Liouville Brownian motion at criticality

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Abstract

In this paper, we construct the Brownian motion of Liouville Quantum gravity when the underlying conformal field theory has a $c = 1$ central charge. Liouville quantum gravity with $c = 1$ corresponds to two-dimensional string theory and is the conjectural scaling limit of large planar maps weighted with a $O(n = 2)$ loop model or a $Q = 4$-state Potts model embedded in a two dimensional surface in a conformal manner.

Following [27], we start by constructing the critical LBM from one fixed point $x \in \mathbb{R}^2$ (or $x \in \mathbb{S}^2$), which amounts to changing the speed of a standard planar Brownian motion depending on the local behaviour of the critical Liouville measure $M'(dx) = -X(x)e^{2X(x)}\,dx$ (where X is a Gaussian Free Field, say on $\mathbb{S}^2$). Extending this construction simultaneously to all points in $\mathbb{R}^2$ requires a fine analysis of the potential properties of the measure M'. This allows us to construct a strong Markov process with continuous sample paths living on the support of M', namely a dense set of Hausdorff dimension 0. We finally construct the Liouville semigroup, resolvent, Green function, heat kernel and Dirichlet form of (critical) Liouville quantum gravity with a $c = 1$ central charge.

In passing, we extend to quite a general setting the construction of the critical Gaussian multiplicative chaos that was initiated in [20, 21].

Key words or phrases: Gaussian multiplicative chaos, critical Liouville quantum gravity, Brownian motion, heat kernel, potential theory.

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Contents

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1 Introduction

1.1 Physics motivations

Liouville field theory is a two dimensional conformal field theory which plays an important part in two dimensional models of quantum gravity. Euclidean quantum gravity is an attempt to quantize general relativity based on Feynman's functional integral and on the Einstein-Hilbert action principle. One integrates over all Riemannian metrics on a d-dimensional manifold Σ.

General relativity is a reparametrization invariant theory which can be formulated with no reference to coordinates at all and this diffeomorphism invariance is a central issue in the quantum theory. The main motivation for considering $2d$ quantum gravity comes from the fact that the Einstein-Hilbert action becomes trivial in $2d$ as it reduces to a topological term and the cosmological constant coupled to the volume of space-time. This is the famous representation of the functional integral over geometries as a Liouville field theory by Polyakov [52] (see also [52, 40, 15]).

More precisely, one couples a Conformal Field Theory (CFT) (or more generally a matter field or quantum field theory) to gravity via any reparametrization invariant action for conformal matter fields with central charge c . A famous example is the coupling of c free scalar matter fields to gravity, which can also be interpreted as an embedding of Σ into a c-dimensional Euclidian space, thus leading to an interpretation of such a specific theory of $2d$-Liouville Quantum Gravity as a bosonic string theory in c dimensions [52].

It is shown in [52, 40, 15]) that the reparametrization invariant action of the CFT can be factorized as a tensor product where the metric is independent of the CFT and roughly takes on the form [52, 40, 15] (we consider an Euclidean background metric for simplicity):

$$g(x) = e^{\gamma X(x)} dx^2, \tag{1.1}$$

where the fluctuations of the field X are governed by the Liouville action and the parameter γ is related to the central charge of the CFT via the famous result in [40] (for $c \leqslant 1$)

$$\gamma = \frac{1}{\sqrt{6}}(\sqrt{25-c} - \sqrt{1-c}). \tag{1.2}$$

Therefore $\gamma \in [0,2]$. When the cosmological constant in the Liouville action is set to 0, one talks about **critical** $2d$-Liouville Quantum Gravity and the Liouville action turns the field X in (1.1) into a Free Field, with appropriate boundary conditions. The reader is referred to [14, 15, 22, 18, 19, 26, 29, 31, 38, 40, 48, 52] for more insights on $2d$-Liouville quantum gravity. For these reasons and though it may be an interesting field in its own right, Liouville field theory (governing the metric in $2d$-Liouville quantum gravity) is an important object in theoretical physics.

In the subcritical case $\gamma < 2, c < 1$, the geometry of the metric tensor (1.1) is mathematically investigated in [22, 27, 28, 55, 57] and the famous KPZ scaling relations [40] are rigorously proved in [4, 22, 56] in a geometrical framework (see also [16] for a non rigorous heat kernel derivation). This paper focuses on the coupling of a CFT with a $c = 1$ central charge to gravity or equivalently $2d$ string theory, i.e. the coupling of one free scalar matter field to gravity: in this case, one has $\gamma = 2$ by relation (1.2). For an excellent review on $2d$ string theory, we refer to Klebanov's lecture notes [38]. As expressed by Klebanov in [38]: "Two-dimensional string theory is the kind of toy model which possesses a remarkably simple structure but at the same time incorporates some of the physics of string theories embedded in higher dimensions". Among the $\gamma \leqslant 2$ theories, the case $\gamma = 2$ probably possesses the richest structure, inherited from its specific status of phase transition. For instance, the construction of the volume form, denoted by M', associated to the metric tensor (1.1) is investigated in [20, 21] where it is proved that it takes on the unusual form:

$$M'(dx) = -X(x)e^{2X(x)}dx, \tag{1.3}$$

which also coincides with a proper renormalization of $e^{2X(x)}dx$. Another specific point, the $\gamma = 2$ theory possesses non trivial conformally invariant gravitationally dressed vertex operators, the so-called **tachyonic fields**: the reader is referred for instance to [38] for more physics insights and to [45] for their rigorous construction and a field derivation of the KPZ formula of these operators. In this paper, we will complete this picture by constructing the Brownian motion (called critical Liouville Brownian motion, critical LBM for short), semi-group, resolvent, Dirichlet form, Green function and heat kernel of the metric tensor (1.1) with $\gamma = 2$. The question of constructing the associated distance is discussed below.

We further point out that Liouville quantum gravity is conjecturally related to discrete and continuum random surfaces. Roughly speaking, when one takes a random two-dimensional manifold and conformally maps it to a disk or the sphere, the image of the metric tensor of the manifold is a metric tensor on the disk or sphere that should correspond to an exponential of a log-correlated Gaussian random variable (some form of the GFF). The reader may consult [22, 60], which contain an extensive overview of the physics literature for a $c < 1$ central charge together with related conjectures and [20] in the case $c = 1$. Discrete critical statistical physical models having $c = 1$ then include one-dimensional matrix models (also called "matrix quantum mechanics" (MQM)) [9, 29, 30, 32, 33, 37, 39, 50, 51, 61], the so-called $O(n)$ loop model on a random planar lattice for $n = 2$ [41, 42, 43, 44], and the Q-state Potts model on a random lattice for $Q = 4$ [8, 13, 24]. The critical LBM is therefore conjectured to be the scaling limit of random walks on large planar maps weighted with a $O(n = 2)$ loop model or a $Q = 4$-state Potts model, which are embedded in a two dimensional surface in a conformal manner. For an introduction to the above mentioned $2d$-statistical models, see, e.g., [49]. We further mention [12, 17] for recent advances on this topic in the context of pure gravity, i.e. with no coupling with a CFT.

To complete this overview of physics literature, we point out that the notions of diffusions or heat kernel are at the core of physics literature about Liouville quantum gravity (see [2, 3, 10, 11, 14, 16, 62] for instance): for more on this, see the subsection on the associated distance.

1.2 Strategy and results

Basically, our approach of the metric tensor $e^{2X(x)}\,dx^2$ (where X is a Free Field) relies on the construction of the associated Brownian motion, called the critical LBM $\mathcal{B}$. Standard results of $2d$-Riemannian geometry tell us that the law of this Brownian motion is a time change of a standard planar Brownian motion B (starting from 0):

$$\mathcal{B}^x_t = x + B_{\langle \mathcal{B}^x\rangle_t} \tag{1.4}$$

where the quadratic variations $\langle \mathcal{B}^x\rangle$ are formally given by:

$$\langle \mathcal{B}^x\rangle_t = F'(x,t)^{-1}, \quad F'(x,t) = \int_0^t e^{2X(x+B_r)}\,dr. \tag{1.5}$$

Put in other words, we should integrate the weight $e^{2X(x)}$ along the paths of the Brownian motion $x+B$ to construct a mapping $t \mapsto F'(x,t)$. The inverse of this mapping corresponds to the quadratic variations of $\mathcal{B}^x$. Of course, because of the irregularity of the field X, giving sense to (1.5) is not straightforward and one has to apply a renormalization procedure: one has to apply a cutoff to the field X (a procedure that smoothes up the field X) and pass to the limit as the cutoff is removed. The procedure is rather standard in this context. Roughly speaking, one introduces an approximating field X_ϵ where the parameter ϵ stands for the extent to which one has regularized the field X (we have $X_\epsilon \to X$ as ϵ goes to 0). One then defines

$$F^\epsilon(x,t) = \int_0^t e^{2X_\epsilon(x+B_r)}\,dr \tag{1.6}$$

and one looks for a suitable deterministic renormalization $a(\epsilon)$ such that the family $a(\epsilon)F^\epsilon(x,t)$ converges towards a non trivial object as $\epsilon \to 0$. In the subcritical case $\gamma < 2$, the situation is rather well understood as the family $a(\epsilon)$ roughly corresponds to

$$a(\epsilon) \simeq \exp(-\frac{\gamma^2}{2}\mathbb{E}[X_\epsilon(x)^2])$$

(the dependence of the point x is usually fictive and may easily get rid of) in such a way that

$$a(\epsilon)\int_0^t e^{\gamma X_\epsilon(x+B_r)}\,dr \tag{1.7}$$

converges towards a non trivial limit. We will call this renormalization procedure standard: it has been successfully applied to construct random measures of the form $e^{\gamma X(x)}dx$ [34, 58, 59] (the reader may consult [55] for an overview on Gaussian multiplicative chaos theory). Though the choice of the cutoff does not affect the nature of the limiting object, a proper choice of the cutoff turns the expression (1.7) into a martingale. This is convenient to handle the convergence of this object.

At criticality ($\gamma = 2$), the situation is conceptually more involved. It is known [20] that the standard renormalization procedure of the volume form yields a trivial object. Logarithmic corrections in the choice of the family $a(\epsilon)$ are necessary (see [20, 21]) and the limiting measure that we get is the same as that corresponding to a metric tensor of the form $-X(x)e^{2X(x)}$. Observe that it is not straightforward to see at first sight that such metric tensors coincide, or even are positive. This subtlety is at the origin of some misunderstandings in the physics literature where the two forms of the tachyon field $e^{2X(x)}$ and $-X(x)e^{2X(x)}$ appear without making perfectly clear that they coincide.

The case of the LBM at criticality obeys this rule too. We will prove that non standard logarithmic corrections are necessary to make the change of time F^ϵ converge and they produce the same limiting change of times as that corresponding to a metric tensor of the form $-X(x)e^{2X(x)}$. This summarizes the almost sure convergence of F^ϵ for one given fixed point $x \in \mathbb{R}^2$. Yet, if one wishes to define a proper Markov process, one has to go one step further and establish that, almost surely, $F^\epsilon(x, \cdot)$ converges simultaneously for all possible starting points $x \in \mathbb{R}^2$: the place of the "almost sure" is important and gives rise to difficulties that are conceptually far different from the construction from one given fixed point. In [27], it is noticed that this simultaneous convergence is possible as soon as the volume form $M_\gamma(dx)$ associated to the metric tensor (1.1) is regular enough so as to make the mapping

$$x \mapsto \int_{\mathbb{R}^2} \ln_+ \frac{1}{|x-y|} M_\gamma(dx)$$

continuous. When $\gamma < 2$, multifractal analysis shows that the measure M_γ possesses a power law decay of the size of balls and this is enough to ensure the continuity of this mapping. In the critical case $\gamma = 2$, the situation is more complicated because the measure M' is rather wild: for instance the Hausdorff dimension of its support is zero [5]. Furthermore, the decay of the size of balls investigated in [5] shows that continuity and even finiteness of the mapping

$$x \mapsto \int_{\mathbb{R}^2} \ln_+ \frac{1}{|x-y|} M'(dx) \tag{1.8}$$

is unlikely to hold on the whole of $\mathbb{R}^2$. Yet we will show that we can have a rather satisfactory control of the size of balls for all x belonging to a set of full M'-measure, call it S:

$$\forall x \in S, \quad \sup_{r \in]0,1]} M'(B(x,r))(-\ln r)^p < +\infty$$

for some p large enough. In particular this estimate shows that the expression (1.8) is finite for every point $x \in S$. Also, this estimate answers a question raised in [5] (a similar estimate was proved by the same authors [6] in the related case of the discrete multiplicative cascades). We will thus construct the change of time $F'(x, \cdot)$ simultaneously for all $x \in S$. What happens on the complement of S does not matter that much since it is a set with null M' measure. Yet we will extend the change of time $F'(x, \cdot)$ to the whole of $\mathbb{R}^2$.

Once F' is constructed, potential theory [25] tells us that the LBM at criticality is a strong Markov process, which preserves the critical measure M'. We will then define the semigroup, resolvent, Green function and heat kernel associated to the LBM at criticality.

We stress that, once all the pieces of the puzzle are glued together, this LBM at criticality appears as a rather weird mathematical object. It is a Markov process with continuous sample paths living on a very thin set, which is dense in $\mathbb{R}^2$ and has Hausdorff dimension 0. And in spite of this rather wild structure, the LBM at criticality is regular enough to possess a (weak form of) heat kernel. Beyond the possible applications in physics, we feel that the study of such an object is a fundamental and challenging mathematical problem, which is far from being settled in this paper.

1.3 Discussion about the associated distance

An important question related to this work is the existence of a distance associated to the metric tensor (1.1) for $\gamma \in [0,2]$. The physics literature contains several suggestions to handle this problem, part of which are discussed along the forthcoming speculative lines. Also since we will base part of our discussion on the results established in [57], we will focus on the non critical case $\gamma < 2$ though a similar discussion should hold in the critical case. In this context, we denote $F(x,t) = \int_0^t e^{\gamma X(x+B_r)}\,dr$ the associated additive functional along the Brownian paths. We know that for all $t > 0$ there exists a Liouville heat kernel $\mathbf{p}_t^X(x,y)$. Many papers in the physics literature have argued that the Liouville heat kernel should have the following representation which is classical in the context of Riemannian geometry

$$\mathbf{p}_t^X(x,y) \underset{d(x,y)\to 0}{\sim} \frac{e^{-d(x,y)^2/t}}{t} \tag{1.9}$$

where $d(x,y)$ is the associated distance, i.e. the "Riemannian distance" defined by (1.1). From the representation (1.9), physicists [2, 3, 62] have derived many fractal and geometrical properties of Liouville quantum gravity. In particular, the paper [62] established a intriguing formula for the dimension d_H of Liouville quantum gravity which can be defined by the heuristic

$$M_\gamma(\{y;\ d(x,y) \leqslant r\}) \sim r^{d_H}$$

where M_γ is the associated volume form. Note that the meaning of the above definition is not obvious since M_γ is a multifractal random measure.

Along the same lines, a recent physics paper [16] establishes an interesting heat kernel derivation of the KPZ formula. The idea behind the paper is that, if relation (1.9) holds, then one can extract the metric from the heat kernel by using the Mellin-Barnes transform given by

$$\int_0^\infty t^{s-1}\mathbf{p}_t^X(x,y)dt.$$

Indeed, a standard computation gives the following equivalent for $s \in]0,1[$

$$\int_0^\infty t^{s-1}\frac{e^{-d(x,y)^2/t}}{t}\,dt \underset{d(x,y)\to 0}{\sim} \frac{C_s}{d(x,y)^{2(1-s)}} \tag{1.10}$$

where C_s is some positive constant. Equivalently, we would have

$$d(x,y) \sim \frac{1}{(\int_0^\infty t^{s-1}\mathbf{p}_t^X(x,y)dt)^{\frac{1}{2(1-s)}}}.$$

Essentially, the authors of [16] prove the following Mellin-Barnes version of the KPZ formula. Let K be some compact set such that the q-Hausdorff measure $\mathcal{H}^q(K)$ is non trivial, i.e. $0 < \mathcal{H}^q(K) < \infty$. Then the authors claim that one should roughly have

$$\int_K\int_K \left(\int_0^\infty t^{-\bar{q}}\mathbf{p}_t^X(x,y)dt\right) e^{\bar{q}\gamma X(x)-\frac{(\bar{q}\gamma)^2}{2}\mathbb{E}[X(x)^2]}\mathcal{H}^q(dx)e^{\bar{q}\gamma X(y)-\frac{(\bar{q}\gamma)^2}{2}\mathbb{E}[X(y)^2]}\mathcal{H}^q(dy) < \infty \tag{1.11}$$

where

$$e^{\bar{q}\gamma X(x)-\frac{(\bar{q}\gamma)^2}{2}\mathbb{E}[X(x)^2]}\mathcal{H}^q(dx) := \lim_{\epsilon\to 0} e^{\bar{q}\gamma X_\epsilon(x)-\frac{(\bar{q}\gamma)^2}{2}\mathbb{E}[X_\epsilon(x)^2]}\mathcal{H}^q(dx)$$

and $\bar{q}$ is related to q by the KPZ relation $(2+\frac{\gamma^2}{2})\bar{q}-\frac{\gamma^2}{2}\bar{q}^2=q$. Thanks to the relation (1.10), the authors claim that this yields a geometrical version of the KPZ equation which does not rely on the Euclidean metric. Recall that the rigorous geometrical derivations of KPZ in [22, 56] rely on the measure M_γ and imply working with Euclidean balls.

All these physicist derivations are a bit puzzling. It seems that the first question that can be settled mathematically is inequality (1.11). Indeed, by the results of [57], one has a very nice representation of a "natural version" of the Mellin -Barnes transform in terms of Brownian bridges; more specifically, one has

$$\int_0^\infty t^{s-1}\mathbf{p}_t^X(x,y)dt=\int_0^\infty \mathbb{E}^x[F(x,t)^{s-1}|B_t=y]\frac{e^{-\frac{|y-x|^2}{2t}}}{2\pi t}dt.$$

Note that we mention a "natural version" because it is not obvious to make sense of the measure $\mathbf{p}_t^X(x,y)dt$. If inequality (1.11) is true, it could be the case that, rather than (1.9), the heat kernel has the following behaviour compatible with the KPZ relation derived in [56]

$$\mathbf{p}_t^X(x,y)\underset{|y-x|\to 0}{\sim}\frac{e^{-M_\gamma(B(x,|y-x|))/t}}{t} \tag{1.12}$$

where $B(x,|y-x|)$ is the Euclidean ball of center x and radius $|y-x|$. In all cases, we believe that these questions are interesting and we hope that this informal discussion will trigger further investigations among mathematicians.

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2 Setup

In this section, we draw up the framework to construct the Liouville Brownian motion at criticality on the whole plane $\mathbb{R}^2$. Other geometries are possible and discussed at the end of the paper.

2.1 Notations

In what follows, we will consider Brownian motions B or $\bar{B}$ on $\mathbb{R}^2$ (or other geometries) independent of the underlying Free Field. We will denote by $\mathbb{E}^Y$ or $\mathbb{P}^Y$ expectations and probability with respect to a field Y. For instance, $\mathbb{E}^X$ or $\mathbb{P}^X$ (resp. $\mathbb{E}^B$ or $\mathbb{P}^B$) stand for expectation and probability with respect to the log-correlated field X (resp. the Brownian motion B). For $d\geqslant 1$, we consider the space $C(\mathbb{R}_+,\mathbb{R}^d)$ of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}^d$ equipped with the topology of uniform convergence over compact subsets of $\mathbb{R}_+$.

2.2 Representation of a log-correlated field

In this section we introduce the log-correlated Gaussian fields X on $\mathbb{R}^2$ that we will work with throughout this paper. One may consider other geometries as well, like the sphere $\mathbb{S}^2$ or the torus

$\mathbb{T}^2$ (in which case an adaptation of the setup and proofs is straightforward). We will represent them via a white noise decomposition.

We consider a family of centered Gaussian processes $((X_\epsilon(x))_{x\in\mathbb{R}^2})_{\epsilon>0}$ with covariance structure given, for $1 \geqslant \epsilon > \epsilon' > 0$, by:

$$K_\epsilon(x,y) = \mathbb{E}[X_\epsilon(x)X_{\epsilon'}(y)] = \int_1^{\frac{1}{\epsilon}} \frac{k(u,x,y)}{u}\,du \tag{2.1}$$

for some family $(k(u,\cdot,\cdot))_{u\,\geqslant\,1}$ of covariance kernels satisfying:

A.1 k is nonnegative, continuous.

A.2 k is locally Lipschitz on the diagonal, i.e. $\forall R>0$, $\exists C_R>0$, $\forall |x|\leqslant R$, $\forall u\geqslant 1$, $\forall y\in\mathbb{R}^2$

$$|k(u,x,x)-k(u,x,y)| \leqslant C_R u|x-y|$$

A.3 k satisfies the integrability condition: for each compact set S,

$$\sup_{x\in S, y\in\mathbb{R}^2} \int_{\frac{1}{|x-y|}}^{\infty} \frac{k(u,x,y)}{u}\,du < +\infty.$$

A.4 the mapping $H_\epsilon(x) = \int_1^{\frac{1}{\epsilon}} \frac{k(u,x,x)}{u}\,du - \ln\frac{1}{\epsilon}$ converges pointwise as $\epsilon\to 0$ and for all compact set K,

$$\sup_{x,y\in K}\sup_{\epsilon\in]0,1]} \frac{|H_\epsilon(x)-H_\epsilon(y)|}{|x-y|} < +\infty.$$

A.5 for each compact set K, there exists a constant $C_K>0$ such that

$$k(u,x,y) \geqslant k(u,x,x)(1-C_K u^{1/2}|x-y|^{1/2})_+$$

for all $x\in K$ and $y\in\mathbb{R}^2$. for all $u\geqslant 1$, $x\in K$ and $y\in\mathbb{R}^2$.

Such a construction of Gaussian processes is carried out in [1, 54] in the translation invariant case. Furthermore, [A.2] implies the following relation that we will use throughout the paper: for each compact set S, there exists a constant $c_S>0$ (only depending on k) such that for all $y\in S$, $\epsilon\in(0,1]$ and $w\in B(y,\epsilon)$, we have

$$\ln\frac{1}{\epsilon} - c_S \leqslant K_\epsilon(y,w) \leqslant \ln\frac{1}{\epsilon} + c_S. \tag{2.2}$$

We denote by $\mathcal{F}_\epsilon$ the sigma algebra generated by $\{X_u(x);\epsilon\leqslant u, x\in\mathbb{R}^2\}$.

2.3 Examples

We explain first a Fourier white noise decomposition of log-correlated translation invariant fields as this description appears rather naturally in physics. Consider a nonnegative even function φ defined on $\mathbb{R}^2$ such that $\lim_{|u|\to\infty}|u|^2\varphi(u)=1$. We consider the kernel

$$K(x,y) = \int_{\mathbb{R}^2} e^{i\langle u,x-y\rangle}\varphi(u)\,du. \tag{2.3}$$

We consider the following cut-off approximations

$$K_\epsilon(x,y)=\frac{1}{2\pi}\int_{B(0,\epsilon^{-1})} e^{i\langle u,x-y\rangle}\varphi(u)\,du. \tag{2.4}$$

The kernel K can be seen as the prototype of kernels of log-type in dimension 2. It has obvious counterparts in any dimension. The cut-off approximation is quite natural, rather usual in physics (sometimes called the ultraviolet cut-off) and has well known analogues on compact manifolds (in terms of series expansion along eigenvalues of the Laplacian for instance). If X has covariance given by (2.3), then X has the following representation

$$X(x)=\int_{\mathbb{R}^2} e^{i\langle u.x\rangle}\sqrt{\varphi(u)}(W_1(du)+iW_2(du))$$

where $W_1(du)$ and $W_2(du)$ are independent Gaussian distributions. The distributions $W_1(du)$ and $W_2(du)$ are functions of the field X (since they are the real and imaginary parts of the Fourier transform of X). The law of $W_1(du)$ is $W(du)+W(-du)$ where W is a standard white noise and the law of $W_2(du)$ is $\widetilde{W}(du)-\widetilde{W}(-du)$ where $\widetilde{W}$ is also standard white noise. One can then consider the following family with covariance (2.4) and which fits into our framework as it corresponds to adding independent fields

$$X_\epsilon(x)=\int_{B(0,\epsilon^{-1})} e^{i\langle u.x\rangle}\sqrt{\varphi(u)}(W_1(du)+iW_2(du))$$

Notice also that the approximations X_ϵ are functions of the original field X since $W_1(du)$ and $W_2(du)$ are functions of the field X.

Notice that K_ϵ can be rewritten as (S stands for the unit sphere and ds for the uniform probability measure on S)

$$K_\epsilon(x,y)=\int_0^{\frac{1}{\epsilon}}\frac{k(r,x,y)}{r}\,dr,\qquad \text{with } k(r,x,y)=r^2\int_S\varphi(rs)\cos(r\langle x-y,s\rangle)\,ds.$$

Let us simplify a bit the discussion by assuming that φ is isotropic. In that case, it is plain to check that assumptions [A.1-5] are satisfied (in the slightly extended context of integration over $[0,\frac{1}{\epsilon}]$ instead of $[1,\frac{1}{\epsilon}]$ but this is harmless as $K_1(x,y)$ is here a very regular Gaussian kernel).

Example 1. Massive Free Field (MFF). *The whole plane MFF is a centered Gaussian distribution with covariance kernel given by the Green function of the operator $2\pi(m^2-\triangle)^{-1}$ on $\mathbb{R}^2$, i.e. by:*

$$\forall x,y\in\mathbb{R}^2,\quad G_m(x,y)=\int_0^\infty e^{-\frac{m^2}{2}u-\frac{|x-y|^2}{2u}}\frac{du}{2u}. \tag{2.5}$$

The real $m>0$ is called the mass. This kernel is of σ-positive type in the sense of Kahane [34] since we integrate a continuous function of positive type with respect to a positive measure. It is furthermore a star-scale invariant kernel (see [1, 54]): it can be rewritten as

$$G_m(x,y)=\int_1^{+\infty}\frac{k_m(u(x-y))}{u}\,du. \tag{2.6}$$

for some continuous covariance kernel $k_m(z)=\frac{1}{2}\int_0^\infty e^{-\frac{m^2}{2v}|z|^2-\frac{v}{2}}\,dv$ and therefore satisfies the assumptions [A.1-5].

One may also choose the Fourier white noise (2.4) decomposition with $\varphi(u) = \frac{1}{|u|^2+m^2}$ or the semigroup covariance structure

$$\mathbb{E}[X_\epsilon(x)X_{\epsilon'}(y)] = \pi \int_{\max(\epsilon,\epsilon')^2}^{\infty} p(u,x,y)e^{-\frac{m^2}{2}u}\,du,$$

which also satisfies assumptions [A.1-5] (modulo a change of variable $u = 1/v^2$ in the above expression: see [21, section D]).

Example 2. Gaussian Free Field (GFF). *Consider a bounded open domain D of $\mathbb{R}^2$. Formally, a GFF on D is a Gaussian distribution with covariance kernel given by the Green function of the Laplacian on D with prescribed boundary conditions. We describe here the case of Dirichlet boundary conditions. The Green function is then given by the formula:*

$$G_D(x,y) = \pi \int_0^\infty p_D(t,x,y)dt \tag{2.7}$$

where p_D is the (sub-Markovian) semi-group of a Brownian motion B killed upon touching the boundary of D, namely for a Borel set $A \subset D$

$$\int_A p_D(t,x,y)\,dy = P^x(B_t \in A,\ T_D > t)$$

with $T_D = inf\{t \geqslant 0,\ B_t \notin D\}$. The most direct way to construct a cut-off family of the GFF on D is then to consider a white noise W distributed on $D \times \mathbb{R}_+$ and to define:

$$X(x) = \sqrt{\pi}\int_{D\times\mathbb{R}_+} p_D(\frac{s}{2},x,y)W(dy,ds).$$

One can check that $\mathbb{E}[X(x)X(x')] = \pi\int_0^\infty p_D(s,x,x')\,ds = G_D(x,x')$. The corresponding cut-off approximations are given by:

$$X_\epsilon(x) = \sqrt{\pi}\int_{D\times[\epsilon^2,\infty[} p_D(\frac{s}{2},x,y)W(dy,ds).$$

They have the following covariance structure

$$\mathbb{E}[X_\epsilon(x)X_{\epsilon'}(y)] = \pi \int_{\max(\epsilon,\epsilon')^2}^{\infty} p_D(u,x,y)\,du, \tag{2.8}$$

which also satisfies assumptions [A.1-5] (on every subdomain of D and modulo a change of variable $u = 1/v^2$ in the above expression: see also [21, section D]).

For some technical reasons, we will sometimes also consider either of the following assumptions:

A.6 $k(v,x,y) = 0$ for $|x-y| \geqslant Dv^{-1}(1+2\ln v)^\alpha$ for some constants $D, \alpha > 0$,

A.6' $(k(v,x,y))_v$ is the family of kernels presented in examples 1 or 2.

2.4 Regularized Riemannian geometry

We would like to consider a Riemannian metric tensor on $\mathbb{R}^2$ (using conventional notations in Riemannian geometry) of the type $e^{2X_\epsilon(x)}dx^2$, where dx^2 stands for the standard Euclidean metric on $\mathbb{R}^2$. Yet, as we will see, such an object has no suitable limit as ϵ goes to 0. So, for future renormalization purposes, we rather consider:

$$g_\epsilon(x)dx^2 = \sqrt{-\ln\epsilon}\,\epsilon^2\,e^{2X_\epsilon(x)}dx^2.$$

2.4.1 Volume form

The Riemannian volume on the manifold $(\mathbb{R}^2, g_\epsilon)$ is given by:

$$M_\epsilon(dx) = \sqrt{-\ln\epsilon}\,\epsilon^2\,e^{2X_\epsilon(x)}\,dx, \tag{2.9}$$

where dx stands for the Lebesgue measure on $\mathbb{R}^2$, and will be called ϵ-regularized critical measure. The study of the limit of the random measures $(M_\epsilon(dx))_\epsilon$ is carried out in [20, 21] in a less general context. It is based on the study of the limit of the family $M'_\epsilon(dx)$ defined by:

$$M'_\epsilon(dx) := (2\,\ln\frac{1}{\epsilon} - X_\epsilon(x))e^{2X_\epsilon(x)-2\ln\frac{1}{\epsilon}}dx.$$

We will extend the results in [20] and prove

Theorem 2.1. *Almost surely, the (locally signed) random measures $(M'_\epsilon(dx))_{\epsilon>0}$ converge as $\epsilon\to 0$ towards a positive random measure $M'(dx)$ in the sense of weak convergence of measures. This limiting measure has full support and is atomless.*

Concerning the Seneta-Heyde norming, we have

Theorem 2.2. *Assume [A.1-5] and either [A.6] or [A.6']. We have the convergence in probability in the sense of weak convergence of measures:*

$$M_\epsilon(dx) \to \sqrt{\frac{2}{\pi}}M'(dx), \qquad \text{as } \epsilon\to 0.$$

The proof of Theorem 2.2 is carried out in [21, section D] (in fact, it is assumed in [21] that the family of kernels $(k(v,x,y))_v$ is translation invariant but adapting the proof is straightforward and thus left to the reader).

Beyond its conceptual importance, the Seneta-Heyde norming is crucial to establish, via Kahane's convexity inequalities [34], the study of moments carried out in [20, 21] and obtain

Proposition 2.3. *Assume [A.1-5] and either [A.6] or [A.6']. For each bounded Borel set A and $q\in]-\infty,1[$, the random variable $M'(A)$ possesses moments of order q. Furthermore, if A has non trivial Lebesgue measure and $x\in\mathbb{R}^2$:*

$$\mathbb{E}[M'(\lambda A+x)^q] \simeq C(q,x)\lambda^{\xi_{M'}(q)}$$

where $\xi_{M'}$ is the power law spectrum of M':

$$\forall q<1, \quad \xi_{M'}(q) = 4q-2q^2.$$

Another important result about the modulus of continuity of the measure M' is established in [5]. We stress here that we pursue the discussion at a heuristic level since the result in [5] is not general enough to apply in our context (to be precise, it is valid for a well chosen family of kernels $(k(v,x,y))_v$ in dimension 1 in order to get nice scaling relations for the associated measure M'). Anyway, we expect this result to be true in greater generality and we will not use it in this paper: we are more interested in its conceptual significance. So, by analogy with [5], the measure M' is expected to possess the following modulus of continuity of "square root of log" type: for all $\gamma < 1/2$, there exists a random variable C almost surely finite such that

$$\forall \text{ ball } B \subset B(0,1), \quad M'(B) \leqslant C(\ln(1+|B|^{-1}))^{-\gamma}. \tag{2.10}$$

Furthermore, the Hausdorff dimension of the carrier of M' is 0. By analogy with the results that one gets in the context of multiplicative cascades [6], one also expects that the above theorem 2.10 cannot be improved. In particular, the measure M' does not possess a modulus of continuity better than a log unlike the subcritical situation explored in [27], where this property turned out to be crucial for the construction of the LBM as a whole Markov process. This remark is at the origin of the further complications arising in our paper (the critical case) in comparison with [27, 28] (the subcritical case).

2.4.2 Regularized Liouville Brownian motion

The main concern of this paper will be the Brownian motion associated with the metric tensor g_ϵ: following standard formulas of Riemannian geometry, one can associate to the Riemannian manifold $(\mathbb{R}^2, g_\epsilon)$ a Brownian motion $\mathcal{B}^\epsilon$:

Definition 2.4 (ϵ-regularized critical Liouville Brownian motion, LBM$_\epsilon$ for short)**.** *For any fixed $\epsilon > 0$, we define the following diffusion on $\mathbb{R}^2$. For any $x \in \mathbb{R}^2$,*

$$\mathcal{B}_t^{\epsilon,x} = x + \int_0^t (-\ln\epsilon)^{-1/4}\,\epsilon^{-1} e^{-X_\epsilon(\mathcal{B}_u^{\epsilon,x})}\, d\bar{B}_u\,. \tag{2.11}$$

where $\bar{B}_t$ is a standard two-dimensional Brownian motion.

We stress that the fact that there is no drift term in the definition of the Brownian motion is typical from a scalar metric tensor in dimension 2. By using the Dambis-Schwarz Theorem, one can define the law of the LBM$_\epsilon$ as

Definition 2.5. *For any $\epsilon > 0$ fixed and $x \in \mathbb{R}^2$,*

$$\mathcal{B}_t^{\epsilon,x} = x + B_{\langle \mathcal{B}^{\epsilon,x}\rangle_t}\,, \tag{2.12}$$

where $(B_r)_{r\geqslant 0}$ is a two-dimensional Brownian motion independent of the field X and the **quadratic variation** *$\langle \mathcal{B}^{\epsilon,x}\rangle$ of $\mathcal{B}^{\epsilon,x}$ is defined as*

$$\langle \mathcal{B}^{\epsilon,x}\rangle_t := \inf\{s \geqslant 0\,:\,(-\ln\epsilon)^{1/2}\,\epsilon^2 \int_0^s e^{2X_\epsilon(x+B_u)}\,du \geqslant t\}\,. \tag{2.13}$$

It will thus be useful to define the following quantity on $\mathbb{R}^2 \times \mathbb{R}_+$:

$$F^\epsilon(x,s) = \epsilon^2 \int_0^s e^{2X_\epsilon(x+B_u)}\,du, \tag{2.14}$$

in such a way that the process $\langle \mathcal{B}^{\epsilon,x} \rangle$ is entirely characterized by:

$$(-\ln \epsilon)^{1/2} F^{\epsilon}(x, \langle \mathcal{B}^{\epsilon,x} \rangle_t) = t. \tag{2.15}$$

Several standard facts can be deduced from the smoothness of X_ϵ. For each fixed $\epsilon > 0$, the LBM$_\epsilon$ a.s. induces a Feller diffusion on $\mathbb{R}^2$, and thus a semi-group $(P^\epsilon_t)_{t \geqslant 0}$, which is symmetric w.r.t the volume form M_ϵ.

We will be mostly interested in establishing the convergence in law of the LBM$_\epsilon$ as $\epsilon \to 0$. Basically, studying the convergence of the LBM$_\epsilon$ thus boils down to establishing the convergence of its quadratic variations $\langle \mathcal{B}^{\epsilon,x} \rangle$.

3 Critical LBM starting from one fixed point

The first section is devoted to the convergence of the ϵ-regularized LBM when starting from one fixed point, say $x \in \mathbb{R}^2$. As in the case of the convergence of measures [20, 21], the critical situation here is technically more complicated than in the subcritical case [27], though conceptually similar. The first crucial step of the construction consists in establishing the convergence towards 0 of the family of functions $(F^\epsilon(x, \cdot))_\epsilon$ and then in computing the first order expansion of the maximum of the field X_ϵ along the Brownian path up to time t, and more precisely to prove that

$$\max_{s \in [0,t]} X_\epsilon(x + B_s) - 2 \ln \frac{1}{\epsilon} \to -\infty, \quad \text{as } \epsilon \to 0. \tag{3.1}$$

This is mainly the content of subsection 3.2, after some preliminary lemmas in subsection 3.1. Then our strategy will mainly be to adapt the ideas related to convergence of critical measures [20, 21].

We further stress that, in the case of measures (see [20]), the content of subsection 3.2 is established thanks to comparison with multiplicative cascades measures and Kahane's convexity inequalities. In our context, no equivalent result has been established in the context of multiplicative cascades in such a way that we have to carry out a direct proof.

3.1 Preliminary results about properties of Brownian paths

Let us consider a standard Brownian motion B on the plane $\mathbb{R}^2$ starting at some given point $x \in \mathbb{R}^2$. Let us consider the occupation measure μ_t of the Brownian motion up to time $t > 0$ and let us define the function

$$\forall \epsilon \in]0,1], \quad h(\epsilon) = \ln \frac{1}{\epsilon} \ln \ln \ln \frac{1}{\epsilon}. \tag{3.2}$$

The following result is proved in [46]

Theorem 3.1. *There exists a deterministic constant $c > 0$ such that $\mathbb{P}^B$-almost surely, the set*

$$E = \{z \in \mathbb{R}^2; \limsup_{\epsilon \to 0} \frac{\mu_t(B(z,\epsilon))}{\epsilon^2 h(\epsilon)} = c\}$$

has full μ_t-measure.

We will need an extra elementary result about the structure of Brownian paths:

Lemma 3.2. *For every $p > 2$, we have almost surely:*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|^2 \ln\big(\frac{1}{|x-y|}+2\big)^p}\mu_t(dx)\mu_t(dy) < +\infty.$$

Proof of Lemma 3.2. We have:

$$\mathbb{E}^B\Big[\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|^2 \ln\big(\frac{1}{|x-y|}+2\big)^p}\mu_t(dx)\mu_t(dy)\Big] = \mathbb{E}^B\Big[\int_{[0,t]^2} \frac{1}{|B_r-B_s|^2 \ln\big(\frac{1}{|B_r-B_s|}+2\big)^p}\,drds\Big]$$
$$= \int_{[0,t]^2} \mathbb{E}^B\Big[\frac{1}{|r-s||B_1|^2 \ln\big(\frac{1}{|r-s|^{1/2}|B_1|}+2\big)^p}\Big]\,drds.$$

Let us compute for $a \leqslant t$ the quantity $\mathbb{E}^B\Big[\frac{1}{a|B_1|^2\ln\big(\frac{1}{a^{1/2}|B_1|}+2\big)^p}\Big]$. By using the density of the Gaussian law, we get:

$$\mathbb{E}^B\Big[\frac{1}{a|B_1|^2 \ln\big(\frac{1}{a^{1/2}|B_1|}+2\big)^p}\Big] = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1}{a|u|^2\ln\big(\frac{1}{a^{1/2}|u|}+2\big)^p}e^{-\frac{|u|^2}{2}}\,du \leqslant \int_0^\infty \frac{1}{ar\ln\big(\frac{1}{a^{1/2}r}+2\big)^p}e^{-\frac{r^2}{2}}\,dr$$
$$\leqslant \frac{1}{a}\int_0^\infty \frac{1}{u\ln\big(\frac{1}{u}+2\big)^p}e^{-\frac{u^2}{2a}}\,du$$
$$\leqslant \frac{1}{a}\int_0^{a^{1/4}} \frac{1}{u\ln\big(\frac{1}{u}+2\big)^p}e^{-\frac{u^2}{2a}}\,du + \frac{1}{a}\int_{a^{1/4}}^{t} \frac{1}{u\ln\big(\frac{1}{u}+2\big)^p}e^{-\frac{u^2}{2a}}\,du + \frac{1}{a}\int_t^\infty \frac{1}{u\ln\big(\frac{1}{u}+2\big)^p}e^{-\frac{u^2}{2a}}\,du$$
$$\leqslant \frac{1}{a}\int_0^{a^{1/4}} \frac{1}{u\ln^p\big(\frac{1}{u}\big)}\,du + \frac{1}{a}\int_0^{t} \frac{1}{u\ln\big(\frac{1}{u}+2\big)^p}e^{-\frac{1}{2a^{1/2}}}\,du + \frac{2}{t^2\ln^p 2}e^{-\frac{t^2}{2a}}$$
$$\leqslant \frac{4^p}{a(p-1)\ln^{p-1}\frac{1}{a}} + \frac{C}{a}e^{-\frac{1}{2a^{1/2}}} + \frac{2}{t^2\ln^p 2}e^{-\frac{t^2}{2a}}.$$

Therefore

$$\mathbb{E}^B\Big[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|^2\ln\big(\frac{1}{|x-y|}+2\big)^p}\mu_t(dx)\mu_t(dy)\Big]$$
$$\leqslant C\int_0^t\int_0^t \Big(\frac{1}{|r-s|\ln^{p-1}\frac{1}{|r-s|}} + \frac{1}{|r-s|}e^{-\frac{1}{2|r-s|^{1/2}}} + e^{-\frac{t^2}{2|r-s|}}\Big)\,drds.$$

As this latter quantity is obviously finite, the proof is complete. ∎

3.2 First order expansion of the maximum of the field along Brownian paths

To begin with, we claim:

Proposition 3.3. *For all $x \in \mathbb{R}^2$, almost surely in X, the family of random mapping $t \mapsto F^\epsilon(x,t)$ converges to 0 in the space $C(\mathbb{R}_+,\mathbb{R}_+)$ as $\epsilon \to 0$.*

Proof. Fix $x \in \mathbb{R}^2$. Observe first that

$$F^\epsilon(x,t) = \epsilon^2\int_0^t e^{2X_\epsilon(x+B_r)}\,dr = \int_0^t e^{g_\epsilon(x+B_r)}e^{2X_\epsilon(x+B_r)-2\mathbb{E}^X[X_\epsilon(x+B_r)^2]}\,dr$$

where

$$g_\epsilon(u) = 2\mathbb{E}^X[X_\epsilon(u)^2] - 2\ln\frac{1}{\epsilon} = \int_1^{1/\epsilon} \frac{k(u,x,x)-1}{u}\,du.$$

By assumption [A.4], the function g_ϵ converges uniformly over the compact subsets of $\mathbb{R}^2$. Furthermore, for each $t>0$, the set $\{x+B_s; s\in[0,t]\}$ is a compact set and g_ϵ converges uniformly over this compact set. So, even if it means considering $\int_0^t e^{2X_\epsilon(x+B_r)-2\mathbb{E}^X[X_\epsilon(x+B_r)^2]}\,dr$ instead of $F^\epsilon(x,t)$, we may assume that $\mathbb{E}^X[X_\epsilon(x+B_r)^2] = \ln\frac{1}{\epsilon}$, in which case $F^\epsilon(x,t)$ is a martingale with respect to the filtration $\mathcal{F}_\epsilon = \sigma\{X_r(x); \epsilon\leqslant r, x\in\mathbb{R}^2\}$. As this martingale is nonnegative, it converges almost surely. We just have to prove that the limit is 0. To this purpose, we use a lemma in [23]. Translated into our context, it reads:

Lemma 3.4. *The almost sure convergence of the family $(F^\epsilon(x,t))_\epsilon$ towards 0 as $\epsilon\to 0$ is equivalent to the fact that $\limsup_{\epsilon\to 0} F^\epsilon(x,t) = +\infty$ under the probability measure defined by:*

$$Q_{|\mathcal{F}_\epsilon} = t^{-1}F^\epsilon(x,t)\,\mathbb{P}^X.$$

The main idea of what follows is to prove that, under Q, $F^\epsilon(x,t)$ is stochastically bounded from below by the exponential of a Brownian motion so that $\limsup_{\epsilon\to 0} F^\epsilon(x,t) = +\infty$ and we apply Lemma 3.4 to conclude .

To carry out this argument, let us define a new probability measure Θ_ϵ on $\mathcal{B}(\mathbb{R}^2)\otimes\mathcal{F}_\epsilon$ by

$$\Theta_{\epsilon|\mathcal{B}(\mathbb{R}^2)\otimes\mathcal{F}_\epsilon} = t^{-1}e^{2X_\epsilon(y)-2\ln\frac{1}{\epsilon}}\,\mathbb{P}^X(d\omega)\mu_t(dy), \tag{3.3}$$

where μ_t stands for the occupation measure of the Brownian motion B^x starting from x. We denote by $\mathbb{E}_{\Theta_\epsilon}$ the corresponding expectation. In fact, since the above definition defines a pre-measure on the ring $\mathcal{B}(\mathbb{R}^2)\otimes\bigcup_\epsilon\mathcal{F}_\epsilon$, one can define a measure Θ on $\mathcal{B}(\mathbb{R}^2)\otimes\mathcal{F}$ by using Caratheodory's extension theorem. We recover the relation $\Theta_{|\mathcal{B}(A)\otimes\mathcal{F}_\epsilon} = \Theta_\epsilon$. Similarly, we construct the probability measure Q on $\mathcal{F} = \sigma\big(\bigcup_\epsilon\mathcal{F}_\epsilon\big)$ by setting:

$$Q_{|\mathcal{F}_\epsilon} = t^{-1}F^\epsilon(x,t)\,d\mathbb{P}^X,$$

which is nothing but the marginal law of $(\omega,y)\mapsto\omega$ with respect to Θ_ϵ. We state a few elementary properties below. The conditional law of y given $\mathcal{F}_\epsilon$ is given by:

$$\Theta_\epsilon(dy|\mathcal{F}_\epsilon) = \frac{e^{2X_\epsilon(y)-2\ln\frac{1}{\epsilon}}}{F^\epsilon(x,t)}\,\mu_t(dy).$$

If Y is a $\mathcal{B}(\mathbb{R}^2)\otimes\mathcal{F}_\epsilon$-measurable random variable then it has the following conditional expectation given $\mathcal{F}_\epsilon$:

$$\mathbb{E}_{\Theta_\epsilon}[Y|\mathcal{F}_\epsilon] = \int_{\mathbb{R}^2} Y(y,\omega)\frac{e^{2X_\epsilon(y)-2\ln\frac{1}{\epsilon}}}{F^\epsilon(x,t)}\,\mu_t(dy).$$

Now we turn to the proof of Proposition 3.3 while keeping in mind this preliminary background. Let us observe that it is enough to prove that the set

$$\{\limsup_{\epsilon\to 0} F^\epsilon(x,t) = +\infty\}$$

has probability 1 conditionally to y under Θ to deduce that it satisfies

$$Q\big(\{\limsup_{\epsilon\to 0} F^\epsilon(x,t) = +\infty\}\big) = 1.$$

So we have to compute the law of $F^\epsilon(x,t)$ under $\Theta(\cdot|y)$. Recall the definition of h in (3.2). We have:

$$F^\epsilon(x,t)=\int_{\mathbb{R}^2} e^{2X_\epsilon(u)-2\ln\frac{1}{\epsilon}}\mu_t(du)\geqslant \int_{\mathbb{R}^2} e^{2X_\epsilon(u)-2\ln\frac{1}{\epsilon}}\mathbb{1}_{B(y,\epsilon)}\mu_t(du).$$

Let us now write

$$X_\epsilon(u)=\lambda_\epsilon(u,y)X_\epsilon(y)+Z_\epsilon(u,y)$$

where $\lambda_\epsilon(u,y)=\frac{K_\epsilon(u,y)}{\ln\frac{1}{\epsilon}}$ and $Z_\epsilon(u,y)=X_\epsilon(u)-\lambda_\epsilon(u,y)X_\epsilon(y)$. Observe that the process $(Z_\epsilon(u,y))_{u\in\mathbb{R}^2}$ is independent of $X_\epsilon(y)$. Therefore

$$\begin{aligned}F^\epsilon(x,t)&\geqslant \int_{\mathbb{R}^2} e^{2X_\epsilon(u)-2\ln\frac{1}{\epsilon}}\mathbb{1}_{B(y,\epsilon)}\mu_t(du)\\ &=e^{2X_\epsilon(y)-2\ln\frac{1}{\epsilon}}\int_{\mathbb{R}^2} e^{2Z_\epsilon(u,y)+2(\lambda_\epsilon(u,y)-1)X_\epsilon(y)}\mathbb{1}_{B(y,\epsilon)}\mu_t(du)\\ &=e^{2X_\epsilon(y)-4\ln\frac{1}{\epsilon}+\ln h(\epsilon)}\times\inf_{u\in B(y,\epsilon)}e^{2(\lambda_\epsilon(u,y)-1)X_\epsilon(y)}\times\frac{1}{\epsilon^2h(\epsilon)}\int_{\mathbb{R}^2}e^{2Z_\epsilon(u,y)}\mathbb{1}_{B(y,\epsilon)}\mu_t(du).\end{aligned}$$

Let us define

$$a_\epsilon(y)=\int_{\mathbb{R}^2}\mathbb{1}_{B(y,\epsilon)}\mu_t(du).$$

With the help of the Jensen inequality, we deduce

$$F^\epsilon(x,t)\geqslant e^{2X_\epsilon(y)-4\ln\frac{1}{\epsilon}+\ln h(\epsilon)}\inf_{u\in B(y,\epsilon)}e^{2(\lambda_\epsilon(u,y)-1)X_\epsilon(y)}\frac{a_\epsilon(y)}{\epsilon^2h(\epsilon)}\exp\Big(\frac{1}{a_\epsilon(y)}\int_{B(y,\epsilon)}2Z_\epsilon(u,y)\mu_t(du)\Big).$$

Let us set

$$Y_\epsilon=a_\epsilon(y)^{-1}\int_{\mathbb{R}^2}2Z_\epsilon(u,y)\mathbb{1}_{B(y,\epsilon)}\mu_t(du).$$

Finally, for all $R>0$, we use the independence of Y_ϵ and $X_\epsilon(y)$ to get

$$\Theta\Big(\{\limsup_{\epsilon\to0}F^\epsilon(x,t)=+\infty\}|y\Big)\tag{3.4}$$
$$\geqslant\Theta\Big(\{\limsup_{\epsilon\to0}e^{2X_\epsilon(y)-4\ln\frac{1}{\epsilon}+\ln h(\epsilon)}\times\inf_{u\in B(y,\epsilon)}e^{2(\lambda_\epsilon(u,y)-1)X_\epsilon(y)}\times\frac{a_\epsilon(y)}{\epsilon^2h(\epsilon)}\times\exp(-R)=+\infty\}|y\Big)$$
$$\times\Theta(Y_\epsilon\geqslant -R|y).\tag{3.5}$$

Now we analyze the behaviour of each term in the above expression.

First, notice that $\Theta(Y_\epsilon\geqslant -R|y)=\mathbb{P}^X(Y_\epsilon\geqslant -R)$ and that Y_ϵ is a centered Gaussian random variable under $\mathbb{P}^X$ with variance

$$a_\epsilon(y)^{-2}\int_{B(y,\epsilon)\times B(y,\epsilon)}\mathbb{E}^X\big[(X_\epsilon(u)-\lambda_\epsilon(u,y)X_\epsilon(y))(X_\epsilon(u')-\lambda_\epsilon(u',y)X_\epsilon(y))\big]\mu_t(du)\mu_t(du').$$

This quantity may be easily evaluated with assumption [A.2] and proved to be less than some constant C, which does not depend on ϵ and $y\in\{x+B_s;s\in[0,t]\}$. Therefore

$$\Theta(Y_\epsilon\geqslant -R|y)\geqslant 1-\rho(R)\tag{3.6}$$

for some nonnegative function ρ that goes to 0 as $R \to \infty$.

Second, from Theorem 3.1, there exists a constant c such that $\mathbb{P}^B$-almost surely, the set

$$E = \{z \in \mathbb{R}^2; \limsup_{\epsilon \to 0} \frac{a_\epsilon(z)}{\epsilon^2 h(\epsilon)} = c\}$$

has full μ_t-measure. Since E has full μ_t measure, even if it means extracting a random subsequence (only depending on B), we may assume that

$$\lim_{\epsilon \to 0} \frac{a_\epsilon(y)}{\epsilon^2 h(\epsilon)} = c. \tag{3.7}$$

Third, under $\Theta(\cdot|y)$, the process $X_\epsilon(y) - 2\ln\frac{1}{\epsilon}$ is a Brownian motion, call it $\bar{B}$, in logarithmic time, i.e.

$$\bar{B}_{\ln\frac{1}{\epsilon}} = X_\epsilon(y) - 2\ln\frac{1}{\epsilon}.$$

We further stress that $\bar{B}$ is independent from B, and thus from the random sequence $(a_\epsilon(y))_\epsilon$. From the law of the iterated logarithm, we deduce that $\Theta(\cdot|y)$-almost surely

$$\limsup_{\epsilon \to 0} e^{2X_\epsilon(y) - 4\ln\frac{1}{\epsilon} + \ln h(\epsilon)} = +\infty. \tag{3.8}$$

Fourth, using assumption [A.2], it is readily seen that there exists some constant C such that for all $\epsilon \in (0,1)$, $y \in \{x + B_s, s \in [0,t]\}$ and all $u \in B(y,\epsilon)$

$$2|\lambda_\epsilon(u,y) - 1| \leqslant C(-\ln\epsilon)^{-1}.$$

Since the process $X_\epsilon(y) - 2\ln\frac{1}{\epsilon}$ is a Brownian motion in logarithmic time under $\Theta(\cdot|y)$, we deduce

$$\liminf_{\epsilon \to 0} \inf_{u \in B(y,\epsilon)} e^{2(\lambda_\epsilon(u,y)-1)X_\epsilon(y)} \geqslant 1. \tag{3.9}$$

By gathering (3.7)+(3.8)+(3.9), we deduce that, under $\Theta(\cdot|y)$:

$$\limsup_{\epsilon \to 0} e^{2X_\epsilon(y) - 4\ln\frac{1}{\epsilon} + \ln h(\epsilon)} \times \inf_{u \in B(y,\epsilon)} e^{2(\lambda_\epsilon(u,y)-1)X_\epsilon(y)} \times \frac{a_\epsilon(y)}{\epsilon^2 h(\epsilon)} \times \exp(-R) = +\infty. \tag{3.10}$$

By plugging (3.6)+(3.10) into (3.5), we get for all $R > 0$:

$$\Theta\Big(\{\limsup_{\epsilon \to 0} F^\epsilon(x,t) = +\infty\}|y\Big) \geqslant 1 - \rho(R).$$

By choosing R arbitrarily large, we complete the proof of Proposition 3.3 with the help of Lemma 3.4. $\square$

Proposition 3.5. *Almost surely in X, for all $x \in \mathbb{R}^2$, $\mathbb{P}^{B^x}$ almost surely, for all $t \geqslant 0$ we have:*

$$\sup_{\epsilon > 0} \sup_{s \in [0,t]} X_\epsilon(B^x_s) - 2\ln\frac{1}{\epsilon} + a\ln\ln\frac{1}{\epsilon} < +\infty \tag{3.11}$$

for all $a \in]0, \frac{1}{4}[$.

Proof. Assume that the kernel $k(u,x,y)$ in (2.1) is given by $k(u(x-y))$ for some continuous covariance kernel k with $k(0)=1$. It is then proved in [20] that, for each fixed $a \in]0,\frac{1}{4}[$, there exists a sequence $(C_n)_n$ of $\mathbb{P}^X$-almost surely finite random variables such that,

$$\sup_{\epsilon>0}\sup_{x\in B(0,n)} X_\epsilon(x) - 2\ln\frac{1}{\epsilon} + a\ln\ln\frac{1}{\epsilon} < C_n. \tag{3.12}$$

Now for each fixed x, the Brownian motion B^x has $\mathbb{P}^x$-almost surely continuous sample paths. Therefore, $\mathbb{P}^x$-almost surely, for any $t \geqslant 0$, we can find n such that $\forall s \in [0,t]$, $B^x_s \in B(0,n)$. Thus the claim follows from (3.12) in this specific case.

One must make some extra effort to extend this result to the more general situation of assumption [A]. It will be convenient to set $t = \ln\frac{1}{\epsilon}$ for $\epsilon \in]0,1]$. For each $x \in \mathbb{R}^2$, we consider the mapping

$$t \mapsto \varphi_x(t) = \mathbb{E}[X_{e^{-t}}(x)^2] = \int_1^{e^t} \frac{k(u,x,x)}{u}\,du.$$

It is continuous and strictly increasing and we denote by $\varphi_x^{-1}(t)$ the inverse mapping. Let us then consider the mapping $t \mapsto T_x(t)$ defined by $\varphi_x(T_x(t)) = t$. We consider the Gaussian process $Y_t(x) = X_{e^{-T_x(t)}}(x)$, which has constant variance t. We have

$$\mathbb{E}[Y_t(x)Y_t(y)] = \int_1^{e^{T_x(t)\wedge T_y(t)}} \frac{k(u,x,y)}{u}\,du \geqslant \int_1^{e^{T_x(t)}} \frac{k(u,x,y)}{u}\,du = \int_1^{e^t} \frac{k(e^{\varphi_x^{-1}(\ln v)},x,y)}{k(e^{\varphi_x^{-1}(\ln v)},x,x)}\,\frac{dv}{v}.$$

By assumption [A.5], for each compact set K, we can find a constant C_K such that

$$\frac{k(u,x,y)}{k(u,x,x)} \geqslant (1 - C_K u^{1/2}|x-y|^{1/2})_+$$

for all $x \in K$ and $y \in \mathbb{R}^2$. We deduce, for all $x \in K$

$$\mathbb{E}[Y_t(x)Y_t(y)] \geqslant \int_1^{e^t} \frac{(1 - C_K e^{\varphi_x^{-1}(\ln v)/2}|x-y|^{1/2})_+}{v}\,dv.$$

By assumption [A.4], one can check that the mapping $x \mapsto \frac{e^{\varphi_x^{-1}(\ln v)}}{v}$ converges uniformly on K towards a bounded strictly positive function. So even if it means changing the constant C_K, we have

$$\mathbb{E}[Y_t(x)Y_t(y)] \geqslant \int_1^{e^t} \frac{(1 - C_K|x-y|^{1/2})_+}{v}\,dv. \tag{3.13}$$

for all $x \in K$ and $y \in \mathbb{R}^2$. From [20], this latter covariance kernel satisfies the estimate (3.11). From [34], the above comparison between covariance kernels (3.13) with equal variance entails that the result also holds for the process Y_t. It is then plain to conclude by noticing that the function $x \mapsto \frac{T_x(t)}{t}$ converges uniformly as $t \to \infty$ over the compact sets towards a strictly positive limit (this results from [A.4]). $\square$

3.3 Limit of the derivative PCAF

Inspired by the construction of measures at criticality [20, 21], it seems reasonable to think that the change of times F^ϵ, when suitable renormalized, should converge towards a random change of times that coincides with the limit of the following process

$$F^{',\epsilon}(x,t) := \int_0^t \big(2\ln\frac{1}{\epsilon} - X_\epsilon(x+B_u)\big)e^{2X_\epsilon(x+B_u)-2\ln\frac{1}{\epsilon}}\,du.$$

Establishing the convergence of the above martingale is the main purpose of this section. Observe that, $\mathbb{P}^B$ almost surely, the family $(F^{',\epsilon}(x,t))_\epsilon$ is almost martingale for each $t>0$ (it is when you replace $\ln\frac{1}{\epsilon}$ by $\mathbb{E}^X[X_\epsilon(x)^2]$). Nevertheless, it is not nonnegative and not uniformly integrable. It is therefore not obvious that such a family almost surely converges towards a (non trivial) positive limiting random variable. The following theorem is the main result of this section:

Theorem 3.6. *Assume [A.1-5] and fix $x\in\mathbb{R}^2$. For each $t>0$, the family $(F^{',\epsilon}(x,t))_\epsilon$ converges almost surely in X and in B as $\epsilon\to 0$ towards a positive random variable denoted by $F'(x,t)$, such that $F'(x,t)>0$ almost surely. Furthermore, almost surely in X and in B, the (non necessarily positive) random mapping $t\mapsto F^{',\epsilon}(x,t)$ converges as $\epsilon\to 0$ in the space $C(\mathbb{R}_+,\mathbb{R}_+)$ towards a strictly increasing continuous random mapping $t\mapsto F'(x,t)$.*

Throughout this section, we will assume that assumptions [A.1-5] are in force. The observation made in the beginning of the proof of Proposition 3.3 remains valid here and, without loss of generality, we may assume that $\ln\frac{1}{\epsilon}=\mathbb{E}^X[X_\epsilon(x)^2]$. In this way, the family $(F^{',\epsilon}(x,t))_\epsilon$ is a martingale. Actually, Proposition 3.5 tells us that it is a positive martingale for t large enough. Therefore it converges almost surely towards a limit $F'(x,t)$. But it is not uniformly integrable so that there are several complications involved in establishing non-triviality of the limit. We have to introduce some further tools to study the convergence. We will introduce a family of auxiliary "truncated" martingales, called below $F^{',\epsilon}_\beta(x,t)$, which are reasonably close to $F^{',\epsilon}(x,t)$ while being square integrable. This will be enough to get the non triviality of $F'(x,t)$.

Given $t>0$, $z,x\in\mathbb{R}^2$ and $\beta>0$, we introduce the random variables

$$\begin{aligned} f^\beta_\epsilon(z) =&\big(2\ln\frac{1}{\epsilon} - X_\epsilon(z)+\beta\big)\mathbb{1}_{\{\tau^\beta_z<\epsilon\}}e^{2X_\epsilon(z)-2\ln\frac{1}{\epsilon}} \\ F^{',\epsilon}_\beta(x,t) =& \int_0^t \big(2\ln\frac{1}{\epsilon} - X_\epsilon(B^x_u)+\beta\big)\mathbb{1}_{\{\tau^\beta_{B^x_u}<\epsilon\}}e^{2X_\epsilon(B^x_u)-2\ln\frac{1}{\epsilon}}\,du = \int_0^t f^\beta_\epsilon(B^x_u)\,du \\ \widetilde{F}^{',\epsilon}_\beta(x,t) =& \int_0^t \big(2\ln\frac{1}{\epsilon} - X_\epsilon(B^x_u)\big)\mathbb{1}_{\{\tau^\beta_{B^x_u}<\epsilon\}}e^{2X_\epsilon(B^x_u)-2\ln\frac{1}{\epsilon}}\,du, \end{aligned} \tag{3.14}$$

where, for each $u\in[0,t]$, τ^β_u is the $(\mathcal{F}_\epsilon)_\epsilon$-stopping time defined by

$$\tau^\beta_z = \sup\{r\leqslant 1, X_r(z)-2\ln\frac{1}{r}>\beta\}.$$

In what follows, we will first investigate the convergence of $(F^{',\epsilon}_\beta(x,t))_{\epsilon\in]0,1]}$ to deduce first the convergence of $(\widetilde{F}^{',\epsilon}_\beta(x,t))_{\epsilon\in]0,1]}$ and then the convergence of $(F^{',\epsilon}(x,t))_{\epsilon\in]0,1]}$. We claim:

Proposition 3.7. *We fix $x\in\mathbb{R}^2$ and $t>0$. Almost surely in B, the process $(F^{',\epsilon}_\beta(x,t))_{\epsilon\in]0,1]}$ is a continuous positive $\mathcal{F}_\epsilon$-martingale and thus converges almost surely in X and B towards a nonnegative random variable denoted by $F'_\beta(x,t)$.*

This proposition is a direct consequence of the stopping time theorem and the martingale convergence theorem. Details are thus left to the reader. What is more involved is the study of the uniform integrability of this martingale:

Proposition 3.8. *We fix $x \in \mathbb{R}^2$ and $t > 0$. Almost surely in B, the martingale $(F_\beta^{',\epsilon}(x,t))_{\epsilon>0}$ is uniformly integrable.*

Proof. Let us first state the following lemma, which will serve in the forthcoming computations.

Lemma 3.9. *Let us denote μ_t the occupation measure of the Brownian motion B_t^x. $\mathbb{P}^{B^x}$-almost surely, for all $\delta > 0$, there are a compact set K and some constant L such that $\mu_t(K^c) \leqslant \delta$ and for all $y \in K$*

$$\sup_{r \leqslant 1} \frac{\mu_t(B(y,r))}{r^2 g(r)} \leqslant L \quad \text{with } g(r) = \ln\big(\frac{1}{r}+2\big)^3.$$

Proof. It is a direct consequence of Lemma 3.2. □

According to this lemma, for each fixed $\delta > 0$, we are given a compact $K = K_\delta$ satisfying the above conditions. We denote by μ_t^K the measure $\mu_t^K(dy) = \mathbb{1}_K(y)\,\mu_t(dy)$ and

$$F_{\beta,K}^{',\epsilon}(x,t) = \int_0^t \big(2\ln\frac{1}{\epsilon} - X_\epsilon(y) + \beta\big)\mathbb{1}_{\{\tau_y^\beta < \epsilon\}} e^{2X_\epsilon(y) - 2\ln\frac{1}{\epsilon}} \mathbb{1}_K(y)\,\mu_t(dy).$$

To prove Proposition 3.8, it suffices to prove that the family $(F_{\beta,K}^{',\epsilon}(x,t))_\epsilon$ is uniformly integrable. Indeed, if true, we have for each $R > 0$ (K^c is the complement of K in $\mathbb{R}^2$)

$$\begin{aligned}
\mathbb{P}^X(F_\beta^{',\epsilon}(x,t)\mathbb{1}_{\{F_\beta^{',\epsilon}(x,t) \geqslant R\}}) &\leqslant \mathbb{P}^X(2F_{\beta,K}^{',\epsilon}(x,t)\mathbb{1}_{\{2F_{\beta,K}^{',\epsilon}(x,t) \geqslant R\}}) + \mathbb{P}^X(2F_{\beta,K^c}^{',\epsilon}(x,t)\mathbb{1}_{\{2F_{\beta,K^c}^{',\epsilon}(x,t) \geqslant R\}}) \\
&\leqslant \mathbb{P}^X(2F_{\beta,K}^{',\epsilon}(x,t)\mathbb{1}_{\{2F_{\beta,K}^{',\epsilon}(x,t) \geqslant R\}}) + 2\mathbb{E}^X[F_{\beta,K^c}^{',\epsilon}(x,t)].
\end{aligned}$$

We deduce

$$\limsup_{R\to\infty} \mathbb{P}^X(F_\beta^{',\epsilon}(x,t)\mathbb{1}_{\{F_\beta^{',\epsilon}(x,t) \geqslant R\}}) \leqslant 2\mathbb{E}^X[F_{\beta,K^c}^{',\epsilon}(x,t)] = 2\beta\delta.$$

By choosing δ arbitrarily small, we prove the uniform integrability of $(F_\beta^{',\epsilon}(x,t))_{\epsilon>0}$.

So, we just have to focus on the uniform integrability of the family $(F_{\beta,K}^{',\epsilon}(x,t))_{\epsilon>0}$. We introduce the annulus $C(y,\epsilon,1) = B(y,1) \setminus B(y,\epsilon)$ for $\epsilon \in (0,1)$. We get:

$$\begin{aligned}
\mathbb{E}^X[F_{\beta,K}^{',\epsilon}(x,t)^2] &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{E}^X[f_\epsilon^\beta(y) f_\epsilon^\beta(w)]\,\mu_t^K(dy)\mu_t^K(dw) \\
&= \int_{\mathbb{R}^2}\int_{B(y,\epsilon)} \mathbb{E}^X[f_\epsilon^\beta(y) f_\epsilon^\beta(w)]\,\mu_t^K(dy)\mu_t^K(dw) \\
&\quad + \int_{\mathbb{R}^2}\int_{C(y,\epsilon,1)} \mathbb{E}^X[f_\epsilon^\beta(y) f_\epsilon^\beta(w)]\,\mu_t^K(dyx)\mu_t^K(dw) \\
&\quad + \int_{\mathbb{R}^2}\int_{B(y,1)^c} \mathbb{E}^X[f_\epsilon^\beta(y) f_\epsilon^\beta(w)]\,\mu_t^K(dy)\mu_t^K(dw) \\
&\overset{def}{=} \Pi_\epsilon^1 + \Pi_\epsilon^2 + \Pi_\epsilon^3.
\end{aligned} \tag{3.15}$$

It is not difficult to see that $\Pi_\epsilon^3 \leqslant Ct^2$ for some constant C independent of ϵ. The main terms are Π_ϵ^1 and Π_ϵ^2. We begin with Π_ϵ^1.

Remark 3.10. *Before going further into details, let us just heuristically explain how to complete the proof. On the ball $B(y,\epsilon)$, the process $X_\epsilon(w)$ is very close to $X_\epsilon(y)$. Therefore, with a good approximation, we can replace $X_\epsilon(w)$ by $X_\epsilon(y)$ and get:*

$$\Pi^1_\epsilon \leqslant C\int_{\mathbb{R}^2}\int_{B(y,\epsilon)} \mathbb{E}^X\Big[(1+(X_\epsilon(y))^2)e^{2X_\epsilon(y)+2\ln\frac{1}{\epsilon}}(\beta - X_\epsilon(y))\mathbb{1}_{\{\sup_{s\in[\epsilon,1]}X_s(y)\leqslant\beta\}}\Big)\Big]\,\mu^K_t(dy)\mu^K_t(dw).$$

Let us define a new probability measure on $\mathcal{F}_\epsilon$ by

$$\mathbb{P}^\beta(A) = \mathbb{E}^X[\mathbb{1}_A(\beta - X_\epsilon(y))\mathbb{1}_{\{\sup_{s\in[\epsilon,1]}X_s(y)\leqslant\beta\}})]$$

and recall that, under $\mathbb{P}^\beta$, the process $(\beta - X_s)_{\epsilon\leqslant s\leqslant 1}$ has the law of $(\beta_{\ln\frac{1}{s}})_{\epsilon\leqslant s\leqslant 1}$ where $(\beta_u)_u$ is a 3-dimensional Bessel process starting from β. Hence

$$\begin{aligned}\Pi^1_\epsilon &\leqslant C\int_{\mathbb{R}^2}\int_{B(y,\epsilon)}\mathbb{E}^\beta\Big[(1+(\beta_{\ln\frac{1}{\epsilon}})^2)e^{-2\beta_{\ln\frac{1}{\epsilon}}+2\ln\frac{1}{\epsilon}}\Big]\,\mu^K_t(B(y,\epsilon))\mu^K_t(dy)\\ &\simeq C\int_{\mathbb{R}^2}(1+\ln\frac{1}{\epsilon})e^{-\sqrt{\ln\frac{1}{\epsilon}}+2\ln\frac{1}{\epsilon}}\mu^K_t(B(y,\epsilon))\mu^K_t(dy)\end{aligned}$$

and this latter quantity goes to 0 as $\epsilon\to 0$ since $\mu^K_t(B(y,\epsilon))\leqslant L\epsilon^2 g(\epsilon)$. Similar ideas will allow us to treat Π^2_ϵ. Nevertheless, details are a bit more tedious.

Let us now try to make rigorous the above remark. Observe that necessarily $K_s(y,w)\leqslant \ln\frac{1}{s}$. Let us define the functions h_1, h_2 and $\bar h$ by:

$$h_1(s) = \ln\frac{1}{s} - K_s(y,w) = h_2(s),\quad \bar h(s) = K_s(y,w). \tag{3.16}$$

By considering 3 independent Brownian motions $B^1, B^2, \bar B$, we further define

$$P^{y,w}_s = B^1_{h_1(s)},\quad P^{w,y}_s = B^2_{h_2(s)},\quad Z_s = \bar B_{\bar h(s)}. \tag{3.17}$$

Observe that the process $(X_s(y), X_s(w))_{0\leqslant s\leqslant 1}$ has the same law as the process $(P^{y,w}_s + Z_s, P^{w,y}_s + Z_s)_{0\leqslant s\leqslant 1}$. Now we compute Π^1_ϵ and then use a Girsanov transform:

$$\begin{aligned}\Pi^1_\epsilon =& \int_{\mathbb{R}^2}\int_{B(y,\epsilon)}\mathbb{E}^X[(\beta - P^{w,y}_\epsilon - Z_\epsilon - 2\ln\frac{1}{\epsilon})\mathbb{1}_{\{\sup_{r\in[\epsilon,1]}P^{w,y}_r + Z_r - 2\ln\frac{1}{r}\leqslant\beta\}}(\beta - P^{y,w}_\epsilon - Z_\epsilon - 2\ln\frac{1}{\epsilon})\times\dots\\ &\dots\mathbb{1}_{\{\sup_{r\in[\epsilon,1]}P^{y,w}_r + Z_r - 2\ln\frac{1}{r}\leqslant\beta\}}e^{2P^{y,w}_\epsilon + 4Z_\epsilon + 2P^{w,y}_\epsilon - 4\ln\frac{1}{\epsilon}}]\,\mu^K_t(dy)\mu^K_t(dw)\\ =& \int_{\mathbb{R}^2}\int_{B(y,\epsilon)}\mathbb{E}^X[(\beta - P^{w,y}_\epsilon - Z_\epsilon)\mathbb{1}_{\{\sup_{r\in[\epsilon,1]}P^{w,y}_r + Z_r\leqslant\beta\}}(\beta - P^{y,w}_\epsilon - Z_\epsilon)\times\dots\\ &\dots\mathbb{1}_{\{\sup_{r\in[\epsilon,1]}P^{y,w}_r + Z_r\leqslant\beta\}}e^{2Z_\epsilon - 2\ln\frac{1}{\epsilon} + 4K_\epsilon(y-w)}]\,\mu^K_t(dy)\mu^K_t(dw).\end{aligned}$$

Let us set:

$$\beta^{y,w}_\epsilon = \beta - \min_{s\in[\epsilon,1]}P^{y,w}_s.$$

Because of assumption [A.2], we have

$$\sup_{\epsilon\in(0,1]}\ \sup_{w\in B(y,\epsilon)}\ \sup_{s\in]\epsilon,1]} h_1(s) + h_2(s)\leqslant c$$

for some constant $c>0$ only depending on k. Therefore we have:

$$\Pi^1_\epsilon \leqslant C\int_{\mathbb{R}^2}\int_{B(y,\epsilon)} \mathbb{E}^X\Big[\Big(1+(Z_\epsilon)^2\Big)e^{2Z_\epsilon+2\ln\frac{1}{\epsilon}}(\beta^{y,w}_\epsilon - Z_\epsilon)\mathbb{1}_{\{\sup_{r\in[\epsilon,1]} Z_r \leqslant \beta^{y,w}_\epsilon\}}\Big]\,\mu^K_t(dy)\mu^K_t(dw).$$

Let us define a new (random) probability measure on $\mathcal{F}_\epsilon$ by

$$\mathbb{P}^{\beta,y,w}(A)=\frac{1}{\beta^{y,w}_\epsilon}\mathbb{E}^Z[\mathbb{1}_A(\beta^{y,w}_\epsilon - Z_\epsilon(y))\mathbb{1}_{\{\sup_{s\in[\epsilon,1]} Z_s(y)\leqslant \beta^{y,w}_\epsilon\}})|\beta^{y,w}_\epsilon]$$

with associated expectation denoted by $\mathbb{E}^{\beta,y,w}$. Recall that, under $\mathbb{P}^{\beta,y,w}$, the process $(\beta^{y,w}_\epsilon - Z_s)_{\epsilon\leqslant s\leqslant 1}$ has the law of $(\beta_{K_s(y,w)})_{\epsilon\leqslant s\leqslant 1}$ where $(\beta_u)_u$ is a 3-dimensional Bessel process starting from $\beta^{y,w}_\epsilon$. Hence:

$$\Pi^1_\epsilon \leqslant C\int_{\mathbb{R}^2}\int_{B(y,\epsilon)} \mathbb{E}^X\Big[\beta^{y,w}_\epsilon\mathbb{E}^{\beta,y,w}\Big[\Big(1+(\beta_{K_\epsilon(y,w)})^2\Big)e^{-2\beta_{K_\epsilon(y,w)}+2\ln\frac{1}{\epsilon}}\Big]\Big]\,\mu^K_t(dy)\mu^K_t(dw).$$

Let us compute the quantity

$$\mathbb{E}^{\beta,y,w}\Big[\Big(1+(\beta_{K_\epsilon(y,w)})^2\Big)e^{-2\beta_{K_\epsilon(y,w)}}\Big].$$

To this purpose, we use the fact that the law of a $3d$-Bessel process starting from $\beta^{y,w}_\epsilon$ is given by $\sqrt{(B^1_t-\beta^{y,w}_\epsilon)^2+(B^2_t)^2+(B^3_t)^2}$ where B^1,B^2,B^3 are three independent Brownian motions. Therefore, by using (2.2) when necessary, we get

$$\begin{aligned}
&\mathbb{E}^{\beta,y,w}\Big[\Big(1+(\beta_{K_\epsilon(y,w)})^2\Big)e^{-2\beta_{K_\epsilon(y,w)}}\Big]\\
&\quad=\int_{\mathbb{R}^3}(1+(u-\beta^{y,w}_\epsilon)^2+v^2+w^2)e^{-2\sqrt{(u-\beta^{y,w}_\epsilon)^2+v^2+w^2}}e^{-\frac{u^2+v^2+w^2}{2K_\epsilon(y,w)}}\,\frac{du dv dw}{(2\pi K_\epsilon(y,w))^{3/2}}\\
&\quad\leqslant C(1+(\beta^{y,w}_\epsilon)^2)e^{2\beta^{y,w}_\epsilon}\int_{\mathbb{R}^3}(1+u^2+v^2+w^2)e^{-2\sqrt{u^2+v^2+w^2}}e^{-\frac{u^2+v^2+w^2}{2K_\epsilon(y,w)}}\,\frac{du dv dw}{(2\pi K_\epsilon(y,w))^{3/2}}\\
&\quad\leqslant C(1+(\beta^{y,w}_\epsilon)^2)e^{2\beta^{y,w}_\epsilon}\int_0^\infty(1+r^2)e^{-2r}e^{-\frac{r^2}{2\ln\frac{1}{\epsilon}}}\,\frac{r^2dr}{(\ln\frac{1}{\epsilon})^{3/2}},
\end{aligned}$$

for some constant $C>0$, which may have changed along lines. Let us set

$$\forall a\geqslant 0,\quad H(a)=\int_0^\infty(1+r^2)e^{-2r}e^{-\frac{r^2}{2a}}\,\frac{r^2dr}{a^{3/2}}.$$

It is plain to check that

$$H(a)\leqslant C(\max(1,a))^{-3/2}$$

for some positive constant C. Hence

$$\mathbb{E}^{\beta,y,w}\Big[\Big(1+(\beta_{K_\epsilon(y,w)})^2\Big)e^{-2\beta_{K_\epsilon(y,w)}}\Big]\leqslant C(1+(\beta^{y,w}_\epsilon)^2)e^{2\beta^{y,w}_\epsilon}H(\ln\frac{1}{\epsilon}).$$

We deduce:

$$\begin{aligned}
\Pi^1_\epsilon&\leqslant C(\ln\frac{1}{\epsilon})^{-3/2}\int_{\mathbb{R}^2}\int_{B(y,\epsilon)}\mathbb{E}^X\Big[\beta^{y,w}_\epsilon(1+(\beta^{y,w}_\epsilon)^2)e^{2\beta^{y,w}_\epsilon}\Big]\,\mu^K_t(dy)\mu^K_t(dw)\\
&\leqslant CH(\ln\frac{1}{\epsilon})\epsilon^{-2}\int_{\mathbb{R}^2}\mu^K_t(B(y,\epsilon))\,\mu^K_t(dy).
\end{aligned}$$

Here we have used the fact $\mathbb{E}^X\Big[C\beta_\epsilon^{y,w}(1+(\beta_\epsilon^{y,w})^2)e^{2\beta_\epsilon^{y,w}}\Big]$ is finite and does not depend on y,w because $h_1(s)$ is bounded on $\mathbb{R}_+$ independently of s,y,w. Because of Lemma 3.9, this latter quantity goes to 0 as $\epsilon\to 0$ $\mathbb{P}^{B^x}$-almost surely, and so does Π^1_ϵ.

Now we treat Π^2_ϵ. We will follow similar arguments as for Π^1_ϵ, though different behaviors are involved. Indeed, in this case, we have to face the possible long range correlations of the kernel k. So we adapt the decomposition of the couple $(X_s(y),X_s(w))_{s\in]0,1]}$ into Wiener integrals as follows. Let us consider a smooth function φ with compact support in the ball $B(0,1)$, such that $0\leqslant\varphi\leqslant 1$ and $\varphi=1$ over a neighborhood of 0. Let us define the functions h_1, h_2, $\bar h$ and $\widehat h$ by:

$$h_1(s)=\ln\frac{1}{s}-K_s(y,w)=h_2(s),\qquad \bar h(s)=\int_1^{\frac1s}\frac{k(u,y,w)\varphi(u(y-w))}{u}\,du,$$
$$\widehat h(s)=\int_1^{\frac1s}\frac{k(u,y,w)\big(1-\varphi(u(y-w))\big)}{u}\,du.$$

By considering 4 independent Brownian motions $B^1,B^2,\bar B,\widehat B$, we further define

$$P_s^{y,w}=B^1_{h_1(s)},\quad P_s^{w,y}=B^2_{h_2(s)},\quad Z_s=\bar B_{\bar h(s)},\quad \widehat Z_s=\widehat B_{\widehat h(s)}.\tag{3.18}$$

An elementary computation of covariance shows that the process $(X_s(y),X_s(w))_{0\leqslant s\leqslant 1}$ has the same law as the process $(P_s^{y,w}+Z_s+\widehat Z_s,P_s^{w,y}+Z_s+\widehat Z_s)_{0\leqslant s\leqslant 1}$. The process Z_s encodes the short-scale correlations of the two Brownian motions $(X_s(y))_s$ and $(X_s(w))_s$, and is the process that will rule the behaviour of the expectation $\mathbb{E}^X[f^\beta_\epsilon(y)f^\beta_\epsilon(w)]$. The remaining terms are just negligible perturbations that we will have to get rid of in the forthcoming computations.

We make first a few elementary remarks. Observe that $\bar h(s)=\bar h(|y-w|)$ for all $s\geqslant|y-w|$ in such a way that $Z_\epsilon=Z_{|y-w|}$ for $\epsilon\leqslant|y-w|$. We also set

$$D:=\sup\{\widehat h(s);s\in]0,1];y,w\in\mathbb{R}^2\}<+\infty.$$

We will often use the elementary relation:

$$\forall a\geqslant 0,\forall x\in\mathbb{R},\quad(\beta-a-x)\mathbb{1}_{\{x\leqslant\beta-a\}}\leqslant(\beta-x)\mathbb{1}_{\{x\leqslant\beta\}}.\tag{3.19}$$

Now we begin the computations related to Π^2_ϵ. So we consider $w\in C(y,\epsilon,1)$ and we have by the Girsanov transform and (3.19):

$$\begin{aligned}
&\mathbb{E}^X[f^\beta_\epsilon(w)f^\beta_\epsilon(y)]\\
&=\mathbb{E}^X\Big[(\beta-P_\epsilon^{y,w}-Z_\epsilon-\widehat Z_\epsilon+2\ln\frac1\epsilon)\mathbb{1}_{\{\sup_{u\in[\epsilon,1]}P_u^{y,w}+Z_u+\widehat Z_u-2\ln\frac1u\leqslant\beta\}}e^{2P_\epsilon^{y,w}+2Z_\epsilon+2\widehat Z_\epsilon-2\ln\frac1\epsilon}\dots\\
&\quad\dots\times(\beta-P_\epsilon^{w,y}-Z_\epsilon-\widehat Z_\epsilon+2\ln\frac1\epsilon)\mathbb{1}_{\{\sup_{u\in[\epsilon,1]}P_u^{w,y}+Z_u+\widehat Z_u-2\ln\frac1u\leqslant\beta\}}e^{2P_\epsilon^{w,y}+2Z_\epsilon+2\widehat Z_\epsilon-2\ln\frac1\epsilon}\Big]\\
&\leqslant e^{8D}\mathbb{E}^X\Big[(\beta-P_\epsilon^{y,w}-Z_\epsilon-\widehat Z_\epsilon+2\ln\frac1\epsilon)\mathbb{1}_{\{\sup_{u\in[\epsilon,1]}P_u^{y,w}+Z_u+\widehat Z_u-2\ln\frac1u\leqslant\beta\}}e^{2P_\epsilon^{y,w}+2Z_\epsilon-2\ln\frac1\epsilon}\dots\\
&\quad\dots\times(\beta-P_\epsilon^{w,y}-Z_\epsilon-\widehat Z_\epsilon+2\ln\frac1\epsilon)\mathbb{1}_{\{\sup_{u\in[\epsilon,1]}P_u^{w,y}+Z_u+\widehat Z_u-2\ln\frac1u\leqslant\beta\}}e^{2P_\epsilon^{w,y}+2Z_\epsilon-2\ln\frac1\epsilon}\Big]\\
&\leqslant e^{8D}\mathbb{E}^X\Big[(\beta-\min_{s\in]0,1]}\widehat Z_s-P_\epsilon^{y,w}-Z_\epsilon+2\ln\frac1\epsilon)\mathbb{1}_{\{\sup_{u\in[\epsilon,1]}P_u^{y,w}+Z_u-2\ln\frac1u\leqslant\beta-\min_{s\in]0,1]}\widehat Z_s\}}e^{2P_\epsilon^{y,w}+2Z_\epsilon-2\ln\frac1\epsilon}\dots\\
&\quad\dots\times(\beta-\min_{s\in]0,1]}\widehat Z_s-P_\epsilon^{w,y}-Z_\epsilon+2\ln\frac1\epsilon)\mathbb{1}_{\{\sup_{u\in[\epsilon,1]}P_u^{w,y}+Z_u-2\ln\frac1u\leqslant\beta-\min_{s\in]0,1]}\widehat Z_s\}}e^{2P_\epsilon^{w,y}+2Z_\epsilon-2\ln\frac1\epsilon}\Big].
\end{aligned}$$

The point is now to see that the above expectation reduces to the same expectation with ϵ replaced by $|y-w|$. First observe that $Z_\epsilon = Z_{|y-w|}$ for $\epsilon \leqslant |y-w|$. Second, from assumption [A.3], we have

$$\sup_{y\in K, w\in\mathbb{R}^2} \int_{\frac{1}{|y-w|}}^{\infty} \frac{k(u,y,w)}{u}\, du = C_K < +\infty.$$

Therefore we can deduce

$$\forall u \leqslant |y-w|, \quad \big|\ln\frac{|y-w|}{u} - (h_1(u) - h_1(|y-w|))\big| \leqslant c$$

for some constant c independent of everything that matters. This means that the quadratic variations of the martingale $(P^{w,y}_u - P^{w,y}_{|y-w|})_{u\,\leqslant\,|y-w|}$ can be identified with $\ln\frac{|y-w|}{\epsilon}$ up to some constant c independent of y, w, ϵ. We further stress that both martingales $P^{y,w}$ and $P^{w,y}$ are independent. Therefore, by conditioning with respect to $\mathcal{F}_{|y-w|}$, the integrand in the above expectation essentially reduces to the product of two independent martingales (recall that, if $X_t = \int_0^t f(r)\, dB_r$ is a Wiener integral, then $(\beta + 2\mathbb{E}[X_t^2] - X_t)\mathbb{1}_{\{\sup_{s\in[0,t]} X_s - 2\mathbb{E}[X_t^2]\,\leqslant\,\beta\}} e^{2X_t - 2\mathbb{E}[X_t^2]}$ is a martingale). By applying the stopping time theorem and by setting $\widehat{\beta} = \beta - \min_{s\in]0,1]} \widehat{Z}_s + c$, we get

$$\begin{aligned}\Pi^2_\epsilon \leqslant C \int_{\mathbb{R}^2}\int_{C(y,\epsilon,1)} &\mathbb{E}^X[(\widehat{\beta} - P^{w,y}_{|y-w|} - Z_{|y-w|} - 2\ln\frac{1}{|y-w|})\mathbb{1}_{\{\sup_{r\in[|y-w|,1]} P^{w,y}_r + Z_r - 2\ln\frac{1}{r}\,\leqslant\,\widehat{\beta}\}}\\ &\dots(\widehat{\beta} - P^{y,w}_{|y-w|} - Z_{|y-w|} - 2\ln\frac{1}{|y-w|})\mathbb{1}_{\{\sup_{r\in[|y-w|,1]} P^{y,w}_r + Z_r - 2\ln\frac{1}{r}\,\leqslant\,\widehat{\beta}\}}\\ &\dots e^{2P^{y,w}_{|y-w|} + 4Z_{|y-w|} + 2P^{w,y}_{|y-w|} - 4\ln\frac{1}{|y-w|}}]\, \mu^K_t(dy)\mu^K_t(dw).\end{aligned}$$

Recall that $\sup_{y,w\in K, s\in]|y-w|,1]} \mathbb{E}[(P^{y,w}_s)^2 + (P^{w,y}_s)^2] \leqslant c'$ and some constant $c' > 0$ only depending on k by assumption [A.2]. Indeed, for $s\in[|y-w|,1]$, we have:

$$\begin{aligned}h_1(s) &= \ln\frac{1}{s} - \int_1^{\frac{1}{s}} \frac{k(u(y-w))}{u}\, du = \int_1^{\frac{1}{s}} \frac{1-k(u(y-w))}{u}\, du\\ &\leqslant \int_1^{\frac{1}{s}} \frac{C_K u|y-w|}{u}\, du \leqslant C_K,\end{aligned}$$

where C_K is the Lipschitz constant given by assumption [A.2]. So, if we use the Girsanov transform and even if it means changing the value of $\widehat{\beta}$ by adding C_K, we get

$$\begin{aligned}\Pi^2_\epsilon \leqslant C \int_{\mathbb{R}^2}\int_{C(y,\epsilon,1)} &\mathbb{E}^X[(\widehat{\beta} - P^{w,y}_{|y-w|} - Z_{|y-w|})\mathbb{1}_{\{\sup_{r\in[|y-w|,1]} P^{w,y}_r + Z_r\,\leqslant\,\widehat{\beta}\}}(\widehat{\beta} - P^{y,w}_{|y-w|} - Z_{|y-w|})\\ &\dots\mathbb{1}_{\{\sup_{r\in[|y-w|,1]} P^{y,w}_r + Z_r\,\leqslant\,\widehat{\beta}\}} e^{2Z_{|y-w|} - 2\ln\frac{1}{|y-w|} + 4K_{|y-w|}(y-w)}]\, \mu^K_t(dy)\mu^K_t(dw).\end{aligned}$$

Let us then define

$$\beta^{y,w} = \widehat{\beta} - \min_{s\in[|y-w|,1]} P^{y,w}_{|y-w|}.$$

We deduce:

$$\begin{aligned}\Pi^2_\epsilon \leqslant C \int_{\mathbb{R}^2}\int_{C(y,\epsilon,1)} &\mathbb{E}^X\Big[\Big(1 + (Z_{|y-w|})^2\Big) e^{2Z_{|y-w|} + 2\ln\frac{1}{|y-w|}}(\beta^{y,w} - Z_{|y-w|}) \times \dots\\ &\dots\times \mathbb{1}_{\{\sup_{r\in[|y-w|,1]} Z_r\,\leqslant\,\beta^{y,w}\}}\Big]\, \mu^K_t(dy)\mu^K_t(dw).\end{aligned}$$

Let us define a new (random) probability measure on $\mathcal{F}_{|y-w|}$ by

$$\mathbb{P}^{\beta,y,w}(A)=\frac{1}{\beta^{y,w}}\mathbb{E}^Z[\mathbb{1}_A(\beta^{y,w}-Z_{|y-w|}(y))\mathbb{1}_{\{\sup_{s\in[|y-w|,1]}Z_s(y)\leqslant\beta^{y,w}\}})|\beta^{y,w}]$$

and recall that, under $\mathbb{P}^{\beta,y,w}$, the process $(\beta^{y,w}-Z_s)_{|y-w|\leqslant s\leqslant 1}$ has the law of $(\beta_{K_s(y-w)})_{|y-w|\leqslant s\leqslant 1}$ where $(\beta_u)_u$ is a 3-dimensional Bessel process starting from $\beta^{y,w}$. Hence

$$\Pi_\epsilon^2\leqslant C\int_{\mathbb{R}^2}\int_{C(y,\epsilon,1)}\mathbb{E}^X\Big[\beta^{y,w}\mathbb{E}^{\beta,y,w}\Big[\Big(1+(\beta_{K_{|y-w|}(y-w)})^2\Big)e^{-2\beta_{K_{|y-w|}(y-w)}+2\ln\frac{1}{|y-w|}}\Big]\Big]\,\mu_t^K(dy)\mu_t^K(dw).$$

From (2.2), we have

$$\ln\frac{1}{|y-w|}-C_K\leqslant K_{|y-w|}(y-w)\leqslant\ln\frac{1}{|y-w|}+C_K.$$

If we proceed along the same lines as we did previously, we get:

$$\begin{aligned}&\mathbb{E}^{\beta,y,w}\Big[\Big(1+(\beta_{K_{|y-w|}(y-w)})^2\Big)e^{-2\beta_{K_{|y-w|}(y-w)}}\Big]\\&\leqslant C(1+(\beta^{y,w})^2)e^{\beta^{y,w}}\int_0^\infty(1+r^2)e^{-r}e^{-\frac{r^2}{2\ln\frac{1}{|y-w|}}}\frac{r^2dr}{(\ln\frac{1}{|y-w|})^{3/2}}\\&\leqslant C(1+(\beta^{y,w})^2)e^{\beta^{y,w}}H(\ln\frac{1}{|y-w|}).\end{aligned}\tag{3.20}$$

Therefore, thanks to Lemma 3.9 (or Lemma 3.2),

$$\sup_{\epsilon\in]0,1]}\Pi_\epsilon^2\leqslant C\int_{\mathbb{R}^2}\int_{B(y,1)}\frac{H(\ln\frac{1}{|y-w|})}{|y-w|^2}\mu_t^K(dw)\mu_t^K(dy)<+\infty.$$

The proof of Proposition 3.8 is complete. ☐

Before proceeding with the proof of Theorem 3.6, let us first state a few corollaries of the previous computations. For $\beta>0$ and $\epsilon\in]0,1]$, define the random measure

$$M_\epsilon^\beta(dx)=f_\epsilon^\beta(x)\,dx.$$

Following [20], the family $(M_\epsilon^\beta)_{\epsilon\in]0,1]}$ almost surely converges as $\epsilon\to0$ in the sense of weak convergence of measure towards a limiting non trivial measure $M^\beta(dx)$ and $M^\beta(dx)=M'(dx)$ on compact sets for β (random) large enough. The following corollary proves Theorem 2.1 and therefore considerably generalizes the results in [20].

Corollary 3.11. *Assume [A.1-5]. Consider the random measure*

$$M_\epsilon^\beta(dx)=f_\epsilon^\beta(x)\,dx.$$

Then for $p<1/2$:

$$\mathbb{E}^X\Big[\int_{B(0,1)}\int_{B(0,1)}\ln^p\frac{1}{|y-w|}M^\beta(dy)M^\beta(dw)\Big]<+\infty.$$

Therefore, the measure M^β (and consequently M') is diffuse.

Proof. If we just replace the occupation measure of the Brownian motion by the Lebesgue measure along the lines of the proof of Proposition 3.8, we get:

$$\begin{aligned}&\mathbb{E}^X\Big[\int_{B(0,1)}\int_{B(0,1)}\ln^p\frac{1}{|y-w|}M^\beta(dy)M^\beta(dw)\Big]\\&\leqslant C\int_{B(0,1)}\int_{B(0,1)}\frac{1}{|y-w|^2}\ln^p\frac{1}{|y-w|}H(\ln\frac{1}{|y-w|})dydw.\end{aligned}$$

This latter quantity is finite for $p<1/2$. The finiteness of such an integral implies that, almost surely, the measure M^β cannot give mass to singletons. □

This result is closely related, though weaker, than (2.10). Yet, the setup of our proof is quite general whereas (2.10) has been proved in [5] for a specific one-dimensional measure that exhibits nice scaling relations. Nonetheless, (2.10) is expected to hold in greater generality but we do not know to which extent the proofs in [5] extend to more general situations. In the same spirit, we claim:

Corollary 3.12. *Fix $t>0$ and $p<1/2$. For each $\delta>0$, there is a compact set K such that $\mu_t(K^c)\leqslant\delta$ and*

$$\mathbb{E}^X\Big[\int_0^t\int_0^t\ln_+^p\frac{1}{|B_r^x-B_s^x|}\mathbb{1}_K(B_r^x)\mathbb{1}_K(B_s^x)F'_\beta(x,dr)F'_\beta(x,ds)\Big]<+\infty.$$

In particular, almost surely in X and in B^x, the random mapping $r\mapsto F'_\beta(x,r)$ does not possess discontinuity point on $[0,t]$.

Proof. For each $\delta>0$, we use once again Lemma 3.9 to find a compact set K and some constant L such that $\mu_t(K^c)\leqslant\delta$ and for all $y\in K$

$$\sup_{r\,\leqslant\,1}\frac{\mu_t(B(y,r))}{r^2g(r)}\leqslant L,\quad\text{with } g(r)=\ln\big(2+\frac{1}{r}\big)^3. \tag{3.21}$$

We denote by μ_t^K the measure

$$\mu_t^K(dy)=\mathbb{1}_K(y)\,\mu_t(dy).$$

Once again, the computations made in proposition 3.8 show that

$$\begin{aligned}&\mathbb{E}^X\Big[\int_0^t\int_0^t\ln_+^p\frac{1}{|B_r^x-B_s^x|}\mathbb{1}_K(B_r^x)\mathbb{1}_K(B_s^x)F'_\beta(x,dr)F'_\beta(x,ds)\Big]\\&\leqslant C\int_0^t\int_0^t\frac{\mathbb{1}_{\{|B_r^x-B_s^x|\,\leqslant\,1\}}}{|B_r^x-B_s^x|^2}\ln^p\frac{1}{|B_r^x-B_s^x|}H\Big(\ln\frac{1}{|B_r^x-B_s^x|}\Big)\mathbb{1}_K(B_r^x)\mathbb{1}_K(B_s^x)drds\\&\quad+C\int_0^t\int_0^t\mathbb{1}_{\{|B_r^x-B_s^x|\,\geqslant\,1\}}\ln_+^p\frac{1}{|B_r^x-B_s^x|}\mathbb{1}_K(B_r^x)\mathbb{1}_K(B_s^x)drds\\&\leqslant C\int_0^t\int_0^t\frac{\mathbb{1}_{\{|u-v|\,\leqslant\,1\}}}{|u-v|^2}\ln^p\frac{1}{|u-v|}H\Big(\ln\frac{1}{|u-v|}\Big)\mu_t^K(du)\mu_t^K(dv).\end{aligned}$$

Because of (3.21), the above integrals are finite for $p<1/2$.

It is plain to deduce the continuity property of the mapping $s\mapsto F'_\beta(x,[0,s])$ since the Brownian motion has continuous sample paths. Indeed, the discontinuity points of this mapping corresponds to the set $\mathcal{A}$ of atoms of the measure $F'_\beta(x,ds)$, which are countable. For each $n\in\mathbb{N}^*$, let us denote by $\mathcal{A}_n$ the set of atoms in $[0,t]$ of this measure that are of size strictly greater than $\int_0^t \mathbb{1}_{K^c_{1/n}}(B^x_s)F'_\beta(x,dr)$. For $s\in\mathcal{A}_n$, we necessarily have $\mathbb{1}_{K^c_{1/n}}(B^x_s)=0$. Also, we necessarily have $\mathbb{1}_{K_{1/n}}(B^x_s)=0$ otherwise the integral $\int_0^t\int_0^t \ln^p \frac{1}{|B^x_r-B^x_s|}\mathbb{1}_{K_{1/n}}(B^x_r)\mathbb{1}_{K_{1/n}}(B^x_s)F'_\beta(x,dr)F'_\beta(x,ds)$ would be infinite. We deduce that $\mathcal{A}_n$ is empty for all $n\in\mathbb{N}^*$, meaning that there is no atom of size greater than $\int_0^t \mathbb{1}_{K^c_{1/n}}(B^x_s)F'_\beta(x,dr)$ for all n. Since

$$\mathbb{E}^x\Big[\int_0^t \mathbb{1}_{K^c_{1/n}}(B^x_s)F'_\beta(x,dr)\Big]=\mu_t(K^c_{1/n})\leqslant 1/n\to 0\quad\text{as } n\to\infty,$$

we deduce that the quantity $\int_0^t \mathbb{1}_{K^c_{1/n}}(B^x_s)F'_\beta(x,dr)$ converges to 0 in probability as $n\to\infty$. We complete the proof. ☐

We are now in position to handle the proof of Theorem 3.6.

Proof of Theorem 3.6. We first observe that the martingale $(F^{',\epsilon}_\beta(x,t))_{\epsilon>0}$ possesses almost surely the same limit as the process $(\widetilde{F}^{',\epsilon}_\beta(x,t))_{\epsilon>0}$ because

$$|F^{',\epsilon}_\beta(x,t)-\widetilde{F}^{',\epsilon}_\beta(x,t)|=\beta\int_0^t \mathbb{1}_{\{\tau^\beta_{B^x_u}<\epsilon\}}e^{2X_\epsilon(B^x_u)-2\ln\frac{1}{\epsilon}}du\leqslant \beta F^\epsilon(x,t) \tag{3.22}$$

and the last quantity converges almost surely towards 0 (see Proposition 3.3). Using Corollary 3.5, we have almost surely in X and in B:

$$\sup_{\epsilon>0}\max_{s\in[0,t]} X_\epsilon(B^x_s)-2\ln\frac{1}{\epsilon}<+\infty,$$

which obviously implies

$$\forall\epsilon>0,\quad F^{',\epsilon}(t,x)=\widetilde{F}^{',\epsilon}_\beta(x,t)$$

for β (random) large enough. We deduce that, almost surely in X and in B, the family $(F^{',\epsilon}(x,t))_{\epsilon>0}$ converges towards a positive random variable.

It is plain to deduce the random measures $(F^{',\epsilon}_\beta(x,dt))_\epsilon$ converges in the sense of weak convergence of measures towards a random measure $F^\beta(x,dt)$. To prove convergence in $C(\mathbb{R}_+,\mathbb{R}_+)$, we just have to prove that the mapping $t\mapsto F^\beta(t,x)$ is continuous. This property is proved in Corollary 3.12. From (3.22) and Proposition 3.3 again, we deduce that the family of random mappings $(t\mapsto F^{',\epsilon}(x,t))_\epsilon$ almost surely converges as $\epsilon\to 0$ in $C(\mathbb{R}_+,\mathbb{R}_+)$ towards a nonnegative nondecreasing mapping $t\mapsto F'(x,t)$.

Let us prove that, almost surely in X and in B, the mapping $t\mapsto F'(x,t)$ is strictly increasing. We first write the relation, for $\epsilon'<\epsilon$,

$$\begin{aligned}F^{',\epsilon'}_\beta(x,dr)=&(2\ln\frac{1}{\epsilon}-X_\epsilon(B^x_r)+\beta)\mathbb{1}_{\{\tau^\beta_{B^x_r}<\epsilon'\}}e^{2X_{\epsilon'}(B^x_r)-2\ln\frac{1}{\epsilon'}}\,dr\\&+(2\ln\frac{\epsilon}{\epsilon'}-X_{\epsilon'}(B^x_r)+X_\epsilon(B^x_r)+\beta)\mathbb{1}_{\{\tau^\beta_{B^x_r}<\epsilon'\}}e^{2X_{\epsilon'}(B^x_r)-2\ln\frac{1}{\epsilon'}}\,dr.\end{aligned} \tag{3.23}$$

By using the same arguments as throughout this section, we pass to the limit in this relation as $\epsilon'\to 0$ and then $\beta\to\infty$ to get

$$F'(x,dr)=e^{2X_\epsilon(B^x_r)-2\ln\frac{1}{\epsilon}}F'_\epsilon(x,dr) \tag{3.24}$$

where $F'_\epsilon(x,dr)$ is almost surely defined as

$$F'_\epsilon(x,dr) = \lim_{\beta\to\infty}\lim_{\epsilon'\to 0} F^{',\epsilon'}_{\beta,\epsilon}(x,dr)$$

and $F^{',\epsilon'}_{\beta,\epsilon}(x,dr)$ is given by

$$(2\ln\frac{\epsilon}{\epsilon'} - X_{\epsilon'}(B^x_r) + X_\epsilon(B^x_r) + \beta)\mathbb{1}_{\{\tau^\beta_{\epsilon,B^x_r}<\epsilon'\}} e^{2(X_{\epsilon'}-X_\epsilon)(B^x_r) - 2\ln\frac{\epsilon}{\epsilon'}}\,dr$$

where

$$\tau^\beta_{\epsilon,z} = \sup\{u\leqslant 1; X_{u\epsilon}(z) - X_\epsilon(z) - 2\ln\frac{1}{u} > \beta - X_\epsilon(z) + 2\ln\frac{1}{\epsilon}\}.$$

Let us stress that we have used the fact that the measure

$$(2\ln\frac{1}{\epsilon} - X_\epsilon(B^x_r) + \beta)\mathbb{1}_{\{\tau^\beta_{B^x_r}<\epsilon'\}} e^{2X_{\epsilon'}(B^x_r) - 2\ln\frac{1}{\epsilon'}}\,dr$$

goes to 0 (it is absolutely continuous w.r.t. to $F^{\epsilon'}(x,dr)$) when passing to the limit in (3.23) as $\epsilon'\to 0$. From (3.24), it is plain to deduce that, almost surely in B, the event $\{F'(x,[s,t])=0\}$ (with $s<t$) belongs to the asymptotic sigma-algebra generated by the field $\{(X_\epsilon(x))_x;\epsilon>0\}$. Therefore it has probability 0 or 1 by the $0-1$ law of Kolmogorov. Since we have already proved that it is not 0, this proves that almost surely in B, $\mathbb{P}^X(F'(x,[s,t])=0)=0$ for any $s<t$. By considering a countable family of intervals $[s_n,t_n]$ generating the Borel sigma field on $\mathbb{R}_+$, we deduce that, almost surely in X and in B, the mapping $t\mapsto F'(x,t)$ is strictly increasing. □

3.4 Renormalization of the change of times

Here we explain the Seneta-Heyde norming for the change of times F^ϵ. Some technical constraints prevents us from claiming that it holds under the only assumptions [A.1-5]. So it is important to stress here that the Seneta-Heyde renormalization is not necessary to construct the critical LBM. It just illustrates that the derivative construction of the change of times $F'(x,t)$ also corresponds to a proper renormalization of F^ϵ.

Theorem 3.13. *Assume [A.1-5] and either [A.6] or [A.6']. Then the conclusions of Theorem 2.2 hold and almost surely in B, we have the following convergence in $\mathbb{P}^X$-probability as $\epsilon\to 0$*

$$\sqrt{\ln\frac{1}{\epsilon}}F^\epsilon(x,t) \to \sqrt{\frac{2}{\pi}}F'(x,t).$$

Proof. The proof is a rather elementary adaptation of the proof in [21, section D]. Just reproduce the proof in [21, section D] while replacing the Lebesgue measure by the occupation measure of the Brownian motion and use Lemma 3.9 when necessary. Details are left to the reader. □

Remark 3.14. *In the case of example (2), it is necessary to consider the occupation measure of the Brownian motion killed upon touching the boundary of the domain D.*

Remark 3.15. *One may wish to extend the Seneta-Heyde renormalization of the change of times for other long-range correlated kernels. Then one has to follow [21, Remark 31]. Basically, this corresponds to the situation when the correlation kernels satisfy $|k(v,x/v,y/v)|+|\partial_x k(v,x/v,y/v)|\leqslant Ce^{-|x|^\beta}$ for some $\beta\in]0,1[$.*

3.5 The LBM does not get stuck

In this subsection, we make sure that the LBM does not get stuck in some area of the state space $\mathbb{R}^2$. Typically, this situation may happen over areas where the field X takes large values, therefore having as consequence to slow down the LBM. Mathematically, this can be formulated as follows: check that the mapping $t \mapsto F'(x,t)$ tends to ∞ as $t \to \infty$.

Theorem 3.16. *Assume [A.1-5] and fix $x \in \mathbb{R}^2$. Almost surely in X and in B,*

$$\lim_{t\to\infty} F'(x,t) = +\infty. \tag{3.25}$$

Proof. It suffices to reproduce the techniques of [27, subsection 2.5]. □

Remark 3.17. *Of course, this statement does not hold in the case of a bounded planar domain when one considers a Brownian motion killed upon touching the boundary of D. In this case, the Liouville Brownian motion (see below) will run until touching the boundary of D.*

3.6 Defining the critical LBM when starting from a given fixed point

We are now in position to define the critical LBM when starting from one fixed point. Indeed, once the change of times F' has been constructed, the strategy is the same as in [27, subsection 2.10].

Definition 3.18. *Assume [A.1-5]. The* **critical Liouville Brownian motion** *is defined by:*

$$\forall t \geqslant 0, \quad \mathcal{B}^x_t = B^x_{\langle \mathcal{B}^x\rangle_t}, \qquad \textit{and} \qquad \forall t \geqslant 0, \quad \sqrt{2/\pi}\, F'(x, \langle \mathcal{B}^x\rangle_t) = t. \tag{3.26}$$

As such, the mapping $t \mapsto \langle \mathcal{B}^x\rangle_t$ is defined on $\mathbb{R}_+$, continuous and strictly increasing.

Theorem 3.19. *Assume [A.1-5] and either [A.6] or [A.6']. Fix $x \in \mathbb{R}^2$. Almost surely in B and in $\mathbb{P}^X$-probability, the family $(B, \langle \mathcal{B}^{\epsilon,x}\rangle, \mathcal{B}^{\epsilon,x})_\epsilon$ converges in the space $C(\mathbb{R}_+,\mathbb{R}^2) \times C(\mathbb{R}_+,\mathbb{R}_+) \times C(\mathbb{R}_+,\mathbb{R}^2)$ equipped with the supremum norm on compact sets towards thetriple $(B, \langle \mathcal{B}^x\rangle, \mathcal{B}^x)$.*

Remark 3.20. *One may wonder whether the process introduced in Definition 2.11 also converges in probability. Actually, the argument carried out in [27, section 2.12] remains true here: almost surely in X, the couple of processes $(\bar{B}, \mathcal{B}^\epsilon)_\epsilon$ in Definition 2.11 converges in law towards a couple $(\bar{B}, \mathcal{B})$ where $\bar{B}$ and $\mathcal{B}$ are independent. Since $\mathcal{B}^\epsilon$ is measurable w.r.t. $\bar{B}$, this shows that the process $\mathcal{B}^\epsilon$ does not converge in probability. This justifies our approach of studying the convergence via the Dambis-Schwarz representation theorem: it leads to studying a process (definition 2.5) that converges in probability.*

4 Critical LBM as a Markov process

Remark 4.1. *In this whole section, we assume that assumptions [A.1-5] are in force. Sometimes, we make a statement to relate our results to the metric tensor g_ϵ. For such a connection to be made, the Seneta-Heyde norming is needed and thus assumption [A.6] or [A.6'] are required. To avoid confusion, this will be explicitly mentioned.*

In this section, we will investigate the critical LBM as a Markov process, meaning that we aim at constructing almost surely in X the critical LBM starting from every point. In the previous section, the guiding line was similar to [27] besides technical difficulties. From now on, the difference will be conceptual too: in the subcritical situation, the issue of constructing the LBM starting from every point is possible because the mapping

$$x \mapsto \int_{\mathbb{R}^2} \ln_+ \frac{1}{|x-y|} M_\gamma(dy)$$

is a continuous function of x, where M_γ stands for the subcritical measure with parameter $\gamma < 2$. This idea is somewhat underlying the theory of traces of Dirichlet forms developed in [25] for instance. At criticality, the main obstacle is pointed out in (2.10): the best modulus of continuity that one may hope for M' is of the type

$$M'(B(x,r)) \leqslant C \frac{1}{\sqrt{\ln(1+r^{-1})}}$$

and cannot be improved as proved in [6] in the context of multiplicative cascades. The mapping

$$x \mapsto \int_{\mathbb{R}^2} \ln_+ \frac{1}{|x-y|} M'(dy) \tag{4.1}$$

is thus certainly not continuous and may even take infinite values. The ideas of [27] need be renewed to face the issues of criticality.

On the other hand, the theory of Dirichlet forms [25, 53] (or potential theory) tells us that one can construct a Positive Continuous Additive Functional (PCAF for short) associated to M' provided that the mapping (4.1) does not take too many infinite values. More precisely, M' is required not to give mass to polar sets of the Brownian motion. The problem of the theory of Dirichlet forms is that one can guarantee the existence of the PCAF but it cannot be identified and all the information that we get by explicitly constructing the PCAF F' is lost in this approach. Also, while being extremely powerful in the description of the Dirichlet form of the LBM, the theory of Dirichlet forms gives much weaker results than the coupling approach developed in [27, 28] concerning the qualitative/quantitative properties of F'. We thus definitely need to gather both of these approaches.

Our strategy will be to identify a large set of points of finiteness of the mapping (4.1) in order to construct a perfectly identified PCAF on the whole space via coupling arguments. Then we will prove that M' does not charge polar sets in order to identify our PCAF with that of the theory of Dirichlet forms in the sense of the Revuz correspondence. Once this gap is bridged, we can apply the full machinery of [25] to get a lot of further information about F': mainly, a full description of the Dirichlet form associated to the critical LBM.

4.1 Background on positive continuous additive functionals and Revuz measures

To facilitate the reading of our results, we summarize here some basic notions of potential theory applied to the standard Brownian $(\Omega, (B_t)_{t \geqslant 0}, (\mathcal{F}_t)_{t \geqslant 0}, (\mathbb{P}^x)_{x \in \mathbb{R}^2})$ in $\mathbb{R}^2$ seen as a Markov process, which is of course reversible for the canonical volume form dx of $\mathbb{R}^2$. These notions can be found with further details in [25, 53]. One may then consider the classical notion of capacity associated to the Brownian motion. In this context, we have the following definition:

Definition 4.2 (Capacity and polar set)**.** *The capacity of an open set $O \subset \mathbb{R}^2$ is defined by*

$$\mathrm{Cap}(O) = \inf\{\int_D |f(x)|^2\,dx + \int_D |\nabla f(x)|^2\,dx; f \in H^1(\mathbb{R}^2, dx),\ f \geqslant 1 \text{ over } O\}.$$

The capacity of a Borel measurable set K is then defined as:

$$\mathrm{Cap}(K) = \inf_{O\,open, K \subset O} \mathrm{Cap}(O).$$

The set K is said polar when $\mathrm{Cap}(K) = 0$.

Definition 4.3 (Revuz measure)**.** *A Revuz measure μ is a Radon measure on $\mathbb{R}^2$ which does not charge the polar sets.*

Then we introduce the notion of PCAF that we will use in the following (see [25, 53]):

Definition 4.4 (PCAF)**.** *A Positive Continuous Additive Functional $(A(x,t,B))_{t \geqslant 0, x \in \mathbb{R}^2/N}$ of the Brownian motion (with $B_0 = 0$) on $\mathbb{R}^2$ is defined by:*
-a polar set N (for the standard Brownian motion),
- for each $x \in \mathbb{R}^2/N$ and $t \geqslant 0$, $A(x,t,B)$ is $\mathcal{F}_t$-adapted and continuous, with values in $[0,\infty]$ and $A(x,0) = 0$,
- almost surely,

$$A(x,t+s,B) - A(x,t,B) = A(x + B_t, s, B_{t+\cdot} - B_t), \quad s,t \geqslant 0.$$

In particular, a PCAF is defined for all starting points $x \in \mathbb{R}^2$ except possibly on a polar set for the standard Brownian motion. One can also work with a PCAF starting from **all** points, that is when the set N in the above definition can be chosen to be empty. In that case, the PCAF is said *in the strict sense*.

Finally, we conclude with the following definition on the support of a PCAF:

Definition 4.5 (support of a PCAF)**.** *Let $(A(x,t,B))_{t \geqslant 0, x \in \mathbb{R}^2/N}$ be a PCAF with associated polar set N. The support of $(A(x,t,B))_{t \geqslant 0, x \in \mathbb{R}^2/N}$ is defined by:*

$$\widetilde{Y} = \Big\{x \in \mathbb{R}^2 \setminus N : \ \mathbb{P}(R(x) = 0) = 1\Big\},$$

where $R(x) = \inf\{t > 0 : \ A(x,t,B) > 0\}$.

From section 5 in [25], there is a one to one correspondence between Revuz measures μ and PCAFs $(A(x,t,B))_{t \geqslant 0, x \in \mathbb{R}^2/N}$ under the Revuz correspondence: for any $t > 0$ and any nonnegative Borel functions f, h:

$$\mathbb{E}_{h.dx}\Big[\int_0^t f(B_r)\,dA_r\Big] = \int_0^t \int_{\mathbb{R}^2} f(x) P_r h(x)\,\mu(dx)\,dr, \tag{4.2}$$

where $(P_r)_{r \geqslant 0}$ stands for the semigroup associated to the planar Brownian motion on $\mathbb{R}^2$.

4.2 Capacity properties of the critical measure

The purpose of this section is to establish some preliminary results in order to apply the theory of Dirichlet forms. In particular, we will establish that the critical measure does not charge polar sets. To this purpose, we will need some fine pathwise properties of Bessel processes. So we first recall a result from [47]:

Theorem 4.6. *Let X be a 3d-Bessel process on $\mathbb{R}_+$ starting from $x \geqslant 0$ with respect to the law $\mathbb{P}_x$.*

1. *Suppose that $\phi \uparrow \infty$ such that $\int_1^\infty \frac{\phi(t)^3}{t} e^{-\frac{1}{2}\phi(t)^2}\, dt < +\infty$. Then*
$$\mathbb{P}_x\Big(X_t > \sqrt{t}\phi(t) \;\; i.o. \text{ as } t \uparrow +\infty\Big) = 0.$$

2. *Suppose that $\psi \downarrow 0$ such that $\int_1^\infty \frac{\psi(t)}{t}\, dt < +\infty$. Then*
$$\mathbb{P}_x\Big(X_t < \sqrt{t}\psi(t) \;\; i.o. \text{ as } t \uparrow +\infty\Big) = 0.$$

Recall that we denote by M_β the measure $M_\beta(dx) = \lim_{\epsilon\to 0} f_\epsilon^\beta(x)\, dx$. The main purpose of this subsection is to prove the following result:

Theorem 4.7. • *Let us consider two functions ϕ, ψ as in the above theorem and define*
$$Y_\epsilon(x) = 2\ln\frac{1}{\epsilon} - X_\epsilon(x).$$
We introduce the set
$$E = \{x \in \mathbb{R}^2; \limsup_{\epsilon\to 0} \frac{Y_\epsilon(x)}{\sqrt{\ln\frac{1}{\epsilon}\phi(\ln\frac{1}{\epsilon})}} \leqslant 1 \quad \text{and} \quad \liminf_{\epsilon\to 0}\frac{Y_\epsilon(x)}{\sqrt{\ln\frac{1}{\epsilon}\psi(\ln\frac{1}{\epsilon})}} \geqslant 1\}.$$
Then M^β gives full mass to E, i.e. $M^\beta(E^c) = 0$.
• *Furthermore, for each ball B, for each $\delta > 0$, there is a compact set $K \subset B$ such that $M_\beta(B \cap K^c) \leqslant \delta$ and for all $p > 0$*
$$\int_K\int_B \Big(\ln\frac{1}{|x-y|}\Big)^p M_\beta(dx)M_\beta(dy) < +\infty.$$

Proof. Let us consider a non empty ball B. We introduce the Peyrière probability measure Q^β on $\Omega \times B$:
$$\int f(x,\omega)\, dQ^\beta = \frac{1}{\beta|B|}\mathbb{E}^X[\int_B f(x,\omega)M^\beta(dx)].$$
Let us consider the random process $t \mapsto \bar{X}_t = X_{e^{-t}}$. The main idea is that under Q^β, the process $(\beta - \bar{X}_t + 2t)_{t\geqslant 0}$ has the law very close to a 3d-Bessel process $(\beta_t)_t$ starting from β. The first claim then follows from Theorem 4.6. We just have to precise the notion of "very close": some negligible terms appear because of the difference between t and $\mathbb{E}^X[\bar{X}_t(x)^2]$ and we have to quantify them.

We consider the measure (convergence is established the same way as for M_β)
$$\bar{M}_\beta(dx) = \lim_{\epsilon\to 0}(2\mathbb{E}^X[\bar{X}_t(x)^2] - X_t(x) + \beta)\mathbb{1}_{\{\sup_{u\in[0,t]}\bar{X}_u - 2\mathbb{E}^X[\bar{X}_u(x)^2] \leqslant \beta\}} e^{2\bar{X}_t - 2\mathbb{E}^X[\bar{X}_t(x)^2]}\, dx$$

and we set $D = \sup_{t \geqslant 0} \sup_{x \in B} |\mathbb{E}^X[\bar{X}_t(x)^2] - t| < +\infty$. We set $H(x) = \lim_{t \to \infty} \mathbb{E}^X[\bar{X}_t(x)^2] - t$ (see assumption [A.4]). Observe that

$$M_\beta(dx) \leqslant e^{H(x)} \bar{M}_{\beta+2D}(dx).$$

Let us set $\hat{\beta} = \beta + 2D$. Therefore, we consider the probability measure $\bar{Q}^{\hat{\beta}}$ on $\Omega \times B$:

$$\int f(x,\omega)\, d\bar{Q}^{\hat{\beta}} = \frac{1}{\hat{\beta}|B|} \mathbb{E}^X\Big[\int_B f(x,\omega) \bar{M}^{\hat{\beta}}(dx)\Big]$$

and Q^β is absolutely continuous with respect to $\bar{Q}^{\hat{\beta}}$. Under $\bar{Q}^{\hat{\beta}}$ (following arguments already seen), the process $(\hat{\beta} - \bar{X}_t + 2\mathbb{E}^X[\bar{X}_t(x)^2])_{t \geqslant 0}$ has the law of $(\mathrm{Bess}_{T_t(x)})_t$ where $(\mathrm{Bess}_t)_t$ a $3d$-Bessel process starting from $\hat{\beta}$ and $T_t(x) = K_{e^{-t}}(x,x)$. The first claim then follows from Theorem 4.6, the fact that $\sup_{x \in B, t \geqslant 0} |K_{e^{-t}}(x,x) - t| < +\infty$ and the absolute continuity of Q^β w.r.t. $\bar{Q}^{\hat{\beta}}$.

Now we prove the second statement, which is more technical. To simplify things a bit, we assume that $\mathbb{E}[X_t(x)^2] = t$. If this not the case, we can apply the same strategy as above, namely considering $\bar{M}_{\beta+2D}$ instead of M_β. We consider now a couple of functions $\phi(t) = (1+t)^\chi$ and $\psi(t) = (1+t)^{-\chi}$ for some small positive parameter χ close to 0. They satisfy the assumptions of Theorem 4.6. Then we consider the random compact sets for $R > 0$ and $t \in [0, +\infty]$,

$$\begin{aligned} K^1_{R,t} =& \{x \in B;\ \sup_{u \in [0,t]} \frac{\beta - \bar{X}_u(x) + 2u}{(1+u)^{1/2+\chi}} \leqslant R\} \qquad K^2_{R,t} = \{x \in B;\ \inf_{u \in [0,t]} \frac{\beta - \bar{X}_u(x) + 2u}{(1+u)^{1/2-\chi}} \geqslant \frac{1}{R}\} \\ K_{R,t} =& K^1_{R,t} \cap K^2_{R,t}. \end{aligned}$$

We will write K_R for $K_{R,\infty}$. From Theorem 4.6, we have $\lim_{R \to \infty} Q^\beta(K_R) = 1$. Therefore, we have $\lim_{R \to \infty} M^\beta(K_R^c \cap B) = 0$ $\mathbb{P}^X$ almost surely.

Let us denote $\mathbb{E}^Q$ expectation with respect to the probability measure Q^β. Without loss of generality, we assume that $\beta|B| = 1$: this avoids to repeatedly write the renormalization constant appearing in the definition of Q^β. To prove the result, we compute

$$\begin{aligned} \mathbb{E}^Q[\mathbb{1}_{K_R}(x) M^\beta(B(x,e^{-t}))] &= \lim_{\epsilon \to 0} \mathbb{E}\Big[\int_B \mathbb{1}_{K_R}(x) M_\beta(B(x,e^{-t})) M_\beta(dx)\Big] \\ &= \lim_{\epsilon \to 0} \mathbb{E}\Big[\int_B \int_{B(x,e^{-t})} \mathbb{1}_{K_R}(x) f^\beta_\epsilon(w)\, dw M_\beta(dx)\Big] \\ &\leqslant \lim_{\epsilon \to 0} \int_B \int_{B(x,e^{-t})} \mathbb{E}[\mathbb{1}_{K_{R,-\ln|x-w|}}(x) f^\beta_\epsilon(w) f^\beta_\epsilon(x)]\, dw dx. \end{aligned}$$

Now we can argue as in the proof of Proposition 3.8 (treatment of Π^2_ϵ) to see that for $\epsilon \leqslant |x-w|$

$$\mathbb{E}[\mathbb{1}_{K_{R,-\ln|x-w|}}(x) f^\beta_\epsilon(w) f^\beta_\epsilon(x)] \leqslant C \mathbb{E}[\mathbb{1}_{K_{R,-\ln|x-w|}}(x) f^{\hat{\beta}}_{|x-w|}(w) f^{\hat{\beta}}_{|w-x|}(x)]$$

for some irrelevant constant C and $\hat{\beta} = \beta - \min_{u \geqslant \ln \frac{1}{|x-w|}} \hat{Z}_u$ where the process $\hat{Z}$ is independent of the sigma algebra $\mathcal{F}_{|x-w|}$ and $\hat{Z}_s = B_{\hat{h}(s)} - B_{\hat{h}(-\ln|x-w|)}$ for some Brownian motion B and $\hat{h}(s) = \int_0^{e^s} \frac{k(u,x,w)(1-\varphi(u(x-w)))}{u}\, du$.

Remark 4.8. *The above technical part is due to the presence of possible long range correlations. Though they do not affect qualitatively our final estimates, getting rid of them may appear technical. The reader who wishes to skip this technical part may instead consider the compact case:* $k(u, x, v) = 0$ *if* $u|x-w| \geqslant 1$*. In that case, we just have*

$$\mathbb{E}[\mathbb{1}_{K_{R,-\ln|x-w|}}(x) f^\beta_\epsilon(w) f^\beta_\epsilon(x)] = \mathbb{E}[\mathbb{1}_{K_{R,-\ln|x-w|}}(x) f^\beta_{|x-w|}(w) f^\beta_{|w-x|}(x)]$$

because of the stopping time theorem and the fact that both martingales $(f^\beta_\epsilon(w) - f^\beta_{|x-w|}(w))_{\epsilon \leqslant |x-w|}$ *and* $(f^\beta_\epsilon(x) - f^\beta_{|x-w|}(x))_{\epsilon \leqslant |x-w|}$ *are independent.*

We are thus left with computing $\mathbb{E}[\mathbb{1}_{K_{R,-\ln|x-w|}}(x) f^{\hat\beta}_{|x-w|}(w) f^{\hat\beta}_{|w-x|}(x)]$. To this purpose, we need the following lemma, the proof of which is straightforward as a simple computation of covariance. Thus, details are left to the reader.

Lemma 4.9. *We consider* $w \neq x$ *such that* $|x-w| \leqslant 1$*. The law of the couple* $(\bar X_t(w), \bar X_t(x))_{t \geqslant 0}$ *can be decomposed as:*

$$(\bar X_t(w), \bar X_t(x))_{t \geqslant 0} = (Z^{x,w}_t + P^{x,w}_t, P^{x,w}_t + D^{x,w}_t)_{t \geqslant 0}$$

where the process $Z^{x,w}, P^{x,w}, D^{x,w}$ *are independent centered Gaussian and, for some independent Brownian motions* B^1, B^2 *independent of* $\bar X$*:*

$$P^{x,w}_t = \int_0^t g'_{x,w}(u)\, d\bar X_u(x), \quad Z^{x,w}_t = \int_0^t (1 - g'_{x,w}(u)^2)^{1/2}\, dB^1_u, \quad D^{x,w}_t = \int_0^t (1 - g'_{x,w}(u)^2)^{1/2}\, dB^2_u.$$

with $g_{x,w}(u) = \int_1^{e^u} \frac{k(y(x-w))}{y}\, dy$*. Moreover*

$$\sup_{t \leqslant \ln \frac{1}{|x-w|}} \mathbb{E}[(Z^{x,w}_t)^2 + (D^{x,w}_t)^2] \leqslant C, \tag{4.3}$$

for some constant C *independent of* x, w *such that* $|x-w| \leqslant 1$*.*

Setting $s_0 = \ln \frac{1}{|x-w|}$, we use this lemma to get

$$\begin{aligned} &\mathbb{E}[\mathbb{1}_{K_{R,-\ln|x-w|}}(x) f^{\hat\beta}_{|w-x|}(w) f^{\hat\beta}_{|w-x|}(x)] \\ &\qquad \leqslant \mathbb{E}\Big[\mathbb{1}_{K_{R,s_0}}(x)(\hat\beta - P^{x,w}_{s_0} - Z^{x,w}_{s_0} - 2s_0)_+ e^{2(P^{x,w}_{s_0} + Z^{x,w}_{s_0}) - 2s_0}(\hat\beta - P^{x,w}_{s_0} - D^{x,w}_{s_0} - 2s_0)_+ \times \dots \\ &\qquad\qquad \mathbb{1}_{\{\sup_{u \in [0,s_0]} P^{x,w}_u + D^{x,w}_u - 2u \leqslant \hat\beta\}} e^{2(P^{x,w}_{s_0} + D^{x,w}_{s_0}) - 2s_0}\Big]. \end{aligned}$$

We get rid of the process $Z^{x,w}$ by using first the Girsanov transform with the corresponding exponential term, and then by estimating the remaining terms containing $Z^{x,w}$ with the help of (4.3). We get for some constant C that may vary along lines but does not depend on x, w:

$$\begin{aligned} &\mathbb{E}[\mathbb{1}_{K_{R,s_0}}(x) f^{\hat\beta}_{|w-x|}(w) f^{\beta}_{|w-x|}(x)] \\ &\leqslant C\mathbb{E}\Big[\mathbb{1}_{K_{R,s_0}}(x)(1+\hat\beta)(1 + |P^{x,w}_{s_0} - 2\mathbb{E}[(P^{x,w}_{s_0})^2]|) e^{2P^{x,w}_{s_0} - 2\mathbb{E}[(P^{x,w}_{s_0})^2]}(\beta - P^{x,w}_{s_0} - D^{x,w}_{s_0} - 2s_0)_+ \times \dots \\ &\quad \dots \times \mathbb{1}_{\{\sup_{u \in [0,s_0]} P^{x,w}_u + D^{x,w}_u - 2u \leqslant \hat\beta\}} e^{2(P^{x,w}_{s_0} + D^{x,w}_{s_0}) - 2s_0}\Big]. \end{aligned}$$

We use the Girsanov transform again to make the term $e^{2(P^{x,w}_{s_0}+D^{x,w}_{s_0})-2\ln\frac{1}{|x-w|}}$ disappear:

$$
\begin{aligned}
&\mathbb{E}[\mathbb{1}_{K_{R,s_0}}(x)f^{\hat{\beta}}_{|w-x|}(w)f^{\hat{\beta}}_{|w-x|}(x)]\\
&\leqslant C\mathbb{E}\Big[\mathbb{1}_{K_{R,s_0}}(x)(1+\hat{\beta})(1+|P^{x,w}_{s_0}|)e^{2P^{x,w}_{s_0}+2\mathbb{E}[(P^{x,w}_{s_0})^2]}(\hat{\beta}-P^{x,w}_{s_0}-D^{x,w}_{s_0})_+\mathbb{1}_{\{\sup_{u\in[0,s_0]}P^{x,w}_u+D^{x,w}_u\leqslant\hat{\beta}\}}\Big].
\end{aligned}
$$

Now we write $P^{x,w}_{s_0}=P^{x,w}_{s_0}+D^{x,w}_{s_0}-D^{x,w}_{s_0}$ and use (4.3) to see that we can replace $\mathbb{E}[(P^{x,w}_{s_0})^2]$ by s_0 even if it means modifying the constant C (which still does not depend on x,w):

$$
\begin{aligned}
&\mathbb{E}[\mathbb{1}_{K_{R,s_0}}(x)f^{\hat{\beta}}_{|w-x|}(w)f^{\hat{\beta}}_{|w-x|}(x)]\\
&\leqslant C\mathbb{E}\Big[\mathbb{1}_{K_R}(x)(1+\hat{\beta})(1+|P^{x,w}_{s_0}+D^{x,w}_{s_0}|+|D^{x,w}_{s_0}|)e^{2(P^{x,w}_{s_0}+D^{x,w}_{s_0})+2s_0}e^{-2D^{x,w}_{s_0}}\times\dots\\
&\qquad(\hat{\beta}-P^{x,w}_{s_0}-D^{x,w}_{s_0})_+\mathbb{1}_{\{\sup_{u\in[0,s_0]}P^{x,w}_u+D^{x,w}_u\leqslant\hat{\beta}\}}\Big].
\end{aligned}
$$

Now we use the fact that for $x\in K_{R,s_0}$:

$$
\sup_{u\leqslant-\ln|x-w|}\frac{\beta-P^{x,w}_u-D^{x,w}_u}{(1+u)^{1/2+\chi}}\leqslant R\qquad\text{and}\qquad\inf_{u\leqslant-\ln|x-w|}\frac{\beta-P^{x,w}_u-D^{x,w}_u}{(1+u)^{1/2-\chi}}\geqslant\frac{1}{R},
$$

to get

$$
\begin{aligned}
&\mathbb{E}[\mathbb{1}_{K_{R,s_0}}(x)f^{\hat{\beta}}_{|w-x|}(w)f^{\hat{\beta}}_{|w-x|}(x)]\\
&\leqslant C\mathbb{E}\Big[\mathbb{1}_{K_{R,s_0}}(x)(1+\hat{\beta})\big(1+R(1+s_0)^{1/2+\chi}+|D^{x,w}_{s_0}|\big)e^{-2R^{-1}(1+s_0)^{1/2-\chi}+2s_0}e^{-2D^{x,w}_{s_0}}\times\dots\\
&\qquad(\hat{\beta}-P^{x,w}_{s_0}-D^{x,w}_{s_0})_+\mathbb{1}_{\{\sup_{u\in[0,s_0]}P^{x,w}_u+D^{x,w}_u\leqslant\hat{\beta}\}}\Big]\\
&\leqslant\frac{C}{|x-w|^2}e^{-2R^{-1}s_0^{1/2-\chi}}(1+Rs_0^{1/2+\chi})\times\dots\\
&\quad\dots\times\mathbb{E}\Big[(1+\hat{\beta})(1+|D^{x,w}_{s_0}|)e^{-2D^{x,w}_{s_0}}(\beta-P^{x,w}_{s_0}-\min_{u\in[0,s_0]}D^{x,w}_u)_+\mathbb{1}_{\{\sup_{u\in[0,s_0]}P^{x,w}_u\leqslant\beta-\min_{u\in[0,s_0]}D^{x,w}_u\}}\Big]\\
&\leqslant\frac{C}{|x-w|^2}e^{-2R^{-1}s_0^{1/2-\chi}}(1+Rs_0^{1/2+\chi})\mathbb{E}\Big[(1+\hat{\beta})(1+|D^{x,w}_{s_0}|)e^{-2D^{x,w}_{s_0}}(\hat{\beta}-\min_{u\in[0,s_0]}D^{x,w}_u)\Big].
\end{aligned}
$$

Because of (4.3), the last expectation is finite and bounded independently of x,w. Indeed, $\hat{\beta}$, $(D^{x,w}_u)_{u\leqslant s_0}$ are independent Wiener integrals with bounded variance (independently of x,w). Therefore we can find $\alpha>0$ such that $\sup_{|x-w|\leqslant1}\mathbb{E}[e^{\alpha(\min_{u\leqslant s_0}D^{x,w}_u)^2}+e^{\alpha\hat{\beta}^2}]<+\infty$. With these estimates, it is plain to see that the above expectation is finite. To sum up, we have proved

$$
\begin{aligned}
&\mathbb{E}^Q[\mathbb{1}_{K_R}(x)M^\beta(B(x,e^{-t}))]\\
&\leqslant\int_{B(x,e^{-t})}\frac{C}{|x-w|^2}e^{-2R^{-1}(\ln\frac{1}{|x-w|})^{1/2-\chi}}(1+R(\ln\frac{1}{|x-w|})^{1/2+\chi})\,dw\\
&=C\int_0^{e^{-t}}\rho^{-1}e^{-2R^{-1}(\ln\frac{1}{\rho})^{1/2-\chi}}(1+R(\ln\frac{1}{\rho})^{1/2+\chi})\,d\rho\\
&=C\int_t^\infty e^{-2R^{-1}y^{1/2-\chi}}(1+Ry^{1/2+\chi})\,dy.
\end{aligned}
$$

Finally we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_{K_R}\int_B\Big(\ln\frac{1}{|x-y|}\Big)^p M_\beta(dx)M_\beta(dy)\Big]\\
&\quad=\mathbb{E}^Q\Big[\mathbb{1}_{K_R}(x)\int_B\Big(\ln\frac{1}{|x-y|}\Big)^p M_\beta(dy)\Big]\\
&\quad\leqslant\sum_{n=1}^\infty\mathbb{E}^Q\Big[\mathbb{1}_{K_R}(x)\int_{B\cap\{2^{-n-1}<|x-y|\leqslant 2^{-n}\}}\Big(\ln\frac{1}{|x-y|}\Big)^p M_\beta(dy)\Big]\\
&\quad\leqslant\sum_{n=1}^\infty(n+1)^p\ln^p 2\,\mathbb{E}^Q\Big[\mathbb{1}_{K_R}(x)M_\beta(B(x,2^{-n}))\Big]\\
&\quad\leqslant C\sum_{n=1}^\infty(n+1)^p\int_{n\ln 2}^\infty e^{-2R^{-1}y^{1/2-\chi}}(1+Ry^{1/2+\chi})\,dy.
\end{aligned}$$

This last series is easily seen to be finite. The proof of Theorem 4.7 is complete. □

It is then a routine trick to deduce

Corollary 4.10. *For each ball B, for all $\delta>0$, there is a compact set $K_\delta\subset B$ such that*

$$M_\beta(B\cap K_\delta^c)\leqslant\delta$$

and for all $p>0$, for all $x\in K_\delta$

$$\sup_{\substack{x\in\mathcal{O}\text{ open }\subset B,\\ \mathrm{diam}(\mathcal{O})\leqslant 1}} M_\beta(\mathcal{O})\big(-\ln\mathrm{diam}(\mathcal{O})\big)^p<+\infty.$$

Corollary 4.11. *1. For each ball B, for each $\delta>0$, there is a compact set $K\subset B$ such that $M'(B\cap K^c)\leqslant\delta$ and for all $p>0$*

$$\int_K\int_B\Big(\ln\frac{1}{|x-y|}\Big)^p M'(dx)M'(dy)<+\infty.$$

2. For each ball B, for all $\delta>0$, there is a compact set $K_\delta\subset B$ such that $M'(B\cap K_\delta^c)\leqslant\delta$ and for all $p>0$, for all $x\in K_\delta$

$$\sup_{\substack{x\in\mathcal{O}\text{ open }\subset B,\\ \mathrm{diam}(\mathcal{O})\leqslant 1}} M'(\mathcal{O})\big(-\ln\mathrm{diam}(\mathcal{O})\big)^p<+\infty.$$

3. Almost surely in X, the Liouville measure M' does not charge the polar sets of the (standard) Brownian motion.

Proof. Both statements results from the fact that M_β coincides with M' over bounded sets for β (random) large enough. □

Remark 4.12. *Observe that the question of the capacity properties of the measure M', i.e. Corollary 4.11, was initially raised in [5] (see at the end of the first section).*

4.3 Defining F' on the whole of $\mathbb{R}^2$

In this subsection, we have two main objectives: to construct the PCAF $F^{'}(x,\cdot)$ on the whole of $\mathbb{R}^2$ and prove the convergence of $(F_\beta^{',\epsilon}(x,\cdot))_\epsilon$ towards $F'(x,\cdot)$. The main difficulty here is the following: in section 3.3 we have proved the almost sure convergence of $(F_\beta^{',\epsilon}(x,\cdot))_\epsilon$ towards F' when the starting point x is fixed. Of course, we can deduce that almost surely, for a countable collection of given starting points, this convergence holds. The main difficulty is to prove that this convergence holds for **all** possible starting points and this definitely requires some further arguments.

For $\mathbf{q}=(q_1,q_2)\in\mathbb{Z}^2$, we denote by $C_{\mathbf{q}}$ the cube $[q_1,q_1+1]\times[q_2,q_2+1]$. We fix $p>1$ and for each $\mathbf{q}\in\mathbb{Z}^2$ and $\delta>0$, we denote by $K^{\mathbf{q}}_{\delta,L}$ the compact set

$$K^{\mathbf{q}}_{\delta,L}=K_\delta\cap\big\{x\in C_{\mathbf{q}};\sup_{\substack{x\in\mathcal{O}\text{ open },\\ \mathrm{diam}(\mathcal{O})\leqslant 1}}M_\beta(\mathcal{O})\big(-\ln\mathrm{diam}(\mathcal{O})\big)^p\leqslant L\big\},\tag{4.4}$$

where K_δ is the compact set given by Corollary 4.10 applied with $B=C_{\mathbf{q}}$. Then we set

$$K^{\mathbf{q}}_{\delta}=\bigcup_{L>0}K^{\mathbf{q}}_{\delta,L},\quad K_{\delta,L}=\bigcup_{\mathbf{q}\in\mathbb{Z}^2}K^{\mathbf{q}}_{\delta,L}\quad\text{and}\quad S=\bigcup_{\delta>0,\mathbf{q}\in\mathbb{Z}^2,L>0}K^{\mathbf{q}}_{\delta,L}.\tag{4.5}$$

From Corollary 4.10, we have $M_\beta((K^{\mathbf{q}}_\delta)^c\cap C_{\mathbf{q}})\leqslant\delta$. Therefore, $M_\beta((\bigcup_{\delta>0}K^{\mathbf{q}}_\delta)^c\cap C_{\mathbf{q}})=0$ for each $\mathbf{q}\in\mathbb{Z}^2$ and thus

$$M_\beta(S^c)=0.\tag{4.6}$$

We consider a Brownian motion B starting from 0 and define a Brownian motion starting from x by $B^x=x+B$ for each point $x\in\mathbb{R}^2$. Following Section 3, we may assume that $(F_\beta^{',\epsilon}(x,\cdot))_\epsilon$ converges almost surely in X and B in $C(\mathbb{R}_+;\mathbb{R}_+)$ towards $F'_\beta(x,\cdot)$ for each rational points $x\in\mathbb{Q}^2$.

For each $\delta,L>0$ and $x\in\mathbb{Q}^2$, we define the adapted continuous random mapping

$$F_\beta^{',\delta,L}(x,t)=\int_0^t\mathbb{1}_{K_{\delta,L}}(B^x_r)F'_\beta(x,dr).\tag{4.7}$$

Proposition 4.13. *Almost surely in X, for each $\delta,L>0$ and $x\in\mathbb{R}^2$, there exists a B^x-adapted continuous random mapping, still denoted by $F_\beta^{',\delta,L}(x,\cdot)$ such that for all sequence of rational points $(x_n)_n$ converging towards x, the sequence $(F_\beta^{',\delta,L}(x_n,\cdot))_n$ converges in $\mathbb{P}^B$-probability in $C(\mathbb{R}_+,\mathbb{R}_+)$ towards $F_\beta^{',\delta,L}(x,\cdot)$.*

Proof. Let us fix $x\in\mathbb{R}^2$. We consider a sequence $(x_n)_n$ of rational points converging towards x. We first establish the convergence in law under $\mathbb{P}^B$ of the sequence $(B^{x_n},F_\beta^{',\delta,L}(x_n,\cdot))_n$ in $C(\mathbb{R}_+,\mathbb{R}^2)\times C(\mathbb{R}_+,\mathbb{R}_+)$. To this purpose, the main idea is an adaptation of [27, section 2.9] with minor modifications. Yet, we outline the proof because we will play with this argument throughout this section. For all $0<s<t$, we write $F_\beta^{',\delta,L}(x,]s,t])$ for $F_\beta^{',\delta,L}(x,t)-F_\beta^{',\delta,L}(x,s)$. The proof relies on two arguments: a coupling argument and the estimate:

$$\sup_{y\in B(0,R)\cap\mathbb{Q}^2}\mathbb{E}^B[F_\beta^{',\delta,L}(y,t)]\to 0,\qquad\text{as }t\to 0.\tag{4.8}$$

We begin with explaining (4.8). For all $y \in \mathbb{Q}^2$, we have:

$$\mathbb{E}^B[F_\beta^{',\delta,L}(y,t)] = \int_{\mathbb{R}^2}\int_0^t p_r(y,z)\,dr\mathbb{1}_{K_{\delta,L}}(z)M_\beta(dz).$$

Furthermore we have for each $p > 1$

$$\sup_{r\in]0,1/2]}\sup_{x\in\mathbb{R}^2}(-\ln r)^p M_\beta(B(x,r)\cap K_{\delta,L}) \leqslant 2^p L. \tag{4.9}$$

Indeed, observe first that $\sup_{r\in]0,1/2]}\sup_{x\in K_{\delta,L}}(-\ln r)^p M_\beta(B(x,r)\cap K_{\delta,L}) \leqslant L$ by definition. To extend this formula to $x \notin K_{\delta,L}$, take $r \leqslant 1/2$ and observe that we have two options: either $B(x,r)\cap K_{\delta,L}$ is empty in which case $M_\beta(B(x,r)\cap K_{\delta,L}) = 0$ or we can find $y \in B(x,r)\cap K_{\delta,L}$ and then $M_\beta(B(x,r)\cap K_{\delta,L}) \leqslant M_\beta(B(y,2r)\cap K_{\delta,L}) \leqslant L(-\ln 2r)^{-p} \leqslant 2^p L(-\ln r)^{-p}$.

We deduce that for all $R > 0$,

$$\sup_{y\in B(0,R)}\int_{\mathbb{R}^2}\int_0^t p_r(y,z)\,dr\mathbb{1}_{K_{\delta,L}}(z)M_\beta(dz) \to 0, \quad \text{ as } t \to 0. \tag{4.10}$$

Indeed, the decay of the size of balls induced by (4.9), used with $p > 2$, is enough to overcome the ln-singularity produced by the heat kernel integral: $\int_0^t p_r(y,z)\,dr$ (to be exhaustive, one should adapt the argument in [27, section 2.7] but this is harmless). Hence (4.8).

Let us now prove that the family $(F_\beta^{',\delta,L}(x_n,\cdot))_n$ is tight in $C(\mathbb{R}_+;\mathbb{R}_+)$. This consists in checking that for all $T,\eta > 0$:

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\mathbb{P}^B\big(\sup_{\substack{0\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F_\beta^{',\delta,L}(x_n,]s,t])| > \eta\big) = 0. \tag{4.11}$$

The control of this supremum uses two arguments: a control of the involved quantities when s,t are closed to 0 via (4.8) and a control of this supremum via a coupling argument when s,t are far enough from 0. To quantify the proximity to 0, we introduce a parameter $\theta > 0$. We have for $\delta < \theta$:

$$\begin{aligned}&\mathbb{P}^B\big(\sup_{\substack{0\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F_\beta^{',\delta,L}(x_n,]s,t])| > \eta\big)\\ &\quad\leqslant \mathbb{P}^B\big(F_\beta^{',\delta,L}(x_n,2\theta) > \eta/4\big) + \mathbb{P}^B\big(\sup_{\substack{\theta\leqslant s,t\leqslant T\\ |t-s|\leqslant\delta}}|F_\beta^{',\delta,L}(x_n,]s,t])| > \eta/2\big).\end{aligned} \tag{4.12}$$

To establish (4.11), it is thus enough to prove that the $\limsup_{\theta\to 0}\limsup_{\delta\to 0}\limsup_{n\to\infty}$ of each term in the right-hand side of the above expression vanishes. The first term is easily treated with the help of the Markov inequality and (4.8) so that we now focus on the second term. To this purpose, we recall the following coupling lemma:

Lemma 4.14. *Fix $x \in \mathbb{R}^2$ and let us start a Brownian motion B^x from x. Let us consider another independent Brownian motion B' starting from 0 and denote by B^y, for a rational $y \in \mathbb{Q}^2$, the Brownian motion $B^y = y + B'$. Let us denote by $\tau_1^{x,y}$ the first time at which the first components of B^x and B^y coincide:*

$$\tau_1^{x,y} = \inf\{u > 0; B_u^{1,x} = B_u^{1,y}\}$$

and by $\tau_2^{x,y}$ the first time at which the second components coincide after $\tau_1^{x,y}$:

$$\tau_2^{x,y} = \inf\{u > \tau_1^{x,y}; B_u^{2,x} = B_u^{2,y}\}$$

The random process $\overline{B}^{x,y}$ defined by

$$\overline{B}_t^{x,y} = \begin{cases} (B_t^{1,x}, B_t^{2,x}) & \text{if} \quad t \leqslant \tau_1^{x,y} \\ (B_t^{1,y}, B_t^{2,x}) & \text{if} \quad \tau_1^{x,y} < t \leqslant \tau_2^{x,y} \\ (B_t^{1,y}, B_t^{2,y}) & \text{if} \quad \tau_2^{x,y} < t. \end{cases}$$

is a new Brownian motion on $\mathbb{R}^2$ starting from x, and coincides with B^y for all times $t > \tau_2^y$. Furthermore, if $y - x \to 0$, we have for all $\eta > 0$:

$$\mathbb{P}(\tau_2^{x,y} > \eta) \to 0.$$

We choose $y \in \mathbb{Q}^2$. We can consider the couple $(F_\beta^{',\delta,L,x_n}(y,\cdot), F_\beta^{',\delta,L}(x_n,\cdot))$ where $F_\beta^{',\delta,L}(x_n,\cdot)$ is the same as that considered throughout this section and $F_\beta^{',\delta,L,x_n}(y,\cdot)$ is constructed as $F_\beta^{',\delta,L}(y,\cdot)$ but we have used the Brownian motion $\bar{B}^{y,x_n}$ of Lemma 4.14 instead of the Brownian motion B^y. The important point to understand is that this couple does not have the same law as the couple $(F_\beta^{',\delta,L}(y,\cdot), F_\beta^{',\delta,L}(x_n,\cdot))$ but it has the same 1-marginal. Furthermore we have $F_\beta^{',\delta,L,x_n}(y,]s,t]) = F_\beta^{',\delta,L}(x_n,]s,t])$ for $\tau_2^{y,x_n} \leqslant s < t$. We deduce:

$$\begin{aligned} &\mathbb{P}^B\big(\sup_{\substack{\theta \leqslant s,t \leqslant T \\ |t-s| \leqslant \delta}} |F_\beta^{',\delta,L}(x_n,]s,t])| > \eta/2\big) \\ &\qquad \leqslant \mathbb{P}^B\big(\sup_{\substack{\theta \leqslant s,t \leqslant T \\ |t-s| \leqslant \delta}} |F_\beta^{',\delta,L}(y,]s,t])| > \eta/2\big) + \mathbb{P}(\tau_2^{y,x_n} > \theta). \end{aligned}$$

It is then obvious to get:

$$\limsup_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}^B\big(\sup_{\substack{\theta \leqslant s,t \leqslant T \\ |t-s| \leqslant \delta}} |F_\beta^{',\delta,L}(x_n,]s,t])| > \eta/2\big) \leqslant \mathbb{P}(\tau_2^{y,x} > \theta). \tag{4.13}$$

Since the choice of y was arbitrary, we can now choose y arbitrarily close to x to make this latter term as close to 0 as we please for a fixed θ. Hence (4.11) and the family $(F_\beta^{',\delta,L}(x_n,\cdot))_n$ is tight in $C(\mathbb{R}_+,\mathbb{R}_+)$.

We can also use the coupling argument to prove that there is only one possible limit in law for all subsequences $(x_n)_n$ such that $x_n \to x$ as $n \to \infty$, thus showing the convergence in law in $C(\mathbb{R}_+,\mathbb{R}_+)$ of the family $(F_\beta^{',\delta,L}(x_n,\cdot))_n$. Here we have only dealt with the convergence of the family $(F_\beta^{',\delta,L}(x_n,\cdot))_n$ but it is straightforward to adapt the argument to the family $(B^{x_n}, F_\beta^{',\delta,L}(x_n,\cdot))_n$.

Now, we come to the convergence in $\mathbb{P}^B$-probability of $(F_\beta^{',\delta,L}(x_n,\cdot))_n$. We fix $t > 0$, we consider the mapping

$$(x,y) \in \mathbb{Q}^2 \times \mathbb{Q}^2 \mapsto \mathbb{E}^B\big[(F_\beta^{',\delta,L}(x,t) - F_\beta^{',\delta,L}(y,t))^2\big].$$

We can expand the square and, following the ideas in [27, section 2.7], use (4.9) to control the ln-singularities to see that the above mapping extends to a continuous function on $\mathbb{R}^2 \times \mathbb{R}^2$, which vanishes on the diagonal $\{(x,x); x \in \mathbb{R}^2\}$. Therefore, if $(x_n)_n$ is a sequence in $\mathbb{Q}^2$ converging towards

x, the sequence $(F_\beta^{',\delta,L}(x_n,t))_n$ converges in L^2 under $\mathbb{P}^B$. We can thus extract a subsequence $(x_{\phi(n)})_n$ such that for all $t\in\mathbb{Q}\cap\mathbb{R}_+$, the sequence $(F_\beta^{',\delta,L}(x_{\phi(n)},t))_n$ converges $\mathbb{P}^B$-almost surely. Also, we have seen that this subsequence converges in law in $C(\mathbb{R}_+,\mathbb{R}_+)$ towards a continuous random mapping. The Dini theorem implies that $(F_\beta^{',\delta,L}(x_{\phi(n)},\cdot))_n$ converges $\mathbb{P}^B$-almost surely in $C(\mathbb{R}_+,\mathbb{R}_+)$ towards a limit denoted by $F_\beta^{',\delta,L}(x,\cdot)$. $\square$

We are now in position to construct F'_β on the whole of $\mathbb{R}^2$.

Theorem 4.15. *Almost surely in X, for all $x\in\mathbb{R}^2$, the random measure $F_\beta^{',\delta,L}(x,dt)$ converges $\mathbb{P}^B$-almost surely as $\delta\to 0$ and $L\to\infty$ in the sense of weak convergence of measures towards a random measure denoted by $F'_\beta(x,dt)$. Furthermore:*

1. *for $x\in\mathbb{Q}^2$, $F'_\beta(x,\cdot)$ coincides with the limit of the family $(F_\beta^{',\epsilon}(x,\cdot))_\epsilon$ as $\epsilon\to 0$ defined in subsection 3.3.*
2. *for $x\in S\cup\mathbb{Q}^2$, convergence of the mapping $t\mapsto F_\beta^{',\delta,L}(x,t)$ towards $t\mapsto F'_\beta(x,t)$ holds $\mathbb{P}^B$-almost surely in $C(\mathbb{R}_+,\mathbb{R}_+)$ as $\delta\to 0$, $L\to\infty$.*
3. *for $x\notin S$, for all $s>0$, convergence of the mapping $t\mapsto F_\beta^{',\delta,L}(x,]s,t])$ towards $t\mapsto F'_\beta(x,]s,t])$ holds $\mathbb{P}^B$-almost surely in $C([s,+\infty[,\mathbb{R}_+)$.*

Proof. Observe that $K_{\delta,L}\subset K_{\delta',L'}$ if $\delta'\leqslant\delta$ and $L'\geqslant L$. Therefore, for all $x\in\mathbb{R}^2$ and for all $t\geqslant 0$, $F_\beta^{',\delta,L}(x,t)\leqslant F_\beta^{',\delta',L'}(x,t)$ if $\delta'\leqslant\delta$. We can thus define the almost sure limit:

$$F_\beta^{',\infty}(x,t)=\lim_{\delta\to 0,L\to\infty}F_\beta^{',\delta,L}(x,t).$$

Actually, this also implies the weak convergence as $\delta\to 0$ and $L\to\infty$ of the measure $F_\beta^{',\delta,L}(x,dt)$ towards a random measure, still denoted by $F_\beta^{',\infty}(x,dt)$.

For $x\in\mathbb{Q}^2$, let us identify $F_\beta^{',\infty}(x,dt)$ with the limit $F'_\beta(x,\cdot)$ of the family $(F_\beta^{',\epsilon}(x,\cdot))_\epsilon$ as $\epsilon\to 0$. By construction we have:

$$F_\beta^{',\delta,L}(x,t)=\int_0^t \mathbb{1}_{K_{\delta,L}}(B_r^x)F'_\beta(x,dr)\leqslant F'_\beta(x,t),$$

in such a way that $F_\beta^{',\infty}(x,t)\leqslant F'_\beta(x,t)$. Second, by the dominated convergence theorem we get

$$\begin{aligned}\mathbb{E}^B\big[|F'_\beta(x,t)-F_\beta^{',\infty}(x,t)|\big]&=\lim_{\delta\to 0,L\to\infty}\mathbb{E}^B\big[\int_0^t\mathbb{1}_{K^c_{\delta,L}}(B_r^x)F'_\beta(x,dr)\big]\\&=\lim_{\delta\to 0,L\to\infty}\int_{\mathbb{R}^2}\int_0^t p_r(x,y)\,dr\mathbb{1}_{K^c_{\delta,L}}(y)M_\beta(dy)\\&=\int_{\mathbb{R}^2}\int_0^t p_r(x,y)\,dr\mathbb{1}_{S^c}(y)M_\beta(dy)\\&=0\end{aligned}$$

because $M_\beta(S^c)=0$. Now that we have identified $F'_\beta(x,\cdot)$ with $F_\beta^{',\infty}(x,\cdot)$ on $x\in\mathbb{Q}^2$, we skip the distinction made with the superscript ∞ and write F'_β for the limit of $F_\beta^{',\delta,L}$.

For $x \in \mathbb{Q}^2$, the continuity of the mapping $t \mapsto F^{'}_\beta(x,t)$ together with the Dini theorem implies the $\mathbb{P}^B$-almost sure convergence of $F^{',\delta,L}_\beta(x,\cdot)$ towards $F^{'}_\beta(x,\cdot)$ in $C(\mathbb{R}_+,\mathbb{R}_+)$. Let us now complete the proof of items 2 and 3. We fix $s>0$. The coupling argument established in the proof of Proposition 4.13 shows that the mapping $t \mapsto F^{'}_\beta(x,]s,t])$ is continuous on $[s,+\infty[$ (it coincides in law with $t \mapsto F^{'}_\beta(y,]s,t])$ with $y \in \mathbb{Q}^2$ as soon as B^x and B^y are coupled before time s, which happens with probability arbitrarily close to 1 provided that y is close enough to x). The Dini theorem again implies item 3. Let us stress that it is not clear that we can take $s=0$ because we need to control the decay of balls at $x \in S^c$ and this decay may happen to be very bad on S^c.

To prove item 2, we also use the Dini theorem but we further need to prove that the mapping $t \mapsto F^{'}_\beta(x,t)$ is continuous at $t=0$ with $F^{'}_\beta(x,0)=0$. This can be done by computing

$$\mathbb{E}^B\big[F^{'}_\beta(x,t)\big] = \int_{\mathbb{R}^2}\int_0^t p_r(x,y)\,dr\mathbb{1}_S(y)M_\beta(dy).$$

Following [27], we observe that the mapping $y \mapsto \int_0^t p_r(x,y)\,dr$ possesses a logarithmic singularity at $y=x$. Furthermore, for $x \in S$, we have

$$\sup_{\substack{x\in\mathcal{O}\text{ open }\subset B,\\ \mathrm{diam}(\mathcal{O})\leqslant 1}} M'(\mathcal{O})\big(-\ln \mathrm{diam}(\mathcal{O})\big)^p < +\infty.$$

Therefore

$$\int_{\mathbb{R}^2}\int_0^1 p_r(x,y)\,dr\mathbb{1}_S(y)M_\beta(dy) < +\infty$$

in such a way that the dominated convergence theorem implies

$$\mathbb{E}^B\big[F^{'}_\beta(x,t)\big] = \int_{\mathbb{R}^2}\int_0^t p_r(x,y)\,dr\mathbb{1}_S(y)M_\beta(dy) \to 0, \qquad \text{as } t\to 0.$$

To sum up, we have proved that the mapping $t \mapsto F^{'}_\beta(x,t)$ is continuous at $t=0$ with $F^{'}_\beta(x,0)=0$ for $x \in S$. We complete the proof of item 2 with the help of the Dini theorem. □

Remark 4.16. *It is important here to stress that the above Proposition shows that for all x, we have defined a mapping $F^{'}_\beta(x,\cdot)$, which is $\mathbb{P}^B$-almost surely continuous with continuous sample paths and satisfies a variant of the additivity property of a PCAF, i.e. for all $s,t \geqslant 0$ we have almost surely*

$$F^{'}_\beta(x,t+s) = F^{'}_\beta(x,t) + \bar{F}^{'}_\beta(x,s),$$

where, conditionally to $\mathcal{F}_t$, the variable $\bar{F}^{'}_\beta(x,s)$ is distributed as $F^{'}_\beta(x+B_t,s)$ under $\mathbb{P}^{x+B_t}$ (measure of a Brownian motion starting from $x+B_t$). Also, we have not so far proved that $F^{'}_\beta$ is a PCAF because it is defined for all x $\mathbb{P}^B$-almost surely whereas we need to define it $\mathbb{P}^B$-almost surely for all x. Yet, we will see that this problem for the construction of a proper PCAF is not too serious.

We claim:

Theorem 4.17. *Almost surely in X, we define*

$$F'(x,t) = \lim_{\beta\to\infty} F^{'}_\beta(x,]0,t])$$

where convergence holds $\mathbb{P}^B$-almost surely in $C(\mathbb{R}_+,\mathbb{R}_+)$. F' is some form of PCAF in the strict sense in $\mathbb{R}^2$: it is defined for all starting points and satisfies the following variant of the additivity property

$$F_\beta^{'}(x,t+s)=F_\beta^{'}(x,t)+\bar{F}_\beta^{'}(x,s),\quad s,t\geqslant 0 \tag{4.14}$$

where, conditionally to $\mathcal{F}_t$, the variable $\bar{F}_\beta^{'}(x,s)$ is distributed as $F_\beta^{'}(x+B_t,s)$ under $\mathbb{P}^{x+B_t}$ (measure of a Brownian motion starting from $x+B_t$).

Furthermore,

1. *for all $x\in\mathbb{R}^2$, F' is continuous, strictly increasing and goes to ∞ as $t\to\infty$.*
2. *F' coincides outside a set of zero capacity with a PCAF of Revuz measure M'.*

Proof. For each $x\in\mathbb{R}^2$, for each $t\geqslant 0$, the mapping $\beta\mapsto F'_\beta(x,]0,t])$ is increasing. We can thus define $F'(x,t)=\lim_{\beta\to\infty}F'_\beta(x,]0,t])$. Furthermore, for each ball B containing x, $F'(x,t)$ coincides with $F'_\beta(x,]0,t])$ for $t<\tau_B(x)=\inf\{u>0;B_u^x\notin B\}$ and β (random) large enough (more precisely for β large enough to make $\sup_{x\in B}\sup_{\epsilon\in]0,1]}X_\epsilon(x)-2\ln\frac{1}{\epsilon}<\beta$, see Proposition 3.5). It is obvious to check that F' satisfies the additivity (4.14).

Now we prove item 1. This results from the coupling argument detailed in Proposition 4.13 as $F'(x,\cdot)$ is strictly increasing and goes to ∞ as $t\to\infty$ for $x\in\mathbb{Q}^2$ (see also [27, Proposition 2.24]).

Finally, we prove item 2, more precisely we establish the relation (4.2) for M' and F'. The construction of F' entails that, for any $0<s<t$

$$\mathbb{E}^{B^x}\Big[\int_s^t f(B_r^x)\,F'(x,dr)\Big]=\int_s^t\int_{\mathbb{R}^2}f(y)p_r(x,y)\,M'(dy)\,dr.$$

Therefore, for any nonnegative Borel functions f,h (P_r stands for the semigroup associated to the planar Brownian motion):

$$\begin{aligned}\int_{\mathbb{R}^2}h(x)\mathbb{E}^{B^x}\Big[\int_s^t f(B_r^x)\,F'(x,dr)\Big]\,dx&=\int_s^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f(y)p_r(x,y)\,M'(dy)h(x)\,dx\,dr\\&=\int_s^t\int_{\mathbb{R}^2}f(y)P_rh(y)\,M'(dy)\,dr.\end{aligned}$$

It suffices to let $s\to 0$ to conclude.

Now, we use [25, Theorem 5.1.3] to prove that there exists a PCAF A associated to M' because M' is a smooth measure in the sense of [25] thanks to Corollary 4.11. Now we have at our disposal a PCAF A with Revuz measure M' and an "almost" PCAF F'. The reader may check that the uniqueness part of [25, Theorem 5.1.4] can be reproduced to prove that F' and A coincide for x outside a set of capacity 0 (just observe that this proof does not use the fact that the set where the PCAF is defined does not depend on x). ☐

4.4 Definition and properties of the critical LBM

Definition 4.18. (critical Liouville Brownian motion). *Almost surely in X, for all $x\in\mathbb{R}^2$, the law of the LBM at criticality, starting from x, is defined by:*

$$\mathcal{B}_t^x=x+B_{\langle\mathcal{B}^x\rangle_t}$$

where $\langle\mathcal{B}^x\rangle$ is defined by

$$\sqrt{2/\pi}F'(x,\langle\mathcal{B}^x\rangle_t)=t.$$

We stress that $\mathcal{B}^x$ is a local martingale.

Proposition 4.19. *The critical LBM is a strong Markov process with continuous sample paths.*

Proof. Strong Markov property results from [25, sect. 6]. Continuity of sample paths results from the fact that F' is strictly increasing. ☐

Theorem 4.20. *Assume further [A.6] or [A.6']. Almost surely in X, for all $x \in S$, the ϵ-regularized Brownian motion $(\mathcal{B}^{\epsilon,x})_\epsilon$ defined by Definition 2.5 converges in law in the space $C(\mathbb{R}_+,\mathbb{R}^2)$ equipped with the supremum norm on compact sets towards $\mathcal{B}^x$.*

Proof. This is just a consequence of Theorems 4.15 and 4.17 as explained in [27]. ☐

From [25, Th. 6.2.1], we claim:

Theorem 4.21. (Dirichlet form). *The critical Liouville Dirichlet form $(\Sigma,\mathcal{F})$ takes on the following explicit form on $L^2(\mathbb{R}^2,M')$:*

$$\Sigma(f,g)=\frac{1}{2}\int_{\mathbb{R}^2}\nabla f(x)\cdot\nabla g(x)\,dx \tag{4.15}$$

with domain

$$\mathcal{F}=\Big\{f\in L^2(\mathbb{R}^2,M')\cap H^1_{loc}(\mathbb{R}^2,dx);\nabla f\in L^2(\mathbb{R}^2,dx)\Big\},$$

Furthermore, it is strongly local and regular.

Let us denote by P^X_t (for $t\geqslant 0$) the mapping

$$f\in C_b(\mathbb{R}^2)\mapsto\big(x\in\mathbb{R}^2\mapsto P^X_t f(x)=\mathbb{E}^B[f(\mathcal{B}^x_t)]\big). \tag{4.16}$$

Similarly we define P^ϵ as the semigroup generated by the Markov process $\mathcal{B}^\epsilon$. From [25, sect. 6], we claim:

Theorem 4.22. (Semigroup). *The linear operator P^X_t, restricted to $C_c(\mathbb{R}^2)$, extends to a linear contraction on $L^p(\mathbb{R}^2,M')$ for all $1\leqslant p<\infty$, still denoted P^X_t. Furthermore:*

- *$(P^X)_{t\geqslant 0}$ is a Markovian strongly continuous semigroup on $L^p(\mathbb{R}^2,M')$ for $1\leqslant p<\infty$.*
- *Assume further [A.6] or [A.6']. Almost surely in X, the ϵ-regularized semigroup $(P^\epsilon)_\epsilon$ converges pointwise for $x\in S$ towards the critical Liouville semigroup. More precisely, for all bounded continuous function f, we have:*

$$\forall x\in S,\forall t\geqslant 0,\quad \lim_{\epsilon\to 0}P^\epsilon_t f(x)=P^X_t f(x).$$

- *P^X is self-adjoint in $L^2(\mathbb{R}^2,M')$.*
- *the measure M' is invariant for P^X_t.*

The **critical Liouville Laplacian** Δ_X is defined as the generator of the critical Liouville semigroup times the usual extra factor 2. The critical Liouville Laplacian corresponds to an operator which can formally be written as

$$\Delta_X=X^{-1}(x)e^{-2X(x)}\Delta$$

and can be thought of as the Laplace-Beltrami operator of $2d$-Liouville quantum gravity at criticality (of course when X is a free field).

One may also consider the resolvent family $(R^X_\lambda)_{\lambda>0}$ associated to the semigroup $(P^X_{\mathbf{t}})_{\mathbf{t}}$. In a standard way, the resolvent operator reads:

$$\forall f \in C_b(\mathbb{R}^2), \forall x \in \mathbb{R}^2, \quad R^X_\lambda f(x) = \int_0^\infty e^{-\lambda t} P^X_t f(x)\, dt. \tag{4.17}$$

Furthermore, the resolvent family $(R^X_\lambda)_{\lambda>0}$ extends to $L^p(\mathbb{R}^2, M')$ for $1 \leqslant p < +\infty$, is strongly continuous for $1 \leqslant p < +\infty$ and is self-adjoint in $L^2(\mathbb{R}^2, M')$. This results from the properties of the semi-group. As a consequence of Theorem 4.22, it is straightforward to see that:

Proposition 4.23. *Assume further [A.6] or [A.6']. Almost surely in X, the ϵ-regularized resolvent family $(R^\epsilon_\lambda)_\lambda$ converges towards the critical Liouville resolvent $(R^X_\lambda)_\lambda$ in the sense that for all function $f \in C_b(\mathbb{R}^2)$:*

$$\forall x \in S, \quad \lim_{\epsilon \to 0} R^\epsilon_\lambda f(x) = R^X_\lambda f(x).$$

Also and similarly to [25, 28], it is possible to get an explicit expression for the resolvent operator:

Proposition 4.24. *Almost surely in X, the resolvent operator takes on the following form for all measurable bounded function f on $\mathbb{R}^2$:*

$$\forall x \in \mathbb{R}^2, \quad R^X_\lambda f(x) = \sqrt{2/\pi}\,\mathbb{E}^B\Big[\int_0^\infty e^{-\lambda\sqrt{2/\pi}F'(x,t)} f(B^x_t)\, F'(x,dt)\Big].$$

The main purpose of this section is to prove the following structure result on the resolvent family:

Theorem 4.25. (massive Liouville Green kernels at criticality). *For every $x \in \mathbb{R}^2$, the resolvent family $(R^X_\lambda)_{\lambda>0}$ is absolutely continuous with respect to the critical Liouville measure. Therefore there exists a family $(\mathbf{r}^X_\lambda(\cdot,\cdot))_\lambda$, called the family of massive critical Liouville Green kernels, of jointly measurable functions such that:*

$$\forall x \in \mathbb{R}^2, \forall f \in B_b(\mathbb{R}^2), \quad R^X_\lambda f(x) = \sqrt{2/\pi}\int_{\mathbb{R}^2} f(y)\mathbf{r}^X_\lambda(x,y)\, M'(dy)$$

and such that:
1) (strict-positivity) for all $\lambda > 0$ and $x \in \mathbb{R}^2$, $M'(dy)$ a.s., $\mathbf{r}^X_\lambda(x,y) > 0$,
2) (symmetry) for all $\lambda > 0$ and $x, y \in \mathbb{R}^2$: $\mathbf{r}^X_\lambda(x,y) = \mathbf{r}^X_\lambda(y,x)$,
3) (resolvent identity) for all $\lambda, \mu > 0$, for all $x, y \in \mathbb{R}^2$,

$$\mathbf{r}^X_\mu(x,y) - \mathbf{r}^X_\lambda(x,y) = (\lambda-\mu)\sqrt{2/\pi}\int_{\mathbb{R}^2} \mathbf{r}^X_\lambda(x,z)\mathbf{r}^X_\mu(z,y)\, M'(dz).$$

4) (λ-excessive) for every y: $e^{-\lambda t}P^X_t(\mathbf{r}_\lambda(\cdot,y))(x) \leqslant \mathbf{r}_\lambda(x,y)$ for M'-almost every x and for all $t > 0$.

Proof. We have to show absolute continuity of the resolvent for $x \in \mathbb{R}^2$. Though inspired by [27, 28], we have to adapt the proof because we do not have "uniform convergence" of the PCAF towards

0 as $t \to 0$. In particular, it is not clear that the resolvent be strong Feller. For $\delta > 0$, we define for $f \in B_b(\mathbb{R}^2)$

$$R^X_{\lambda,\delta} f(x) = \sqrt{2/\pi}\,\mathbb{E}^B\Big[\int_\delta^\infty e^{-\lambda\sqrt{2/\pi}F'(x,]\delta,t])} f(B^x_t)\,F'(x,dt)\Big],$$

where $F'(x,dr)$ stands for the random measure associated to the increasing function $t \mapsto F'(x,t)$. Once again, the coupling argument of Proposition 4.13, it is plain to see that the mapping

$$x \in \mathbb{R}^2 \mapsto R^X_{\lambda,\delta} f(x)$$

is continuous. Now we claim

Lemma 4.26. *For every $x \in \mathbb{R}^2$, $\delta > 0$ and all nonnegative bounded Borelian function f, we have*

$$R^X_\lambda f(x) = 0 \Longrightarrow R^X_{\lambda,\delta} f(x) = 0.$$

Lemma 4.27. *For every $x \in \mathbb{R}^2$ and all nonnegative bounded Borelian function f, we have*

$$\forall \delta > 0, \quad R^X_{\lambda,\delta} f(x) = 0 \Longrightarrow R^X_\lambda f(x) = 0.$$

We postpone the proofs of the above two lemmas. If A is a measurable set such that $M'(A) = 0$ then by invariance of M' for the resolvent family, we deduce that $R_\lambda \mathbb{1}_A(x) = 0$ for M'-almost every $x \in \mathbb{R}^2$. Since S has full M'-measure and because M' has full support, we deduce that $R^X_\lambda f(x) = 0$ for x belonging to a dense subset of $\mathbb{R}^2$. From Lemma 4.26, $R^X_{\lambda,\delta} f(x) = 0$ for x belonging to a dense subset of $\mathbb{R}^2$. Continuity of $R^X_{\lambda,\delta} f$ entails that this function identically vanishes on $\mathbb{R}^2$ for every $\delta > 0$. With the help of Lemma 4.27, we deduce that $R_\lambda \mathbb{1}_A(x) = 0$ for $x \in \mathbb{R}^2$. Therefore, for all $x \in \mathbb{R}^2$, $R_\lambda \mathbb{1}_A(x) = 0$, thus showing that the measure $A \mapsto R^X_\lambda \mathbb{1}_A(x)$ is absolutely continuous with respect to M'. □

Proof of Lemmas 4.26 and 4.27. For $x \in \mathbb{R}^2$ and all bounded nonnegative Borelian function f, we have (see Proposition 4.24)

$$\begin{aligned}
R^X_\lambda f(x) =& \sqrt{2/\pi}\,\mathbb{E}^{B^x}\Big[\int_0^\infty e^{-\lambda\sqrt{2/\pi}F'(x,t)} f(B^x_t)\,F'(x,dt)\Big] \\
=& \sqrt{2/\pi}\,\mathbb{E}^{B^x}\Big[\int_0^\delta e^{-\lambda\sqrt{2/\pi}F'(x,t)} f(B^x_t)\,F'(x,dt)\Big] \\
&+ \sqrt{2/\pi}\,\mathbb{E}^{B^x}\Big[e^{-\lambda F'(x,\delta)}\int_\delta^\infty e^{-\lambda\sqrt{2/\pi}F'(x,]\delta,t])} f(B^x_t)\,F'(x,dt)\Big].
\end{aligned} \tag{4.18}$$

Therefore

$$R^X_\lambda f(x) \geqslant \sqrt{2/\pi}\,\mathbb{E}^{B^x}\Big[e^{-\lambda F'(x,\delta)}\int_\delta^\infty e^{-\lambda\sqrt{2/\pi}F'(x,]\delta,t])} f(B^x_t)\,F'(x,dt)\Big].$$

For $x \in \mathbb{R}^2$, we have $e^{-\lambda F'(x,\delta)} > 0$ $\mathbb{P}^{B^x}$-almost surely. The proof of Lemma 4.26 follows.

Furthermore, for $x \in \mathbb{R}^2$, we have

$$\sqrt{2/\pi}\,\mathbb{E}^{B^x}\Big[\int_0^\delta e^{-\lambda\sqrt{2/\pi}F'(x,t)} f(B^x_t)\,F'(x,dt)\Big] \leqslant \lambda^{-1}\|f\|_\infty \mathbb{E}^{B^x}\Big[1 - e^{-\lambda\sqrt{2/\pi}F'(x,\delta)}\Big].$$

Since $F'(x,\delta)$ converges in law towards 0 as $\delta \to 0$, we deduce that

$$\lim_{\delta\to 0}\sqrt{2/\pi}\,\mathbb{E}^{B^x}\Big[\int_0^\delta e^{-\lambda\sqrt{2/\pi}F'(x,t)}f(B_t^x)\,F'(x,dt)\Big]=0.$$

From (4.18), we deduce:

$$R_\lambda^X f(x)\leqslant R_{\lambda,\delta}^X f(x)+\sqrt{2/\pi}\,\mathbb{E}^{B^x}\Big[\int_0^\delta e^{-\lambda\sqrt{2/\pi}F'(x,t)}f(B_t^x)\,F'(x,dt)\Big].$$

The proof of Lemma 4.27 follows. ∎

As prescribed in [25, section 1.5], let us define the Green function for $f\in L^1(D,M')$ by

$$Gf(x)=\lim_{t\to\infty}\int_0^t P_r^X f(x)\,dr.$$

We further denote g the standard Green kernel on $\mathbb{R}^2$. Following [25], we say that the semi-group $(P_t^X)_t$, which is symmetric w.r.t. the measure M', is *irreducible* if any P_t^X-invariant set B satisfies $M'(B)=0$ or $M'(B^c)=0$. We say that (P_t^X) is recurrent if, for any $f\in L^1_+(D,M')$, we have $Gf(x)=0$ or $Gf(x)=+\infty$ M'-almost surely.

Theorem 4.28. (Liouville Green function at criticality). *The critical Liouville semi-group is irreducible and recurrent. The critical Liouville Green function, denoted by G^X, is given for every $x\in S$ by*

$$G^X f(x)=\sqrt{2/\pi}\int_{\mathbb{R}^2}\frac{1}{\pi}\ln\frac{1}{|x-y|}f(y)\,M'(dy)$$

for all functions $f\in L^1(\mathbb{R}^2,M')$ such that

$$\int_{\mathbb{R}^2}f(y)\,M'(dy)=0.$$

Proof. Irreducibility is a straightforward consequence of Theorem 4.25 and the remaining part of the statement is a straightforward adaptation of [27, 28] for $x\in S$. ∎

We investigate now the existence of probability densities of the critical Liouville semi-group with respect to the critical Liouville measure.

Theorem 4.29. (Critical Liouville heat kernel). *The critical Liouville semigroup $(P_{\mathbf{t}}^X)_{\mathbf{t}>0}$ is absolutely continuous with respect to the critical Liouville measure. There exists a family of nonnegative functions $(\mathbf{p}_t^X(\cdot,\cdot))_{t\geqslant 0}$, which we call the critical Liouville heat kernel, such that:*

$$\forall x\in\mathbb{R}^2, dt\ a.s., \forall f\in B_b(\mathbb{R}^2),\quad P_t^X f(x)=\sqrt{2/\pi}\int_{\mathbb{R}^2}f(y)\mathbf{p}_t^X(x,y)\,M'(dy).$$

Proof. From Theorems 4.25 and [25, Theorems 4.1.2 and 4.2.4], the Liouville semi-group is absolutely continuous with respect to the Liouville measure. ∎

Remark 4.30. *Though we call the family $(\mathbf{p}_t^X(\cdot,\cdot))_{t\geqslant 0}$ heat kernel, we are not in position to establish most of the regularity properties expected from a heat kernel because we do not know how to construct a measurable version. For instance, in the subcritical situation [28, 57], it can be given* *sense to a measurable version by considering it as a measure* $\mathbf{p}(t,x,y)\,dt$*: since the massive Green kernel is completely monotone it is the Laplace transform of some measure denoted by* $\mathbf{p}(t,x,y)\,dt$*. In that case, it is proved in [57] that this measure is continuous with respect to* x,y *in the sense of weak convergence of measures and this allows us to make sense of the heat kernel on the diagonal* $\mathbf{p}(t,x,x)\,dt$*. A weak form of the notion of spectral dimension is then obtained in [57], which is* 2*. We do not know how to adapt the argument in the critical case as we cannot prove the continuity of the mapping* $(x,y)\mapsto \mathbf{p}(t,x,y)\,dt$*.*

Let us consider the set E defined in Theorem 4.7. Recall that $M'(E^c)=0$. As a consequence of Theorem 4.25, we obtain the following result where λ is the Lebesgue measure:

Corollary 4.31. *Almost surely in* X*, for all starting points* $x\in\mathbb{R}^2$*, the critical LBM spends Lebesgue-almost all the time in the set* E*:*

$$\text{a.s. in } X, \forall x\in\mathbb{R}^2,\ \text{a.s. under } \mathbb{P}^{B^x},\quad \lambda\{t\geqslant 0;\ \mathcal{B}^x_t\in E^c\}=0.$$

If one applies Theorem 4.29 instead, one obtains the similar but different result:

Corollary 4.32. *Almost surely in* X*, for all* $t>0$

$$\mathbb{P}^{B^x}\ \text{a.s.},\quad \mathcal{B}^x_t\in E.$$

4.5 Remarks about associated Feynman path integrals

Remind that the standard Wiener measure gives a rigorous interpretation of the heuristic path integral on $\mathbb{R}^2$

$$\frac{1}{Z_0}\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{2}\int_0^T|\sigma'(s)|^2\,ds\Big)\mathcal{D}\sigma \tag{4.19}$$

where Z_0 appears as a normalization constant and $f: C([0,T],\mathbb{R}^2)\to\mathbb{R}$ is a bounded continuous function. The construction of the critical LBM allows us to make sense of several other Feynman path integrals, which we discuss below. The "Wiener measure" associated to the critical LBM has the following equivalent path integral interpretations:

$$\begin{aligned}
&\mathbb{E}^B\big[f\big((\mathcal{B}_t)_{0\leqslant t\leqslant T}\big)\big]\\
=&\frac{1}{Z_1}\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{2}\int_0^T -X(\sigma_s)e^{2X(\sigma_s)}|\sigma'_s|^2\,ds\Big)\mathcal{D}\sigma\\
=&\frac{1}{Z_1}\int_{C([0,T];\mathbb{R}^2)} f(\sigma)\exp\Big(-\frac{1}{\sqrt{2\pi}}\int_0^T -X(\sigma_s)e^{2X(\sigma_s)}|\sigma'_s|^2\,ds\Big)\mathcal{D}\sigma,
\end{aligned}$$

where Z_1, Z'_1 are normalization constants, valid for all bounded continuous function $f: C([0,T],\mathbb{R}^2)\to\mathbb{R}$.

5 Further remarks and GFF on other domains

So far, we constructed in detail the LBM on (subdomains of) $\mathbb{R}^2$. This construction may be adapted to other geometries like the sphere $\mathbb{S}^2$ or torus $\mathbb{T}^2$ (equipped with a standard Gaussian Free Field (GFF) with vanishing average for instance). Actually, our techniques can be adapted to other 2-dimensional Riemannian manifolds with a scalar metric tensor. The main reason is that a Riemannian manifold is locally isometric to $\mathbb{R}^2$. We will not detail the proofs since the whole machinery works essentially the same as in the plane: it suffices to have at our disposal a white noise decomposition of the underlying Gaussian distribution and to adapt properly our assumptions.

We rather give here further details in the case of the GFF on planar domains as the associated Brownian motion possesses important conformal invariance properties. We consider a bounded planar domain D. The Liouville Brownian motion on D is defined as follows:
• consider a white noise cut-off approximation $(X_\epsilon)_\epsilon$ of the GFF on D with Dirichlet boundary conditions: see equation (2.8).
• first define the time change as the limit

$$F'(x,t)=\lim_{\epsilon\to 0}\epsilon^2\int_0^t(2\mathbb{E}[X_\epsilon(x+B_u)^2]-X_\epsilon(x+B_u))e^{2X_\epsilon(x+B_u)}\,du$$

for all $t<\tau_x^D$ where B is a standard planar Brownian motion and τ_x^D its first exit time out of D. This limit turns out to be the same as

$$\int_0^t C(x+B_u,D)^2(2\mathbb{E}[X(x+B_u)^2]-X(x+B_u))e^{2X(x+B_u)-2\mathbb{E}[X(x+B_u)^2]}\,du$$

where $C(x,D)$ is the conformal radius at x in the domain D (see [45] or [22] in a different context).
• extend this construction to all possible starting points as in section 4. Define the exit time of this LBM out of D by $\hat{\tau}_x^D=\sqrt{2/\pi}F'(x,\tau_x^D)$ and then define the Liouville Brownian motion as in Definition 4.18 for all time $t<\hat{\tau}_x^D$.
• Observe that this Liouville Brownian motion is invariant under conformal reparametrization. This means that for all conformal map $\psi:D'\to D$ the process $\psi^{-1}(\mathcal{B}^x)$ has the law of the Liouville Brownian motion on D' where in the construction of F' we use the standard reparametrization rule of Liouville field theory $X\to X\circ\psi+Q\ln|\psi'|$ where $Q=\frac{2}{\gamma}+\frac{\gamma}{2}$ for a subdomain of $\mathbb{C}$ (or $Q=\frac{2}{\gamma}$ for a GFF on the sphere or the torus with vanishing mean).

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