is strongly radially porous.

Proof of Lemma 2. The idea is that it is enough to prove that the adherence of $\mathcal{D}_K$ contains every $C \in \mathcal{B}_0$ who has curvature bounded by two positive real numbers, and then to use the radial porosity of $\Phi_c(K)$ to get $C' \in \mathcal{D}_K$ near from C by intersecting C with some large balls B_i whose boundary spheres don't meet $\Phi_c(K)$, but in such a way that $\Phi_c(K) \subset C \setminus C'$. We give however more details though the reader may find them boring (or clumsy!).

Convex polytopes constitute a dense set in $\mathcal{B}_0$, idem for finite intersections C of closed balls B_i with radius r_i, and also do their Minkovski closed ball $C_r = \{x \in \mathbb{E}^{d+1} \mid \mathrm{dist}(x, C) \leq r\}$, with $r > 0$. So for such a C_r and a given $\varepsilon > 0$, we just need to find some $C' \in \mathcal{D}_K$ with $d_{\mathrm{H}}(C_r, C') < \varepsilon$.

We observe the following curvature estimation: if $a \in \Sigma = \partial C_r$, there are two Balls B and B' with radius (not depending on a) r and $R = r + \max r_i$, such that $B \subset C_r \subset B'$ and $a \in \partial B \cap \partial B'$.

Let $x_a \in S^d$ be such that $a = \mathrm{argmax}_{C_r}(. \mid x_a)$. Let $b \in \Sigma$, $\rho = \|b - a\|$ and r_b be the radius of the sphere containing b and tangent in a to Σ. Let $t = (a - b \mid x_a)$, we have then $r_b^2 - (r_b - t)^2 = \rho^2 - t^2$, thus $2r_b t = \rho^2$. But $r \leq r_a \leq R$, so we have: $\forall a, b \in \Sigma, \sqrt{2r(a - b \mid x_a)} \leq \|b - a\| \leq \sqrt{2R(a - b \mid x_a)}$. As $\Phi_{C_r}(K)$ is porous for the Euclidean metric, it is pertinent to consider some radius $\delta < \varepsilon$ and $n \geq 4\sqrt{R/r}$ such that in $\mathbb{E}^d$ we have:

$$\Phi_C(K) \cap \mathrm{B}(a, \delta) \subset \Phi_C(K) \cap \mathrm{B}(a, \delta/n)$$

Then the half space $H = \{c \in \mathbb{E}^{d+1} \mid (a - c \mid x_a) \geq \delta^2/8R\}$ satisfies $c \in \Sigma \cap \partial H \Rightarrow \|c - a\| \leq \delta/2$ and also $c \in \Sigma \cap H \Rightarrow 2\delta/n \leq \|c - a\|$. Thus we have $\emptyset = \Phi_C(K) \cap \partial H$ and $\emptyset = H \cap \Phi_C(K) \cap \mathrm{B}(a, \delta)$.

Thus we can find a finite family of closed half spaces H_i such that $C'' = C_r \cap \bigcap H_i$ satisfies $C'' \in \mathcal{B}_0$, $d_{\mathrm{H}}(C, C') < \varepsilon$, $\emptyset = \Phi_C(K) \cap \partial H_i$ and $\emptyset = C'' \cap \Phi_C(K)$.

We then obtain the wanted $C' = C \cap \bigcap H_i' \in \mathcal{B}_0$ by replacing each H_i by some well chosed close ball H_i' (with large radius).

Proof of Proposition 1. Because K is strongly porous, it is enough by Lemma 2 to check that a given $C \in \mathcal{D}_K$ necessarily belongs to the adherence $\overline{G_{K,\varepsilon}}$ of $G_{K,\varepsilon}$. But because K is h-radially porous and because Φ_c is bi-Lipschitzian, this follows from the estimations of Section 2 applied in each ∂B_i instead of Σ, and with a finite number of conical points created in each $\partial C \cap \partial B_i$.

We now state a slightly stronger version of Theorem 1 (implying for instance that typically, $\mathcal{E}_C$ is not a countable union of sets of dimension $< d/3$):

THEOREM 2 *let h be some doubling dimension function such that $t^{d/3} = \mathrm{o}(h(t))$. Then for most $C \in \mathcal{B}$ and for every no empty open subset ω of $\Sigma = \partial C$, we have $\mathcal{H}^h(\omega \cap \mathcal{E}_\Sigma) = \infty$ and thus $\dim_{\mathrm{H}}(\omega \cap \mathcal{E}_\Sigma) \geq d/3$.*

Thus we also have $\dim_{\mathrm{H}}(\mathcal{E}_\Sigma) \geq d/3$

The estimations of the Hausorff dimensions are obtained by taking for instance $h(t) = -t^{d/3} \ln t$ for all $t > 0$ small enough. Actually to prove that most C satisfy $\dim_{\mathrm{H}}(\mathcal{E}_\Sigma) \geq d/3$, it is enough to consider the simplest case where $h(t) = t^\alpha$ for some $0 < \alpha < d/3$.

Proof of Theorem 2. As mentioned above, it is enough to prove the properties for most $C \in \mathcal{B}_0$. This follows from:

LEMMA 3 *For h as in Theorem 2, we can find a compact subset of S^d satisfying $\mathcal{H}^h(K) > 0$ and with K φ-radially porous for some dimension function φ with $\varphi(t) = \mathrm{o}(t^3)$.*

Indeed by Proposition 1, G_K is a G_δ dense in $\mathcal{B}_0$. But this is also true for any image K' of K by an isometry of S^d. Now if we choose a countable dense set $\mathcal{D}$ of such isometries, then the intersection G of all G_L, for $L \in \mathcal{D}(K)$, will be a G_δ dense subset in $\mathcal{B}_0$ and every $C \in G$ satisfies the properties wanted in Theorem 2.

Remark. Lemma 3 implies that we can have $\dim_H K \geq d/3$. Conversly it is easily seen that the φ–radial porosity in Lemma 3 implies that $\mathcal{H}^{d/3}(K) = 0$ and hence $\dim_H K \leq d/3$.

Proof of Lemma 3. It is enough to define such a K in $\mathbb{R}^d$, endowed with the norm $\|.\|_\infty$. We use for this a quite standard construction of a rather regular Cantor set.

K will be the intersection of a decreasing sequence of compact sets (K_n). Each K_n is the finite union of pairwise disjointed cubes of side r_n, (more precisely closed balls of radius $r_n/2$). $K_0 = [0,1]^d$. If C is one of the cube of K_{n-1} we divide it in N_n^d cubes C_i with sides $\rho_n = r_{n-1}/N_n$ and $C \cap K_n$ will be the union of the cubes C'_i where C'_i is the closed ball of radius r_n and with same center as C_i. r_n, and thus K_n is defined by N_n and by the condition of $\mathcal{H}^h$-mass repartition:

$$h(r_{n-1}) = N_n^d h(r_n). \tag{1}$$

Now there only remains to explain how we choose the integers $N_n \geq 2$ at the step n. For this we just need the following lemma, with $s = d/3$:

LEMMA 4 *Let $0 < s \leq d$ and K be define as above, with some doubling function dimension $h = t^s\theta(t)$ such that $\theta(t)$ is decreasing on $\mathbb{R}_+^*$.*

Then K is an h-set, i. e. $0 < \mathcal{H}^h(K) < \infty$.

Moreover if θ has infinite limit in zero and if the sequence (N_n) increases quickly enough, then we also have $r_n \leq \rho_n^{d/s}/n$ and thus K is φ-radially porous for a dimension function satisfying $\varphi(t) = \mathrm{o}(t^{d/s})$.

Actually he second point does not require θ to be decreasing, nor h to be doubling, but only that θ has infinite limit in zero.

Proof of Lemma 4. We begin by the second point which is quite easy: (1) means now:

$$r_{n-1}^s\theta(r_{n-1}) = N_n^d r_n^s\theta(r_n)$$

that is

$$r_n = \left(\frac{r_{n-1}}{N_n}\right)^{d/s}\left(\frac{\theta(r_{n-1})}{r_{n-1}^{d-s}\theta(r_n)}\right)^{1/s}$$

where $r_n \leq r_{n-1}/N_n$. So we can take N_n large enough so that the second term of the right product is less then $1/n$. The function φ is then choosed so that $\varphi(\rho_n) = \rho_n^{d/s}/n$.

To prove the first point, we can suppose that $s = d$, by changing $\theta(t)$ with $t^{s-d}\theta(t)$, moreover $\mathcal{H}^h(K) \leq h(1)$ is immediate from the construction. The proof of the contrary inequality is more tricky[1] but standard: it is enough to check that for every covering of K by closed balls C_i with side R_i we have:

$$\sum_{i\in I} h(R_i) \geq h(1)/R \tag{2}$$

Actually, considering slight increase of the sides R_i, one sees that it is enough to check (2) when K is covered by the interiors $(\operatorname{int} C_i)$. As K is compact, finite such coverings are enough. Because each of the d projections of K are closed sets of empty interior in $\mathbb{R}$, up to another slight change of (C_i) we can also suppose that $K \cap \partial C_i = \emptyset$, and thus that for some n, the boundaries ∂C_i never meet K_n. We have then for every cube of K_n (with side r_n):

$$C \cap C_i \neq \emptyset \Rightarrow C \subset C_i \tag{3}$$

[1] when $h(t) = t^\alpha$ one can get a similar $K = L^d$ for some compact L satisfying $\dim_{\mathrm{H}} L = \dim_{\mathrm{B}} L = \alpha/d$, see Falconer [5] exemple 4.7 p.59 and thus by corollary 7.4 p.95 ibid. get $\dim_{\mathrm{H}} L^d = d\dim_{\mathrm{H}} L = \alpha$.

Finally we just have to check (2) for a finite covering of K by closed balls satisfying (3) for some n. We can also suppose the left member of (2) minimal with respect to n. We observe then that we get a covering (C'_i) of $[0,1]^d$ by cubes of sides denoted R'_i, if we set:

If for some integer $1 \leq k \leq n$ we have $R_i = r_k$, then C'_i is the larger cube with side ρ_k and with same center as C_i; in that case we said that $i \in A$ and we remark that

$$h(R_i) = \frac{h(r_{k-1})}{N_k^d} = h(r_{k-1})\frac{\rho_k^d}{r_{k-1}^d} = \rho_k^d \theta(r_{k-1}) \geq {R'_i}^d \theta(1).$$

Else we said that $i \in B$ and C'_i is the larger cube with side $2R_i$ and with same center as C_i.

With this in mind, we have

$$\sum_{i\in I} h(R_i) \geq \sum_{i\in A} {R'_i}^d \theta(1) + \frac{1}{R}\sum_{i\in B} h(R'_i) \geq \theta(1)\sum_{i\in A} {R'_i}^d + \frac{1}{R}\sum_{i\in B} {R'_i}^d \theta(R'_i)$$

$$\geq \theta(1)\sum_{i\in A} {R'_i}^d + \frac{1}{R}\sum_{i\in B} {R'_i}^d \theta(1) \geq \frac{\theta(1)}{R}\sum_{i\in I} {R'_i}^d \geq \frac{\theta(1)}{R} = \frac{h(1)}{R}$$

This proves Lemma 4. To conclude the proof of Lemma 3 we must observe that we are concerned with a dimension function as in Theorem 2, that is $h(t) = t^{d/3}\theta(t)$, with θ of limite infinite in zero. To apply Lemma 4, we must explain that we can also suppose that θ is decreasing. This is true because else we can set $\theta'(t) = \inf_{]0,t]} \theta$ and we can replace h with $h'(t) = t^{d/3}\theta'(t) \leq h(t)$. For this we mainly have to check that h' is nondecreasing. So we choose $0 < t < t'$ and then $t'' \leq t'$ such that $\theta(t'') = \theta'(t')$; we can suppose $\theta'(t') \neq \theta'(t)$ and we have then $t'' > t$ and thus

$$h'(t') = {t'}^{d/3}\theta(t'') \geq h(t'') \geq h(t).$$

Finally it is easily seen that h' is doubling when h is.
Remark. We could try to reinforce Proposition 1 by taking $h(t) = t^{d/3}$ and replacing $\mathcal{B}_0$ with $\mathcal{B}_{0,R} = \{C \in \mathcal{B} \mid \mathrm{B}(0, 1/R) \subset C \subset \mathrm{B}(0,R)\}$ (to get a uniform bilipschitz born for all the maps Φ_C we are concerned with). But

in the second point of Lemma 4, if we take $h(t) = t^{d/3}$, then we get by the same calculus $\rho_n^3 = \mathrm{o}(r_n)$. Because of this the reinforcement seems of no use for our estimation of the size of $\mathcal{E}_C$.

Another canonical idea to slightly deform a ball B in view of creating endpoints is to consider for instance the convex hull C of some compact set $K \subset \partial B$, large enough in dimension and porous enough. But it seems to us that it should not give better borns and that the proofs would be more difficult, if not impossible to us, because we would not able to make estimations analogous to the very simple ones of Section 2.

We end with a property involving this last idea. Let $\mathcal{K}$ denote the space of all nonempty compact subsets K of $\mathbb{E}^{d+1}$, endowed with the Pompeiu-Hausdorff metric d_{H}. Then we have:

PROPOSITION 2 *For most $K \in \mathcal{K}$, if Σ denotes the boundary of the convex hull* $\operatorname{conv} K$ *of K, then we have $K \cap \Sigma = \mathcal{E}_\Sigma$.*

Proof We observe first that $K \mapsto \operatorname{conv} K$ is continuous and thus that for $\varepsilon > 0$, the set

$$F_\varepsilon = \{K \in \mathcal{K} \mid K \cap \partial \operatorname{conv} K \subset \mathcal{E}_{\partial \operatorname{conv} K, \varepsilon}\}$$

is open in $\mathcal{K}$ (because of Lemma 1). It is also dense because it contains all finite K. From this we get that $K \cap \Sigma \subset \mathcal{E}_\Sigma$ for a typical $K \in \mathcal{K}$. But it is also known that each point $a \in \Sigma \setminus K$ lies in the interior of some linear segment $[xy] \subset \Sigma$ joining points of $K \cap \Sigma$, and thus $a \notin \mathcal{E}_\Sigma$.

Remark. Typical $K \in \mathcal{K}$ share rather surprising properties, for instance they are Cantor sets of null Hausdorff dimension, but they also have a kind of smoothness which implies the smoothness of $\operatorname{conv} K$, see [15]. One could said that for a typical $K \in \mathcal{K}$, $\operatorname{conv} K$ is nearer from a convex polyhedron P that is a typical convex body C. Some properties of $\operatorname{conv} K$ are studied in Gruber's survey [6], in Wiaecker [11] and also in [9].

In a recent work Rouyer [10] has also studied main properties of compact sets, who are typical in the Gromov-Hausdorff metric space of all metric non empty compact spaces, up to isometry, endowed with the Gromov-Hausdorff metric

$$d_{\mathrm{GH}}(A, B) = \inf d_{\mathrm{H}}(f(A), g(B))$$

for all isometric injections $f : A \to E$, $g : B \to E$ of A and B in a same metric space E.

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