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Complex Gaussian multiplicative chaos

Hubert Lacoïn ¹, Rémi Rhodes ¹², Vincent Vargas ¹²

Abstract

In this article, we study complex Gaussian multiplicative chaos. More precisely, we study the renormalization theory and the limit of the exponential of a complex log-correlated Gaussian field in all dimensions (including Gaussian Free Fields in dimension 2). Our main working assumption is that the real part and the imaginary part are independent. We also discuss applications in $2D$ string theory; in particular we give a rigorous mathematical definition of the so-called Tachyon fields, the conformally invariant operators in critical Liouville Quantum Gravity with a $c = 1$ central charge, and derive the original KPZ formula for these fields.

Key words or phrases: Random measures, complex Gaussian multiplicative chaos, tachyon fields, multifractal.

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1 Introduction

In dimension d , a real Gaussian multiplicative chaos is a random measure on a given domain D of \mathbb{R}^d that can be formally written, for any Borel set $A \subset D$ as:

$$M^\gamma(A) = \int_A e^{\gamma X(x) - \frac{\gamma^2}{2} \mathbb{E}[X^2(x)]} dx, \quad (1.1)$$

where dx stands for the Lebesgue measure on D (or more generally any Radon measure instead of dx : see [49] for a recent review on Gaussian multiplicative chaos) and X is a centered Gaussian distribution possessing a covariance kernel of the form:

$$\mathbb{E}[X(x)X(y)] = \ln_+ \frac{1}{|x - y|} + g(x, y), \quad (1.2)$$

with $\ln_+(u) = \max(\ln u, 0)$ and g a continuous function over $D \times D$. The covariance kernel thus possesses a singularity along the diagonal and it is clear that giving sense to (1.1) is not straightforward (how do you define the exponential of a distribution?). The standard approach consists in applying a “cut-off” to the distribution X , that is in regularizing the field X in order to get rid of the singularity of the covariance kernel and get a nicer field. The regularization usually depends on a small parameter, call it ε , that stands for the extent to which the field has been regularized. The measure (1.1) is naturally understood as the limit of the random measures:

$$M_\varepsilon^\gamma(A) = \int_A e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2} \mathbb{E}[X_\varepsilon^2(x)]} dx \quad (1.3)$$

when the regularization parameter ε goes to 0. It is well known [35, 3, 49] that this procedure produces non trivial limiting objects when the real parameter γ is strictly less than some critical value $\gamma_c = \sqrt{2d}$.

The critical case, i.e. $\gamma = \gamma_c$, has been investigated in [22, 23]. An extra renormalization term is necessary to obtain a random measure M' . This limiting measure is also called the *derivative Gaussian multiplicative chaos* because it can also be obtained by differentiating (1.1) with respect to the parameter γ :

$$M'(A) = \lim_{\varepsilon \rightarrow 0} \int_A (\gamma_c \mathbb{E}[X_\varepsilon^2(x)] - X_\varepsilon(x)) e^{\gamma_c X_\varepsilon(x) - \frac{\gamma_c^2}{2} \mathbb{E}[X_\varepsilon^2(x)]} dx. \quad (1.4)$$

This object has recently received much attention because of its fundamental role in analyzing the behaviour of the maximum of log-correlated Gaussian fields. The reader is referred to the papers [10, 9, 43] for substantial recent advances on this topic. The super-critical case $\gamma > \gamma_c$ is still open but some conjectures are stated in [6, 22, 49], based on recent results [8, 42, 56] obtained in simpler but related models (see Section 5).

Standard theory of Gaussian multiplicative chaos has found many applications in finance, Liouville Quantum Gravity or turbulence (see [49] and references therein). Yet, the need of understanding the renormalization theory of Gaussian multiplicative chaos with a complex value of the parameter γ has emerged. This is for instance the case in $2D$ -string theory and more precisely when looking at conformal matter fields coupled to gravity (see below).

In this paper, we consider two independent identically distributed centered Gaussian distributions X and Y , each of which with covariance kernel of the type (1.2). By considering their respective regularizations $(X_\varepsilon)_\varepsilon$ and $(Y_\varepsilon)_\varepsilon$, the problem addressed here is to find a proper renormalization as well as the limit of the family of complex random measures :

$$M_\varepsilon^{\gamma, \beta}(A) = \int_A e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx \quad (1.5)$$

where γ, β are real constants. Notice that we may restrict to the case when γ, β are nonnegative by symmetry of the Gaussian law. We will see that the renormalization theory of these measures presents three phases, summarized in Figure 1, depending on the considered values of γ and β .

1.1 Previous related works

We first mention the recent work [34] where the authors conduct a thorough study of all phases (inner and frontier) in the simpler context of the Random Energy Model (REM) partition function. They give the precise asymptotics of all phases. Note that in this context, phase I (inner and frontier) is trivial at order 1 as the (mean) renormalized partition function converges to a non

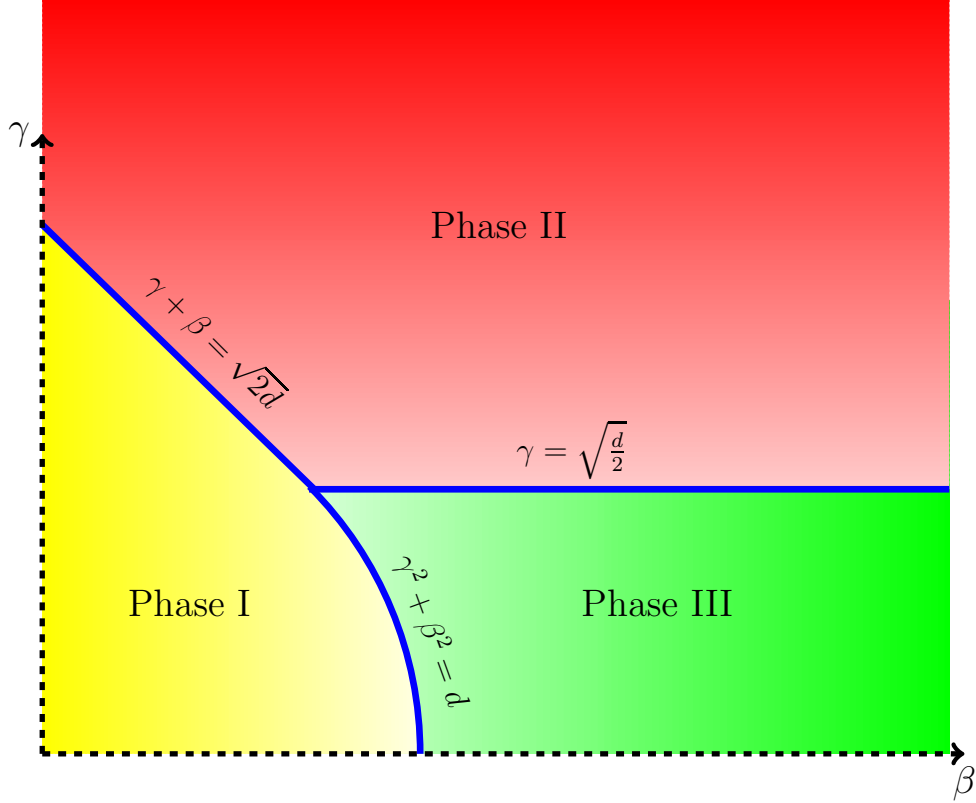


Figure 1: Phase diagram

vanishing constant: this is due to the lack of correlations in the model. Hence, in [34], the authors go one step further as they give the fluctuations.

As is now well known, correlations may be added for instance on a tree structure like Mandelbrot multiplicative cascades. In this context, this problem is investigated in [15, 4, 5]. In the pioneering work [15], the authors computed the free energy and deduced a phase diagram similar to our Figure 1. In [4], the authors treat the case of dyadic multiplicative cascades. Since this model is 1-dimensional, we may see the complex random measure $M_\varepsilon^{\gamma,\beta}(A)$ as a random function $t \mapsto M_\varepsilon^{\gamma,\beta}([0, t])$. Translated in our context, the results in [4] are the following:

- Phase I: the authors prove that there is almost sure convergence in the space of continuous functions. In fact, in the companion paper [5], the authors show almost sure convergence in the space of continuous functions for a class of models that includes the one we consider here (except for free fields).
- Phase II, frontier I/II, frontier II/III and triple point: not investigated.
- Phase III and frontier I/III: the authors show that the sequence is tight when properly renormalized. Convergence is not investigated.

We further stress that the authors in [4, 5] do not prove the convergence in law in phase III but claim that if convergence holds then every possible limit is a Brownian motion in multifractal time. The argument is based on the uniqueness property of the solution of some fixed point equation, the star equation for multiplicative cascades. The corresponding equation for Gaussian multiplicative chaos has been introduced in [1] but uniqueness has only been established in the

real subcritical case so far. So the same argument cannot be used in the context of Gaussian multiplicative chaos. On the other hand, we also point out that the approach developed in [4, 5] is general enough in the inner phase I to treat situations where the real part and imaginary part are not necessarily independent.

1.2 Content of the paper

The purpose of this manuscript is not only to investigate the phase diagram in the context of Gaussian multiplicative chaos but also to describe the limiting object that we obtain when renormalizing properly the family $(M_\varepsilon^{\gamma,\beta})_\varepsilon$ as $\varepsilon \rightarrow 0$. We will first show that the model exhibits three phases, which are represented in Figure 1. We will also describe the limiting objects after renormalization. Apart from the inner phase I and the frontier of phases I/II where the renormalization procedure produces new objects, we prove that the renormalization procedure leads to objects that can be described in terms of the limiting measures that we get on the real line $\beta = 0$. Roughly speaking, we get a complex Gaussian random measure with a random intensity: the real part X_ε governs the description of the intensity whereas all the information about the field Y is lost into a white noise (a similar phenomenon is observed in [18] for a different model). Figure 2 is a brief yet complete description of the picture we draw. Actually, we do not treat the triple point and the description we give in the inner phase II is a conjecture: our method to explain how the complex Gaussian random measure appears indicates that the inner phase II can be described by a complex Gaussian random measure with random intensity given by the objects we get in the real supercritical case ($\gamma > \sqrt{2d}$), which remains conjectural so far. We will also detail applications in Conformal Field Theories (CFT) and 2D-Liouville Quantum Gravity (LQG), which are summarized below.

1.3 Applications in 2D-string theory

Polyakov [47] showed that 2D string theory could be interpreted as a theory of two dimensional quantum gravity, where the string coordinate is considered as a c -dimensional matter field defined on some two dimensional worldsheet Σ equipped with a metric g . The coupling between the matter field and the metric is governed by the Polyakov action, which factorizes as a tensor product of the classical Liouville action and that of a Gaussian Free Field Y . The metric on Σ is thus a random variable, which roughly takes on the form [47, 40, 12] (we consider an Euclidean background metric for simplicity):

$$g(z) = e^{bX(z)} dz^2, \quad (1.6)$$

where b is a coupling constant expressed in terms of the central charge c of the matter field and X is a random field, the fluctuations of which are governed by the Liouville action. In critical 2D-Liouville Quantum Gravity, this action turns the field X into a Free Field, with appropriate mean and boundary conditions. Two-dimensional string theory corresponds to the case $c = 1, b = 2$. The Liouville Quantum Gravity with $c = 1$ is the conjectured scaling limit of critical statistical physical models having a $c = 1$ central charge (like the $O(n = 2)$ loop model or the $Q = 4$ -states Potts model) defined on random lattices. We do not review here the huge amount of works on this topic and we refer the reader to [13, 12, 25, 16, 17, 28, 30, 31, 38, 40, 45, 47] for further insights.

In critical 2D-Liouville Quantum Gravity, the so-called **tachyon fields** T are the operators which are conformally invariant within the theory (see the excellent reviews [38, 45]). In this paper, we will mathematically construct the tachyon fields for a $c = 1$ central charge

$$e^{\gamma X(x) + i\beta Y(x)} dx$$

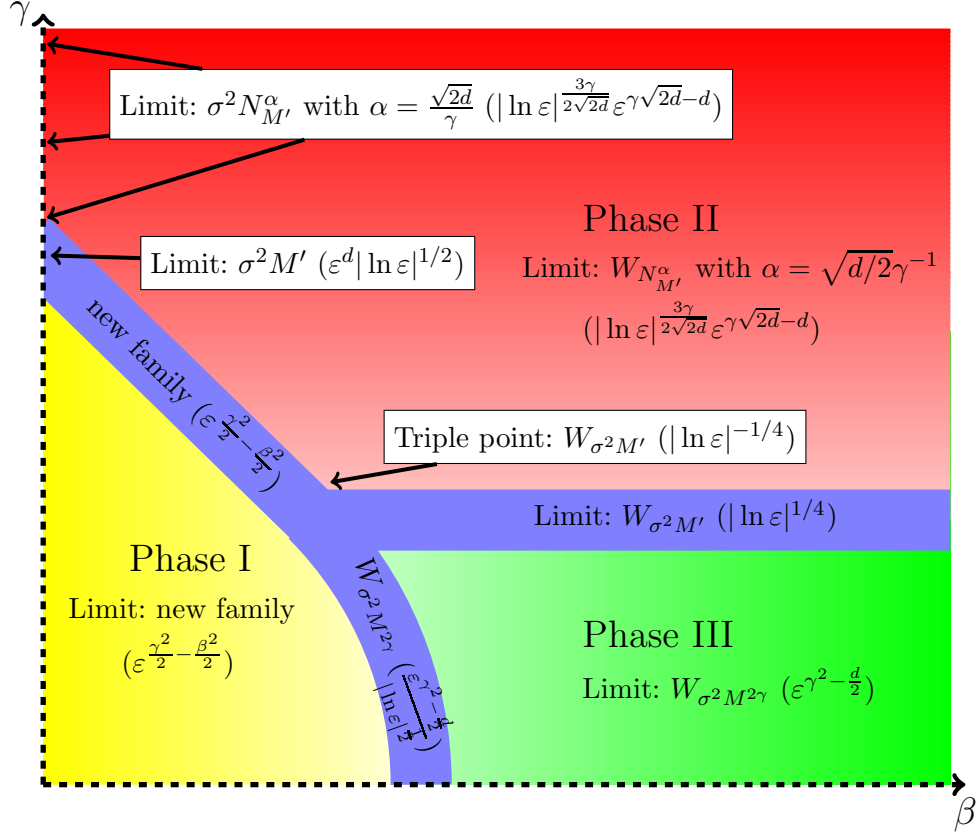


Figure 2: Limiting measure diagram. We indicate between brackets how to renormalize the field $M_\varepsilon^{\gamma,\beta}$ and the limiting field. For instance in phase III, "Limit $W_{\sigma^2 M^{2\gamma}}$ ($\varepsilon^{\gamma^2 - \frac{d}{2}}$)" means that the field $\varepsilon^{\gamma^2 - \frac{d}{2}} M_\varepsilon^{\gamma,\beta}$ converges as $\varepsilon \rightarrow 0$ towards $W_{\sigma^2 M^{2\gamma}}$. Now we explain the description of the limiting law: conditionally on μ , W_μ stands for a complex Gaussian random measure with intensity μ . Conditionally on μ , N_μ^α is a α -stable Poisson random measure with intensity μ . M' is the derivative martingale and $M^{2\gamma}$ is a standard Gaussian multiplicative chaos with intermittency parameter 2γ . The constant σ^2 depends on γ and β .

for $\gamma \pm \beta = 2$ and $\gamma \in]1, 2[$, where following the above discussion X and Y are two independent Gaussian Free Fields. We will further argue that the Wick ordering (see section 7 for a discussion about this notion) of the above field does not produce tachyons for $\gamma \pm \beta = 2$ and $\gamma \leq 1$. The main reason is that, below the threshold $\gamma = 1$, a nonstandard renormalization procedure is necessary (we enter phase III on Figure 2), and this deeply modifies the conformal dimension of these fields.

Finally, we will also derive the corresponding KPZ formula (see [40])

$$\Delta_{i\beta}^0 = \Delta_{i\beta}^q + \frac{b^2}{4} \Delta_{i\beta}^q (\Delta_{i\beta}^q - 1),$$

which is a relation between the conformal dimension $\Delta_{i\beta}^0$ of the spinless vertex operator $e^{i\beta Y}$ and the quantum dimension $\Delta_{i\beta}^q$ (or gravitational dimension) of this operator gravitationally dressed. The reader is referred to section 7 for further details.

1.4 Chodos-Thorn/Feigin-Fuks Theory

In subsection 7.4, we will also discuss some connections between our work and the vertex operators of the so-called Chodos-Thorn/Feigin-Fuks Theory (CTFF), which is a Gaussian free field conformal theory where a background charge is inserted in order to lower the central charge below the $c = 1$ value (see [11, 19, 20, 30, 36]). Connections (and questions) with the imaginary geometry developed in [44, 54] are also mentioned.

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2 Setup

Let us introduce a canonical family of log-correlated Gaussian distributions, called star scale invariant, and their cut-off approximations, which we will work with in the first part of this paper. Of course, other natural choices are possible and they are discussed in subsection 2.1 below.

Let us consider a continuous covariance kernel k on \mathbb{R}^d such that:

Assumption (A). *The kernel k satisfies the following assumptions, for some constant C_k independent of $x \in \mathbb{R}^d$:*

A1. k is normalized by the condition $k(0) = 1$,

A2. $k(x) \leq C_k(1 + |x|)^{-\nu}$ for some $\nu > d$.

A3. $|k(x) - k(0)| \leq C_k|x|$ for all $x \in \mathbb{R}^d$.

We set for $\varepsilon \in]0, 1]$ and $x \in \mathbb{R}^d$

$$K_\varepsilon(x) = \int_1^{\frac{1}{\varepsilon}} \frac{k(xu)}{u} du$$

and

$$G_\varepsilon(x) = e^{-K_\varepsilon(x)}.$$

We consider two independent families of centered Gaussian processes $(X_\varepsilon(x))_{x \in \mathbb{R}^d, \varepsilon \in]0, 1]}$ and $(Y_\varepsilon(x))_{x \in \mathbb{R}^d, \varepsilon \in]0, 1]}$ with covariance kernel given by:

$$\forall \varepsilon, \varepsilon' \in]0, 1], \quad \mathbb{E}[X_\varepsilon(x)X_{\varepsilon'}(y)] = \mathbb{E}[Y_\varepsilon(x)Y_{\varepsilon'}(y)] = K_{\varepsilon \vee \varepsilon'}(y - x),$$

where $\varepsilon \vee \varepsilon' := \sup(\varepsilon, \varepsilon')$. The construction of such fields is possible via a white noise decomposition as explained in [1]. We set:

$$\begin{aligned} \mathcal{F}_\varepsilon^X &= \sigma\{X_u(x); x \in \mathbb{R}^d, u \geq \varepsilon\} & \text{and} & & \mathcal{F}_\varepsilon^Y &= \sigma\{Y_u(x); x \in \mathbb{R}^d, u \geq \varepsilon\} \\ \mathcal{F}^X &= \sigma\{X_u(x); x \in \mathbb{R}^d, u \in]0, 1]\} & \text{and} & & \mathcal{F}^Y &= \sigma\{Y_u(x); x \in \mathbb{R}^d, u \in]0, 1]\} \end{aligned}$$

and

$$\mathcal{F}_\varepsilon = \sigma\{X_u(x), Y_u(x); x \in \mathbb{R}^d, u \geq \varepsilon\}.$$

We stress that, for $u < \varepsilon$, the field $(X_u(x) - X_\varepsilon(x))_{x \in \mathbb{R}^d}$ is independent from \mathcal{F}_ε . Then we consider the following locally finite complex measure:

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \quad M_\varepsilon^{\gamma, \beta}(A) = \int_A e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx. \quad (2.1)$$

Let us finally notice that K can be approximated as follows

$$\forall x \in B(0, R), \quad |K_\varepsilon(x) - |\ln(|x| \vee \varepsilon)|| \leq C_R. \quad (2.2)$$

2.1 Examples

Let us also mention here some important Gaussian fields covered by our methods:

1. Exact scale invariant kernels like the one studied in [2, 50]). At first sight, these kernels do not satisfy Assumption (A) as they cannot be written as $\int_1^\infty \frac{k(ux)}{u} du$. Yet, they can be written as $\int_1^\infty \frac{k(ux)}{u} du + H(x)$ for some continuous translation invariant covariance kernel H . And our proof are easy to adapt if one adds a smooth field to X_ε and Y_ε .
2. Massive Gaussian Free Field (MFF for short) on \mathbb{R}^2 . The whole plane MFF is a centered Gaussian distribution with covariance kernel given by the Green function of the operator $2\pi(m^2 - \Delta)^{-1}$ on \mathbb{R}^2 , i.e. by:

$$\forall x, y \in \mathbb{R}^2, \quad G_m(x, y) = \int_0^\infty e^{-\frac{m^2}{2}u - \frac{|x-y|^2}{2u}} \frac{du}{2u}. \quad (2.3)$$

The real $m > 0$ is called the mass. This kernel is of σ -positive type in the sense of Kahane [35] since we integrate a continuous function of positive type with respect to a positive measure. It is furthermore a star-scale invariant kernel (see [1, 48]): it can be rewritten as

$$G_m(x, y) = \int_1^{+\infty} \frac{k_m(u(x-y))}{u} du. \quad (2.4)$$

for some continuous covariance kernel $k_m = \frac{1}{2} \int_0^\infty e^{-\frac{m^2}{2v}|z|^2 - \frac{v}{2}} dv$.

3. Gaussian Free Fields in a compact domain. In dimension 2, an important family of Gaussian distributions is the family of Gaussian free fields (see [31, 53] for instance). They do not satisfy Assumption (A) (in particular they are not translation invariant) and substantial modifications are needed to adapt the proofs to this case. Because of the importance of applications in this context, we treat specifically these fields in section 6. Applications are given in section 7.
4. Log-correlated Gaussian Fields (LGF) with covariance kernel given by $(m^2 - \Delta)^{-d/2}$ in any dimension d for $m \geq 0$ (see [24, 41] for instance). Furthermore, in the case of the whole plane and $m > 0$, the Green function of the operator $(m^2 - \Delta)^{-d/2}$ is a star scale invariant kernel.

Remark 2.1. *The reader may skip this remark upon the first reading. One may also wish to extend our methods to kernels possibly depending on the scale and non stationary, i.e. of the type*

$$K(x, y) = \int_1^\infty \frac{k(ux, uy, u)}{u} du$$

where $((x, y) \mapsto k(x, y, u))_{u \geq 1}$ is a family of (non necessarily stationary) covariance kernels. By modifying properly Assumption (A) to fit to this case, one can see that our methods apply provided that one takes care of the following subtlety. In the case of non-stationary kernels: the Wick ordering of the field $e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx$ yields a different limit from the martingale renormalization. Let us illustrate this with Theorem 3 for instance. This theorem may be directly applied to the martingale

$$\int_A e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x) - \left(\frac{\gamma^2}{2} - \frac{\beta^2}{2}\right) \mathbb{E}[X_\varepsilon(x)^2]} dx,$$

which converges towards a limit $M^{\gamma, \beta}$. Notice the renormalization by the variance instead of the appropriate power of ε . Yet, one may be instead interested in the Wick ordering of the field $e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx$, i.e. in the limit of the field

$$\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_A e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx.$$

By evaluating the difference between $\ln \frac{1}{\varepsilon}$ and $\mathbb{E}[X_\varepsilon(x)^2]$, formally, we have

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_A e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx = \int_A e^{-\left(\frac{\gamma^2}{2} - \frac{\beta^2}{2}\right) \int_1^\infty \frac{1 - k(ux, ux, u)}{u} du} M^{\gamma, \beta}(dx).$$

In the important case of Gaussian Free Fields, this exponential term in the Wick ordering limit makes the conformal radius appear. Details are given in the case of the GFF in Section 6. We let the reader adapt the argument to the other fields he might be interested in.

2.2 Notations

We will further denote by $C(E, F)$ the space of continuous functions from E to F . The notation $f(x) \simeq_{x \rightarrow x_0} g(x)$ means that

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1.$$

$C^k(\mathbb{R}^d)$ (resp. $C_c^k(\mathbb{R}^d)$) denotes the space of functions defined on \mathbb{R}^d that are k times continuously differentiable (resp. k times continuously differentiable with a compact support) on D equipped with the topology of uniform convergence on D for the derivatives up to order k . The random variables in this paper are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we denote by \mathbb{E} the corresponding expectation. The space of random variables with integrable p -th power is denoted \mathbb{L}_p . The space of measurable functions defined on a Borel set D with integrable p -th power will be denoted $L^p(D)$.

When we make use of one of the two following inequalities

$$\left(\sum_{i \in I} a_i \right)^\theta \geq \sum_{i \in I} a_i^\theta \text{ when } \theta > 1, \quad \left(\sum_{i \in I} a_i \right)^\theta \leq \sum_{i \in I} a_i^\theta \text{ when } \theta \leq 1 \quad (2.5)$$

which are valid for any collection of positive numbers $(a_i)_{i \in I}$ we will simply say, by *superadditivity* or by *subadditivity*.

2.3 A toolbox of useful results

Let us introduce here some useful result that we will massively use in the proofs. The following convexity inequality is proved in [35] by interpolation and Gaussian integration by parts, we present also a special consequence of it.

Proposition 2.2 (Kahane's convexity inequality). *Let Z_1 and Z_2 be two centered Gaussian fields on \mathbb{R}^d (or on any metric space) with covariance kernels $K_1(x, y)$ and $K_2(x, y)$ respectively.*

If $\forall x, y \in \mathbb{R}^d$, $K_1(x, y) \leq K_2(x, y)$ then, for all real convex function F and all positive measure σ on \mathbb{R}^d , we have

$$\mathbb{E} \left[F \left(\int_{\mathbb{R}^d} e^{Z_1(x) - \mathbb{E}[Z_1^2(x)]/2} \sigma(dx) \right) \right] \leq \mathbb{E} \left[F \left(\int_{\mathbb{R}^d} e^{Z_2(x) - \mathbb{E}[Z_2^2(x)]/2} \sigma(dx) \right) \right]. \quad (2.6)$$

As a consequence if $\forall x, y \in \mathbb{R}^d$, $K_1(x, y) \leq K_2(x, y) + \alpha$ for some $\alpha \in \mathbb{R}$ then we have for $p > 0$

$$\begin{aligned} \mathbb{E} \left[\left(\int_{\mathbb{R}^d} e^{Z_1(x) - \mathbb{E}[Z_1^2(x)]/2} \sigma(dx) \right)^p \right] &\leq e^{\frac{1}{2}\alpha p(p-1)} \mathbb{E} \left[\left(\int_{\mathbb{R}^d} e^{Z_2(x) - \mathbb{E}[Z_2^2(x)]/2} \sigma(dx) \right)^p \right] && \text{if } p > 1 \\ \mathbb{E} \left[\left(\int_{\mathbb{R}^d} e^{Z_1(x) - \mathbb{E}[Z_1^2(x)]/2} \sigma(dx) \right)^p \right] &\geq e^{\frac{1}{2}\alpha p(p-1)} \mathbb{E} \left[\left(\int_{\mathbb{R}^d} e^{Z_2(x) - \mathbb{E}[Z_2^2(x)]/2} \sigma(dx) \right)^p \right] && \text{if } p < 1. \end{aligned} \quad (2.7)$$

Proof of (2.7). We consider the case $\alpha \geq 0$. In that case we apply (2.6) to the fields Z_1 and $Z_2 + \Omega$ where Ω a Gaussian of variance α which is independent of Z_2 (the kernel of $Z_2 + \Omega$ is $K_2 + \alpha$). Then the inequalities (2.7) are obtained by integrating over the variable Z . When $\alpha < 0$ we consider $Z_1 + \Omega$ and Z_2 instead. \square

The above proposition allows us to compare moments of order p for two different log-normal multiplicative chaos integrated on a measure σ . We will sometimes use it to make comparisons with a chaos which present a nice property of stochastic scale invariance and is constructed in [50, Proposition 2.9]

Proposition 2.3. *For every dimension d and $T > 0$, one can construct a sequence of Gaussian fields $\{(X_\varepsilon(x))_{x \in \mathbb{R}^d}, \varepsilon \geq 0\}$ whose covariance structures are given by*

$$\mathbb{E}[X_\varepsilon(x)X_\varepsilon(y)] = \int_{m \in O(d)} g_\varepsilon(m(x-y)) \sigma_d(dm), \quad (2.8)$$

where $O(d)$ is the orthogonal group on \mathbb{R}^d , σ_d is the Haar measure on it and

$$g_\varepsilon(x) := \begin{cases} \ln(T/\varepsilon) + 1 - \frac{|x_1|}{\varepsilon} & \text{when } |x_1| \leq \varepsilon, \\ \ln_+(T/|x_1|) & \text{when } |x_1| \geq \varepsilon. \end{cases} \quad (2.9)$$

where x_1 is the first real-coordinate of x in \mathbb{R}^d .

For any fixed $\lambda \in (0, 1)$,

$$(X_{\lambda\varepsilon}(\lambda x))_{x \in B(0, T/2)} = \Omega_\lambda + (X_\varepsilon(x))_{x \in B(0, T/2)}, \quad (2.10)$$

where Ω_λ is a Gaussian variable of variance $|\ln \lambda|$ which is independent of X_ε . Finally given $T \geq 0$, $R > 0$ there exists a constant C such that for all $z \in B(0, R)$

$$|\ln(|z| \vee \varepsilon)| - C \leq \mathbb{E}[X_\varepsilon(x)X_\varepsilon(x+z)] \leq |\ln(|z| \vee \varepsilon)| + C. \quad (2.11)$$

3 Study of phase I and its I/II boundary

3.1 Study of the inner phase

We study the inner phase I, namely

$$\mathcal{P}_I := \left\{ \gamma + \beta < \sqrt{2d}, \gamma \in \left[\sqrt{\frac{d}{2}}, \sqrt{2d} \right] \right\} \cup \{ \gamma^2 + \beta^2 < d \}. \quad (3.1)$$

Throughout the paper, we will use the terminology "inner phase I" to denote the couples of parameters (γ, β) satisfying (3.1) in order to avoid heavy notations.

We are interested in the martingale $(\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}(dx))_\varepsilon$. The reader can check that (3.1) is equivalent to the existence of some $p \in]1, 2[$ such that $\zeta(p) > d$ where:

$$\zeta(p) = \left(d + \frac{\gamma^2}{2} - \frac{\beta^2}{2} \right) p - \frac{\gamma^2}{2} p^2.$$

As a warm-up, the reader may check that if one considers $p \geq 2$ such that $\zeta(p) > d$ then the martingale is bounded in \mathbb{L}_2 . This corresponds to the parameters (γ, β) such that $\gamma^2 + \beta^2 < d$. This \mathbb{L}_2 phase is rather straightforward to study. Also, if one introduces

$$p_c(\gamma, \beta) := \sup\{p > 1; \zeta(p) > d\}, \quad (3.2)$$

one gets that $p_c \in]\frac{\sqrt{2d}}{\gamma}, \frac{2d}{\gamma^2}]$. We have the following behaviour inside phase I:

Theorem 3.1. (Convergence) *Let (β, γ) belongs to inner phase I. Consider $p \in]1, 2[$ such that $\zeta(p) > d$.*

1. *For all compactly supported bounded measurable function f , the martingale*

$$\left(\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_{\mathbb{R}^d} f(x) M_\varepsilon^{\gamma, \beta}(dx) \right)_\varepsilon$$

is uniformly bounded in \mathbb{L}_p . Furthermore, for all $R > 0$, there exists a constant $C_{p,R}$ (only depending on p, R) such that for all bounded measurable function f with compact support in $B(0, R)$:

$$\mathbb{E} \left[\sup_{\varepsilon \in]0, 1]} \left| \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_{\mathbb{R}^d} f(x) M_\varepsilon^{\gamma, \beta}(dx) \right|^p \right] \leq C_{p,R} \|f\|_\infty^p.$$

2. *The $\mathcal{D}'(\mathbb{R}^d)$ -valued martingale:*

$$\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta} : \varphi \rightarrow \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_{\mathbb{R}^d} \varphi(x) e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx$$

converges almost surely in the space $\mathcal{D}'_d(\mathbb{R}^d)$ of distributions of order d towards a non trivial limit $M^{\gamma, \beta}$. More precisely, for each $R > 0$, there exists a random variable $Z^R \in \mathbb{L}_p$ such that for all functions $\varphi \in C_c^d(B(0, R))$:

$$|M^{\gamma, \beta}(\varphi)| \leq Z^R \sup_{x \in B(0, R)} \left| \frac{\partial^d \varphi(x)}{\partial x_1 \cdots \partial x_d} \right|.$$

3. *In dimension 1, we have convergence of $(\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}[0, t])_{t \in [0, T]}$ in the space of continuous functions.*

Remark 3.2. For the reader who wishes to skip the proofs, we stress here that item 1. of Theorem 3.1 is proved in dimension 1 in [5, Prop. 3.1] in greater generality. Actually, their argument is quite elegant and flexible: it may be extended to treat situations like

$$\int_{\mathbb{R}^2} f(x) e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} \sigma(dx)$$

for general possibly correlated X_ε and Y_ε and σ a Radon measure. In our context, their main assumption is that the kernel k introduced in section 2 has compact support. Their proof is written in dimension 1 but clearly adapts to higher dimensions. However, the proofs of their paper can not be adapted to the case of long range correlated fields like Gaussian Free Fields (see section 6). Furthermore their tightness criterion clearly works in dimension 1 but extension to higher dimension does not make sense. Here, we suggest to study tightness in the space of distributions of order d as this will turn out to be important in view of the applications in Euclidean Field Theory. This step is carried out via a sharp analysis of capacity of the involved measures (see Lemma 3.13). Other choices of spaces for tightness may be investigated as well.

Remark 3.3. Important enough, we point out that the strategy developed in [5] is not robust enough to treat the frontier of phases 1 and 2. This point will be developed in subsection 3.2 via a lemma that guarantees at least the existence of some $p > 1$ (maybe not sharp) such that the martingale is bounded in \mathbb{L}_p .

Remark 3.4. Item 1. of the above theorem describes a sufficient condition on $p \in]1, 2[$ in order for the martingale to be uniformly bounded in \mathbb{L}_p . We do not know if this condition is sharp as in the real case $\beta = 0$. We prove a weaker statement in Proposition 3.5 by proving that the condition $\xi(p) \geq d$ is necessary when $p \geq 2$.

The fact that for any f

$$\left(\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_{\mathbb{R}^d} f(x) M_\varepsilon^{\gamma, \beta}(dx) \right)_{\varepsilon \in [0, 1]}$$

is a martingale for decreasing ε simply follows from our construction of the X_ε , which are sums of independent infinitesimal fields, and the choice of the renormalization which guarantees that the mean is constant. The martingale property is not required to prove convergence in \mathbb{L}^p (see for instance the circle average construction of the GFF exponential in Section 6) but it allows to have shorter and perhaps more elegant proofs. An important step in the proof of the Theorem is the uniform control of the capacity of the measure $M_\varepsilon^{\gamma, 0}$ which we prove only later in Lemma 3.10.

Proof of Theorem 3.1.

Item 1. We do not follow the proof of [5, Prop. 3.1] as we want to give a proof that is also valid for fields with long range correlations. Let us consider a bounded measurable function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ with support included in $B(0, 1)$ and $p \in]1, 2[$ such that $\zeta(p) > d$. We have by Jensen's inequality:

$$\begin{aligned} \mathbb{E} \left[\left| \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_{\mathbb{R}^d} \varphi(x) M_\varepsilon^{\gamma, \beta}(dx) \right|^p \right] &\leq \mathbb{E} \left[\mathbb{E} \left[\varepsilon^{\gamma^2 - \beta^2} \int_{B(0, 1)^2} \varphi(x) \varphi(y) M_\varepsilon^{\gamma, \beta}(dx) M_\varepsilon^{\gamma, \beta}(dy) \middle| \mathcal{F}^X \right]^{p/2} \right] \\ &\leq \|\varphi\|_\infty^p \mathbb{E} \left[\mathbb{E} \left[\left(\varepsilon^{\gamma^2} \int_{B(0, 1)^2} \frac{1}{|x - y|^{\beta^2}} M_\varepsilon^{\gamma, 0}(dx) M_\varepsilon^{\gamma, 0}(dy) \right)^{p/2} \right] \right]. \end{aligned}$$

We can then conclude with Lemma 3.10.

Item 2. We consider the mapping

$$x = (x_1, \dots, x_d) \in \mathbb{R}^d \mapsto F_\varepsilon^{\gamma, \beta}(x_1, \dots, x_d) := \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_{[0, x_1] \times \dots \times [0, x_d]} \varphi(x) e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx.$$

Let φ be a smooth test function with support in $]0, 1[^d$. By integration by parts, we get:

$$\begin{aligned} & \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_{]0, 1[^d} \varphi(x) e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx \\ &= (-1)^d \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_{]0, 1[^d} \frac{\partial^d \varphi(x)}{\partial x_1 \dots \partial x_d} \left(\int_{[0, x_1] \times \dots \times [0, x_d]} e^{\gamma X_\varepsilon(u) + i\beta Y_\varepsilon(u)} du_1 \dots du_d \right) dx, \end{aligned}$$

where $u = (u_1, \dots, u_d)$. Therefore, we conclude that:

$$|M_\varepsilon^{\gamma, \beta}(\varphi)| \leq \sup_{x \in]0, 1[^d} \left| \frac{\partial^d \varphi(x)}{\partial x_1 \dots \partial x_d} \right| Z_\varepsilon,$$

where:

$$Z_\varepsilon = \int_{]0, 1[^d} |F_\varepsilon^{\gamma, \beta}(x_1, \dots, x_d)| dx.$$

Observe that $(Z_\varepsilon)_\varepsilon$ is a positive submartingale. Furthermore, from Item 1, we deduce that

$$\mathbb{E}[|Z_\varepsilon|^p] \leq \int_{]0, 1[^d} \mathbb{E}[|F_\varepsilon^{\gamma, \beta}(x_1, \dots, x_d)|^p] dx \leq C_{p, 1}.$$

Item 3. One applies [5, Prop 3.2]. □

Proposition 3.5. (Necessary conditions for \mathbb{L}_p convergence for $p \geq 2$) *If the martingale*

$$\left(\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}([0, 1]^d) \right)_\varepsilon$$

is bounded in \mathbb{L}_p for some $p \geq 2$ then $\xi(p) \geq d$.

Proof. We consider $p \geq 2$ such that $\mathbb{E}[|M_\varepsilon^{\gamma, \beta}([0, 1]^d)|^p] < \infty$. We have the following inequalities:

$$\begin{aligned} \mathbb{E}[|\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}([0, 1]^d)|^p] &= \mathbb{E}[\mathbb{E}[|\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}([0, 1]^d)|^2]^{p/2} | \mathcal{F}^X]] \\ &\geq \mathbb{E}[\mathbb{E}[|\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}([0, 1]^d)|^2] | \mathcal{F}^X]]^{p/2} \\ &= \mathbb{E}\left[\left(\int_{[0, 1]^d \times [0, 1]^d} \varepsilon^{\gamma^2} \frac{M_\varepsilon^{\gamma, 0}(dx) M_\varepsilon^{\gamma, 0}(dy)}{G_\varepsilon(x - y)^{\beta^2}} \right)^{p/2} \right] \\ &\geq n^d \mathbb{E}\left[\left(\int_{[0, 1/n]^d \times [0, 1/n]^d} \varepsilon^{\gamma^2} \frac{M_\varepsilon^{\gamma, 0}(dx) M_\varepsilon^{\gamma, 0}(dy)}{G_\varepsilon(x - y)^{\beta^2}} \right)^{p/2} \right] \end{aligned}$$

where the first inequality is Jensen's inequality for the conditional expectation and in the last one we have used super-additivity and stationarity. Now from Kahane's inequality (2.7), we can, at the cost of a multiplicative constant, replace X in the last line by the scale invariant field given by Proposition 2.3. This gives

$$\mathbb{E}[|M_\varepsilon^{\gamma, \beta}[0, 1]^p|] \geq C n^{d - \zeta(p)} \mathbb{E}\left[\left(\int_{[0, 1]^d \times [0, 1]^d} \frac{M_\varepsilon^{\gamma, 0}(dx) M_\varepsilon^{\gamma, 0}(dy)}{|y - x|^{\beta^2}} \right)^{p/2} \right],$$

for some fixed constant $C > 0$. Hence the desired result by letting $n \rightarrow \infty$. □

Theorem 3.6. (Multifractal spectrum) *We consider $p \in]1, 2]$ such that $\zeta(p) > d$. Then for all $q \in [0, p]$ and some constant $C_q > 0$:*

$$\mathbb{E}[|M^{\gamma, \beta}(B(0, r))|^q] \underset{r \rightarrow 0}{\simeq} C_q r^{\zeta(q)}.$$

Remark 3.7. *We stress here that the above result is standard in the real case $\beta = 0$. Yet, in the general case $\beta \neq 0$, the proof is far from straightforward. The difficulty here is that the fluctuations of the process Y in the term $e^{i\beta Y_\varepsilon}$ may cause a faster decay of $M_\varepsilon^{\gamma, \beta}$ than expected, a kind of Riemann-Lebesgue averaging to 0. We have to make sure that this does not happen. We further stress that this averaging to 0 occurs outside the phase I and that the phases II and III may be seen as the study of the fluctuations along this averaging.*

Furthermore, we stress that the results holds as well if we replace the ball $B(0, r)$ by $\varphi(\cdot/r)$ for some continuous and compactly supported function φ .

Proof. We carry out the proof in dimension 1. Observe that the martingale $(\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}(K))_\varepsilon$ is uniformly bounded in \mathbb{L}_p for all compact sets K . Let us fix $r > 0$. Let us consider a family of complex random measures for $r > \varepsilon > 0$:

$$M_{r, \varepsilon}^{\gamma, \beta}(dx) = e^{\gamma(X_\varepsilon - X_r)(x) + i\beta(Y_\varepsilon - Y_r)(x) - (\frac{\gamma^2}{2} - \frac{\beta^2}{2}) \ln \frac{r}{\varepsilon}} dx.$$

Observe that $M_{r, \varepsilon}^{\gamma, \beta}(dx)$ is independent of the fields X_r and Y_r and has the same law as $r \left(\frac{\varepsilon}{r}\right)^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_{\frac{\varepsilon}{r}}^{\gamma, \beta}(dx/r)$.

For $\varepsilon < r$, we can decompose $M_\varepsilon^{\gamma, \beta}$ as

$$\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}(dx) = e^{\gamma X_r(x) + i\beta Y_r(x) - (\frac{\gamma^2}{2} - \frac{\beta^2}{2}) \ln \frac{1}{r}} M_{r, \varepsilon}^{\gamma, \beta}(dx). \quad (3.3)$$

Therefore we have:

$$\begin{aligned} & \mathbb{E}\left[\left|\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}([0, r])\right|^q\right] \\ &= \mathbb{E}\left[\left|\int_{[0, r]} e^{\gamma X_r(x) + i\beta Y_r(x) - (\frac{\gamma^2}{2} - \frac{\beta^2}{2}) \ln \frac{1}{r}} M_{r, \varepsilon}^{\gamma, \beta}(dx)\right|^q\right] \\ &= r^{q(\frac{\gamma^2}{2} - \frac{\beta^2}{2}) - \frac{q^2 \gamma^2}{2}} \mathbb{E}\left[e^{q\gamma X_r(0) - \frac{q^2 \gamma^2}{2} \ln \frac{1}{r}} \left|\int_{[0, r]} e^{\gamma(X_r(x) - X_r(0)) + i\beta(Y_r(x) - Y_r(0))} M_{r, \varepsilon}^{\gamma, \beta}(dx)\right|^q\right] \\ &= r^{q(1 + \frac{\gamma^2}{2} - \frac{\beta^2}{2}) - \frac{q^2 \gamma^2}{2}} \mathbb{E}\left[e^{q\gamma X'_r(0) - \frac{q^2 \gamma^2}{2} \ln \frac{1}{r}} \left|\int_{[0, r]} e^{\gamma(X'_r(x) - X'_r(0)) + i\beta(Y'_r(x) - Y'_r(0))} \left(\frac{\varepsilon}{r}\right)^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_{\frac{\varepsilon}{r}}^{\gamma, \beta}(dx/r)\right|^q\right], \end{aligned}$$

where X' and Y' are fields that are independent of X and Y with the same law (and hence $(r \left(\frac{\varepsilon}{r}\right)^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_{\frac{\varepsilon}{r}}^{\gamma, \beta}(dx/r), X'_r, Y'_r)$ has the same law as $(M_{r, \varepsilon}^{\gamma, \beta}(dx), X_r, Y_r)$).

Now we make a change of variables in the integral and the use the Girsanov transform to get:

$$\begin{aligned} & \mathbb{E}\left[\left|\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}([0, r])\right|^q\right] \\ &= r^{\zeta(q)} \mathbb{E}\left[e^{q\gamma X'_r(0) - \frac{q^2 \gamma^2}{2} \ln \frac{1}{r}} \left|\int_{[0, 1]} e^{\gamma(X'_r(rx) - X'_r(0)) + i\beta(Y'_r(rx) - Y'_r(0))} \left(\frac{\varepsilon}{r}\right)^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_{\frac{\varepsilon}{r}}^{\gamma, \beta}(dx)\right|^q\right] \\ &= r^{\zeta(q)} \mathbb{E}\left[\left|\int_{[0, 1]} e^{\gamma Z_r^X(x) + i\beta Z_r^Y(x) - \frac{\gamma^2 - \beta^2}{2} \mathbb{E}[Z_r^X(x)] + f_r(x)} \left(\frac{\varepsilon}{r}\right)^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_{\frac{\varepsilon}{r}}^{\gamma, \beta}(dx)\right|^q\right] \quad (3.4) \end{aligned}$$

where we have set $Z_r^X(x) = X_r'(rx) - X_r'(0)$, $Z_r^Y(x) = Y_r'(rx) - Y_r'(0)$ and

$$f_r(x) = \frac{(2q-1)\gamma^2 + \beta^2}{2} \int_r^1 \frac{k(ux) - 1}{u} du.$$

To conclude the proof, we have to show that

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| \int_{[0,1]} e^{\gamma Z_r^X(x) + i\beta Z_r^Y(x) - \frac{\gamma^2 - \beta^2}{2} \mathbb{E}[Z^X(x)] + f_r(x)} \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}(dx) \right|^q \right] \quad (3.5)$$

exists and is non-zero. We do it using a martingale argument. The covariance kernel of Z is given by,

$$\mathbb{E}[Z_r^X(x) Z_r^X(x')] = \int_1^{1/r} \frac{k(r(x-x')u) - k(rx) - k(rx'u) + 1}{u} du \quad (3.6)$$

$$= \int_r^1 \frac{k((x-x')v) - k(xv) - k(x'v) + 1}{v} dv. \quad (3.7)$$

Due to this structure of covariance, one can construct a process whose marginal have the same law (we also name it $Z = (Z^X, Z^Y)$ as it brings no confusion) indexed by $r \leq 1$ such that for each x and $r' < r$, $\mathbb{E}[Z_{r'}(x) | Z_r(x)] = Z_r(x)$. With this construction the process

$$A_{\varepsilon, r} = \left(\int_{[0,1]} e^{\gamma Z_r^X(x) + i\beta Z_r^Y(x) - \frac{\gamma^2 - \beta^2}{2} \mathbb{E}[Z^X(x)] + f_0(x)} \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_\varepsilon^{\gamma, \beta}(dx) \right)_{\varepsilon \in [0,1], r \in [0,1]}$$

is a doubly indexed martingale (note that f_r has been changed to f_0) and thus

$$\lim_{r \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \mathbb{E}[|A_{\varepsilon, r}|^q] = \sup_{\varepsilon, r} \mathbb{E}[|A_{\varepsilon, r}|^q] > 0 \quad (3.8)$$

exists and we just have to show uniform boundedness of $|A_{\varepsilon, r}|^q$. Let \mathcal{F}^Z denote the sigma algebra generated by Z , by Jensen's inequality

$$\begin{aligned} \mathbb{E}[|A_{\varepsilon, r}|^q] &\leq \mathbb{E} \left[\left(\mathbb{E}[|A_{\varepsilon, r}|^2 | \mathcal{F}^Y, \mathcal{F}^Z] \right)^{q/2} \right] \\ &\leq C \mathbb{E} \left[\left(\varepsilon^{\gamma^2} \int_{[0,1]^2} \frac{1}{|x-y|^{\beta^2}} M_\varepsilon^{\gamma, 0}(dx) M_\varepsilon^{\gamma, 0}(dy) \right)^{q/2} \right]. \end{aligned} \quad (3.9)$$

for some universal constant C (where the second inequality is obtained by computing explicitly the average as in the proof of Theorem 3.1) and we conclude using Lemma 3.10. What remains to show is that replacing f_0 by f_r does not change the limit. This is easy: if $\hat{A}_{\varepsilon, r}$ denotes the process where f_0 is replaced by f_r we obtain after redoing the same computation with an extra $e^{f_r - f_0} - 1$ factor that

$$\mathbb{E}[|A_{\varepsilon, r} - \hat{A}_{\varepsilon, r}|^q] = o(1) \mathbb{E} \left[\left(\varepsilon^{\gamma^2} \int_{[0,1]^2} \frac{1}{|x-y|^{\beta^2}} M_\varepsilon^{\gamma, 0}(dx) M_\varepsilon^{\gamma, 0}(dy) \right)^{q/2} \right] \quad (3.10)$$

when r tends to zero, and conclude. \square

We also have:

Theorem 3.8. (Star scale invariance). *Assume that the kernel k is of class C^{2d} with derivatives of order $2d$ Hölder. The distribution of order d $M^{\gamma,\beta}$ is star scale invariant in the sense that it can be written as*

$$M^{\gamma,\beta}(dx) = e^{\gamma X_r(x) + i\beta Y_r(x) - (\frac{\gamma^2}{2} - \frac{\beta^2}{2}) \ln \frac{1}{r}} \widetilde{M}^{\gamma,\beta}(dx)$$

where $\widetilde{M}^{\gamma,\beta}$ is a distribution of order d , is independent of the fields X_r and Y_r and has the same law as $r^d M^{\gamma,\beta}(dx/r)$.

Remark 3.9. *The scaling relation (3.3) or that of Theorem 3.8 is only valid for star scale invariant kernels as those described in section 2. If one wishes to apply this argument to more general situations than those described in sections 2.1 or 6, the difference is that we have a decomposition*

$$M^{\gamma,\beta}(dt) = r^d e^{\gamma X_r(t) - (\frac{\gamma^2}{2} - \frac{\beta^2}{2}) \mathbb{E}[X_r(t)^2]} \overline{M}^{\gamma,\beta,r} \left(\frac{dt}{r} \right)$$

where the measure $\overline{M}^{\gamma,\beta,r}$ is independent of X_r, Y_r and the family of complex valued distributions $(\overline{M}^{\gamma,\beta,r})_r$ is tight in the space of distributions of order d .

Proof. We stick to the notations of the beginning of the proof of Theorem 3.6 and write

$$\varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_{\varepsilon}^{\gamma,\beta}(dx) = e^{\gamma X_r(x) + i\beta Y_r(x) - (\frac{\gamma^2}{2} - \frac{\beta^2}{2}) \ln \frac{1}{r}} M_{r,\varepsilon}^{\gamma,\beta}(dx). \quad (3.11)$$

where $M_{r,\varepsilon}^{\gamma,\beta}(dx)$ is independent of the fields X_r and Y_r and has the same law as $r^d \left(\frac{\varepsilon}{r}\right)^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_{\frac{\varepsilon}{r}}^{\gamma,\beta}(dx/r)$. From item 2 of Theorem 3.1, the left-hand side converges in the sense of distributions of order d towards $M^{\gamma,\beta}(dx)$. Concerning the right-hand side, for all $r > 0$, the family $r^d \left(\frac{\varepsilon}{r}\right)^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_{\frac{\varepsilon}{r}}^{\gamma,\beta}(dx/r)$ converges in law as $\varepsilon \rightarrow 0$ in the sense of distributions of order d towards $M_r^{\gamma,\beta}(dx)$, which has the same law as $r^d M^{\gamma,\beta}(dx/r)$ and is independent of X_r, Y_r . Because of our assumption on the regularity of k , we can apply Proposition B.2 to prove that both processes X_r, Y_r are almost surely of class C^d . We can then pass to the limit in (3.11) to complete the proof of Theorem 3.8. \square

Analysis of the capacity

The following lemma settles the case where (β, γ) belongs to the inner phase I. Recall that this implies the existence of $p \in]1, 2[$ such that $\zeta(p) > d$.

Lemma 3.10. *Let (β, γ) belong to the inner phase I. If $p \in]1, 2[$ is such that $\zeta(p) > d$, there exists $C > 0$ such that we have for all $\varepsilon < 1$:*

$$\mathbb{E} \left[\left(\int_{x,y \in [0,1]^d} \varepsilon^{\gamma^2} \frac{M_{\varepsilon}^{\gamma,0}(dx) M_{\varepsilon}^{\gamma,0}(dy)}{|y-x|^{\beta^2}} \right)^{p/2} \right] \leq C.$$

Proof. For simplicity, we suppose that $d = 1$. From Proposition 2.2, (2.2) and (2.11), it is sufficient to prove the result when $X_{\varepsilon}(x) = Z$ is the scale invariant Gaussian log-correlated fields described in Proposition 2.3 (we apply (2.7) to the field $X_{\varepsilon}(x) + X_{\varepsilon}(y)$ indexed by \mathbb{R}^2) with $T = 2$. Now,

by subadditivity of $x \mapsto x^{p/2}$, we get:

$$\begin{aligned} \mathbb{E} \left[\left(\int_{x,y \in [0,1]} \left(\frac{\varepsilon}{2}\right)^{\gamma^2} \frac{M_{\varepsilon/2}^{\gamma,0}(dx) M_{\varepsilon/2}^{\gamma,0}(dy)}{|y-x|^{\beta^2}} \right)^{\frac{p}{2}} \right] &\leq 2\mathbb{E} \left[\left(\int_{x,y \in [0, \frac{1}{2}]} \left(\frac{\varepsilon}{2}\right)^{\gamma^2} \frac{M_{\varepsilon/2}^{\gamma,0}(dx) M_{\varepsilon/2}^{\gamma,0}(dy)}{|y-x|^{\beta^2}} \right)^{\frac{p}{2}} \right] \\ &+ 2\mathbb{E} \left[\left(\int_{x \in [0, \frac{1}{2}], y \in [\frac{1}{2}, 1]} \left(\frac{\varepsilon}{2}\right)^{\gamma^2} \frac{M_{\varepsilon/2}^{\gamma,0}(dx) M_{\varepsilon/2}^{\gamma,0}(dy)}{|y-x|^{\beta^2}} \right)^{\frac{p}{2}} \right]. \end{aligned}$$

We first handle the second term in the above sum. We have by Jensen's Inequality that:

$$\begin{aligned} \mathbb{E} \left[\left(\int_{(x,y) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1]} \left(\frac{\varepsilon}{2}\right)^{\gamma^2} \frac{M_{\varepsilon/2}^{\gamma,0}(dx) M_{\varepsilon/2}^{\gamma,0}(dy)}{|y-x|^{\beta^2}} \right)^{\frac{p}{2}} \right] &\leq C \left(\int_{(x,y) \in [0, \frac{1}{2}] \times [\frac{1}{2}, 1]} \frac{dxdy}{|y-x|^{\gamma^2 + \beta^2}} \right)^{\frac{p}{2}} \\ &\leq C \int_0^1 \frac{du}{u^{\gamma^2 + \beta^2}} \int_0^u dv. \end{aligned}$$

This latter quantity is finite since $\gamma^2 + \beta^2 < 2$. Hence, we get the existence of some constant $C > 0$ such that:

$$\mathbb{E} \left[\left(\int_{x,y \in [0,1]} \left(\frac{\varepsilon}{2}\right)^{\gamma^2} \frac{M_{\varepsilon/2}^{\gamma,0}(dx) M_{\varepsilon/2}^{\gamma,0}(dy)}{|y-x|^{\beta^2}} \right)^{\frac{p}{2}} \right] \leq 2\mathbb{E} \left[\left(\int_{x,y \in [0, \frac{1}{2}]} \left(\frac{\varepsilon}{2}\right)^{\gamma^2} \frac{M_{\varepsilon/2}^{\gamma,0}(dx) M_{\varepsilon/2}^{\gamma,0}(dy)}{|y-x|^{\beta^2}} \right)^{\frac{p}{2}} \right] + C. \quad (3.12)$$

By stochastic scale invariance (2.10) for $\lambda = 1/2$, we have

$$\begin{aligned} &\mathbb{E} \left[\left(\int_{(x,y) \in [0, \frac{1}{2}]^2} \left(\frac{\varepsilon}{2}\right)^{\gamma^2} \frac{M_{\varepsilon/2}^{\gamma,0}(dx) M_{\varepsilon/2}^{\gamma,0}(dy)}{|y-x|^{\beta^2}} \right)^{p/2} \right] \\ &= \frac{1}{2^{p(1+\gamma^2/2-\beta^2/2)}} \mathbb{E} \left[\left(\int_{(x,y) \in [0,1]^2} \varepsilon^{\gamma^2} \frac{e^{\gamma X_{\varepsilon/2}(x/2) + \gamma X_{\varepsilon/2}(y/2)} dxdy}{|y-x|^{\beta^2}} \right)^{p/2} \right] \\ &= \frac{1}{2^{p(1+\gamma^2/2-\beta^2/2)}} \mathbb{E}[e^{p\gamma\Omega_{1/2}}] \mathbb{E} \left[\left(\int_{(x,y) \in [0,1]^2} \varepsilon^{\gamma^2} \frac{e^{\gamma X_{\varepsilon}(x) + \gamma X_{\varepsilon}(y)} dxdy}{|y-x|^{\beta^2}} \right)^{p/2} \right] \\ &= \frac{1}{2^{\zeta(p)}} \mathbb{E} \left[\left(\int_{(x,y) \in [0,1]^2} \varepsilon^{\gamma^2} \frac{e^{\gamma X_{\varepsilon}(x) + \gamma X_{\varepsilon}(y)} dxdy}{|y-x|^{\beta^2}} \right)^{p/2} \right]. \end{aligned}$$

If we set $u_n = \mathbb{E} \left[\left(\int_{(x,y) \in [0,1]^2} \varepsilon_n^{\gamma^2} \frac{e^{\gamma X_{\varepsilon_n}(x) + \gamma X_{\varepsilon_n}(y)} dxdy}{|y-x|^{\beta^2}} \right)^{p/2} \right]$ where $\varepsilon_n = \frac{1}{2^n}$ then inequality (3.12) amounts to $u_{n+1} \leq \rho u_n + C$ where $\rho = \frac{2}{2^{\zeta(p)}} < 1$. Hence the sequence $(u_n)_{n \geq 1}$ is bounded, yielding the result. \square

3.2 Phase transition I/II

Theorem 3.11. *Let us consider the frontier I/II: $\beta + \gamma = \sqrt{2d}$ and $\gamma \in]\sqrt{\frac{d}{2}}, \sqrt{2d}[$. We further consider any $p \in]1, \frac{\sqrt{2d}}{\gamma}[$.*

1. For all compact compactly supported bounded measurable function f , the martingale

$$(\varepsilon^{\frac{\gamma^2}{2}-\frac{\beta^2}{2}} \int_{\mathbb{R}^d} f(x) M_\varepsilon^{\gamma,\beta}(dx))_\varepsilon$$

is uniformly bounded in \mathbb{L}_p . Furthermore, for all $R > 0$, there exists a constant $C_{p,R}$ (only depending on p, R) such that for all bounded measurable function f with compact support in $B(0, R)$:

$$\mathbb{E} \left[\sup_{\varepsilon \in]0,1]} \left| \varepsilon^{\frac{\gamma^2}{2}-\frac{\beta^2}{2}} \int_{\mathbb{R}^d} f(x) M_\varepsilon^{\gamma,\beta}(dx) \right|^p \right] \leq C_{p,R} \|f\|_\infty^p.$$

2. The $\mathcal{D}'(\mathbb{R}^d)$ -valued martingale:

$$M_\varepsilon^{\gamma,\beta} : \varphi \rightarrow \varepsilon^{\frac{\gamma^2}{2}-\frac{\beta^2}{2}} \int_{\mathbb{R}^d} \varphi(x) e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx$$

converges almost surely in the space $\mathcal{D}'_d(\mathbb{R}^d)$ of distributions of order d towards a non trivial limit $M^{\gamma,\beta}$. More precisely, for each $R > 0$, there exists a random variable $Z^R \in \mathbb{L}_p$ such that for all functions $\varphi \in C_c^d(B(0, R))$:

$$|M^{\gamma,\beta}(\varphi)| \leq Z^R \sup_{x \in B(0,R)} \left| \frac{\partial^d \varphi(x)}{\partial x_1 \cdots \partial x_d} \right|.$$

3. For all $p < \frac{\sqrt{2d}}{\gamma}$, we set $\zeta(p) = (d + \frac{\gamma^2}{2} - \frac{\beta^2}{2})p - \frac{\gamma^2}{2}p^2 = \sqrt{2d}\gamma p - \frac{\gamma^2}{2}p^2$. There exists some constant $C_p > 0$ such that:

$$\mathbb{E}[|M^{\gamma,\beta}(B(0, r))|^p] \underset{r \rightarrow 0}{\simeq} C_p r^{\zeta(p)}.$$

4. Assume that the kernel k is of class C^{2d} with derivatives of order $2d$ Hölder. The distribution of order d , $M^{\gamma,\beta}$ is star scale invariant in the sense that it can be written as

$$M^{\gamma,\beta}(dx) = e^{\gamma X_r(x) + i\beta Y_r(x) - (\frac{\gamma^2}{2} - \frac{\beta^2}{2}) \ln \frac{1}{r}} \widetilde{M}_r^{\gamma,\beta}(dx)$$

where $\widetilde{M}_r^{\gamma,\beta}$ is a distribution of order d , is independent of the fields X_r and Y_r and has the same law as $rM^{\gamma,\beta}(dx/r)$.

Remark 3.12. In this case, ζ is increasing on $]0, \frac{\sqrt{2d}}{\gamma}[$ with $\zeta(\frac{\sqrt{2d}}{\gamma}) = d$ and $\zeta'(\frac{\sqrt{2d}}{\gamma}) = 0$. Hence the continuity of $M^{\gamma,\beta}$ in dimension 1 is not obvious.

Proof. The argument for item 1 and 2 is the same as in the proof of Theorem 3.1 except that we use Lemma 3.13 below instead of Lemma 3.10 (indeed, we can choose p such that $\alpha = p/2 \in]\frac{\sqrt{2d}}{3\gamma}, \sqrt{\frac{d}{2}}\frac{1}{\gamma}[$ and $3\alpha\gamma/\sqrt{2d} > 1$). Items 3 and 4 are proved as in Theorem 3.6 and 3.8. \square

Analysis of the capacity

The following lemma settles the case $\beta + \gamma = \sqrt{2d}$ and $\gamma \in]\sqrt{\frac{d}{2}}, \sqrt{2d}[$:

Lemma 3.13. Let $\gamma \in]\sqrt{\frac{d}{2}}, \sqrt{2d}[$. Let $l \geq 0$. For all $\alpha \in]\frac{\sqrt{2d}}{3\gamma}, \sqrt{\frac{d}{2}}\frac{1}{\gamma}[$, there exists $C > 0$ such that for all $\varepsilon, \varepsilon' \leq \frac{1}{2^l}$:

$$\mathbb{E} \left[\left(\int_{x \in [0,1]^d, |y-x| \leq \frac{1}{2^l}} (\varepsilon')^{\gamma^2/2} \varepsilon^{\gamma^2/2} \frac{M_{\varepsilon'}^{\gamma,0}(dx) M_\varepsilon^{\gamma,0}(dy)}{|y-x|^{(\sqrt{2d}-\gamma)^2}} \right)^\alpha \right] \leq C \sum_{j \geq l} \frac{1}{j^{\frac{3}{\sqrt{2d}}\alpha\gamma}}.$$

Proof. For simplicity, we suppose that $\varepsilon = \varepsilon_k = \frac{1}{2^k}$ and $\varepsilon' = \varepsilon_{k'} = \frac{1}{2^{k'}}$ with $k \leq k'$. We set $X_k(x) = X_{\varepsilon_k}(x)$ ($X_{k'}(x) = X_{\varepsilon_{k'}}(x)$), $M_k(x) = M_{\varepsilon_k}^{\gamma,0}(x)$ ($M_{k'}(x) = M_{\varepsilon_{k'}}^{\gamma,0}(x)$) and $\mathcal{F}_k = \mathcal{F}_{\varepsilon_k}$ ($\mathcal{F}_{k'} = \mathcal{F}_{\varepsilon_{k'}}$). We further define

$$A_j = \{(x, y) \in ([0, 1]^d)^2; |x - y| \leq 2^{-j}\} \quad D_j = \{(x, y) \in ([0, 1]^d)^2; 2^{-j} < |y - x| < 2^{-j+1}\}$$

We have:

$$\begin{aligned} \int_{A_l} \frac{1}{2^{k\gamma^2/2}} \frac{1}{2^{k'\gamma^2/2}} \frac{M_k(dx)M_{k'}(dy)}{|y-x|^{(\sqrt{2d}-\gamma)^2}} &\leq C \sum_{j=l+1}^k 2^{j(\sqrt{2d}-\gamma)^2} \int_{D_j} \frac{1}{2^{k\gamma^2/2}} \frac{1}{2^{k'\gamma^2/2}} M_k(dx)M_{k'}(dy) \\ &+ \int_{A_k} \frac{1}{2^{k\gamma^2/2}} \frac{1}{2^{k'\gamma^2/2}} \frac{M_k(dx)M_{k'}(dy)}{|y-x|^{(\sqrt{2d}-\gamma)^2}}. \end{aligned}$$

Therefore, if $\alpha < 1$, we get the following inequality:

$$\begin{aligned} \left(\int_{A_l} \frac{1}{2^{k\gamma^2/2}} \frac{1}{2^{k'\gamma^2/2}} \frac{M_k(dx)M_{k'}(dy)}{|y-x|^{(\sqrt{2d}-\gamma)^2}} \right)^\alpha &\leq C \sum_{j=l+1}^k 2^{\alpha j(\sqrt{2d}-\gamma)^2} \left(\int_{D_j} \frac{1}{2^{k\gamma^2/2}} \frac{1}{2^{k'\gamma^2/2}} M_k(dx)M_{k'}(dy) \right)^\alpha \\ &+ \left(\int_{A_k} \frac{1}{2^{k\gamma^2/2}} \frac{1}{2^{k'\gamma^2/2}} \frac{M_k(dx)M_{k'}(dy)}{|y-x|^{(\sqrt{2d}-\gamma)^2}} \right)^\alpha. \end{aligned} \quad (3.13)$$

Now, we estimate each quantity in the sum on the right hand side. We start with the last term above. We have by Jensen's inequality

$$\begin{aligned} \mathbb{E} \left[\left(\int_{A_k} \frac{1}{2^{k\gamma^2/2}} \frac{1}{2^{k'\gamma^2/2}} \frac{M_k(dx)M_{k'}(dy)}{|y-x|^{(\sqrt{2d}-\gamma)^2}} \right)^\alpha \right] &\leq \mathbb{E} \left[\left(\mathbb{E} \left[\int_{A_k} \frac{1}{2^{k\gamma^2/2}} \frac{1}{2^{k'\gamma^2/2}} \frac{M_k(dx)M_{k'}(dy)}{|y-x|^{(\sqrt{2d}-\gamma)^2}} \middle| \mathcal{F}_k \right] \right)^\alpha \right] \\ &= \mathbb{E} \left[\left(\int_{A_k} \frac{1}{2^{k\gamma^2}} \frac{M_k(dx)M_k(dy)}{|y-x|^{(\sqrt{2d}-\gamma)^2}} \right)^\alpha \right]. \end{aligned}$$

Now we use the inequality $ab \leq a^2 + b^2/2$ and obtain (for some constant $C(\gamma, d)$)

$$\begin{aligned} \mathbb{E} \left[\left(\int_{A_k} \frac{1}{2^{k\gamma^2/2}} \frac{1}{2^{k'\gamma^2/2}} \frac{M_k(dx)M_{k'}(dy)}{|y-x|^{(\sqrt{2d}-\gamma)^2}} \right)^\alpha \right] &\leq \mathbb{E} \left[\left(\int_{A_k} \frac{1}{2^{k\gamma^2}} \frac{e^{2\gamma X_k(x)} dx dy}{|y-x|^{(\sqrt{2d}-\gamma)^2}} \right)^\alpha \right] \\ &= C 2^{\alpha k((\sqrt{2d}-\gamma)^2 - d)} \mathbb{E} \left[\left(\int_{[0,1]^d} \frac{1}{2^{k\gamma^2}} e^{2\gamma X_k(x)} dx \right)^\alpha \right] \\ &= C \mathbb{E} \left[\left(\int_{[0,1]^d} 2^{dk} e^{2\gamma(X_k(x) - \sqrt{2d} \ln 2^k)} dx \right)^\alpha \right]. \end{aligned}$$

To conclude the proof we need the following result which is a refined version of [33, Theorem 1.6] coming from [42, Prop 2.1] or [7, Lemma 9], the proof of which is postponed to the end of this section.

Lemma 3.14. *For any $\gamma > \sqrt{\frac{d}{2}}$ and any $\alpha < \sqrt{\frac{d}{2}} \frac{1}{\gamma}$, there exists C (depending on γ and α) such that for all ε we have*

$$\mathbb{E} \left[\left(\int_{[0,1]^d} \varepsilon^{-d} e^{2\gamma(X_\varepsilon(x) - \sqrt{2d} \ln \frac{1}{\varepsilon})} dx \right)^\alpha \right] \leq \frac{C}{|\ln \varepsilon|^{\frac{3}{\sqrt{2d}} \alpha \gamma}}. \quad (3.14)$$

Let us admit this lemma for a while. We get the bound

$$\mathbb{E} \left[\left(\int_{[0,1]^d} 2^{dk} e^{2\gamma(X_k(x) - \sqrt{2d} \ln 2^k)} dx \right)^\alpha \right] \leq \frac{C}{k^{\frac{3}{\sqrt{2d}} \alpha \gamma}}.$$

Now, we handle the terms in the sum (3.13). Without loss of generality, we may assume that the kernel k appearing in Assumption (A) has a compact support included in the ball $B(0, 1)$. Indeed if not, we can use Proposition 2.2 and (2.2) to get a comparison with the compactly supported case. In particular, we will use the fact that, conditionally to \mathcal{F}_j , the sigma algebras $\sigma\{M_k(A); A \subset B_1\}$ and $\sigma\{M_k(A); A \subset B_2\}$ are independent as soon as $\text{dist}(B_1, B_2) > 2^{-j}$. Thus we get

$$\begin{aligned} \mathbb{E} \left[\left(\int_{D_j} \frac{1}{2^{k\gamma^2}} M_k(dx) M_k(dy) \right)^\alpha \right] &\leq \mathbb{E} \left[\left(\mathbb{E} \left[\int_{D_j} \frac{1}{2^{k\gamma^2}} M_k(dx) M_k(dy) \middle| \mathcal{F}_j \right] \right)^\alpha \right] \\ &\leq \mathbb{E} \left[\left(\int_{D_j} \frac{1}{2^{j\gamma^2}} M_j(dx) M_j(dy) \right)^\alpha \right] \\ &\leq \mathbb{E} \left[\left(\int_{|y-x| < \frac{1}{2^{j-1}}} \frac{1}{2^{j\gamma^2}} M_j(dx) M_j(dy) \right)^\alpha \right]. \end{aligned}$$

Now, one can conclude similarly to the previous term. \square

Proof of Lemma 3.14. We assume that $\varepsilon = \frac{1}{2^k}$ (this brings no loss of generality since by Proposition 2.2, one can compare the l.h.s. of 3.14 for two ε within a factor of 2).

For $r \in \mathbb{N}$ let us cover the interval $[0, 1]^d$ by the dyadic cubes of the form:

$$I_{\mathbf{z}}^{(r)} := \prod_{1 \leq i \leq d} \left[\frac{z_i}{2^r} + \frac{z_i + 1}{2^r} \right],$$

where $\mathbf{z} = (z_1, \dots, z_d) \in \{0, \dots, 2^r - 1\}^d$. Let $(Y^{(k)}(x))_{x \in [0,1]^d}$ be a standard Gaussian 2^d -adic cascade defined on $[0, 1]^d$ by the covariance function

$$\mathbb{E}[Y^{(k)}(x)Y^{(k)}(y)] := \ln \frac{1}{d_2(x, y) \vee 2^{-k}},$$

where $d_2(x, y)$ is the dyadic distance defined on $[0, 1]^d$, i.e. $d_2(x, y)$ is an inverse power of 2 and we have

$$d_2(x, y) \leq 2^{-r} \Leftrightarrow (\exists \mathbf{z}, x, y \in I_{\mathbf{z}}^{(r)}).$$

Note that the process Y is a particular case of branching random walk (studied in e.g. [33, 42]) with i.i.d Gaussian step, and deterministic branching number 2^d .

From (2.2), there exists C such that

$$\mathbb{E}[X_k(x)X_k(y)] + C \geq \mathbb{E}[Y^{(k)}(x)Y^{(k)}(y)].$$

Hence, by Proposition 2.2, we get that there exists some $C > 0$ such that:

$$\begin{aligned} \mathbb{E} \left[\left(\int_{[0,1]^d} 2^{dk} e^{2\gamma(X_k(x) - \sqrt{2d} \ln 2^k)} dx \right)^\alpha \right] &\leq CE \left[\left(\int_{[0,1]^d} 2^{dk} e^{2\gamma(Y^{(k)}(x) - \sqrt{2d} \ln 2^k)} dx \right)^\alpha \right] \\ &= CE \left[\left(\sum_{\mathbf{z} \in \{0, \dots, 2^k - 1\}^d} e^{2\gamma(Y^{(k)}(2^{-k}\mathbf{z}) - \sqrt{2d} \ln 2^k)} \right)^\alpha \right]. \end{aligned}$$

Now, by using [42, Prop 2.1] or [7, Lemma 9], we know that for all $\beta > 1$ and $\alpha < \frac{1}{\beta}$:

$$\mathbb{E} \left[\left(\sum_{\mathbf{z} \in \{0, \dots, 2^k - 1\}^d} e^{\beta(\sqrt{2d}Y^{(k)}(2^{-k}\mathbf{z}) - 2d \ln 2^k)} \right)^\alpha \right] \leq \frac{C}{k^{\frac{3}{2}\alpha\beta}}$$

Now, recall that $\gamma > \sqrt{\frac{d}{2}}$ and hence for all $\alpha < \sqrt{\frac{d}{2}} \frac{1}{\gamma}$:

$$\mathbb{E} \left[\left(\sum_{\mathbf{z} \in \{0, \dots, 2^k - 1\}^d} e^{2\gamma(Y^{(k)}(2^{-k}\mathbf{z}) - \sqrt{2d} \ln 2^k)} \right)^\alpha \right] \leq \frac{C}{k^{\frac{3}{\sqrt{2d}}\alpha\gamma}}.$$

□

4 Study of the phase III and its I/III and II/III boundaries

4.1 Results and organization of the proofs

In the inner phase III, the limit object describing the renormalized measure is a complex white-noise whose intensity depends on X and is given by $M^{2\gamma,0}(dx)$ (all the information about Y is lost in the limit).

The frontiers I/III and II/III present similar behaviors but there are additional technical difficulties especially for the frontier II/III. Indeed, in that case, the intensity measure $M^{2\gamma,0}(dx)$ is the so-called derivative martingale (see [22, 23]).

Now, we define the function σ^2 that will appear throughout the following convergence results

Definition 4.1. Function σ^2 . *In dimension d , we define the function the function of the parameters (γ, β) :*

$$\sigma^2(\beta^2 + \gamma^2) := \begin{cases} \int_{\mathbb{R}^d} \exp\left(-(\gamma^2 + \beta^2) \int_0^1 \frac{1-k(uz)}{u} du\right) dz & \text{if } \beta^2 + \gamma^2 > d \\ \int_{z \in \mathbb{S}^{d-1}} \nu_{d-1}(dz) \exp\left(\int_0^\infty \frac{k(uz) - \mathbb{1}_{[0,1]}(u)}{u} du\right) & \text{if } \beta^2 + \gamma^2 = d \end{cases} \quad (4.1)$$

where $\nu_{d-1}(z)$ denotes the surface measure on \mathbb{S}^{d-1} (the $d-1$ dimensional unit sphere).

Observe that in dimension 1, for $\beta^2 + \gamma^2 = 1$, σ^2 takes the simpler form

$$\sigma^2(1) = 2 \exp\left(\int_0^\infty \frac{k(u) - \mathbb{1}_{[0,1]}(u)}{u} du\right).$$

We also stress that we have the following asymptotic behaviour for large z and $\beta^2 + \gamma^2 > d$:

$$\exp\left(-(\gamma^2 + \beta^2) \int_0^1 \frac{1-k(uz)}{u} du\right) \simeq \frac{1}{(1+|z|)^{\gamma^2+\beta^2}}.$$

Therefore σ^2 is well defined for $\beta^2 + \gamma^2 > d$.

We are now in position to state the main results of this section:

Theorem 4.2. • When $\gamma \in [0, \sqrt{\frac{d}{2}}[$ and $\beta^2 + \gamma^2 > d$, we have

$$\left(\varepsilon^{\gamma^2 - \frac{d}{2}} M_\varepsilon^{\gamma, \beta}(A) \right)_{AC\mathbb{R}^d} \Rightarrow (W_{\sigma^2 M^{2\gamma, 0}}(A))_{AC\mathbb{R}^d}. \quad (4.2)$$

where $\sigma^2 = \sigma^2(\beta^2 + \gamma^2)$ and W is a standard complex Gaussian measure on \mathbb{R}^d with intensity $\sigma^2 M^{2\gamma, 0}$. The above convergence holds in the sense of convergence in law of the finite dimensional distributions.

• When $\gamma \in [0, \sqrt{\frac{d}{2}}[$ and $\beta^2 + \gamma^2 = d$, we have

$$\left(\varepsilon^{\gamma^2 - \frac{d}{2}} |\ln \varepsilon|^{-1/2} M_\varepsilon^{\gamma, \beta}(A) \right)_{AC\mathbb{R}^d} \Rightarrow (W_{\sigma^2 M^{2\gamma, 0}}(A))_{AC\mathbb{R}^d}. \quad (4.3)$$

where $\sigma^2 = \sigma^2(d)$ and W is a standard complex Gaussian measure on \mathbb{R}^d with intensity $\sigma^2 M^{2\gamma, 0}$. The above convergence holds in the sense of convergence in law of the finite dimensional distributions.

Concerning the frontier II/III, we will need to use the results in [22, 23]. Further assumptions must be made in order to make sure that one can construct the derivative martingale and prove the Seneta-Heyde renormalization (see [23, Section D and Remark 31]):

Assumption (A). We consider the following assumptions:

A4. k is nonnegative,

A5. k has compact support,

A5'. k admits a nonnegative convolution square root g , i.e. $k(x) = \int_{\mathbb{R}^d} g(y)g(x+y) dy$, such that g is nonnegative, g and ∂g are integrable, and for some constants $C > 0$, $\alpha > 1$

$$\sup_{|x| \geq 2} g(x) + |\partial g(x)| < +\infty, \quad \int_1^\infty v^{-1} \int_{B(0, \ln^\alpha v)} g^2(y) dy dv < \infty, \quad k(x) + |\partial k(x)| \leq C e^{-|x|^{1/\alpha}}.$$

Theorem 4.3. Assume that k satisfies assumptions A.1-4 and either A.5 or A.5'. When $\gamma = \sqrt{d/2}$ and $\beta^2 + \gamma^2 > d$, we have

$$\left((-\ln \varepsilon)^{1/4} M_\varepsilon^{\gamma, \beta}(A) \right)_{AC\mathbb{R}^d} \Rightarrow (W_{\sigma^2 M'}(A))_{AC\mathbb{R}^d}. \quad (4.4)$$

with

$$\sigma^2 = \sqrt{\frac{2}{\pi}} \sigma^2(\beta^2 + d/2),$$

and the law of $W_{\sigma^2 M'}(\cdot)$ is, conditionally to X , that of a complex Gaussian random measure with intensity $\sigma^2 M'$. The above convergence holds in the sense of convergence in law of the finite dimensional distributions.

In dimension 1, the law of $(W_{\sigma^2 M'}([0, t]))_{t \geq 0}$ coincides with $(B_{\sigma^2 M'}([0, t]))_t$, where B is a complex Brownian motion independent of X .

To carry out the proof, we compute the limit of the moments conditionally to X and prove that they match those of the prescribed Gaussian random measure. The first important step is to identify the second moment of $\mathbb{E} \left[|M_\varepsilon^{\gamma, \beta}(A)|^2 | X \right]$ in each case. This work is done in Sections 4.2 to 4.4.

Then in a second step, we conclude the proof of the above theorems in Section 4.5. The main idea is to consider the higher order moments and to show that they are those of a Gaussian variable. The main technical material on these moments is gathered in appendix A.

Remark 4.4. *In dimension 1, it is enough to assume k to be twice differentiable on some interval $[0, \delta]$ for some $\delta > 0$. The left and right derivative at 0 need not be the same. For instance, one can consider the kernel $k(u) = (1 - |u|)_+$ in dimension 1.*

4.2 The second moments

Let us first focus on computing the second moment. We want to prove the following:

Proposition 4.5. *Let A be some compact ball with a non-empty interior.*

- *When $\gamma^2 + \beta^2 > d$, $\gamma < \sqrt{\frac{d}{2}}$, we have the following \mathbb{L}_1 -convergence*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| \varepsilon^{\gamma^2 - d/2} M_\varepsilon^{\gamma, \beta}(A) \right|^2 \mid X \right] = \sigma^2(\gamma^2 + \beta^2) M^{2\gamma, 0}(A). \quad (4.5)$$

where

$$\sigma^2(\gamma^2 + \beta^2) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{\gamma^2 + \beta^2 - d} \int_{|r| \leq 1} \frac{dr}{G_\varepsilon(r)^{\beta^2 + \gamma^2}}. \quad (4.6)$$

- *When $\gamma^2 + \beta^2 = d$, $\gamma < \sqrt{\frac{d}{2}}$, we have the following \mathbb{L}_1 -convergence*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| \varepsilon^{\gamma^2 - d/2} |\ln(\varepsilon)|^{-1/2} M_\varepsilon^{\gamma, \beta}(A) \right|^2 \mid X \right] = \sigma^2(1) M^{2\gamma, 0}(A). \quad (4.7)$$

where

$$\sigma^2(1) = 2 \lim_{\varepsilon \rightarrow \infty} |\ln \varepsilon|^{-1} \int_0^1 \frac{dr}{G_\varepsilon(r)}. \quad (4.8)$$

Proposition 4.6. *Let A be some compact ball with non empty interior. When $\gamma^2 + \beta^2 > d$, $\gamma = \sqrt{\frac{d}{2}}$, we have the following convergence in probability*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\left| |\ln \varepsilon|^{1/4} M_\varepsilon^{\gamma, \beta}(A) \right|^2 \mid X \right] = \sigma^2(\gamma^2 + \beta^2) \sqrt{\frac{2}{\pi}} M'(A). \quad (4.9)$$

First, we point out that computing the limits (4.6) and (4.8) yields the expression of Definition 4.1. This point will be shown in our proofs.

The proof of the above two propositions will be the object of Sections 4.2 to 4.4. For simplicity, we will consider the case $d = 1$ since the general case is a straightforward adaptation of this case. Computing the variance of $M(A)$ for an arbitrary interval is not more difficult than dealing with the case $A = [0, 1]$ so that we will only consider this case in the proof. Before going to the core of the proof (in which we will have to deal with the II/III case separately) let us exhibit a more user-friendly expression for $\mathbb{E}[|M_\varepsilon^{\gamma, \beta}(A)|^2 \mid \mathcal{F}^X]$:

$$\mathbb{E} \left[\left| \varepsilon^{\gamma^2 - 1/2} M_\varepsilon^{\gamma, \beta}([0, 1]) \right|^2 \mid \mathcal{F}^X \right] = \varepsilon^{2\gamma^2 + \beta^2 - 1} \int_{[0, 1]^2} \frac{M_\varepsilon^{\gamma, 0}(dx) M_\varepsilon^{\gamma, 0}(dy)}{G_\varepsilon(x - y)^{\beta^2}}. \quad (4.10)$$

The right-hand side of (4.10) can be rewritten as

$$\varepsilon^{\gamma^2 + \beta^2 - 1} \int_{[0, 1]^2} \frac{\exp\left(\gamma(X_\varepsilon(x) + X_\varepsilon(y)) - \frac{\gamma^2}{2} \mathbb{E}[(X_\varepsilon(x) + X_\varepsilon(y))^2]\right)}{G_\varepsilon(x - y)^{\beta^2 + \gamma^2}} dx dy. \quad (4.11)$$

By symmetry the integral is equal to $2 \int_{\{y \geq x\}}$ (of course we cannot do anything of the kind when $d \geq 2$ but this step is just performed for notational convenience here and there is nothing crucial about it). With the change of variables $y - x \rightarrow r$ and setting

$$X_{\varepsilon,r}(z) = X_\varepsilon(z+r) + X_\varepsilon(z), \quad (4.12)$$

we obtain that (4.11) is equal to

$$2 \int_0^1 \frac{1}{G_\varepsilon(r)^{\beta^2+\gamma^2}} \left(\int_0^{1-r} \exp \left(\gamma X_{\varepsilon,r}(z) - \frac{\gamma^2}{2} \mathbb{E}[(X_{\varepsilon,r}(z))^2] \right) dz \right) dr = 2 \int_0^1 \frac{1}{G_\varepsilon(r)^{\beta^2+\gamma^2}} \hat{M}_{\varepsilon,r}^\gamma dr. \quad (4.13)$$

where we have defined

$$\begin{aligned} \hat{M}_{\varepsilon,r}^\gamma &= \int_0^{1-r} \exp \left(\gamma X_{\varepsilon,r}(z) - \frac{\gamma^2}{2} \mathbb{E}[(X_{\varepsilon,r}(z))^2] \right) dz, \\ \text{and } M_{\varepsilon,r}^\gamma &= \int_0^1 \exp \left(\gamma X_{\varepsilon,r}(z) - \frac{\gamma^2}{2} \mathbb{E}[(X_{\varepsilon,r}(z))^2] \right) dz. \end{aligned} \quad (4.14)$$

As a conclusion we have

$$\mathbb{E} \left[\left| \varepsilon^{\gamma^2-1/2} M_\varepsilon^{\gamma,\beta}([0,1]) \right|^2 \mid \mathcal{F}^X \right] = \varepsilon^{\gamma^2+\beta^2-1} \int_0^1 \frac{2}{G_\varepsilon(r)^{\beta^2+\gamma^2}} \hat{M}_{\varepsilon,r}^\gamma dr. \quad (4.15)$$

Similarly when $\gamma^2 + \beta^2 = 1$, we have

$$\mathbb{E} \left[\left| \varepsilon^{\gamma^2-1/2} |\ln \varepsilon|^{-1/2} M_\varepsilon^{\gamma,\beta}([0,1]) \right|^2 \mid \mathcal{F}^X \right] = |\ln \varepsilon|^{-1} \int_0^1 \frac{2}{G_\varepsilon(r)} \hat{M}_{\varepsilon,r}^\gamma dr. \quad (4.16)$$

The first key tool of the argument relies on the following lemma, the proof of which is straightforward and thus left to the reader:

Lemma 4.7. *We can write:*

$$G_\varepsilon(\varepsilon t) = \varepsilon f(t) g_\varepsilon(t)$$

with

$$f(t) = e^{\int_0^{|t|} \frac{1-k(u)}{u} du} \quad \text{and} \quad g_\varepsilon(t) = e^{-\int_0^{\varepsilon t} \frac{1-k(u)}{u} du}.$$

The function f is continuous on \mathbb{R} and $f(t) \simeq \ln |t|$ for t large. Furthermore, for each fixed t , $g_\varepsilon(t) \rightarrow 1$ as $\varepsilon \rightarrow 0$ and $\sup_{\varepsilon|t| \leq 1} |\ln g_\varepsilon(t)| = C < \infty$.

In (4.15) and (4.16), we integrate $\hat{M}_{\varepsilon,r}^\gamma$ w.r.t. to a measure on $[0,1]$: either $\varepsilon^{\gamma^2+\beta^2-1} \frac{2}{G_\varepsilon(r)^{\beta^2+\gamma^2}} dr$ or $|\ln \varepsilon|^{-1} \frac{2}{G_\varepsilon(r)} dr$. In both cases the total mass of the measure is of order one and converges when $\varepsilon \rightarrow 0$. Furthermore when $\varepsilon \rightarrow 0$ most of the mass of these measures is supported by small r , even by r that are of order ε in the case $\gamma^2 + \beta^2 > 1$.

Now the idea is that $X_{\varepsilon,r}(z) \approx 2X_\varepsilon$ when r is small and hence that $\hat{M}_{\varepsilon,r}^\gamma$ (or $M_{\varepsilon,r}^\gamma$) can be replaced by $M_\varepsilon^{2\gamma}([0,1])$ in the integral. This corresponds to proving the following lemma

Lemma 4.8. *For $\gamma < 1/\sqrt{2}$, we have the following convergence in \mathbb{L}_1 ,*

$$\lim_{\varepsilon,s \rightarrow 0} \hat{M}_{\varepsilon,s}^\gamma = \lim_{\varepsilon,s \rightarrow 0} M_{\varepsilon,s}^\gamma = M^{2\gamma,0}([0,1]) \quad (4.17)$$

When $\gamma = 1/\sqrt{2}$, $M^{2\gamma,0}([0,1])$ has to be replaced by the derivative martingale, which is not in \mathbb{L}_1 so that there is no hope for an equivalent statement to hold. The case is treated separately in Section 4.4.

Proof of Proposition 4.5 using Lemma 4.8. We first show (4.6) and (4.7). Using the definition of G_ε we can write

$$\varepsilon^{\gamma^2+\beta^2-1} \int_0^1 \frac{dr}{G_\varepsilon(r)^{\beta^2+\gamma^2}} = \varepsilon^{-1} \int_0^1 dr \exp\left((\gamma^2 + \beta^2) \int_1^{\varepsilon^{-1}} \frac{k(ur) - 1}{u} du\right). \quad (4.18)$$

Then we use first the change of variables $r \rightarrow r' = r\varepsilon^{-1}$ and $u \rightarrow u' = u\varepsilon$ and the above integral becomes

$$\int_0^{1/\varepsilon} dr' \exp\left((\gamma^2 + \beta^2) \int_\varepsilon^1 \frac{k(u'r') - 1}{u'} du'\right), \quad (4.19)$$

and then we obtain (4.6) by taking the limit when $\varepsilon \rightarrow 0$ and applying standard integration theorems.

For (4.7), we have

$$\begin{aligned} |\ln \varepsilon|^{-1} \int_0^1 \frac{dr}{G_\varepsilon(r)} &= |\ln \varepsilon|^{-1} \int_0^{|\ln \varepsilon|^{1/2}} \frac{dr}{G_\varepsilon(r)} + |\ln \varepsilon|^{-1} \int_{|\ln \varepsilon|^{-1/2}}^1 \frac{dr}{G_\varepsilon(r)} \\ &\quad + |\ln \varepsilon|^{-1} \int_{|\ln \varepsilon|^{1/2}}^{|\ln \varepsilon|^{-1/2}} \frac{dr}{r} \exp\left(\int_1^{\varepsilon^{-1}} \frac{k(ur) - \mathbb{1}_{[0,r^{-1}]}(u)}{u} du\right). \end{aligned} \quad (4.20)$$

The first term tends to zero because $G_\varepsilon(r) \geq C\varepsilon$, the second term as well because $G_\varepsilon(r) \geq C/r$. To prove the convergence of the third term, we perform the change of variables $u \rightarrow u' = ur$ to obtain

$$\int_{|\ln \varepsilon|^{1/2}}^{|\ln \varepsilon|^{-1/2}} \frac{dr}{r} \exp\left(\int_r^{\varepsilon^{-1}r} \frac{k(u') - \mathbb{1}_{[0,1]}(u')}{u'} du'\right). \quad (4.21)$$

The term in the exponential converges uniformly to

$$\int_0^\infty \frac{k(u') - \mathbb{1}_{[0,1]}(u')}{u'} du'$$

on the domain of integration when ε tends to zero and then integrating with respect to r cancels the $|\ln \varepsilon|^{-1}$ in front of the integral.

Now what remains to prove is that in \mathbb{L}_1 :

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^{\gamma^2+\beta^2-1} \int_0^1 \frac{1}{G_\varepsilon(r)^{\beta^2+\gamma^2}} |\hat{M}_{\varepsilon,r}^\gamma - M^{2\gamma,0}([0,1])| dr &= 0, \quad \text{when } \gamma + \beta < 1 \\ \lim_{\varepsilon \rightarrow 0} |\ln \varepsilon|^{-1} \int_0^1 \frac{1}{G_\varepsilon(r)} |\hat{M}_{\varepsilon,r}^\gamma - M^{2\gamma,0}([0,1])| dr &= 0 \quad \text{when } \gamma + \beta = 1 \end{aligned} \quad (4.22)$$

Let δ be arbitrarily small. According to Lemma 4.8 we can find η such that if $\max(r, \varepsilon) \leq \eta$ and

$$|\hat{M}_{\varepsilon,r}^\gamma - M^{2\gamma,0}([0,1])| \leq \delta$$

Then for all $\varepsilon < \eta$ we have

$$\begin{aligned} \mathbb{E} \left[\varepsilon^{\gamma^2 + \beta^2 - 1} \int_0^1 \frac{2}{G_\varepsilon(r)^{\beta^2 + \gamma^2}} |\hat{M}_{\varepsilon,r}^\gamma - M^{2\gamma,0}([0,1])| dr \right] \\ \leq \delta \varepsilon^{\gamma^2 + \beta^2 - 1} \int_0^1 \frac{2}{G_\varepsilon(r)^{\beta^2 + \gamma^2}} dr + \int_{r \in [\eta,1]} \frac{2}{G_\varepsilon(r)^{\beta^2 + \gamma^2}} dr. \end{aligned} \quad (4.23)$$

According the proof of (4.6), the first term is smaller than a constant times δ and the second one can be made arbitrary small for a fixed η if ε is taken sufficiently small. is enough to conclude as δ is arbitrary. The other limit can be proven in the same manner. \square

4.3 Proof of Lemma 4.8

First let us notice that it is sufficient to prove the result for $M_{\varepsilon,s}$ as

$$\mathbb{E} \left[|M_{\varepsilon,s} - \hat{M}_{\varepsilon,s}| \right] = s \quad (4.24)$$

and thus tends to zero.

If $\varepsilon \leq \sqrt{s}$, we write

$$|M_{\varepsilon,s}^\gamma - M^{2\gamma,0}| \leq |M_{\varepsilon,s}^\gamma - M_{\sqrt{s},s}^\gamma| + |M_{\sqrt{s},s}^\gamma - M_{\sqrt{s},s}^{2\gamma,0}[0,1]| + |M_{\sqrt{s},s}^{2\gamma,0}[0,1] - M^{2\gamma,0}[0,1]| \quad (4.25)$$

and if $\varepsilon > \sqrt{s}$,

$$|M_{\varepsilon,s}^\gamma - M^{2\gamma,0}[0,1]| \leq |M_{\varepsilon,s}^\gamma - M_\varepsilon^{2\gamma,0}[0,1]| + |M_\varepsilon^{2\gamma,0}[0,1] - M^{2\gamma,0}[0,1]|. \quad (4.26)$$

The last terms in the r.h.s. of both (4.25) and (4.26) converge to 0 in \mathbb{L}_1 as the martingale $(M_{e^{-t}}^{2\gamma,0})_{t \geq 0}$ is uniformly integrable (see e.g. [35]) and we just have to care about the other terms. For the second term in (4.25), we use the following result

Lemma 4.9. *Let (X, Y) be a centered Gaussian vector. There exists a universal constant C such that*

$$\mathbb{E} \left[\left| e^{X - \mathbb{E}[X^2]/2} - e^{Y - \mathbb{E}[Y^2]/2} \right| \right] \leq C \sqrt{\mathbb{E}[(X - Y)^2]}. \quad (4.27)$$

Proof. We use the Girsanov formula for the measure tilted by X and we obtain

$$\begin{aligned} \mathbb{E} \left[\left| e^{X - \mathbb{E}[X^2]/2} - e^{Y - \mathbb{E}[Y^2]/2} \right| \right] &= \mathbb{E} \left[\left| e^{X - \mathbb{E}[X^2]} \left| 1 - e^{Y - X - \mathbb{E}[Y^2]/2 + \mathbb{E}[X^2]/2} \right| \right| \right] \\ &= \mathbb{E} \left[\left| 1 - e^{Y + \mathbb{E}[XY] - X - \mathbb{E}[X^2] - \mathbb{E}[Y^2]/2 + \mathbb{E}[X^2]/2} \right| \right] = \mathbb{E} \left[\left| 1 - e^{(X - Y) - \mathbb{E}[(X - Y)^2]/2} \right| \right]. \end{aligned} \quad (4.28)$$

Then the reader can check that the last term can be bounded by $C \sqrt{\mathbb{E}[(X - Y)^2]}$ (it is sufficient to check it when $(X - Y)$ has small variance because it is always smaller than 2). \square

By Jensen's inequality and stationarity, we have

$$|M_{\sqrt{s},s}^\gamma - M_{\sqrt{s}}^{2\gamma,0}[0,1]| \leq \int_0^1 \mathbb{E} \left[\left| e^{\gamma X_{\sqrt{s},s}(x) - \frac{\gamma^2}{2} \mathbb{E}[X_{\sqrt{s},s}(x)^2]} - e^{2\gamma X_{\sqrt{s}}(x) - 2\gamma^2 \mathbb{E}[X_{\sqrt{s}}(x)^2]} \right| \right] dx \quad (4.29)$$

$$\leq C \sqrt{\mathbb{E} \left[X_{\sqrt{s},s}(0) - X_{\sqrt{s}}(0) \right]} \leq C' \sqrt{s}, \quad (4.30)$$

which tends to zero. The same computation allow to control also the first term in (4.26)

Finally, we control the $|M_{\varepsilon',s}^\gamma - M_{\sqrt{s},s}^\gamma|$ term in (4.25) by proving the following Lemma that ensures that the $(M_{\varepsilon',s}^\gamma)_{\varepsilon \geq 0}$ is uniformly Cauchy in \mathbb{L}^q for some $q > 1$ (and hence in \mathbb{L}^1) when $\varepsilon \rightarrow 0$.

Lemma 4.10. *For any $\gamma < 1/\sqrt{2}$, for any $q \in (1, \min(2, \frac{1}{2\gamma^2}))$, there exists $C > 0$ and $\alpha(q, \gamma) > 0$ (not depending on s , but only on q) such that for any $\varepsilon > \varepsilon' > 0$ we have*

$$\mathbb{E} \left[|M_{\varepsilon',s}^\gamma - M_{\varepsilon,s}^\gamma|^q \right]^{1/q} \leq C\varepsilon^\alpha \quad (4.31)$$

Proof. In the proof we always consider that ε is small enough. It is sufficient to prove the result for $\varepsilon' \in [\varepsilon/2, \varepsilon)$ as then we can use telescopic sums i.e. if $\varepsilon' \in [2^{-(m+1)}\varepsilon, 2^{-m}\varepsilon)$, we have

$$\begin{aligned} \mathbb{E} \left[|M_{\varepsilon',s}^\gamma - M_{\varepsilon,s}^\gamma|^q \right]^{1/q} &\leq \sum_{k=0}^{m-1} \mathbb{E} \left[|M_{2^{-(k+1)}\varepsilon,s}^\gamma - M_{2^{-k}\varepsilon,s}^\gamma|^q \right]^{1/q} + \mathbb{E} \left[|M_{\varepsilon',s}^\gamma - M_{2^{-m}\varepsilon,s}^\gamma|^q \right]^{1/q} \\ &\leq C\varepsilon^\alpha \sum_{k=0}^m 2^{-k\alpha}. \end{aligned} \quad (4.32)$$

Recall that \mathcal{F}_ε is the σ -algebra generated by $X_t, t \geq \varepsilon$. We have for all $q \in (1, 2)$:

$$\mathbb{E} \left[|M_{\varepsilon',s}^\gamma - M_{\varepsilon,s}^\gamma|^q \right] \leq \mathbb{E} \left[\mathbb{E} \left[(M_{\varepsilon',s}^\gamma - M_{\varepsilon,s}^\gamma)^2 \mid \mathcal{F}_\varepsilon \right]^{q/2} \right] \quad (4.33)$$

The conditional expectation $\mathbb{E} \left[(M_{\varepsilon',s}^\gamma - M_{\varepsilon,s}^\gamma)^2 \mid \mathcal{F}_\varepsilon \right]$ is in fact a conditional variance as $\mathbb{E}[M_{\varepsilon',s}^\gamma \mid \mathcal{F}_\varepsilon] = M_{\varepsilon,s}^\gamma$. Then the conditional variance can be expanded as a double integral:

$$\begin{aligned} &\int_{[0,1]^2} dx dy e^{\gamma(X_{\varepsilon,s}(x) + X_{\varepsilon,s}(y)) - \frac{\gamma^2}{2}\mathbb{E}[X_{\varepsilon,s}^2(x) + X_{\varepsilon,s}^2(y)]} \\ &\times \text{Cov} \left(e^{\gamma(X_{\varepsilon',s} - X_{\varepsilon,s})(x) - \frac{\gamma^2}{2}\mathbb{E}[(X_{\varepsilon',s} - X_{\varepsilon,s})^2(x)]}, e^{\gamma(X_{\varepsilon',s} - X_{\varepsilon,s})(y) - \frac{\gamma^2}{2}\mathbb{E}[(X_{\varepsilon',s} - X_{\varepsilon,s})^2(y)]} \right) \end{aligned} \quad (4.34)$$

Now we try to control the covariance term in x and y , with a simple function. We have

$$\begin{aligned} &\text{Cov} \left(e^{\gamma(X_{\varepsilon',s} - X_{\varepsilon,s})(x) - \frac{\gamma^2}{2}\mathbb{E}[(X_{\varepsilon',s} - X_{\varepsilon,s})^2(x)]}, e^{\gamma(X_{\varepsilon',s} - X_{\varepsilon,s})(y) - \frac{\gamma^2}{2}\mathbb{E}[(X_{\varepsilon',s} - X_{\varepsilon,s})^2(y)]} \right) \\ &= e^{\gamma^2\mathbb{E}((X_{\varepsilon',s} - X_{\varepsilon,s})(x)(X_{\varepsilon',s} - X_{\varepsilon,s})(y))} - 1 \\ &\leq C\mathbb{E}((X_{\varepsilon',s} - X_{\varepsilon,s})(x)(X_{\varepsilon',s} - X_{\varepsilon,s})(y)), \end{aligned} \quad (4.35)$$

where the last inequality uses the fact that $\varepsilon/2 \leq \varepsilon'$ (so that the covariances are uniformly bounded by $4\ln 2$). Then using translation invariance we have

$$\begin{aligned} &\mathbb{E}((X_{\varepsilon',s} - X_{\varepsilon,s})(x)(X_{\varepsilon',s} - X_{\varepsilon,s})(y)) \\ &= 2\mathbb{E}((X_{\varepsilon'} - X_\varepsilon)(x)(X_{\varepsilon'} - X_\varepsilon)(y)) + \mathbb{E}((X_{\varepsilon'} - X_\varepsilon)(x+s)(X_{\varepsilon'} - X_\varepsilon)(y)) \\ &\quad + \mathbb{E}((X_{\varepsilon'} - X_\varepsilon)(x-s)(X_{\varepsilon'} - X_\varepsilon)(y)). \end{aligned} \quad (4.36)$$

The first term is equal to

$$\int_{1/\varepsilon}^{1/\varepsilon'} \frac{k(u(x-y))}{u} du \leq \ln 2 \max_{v \geq \frac{|x-y|}{\varepsilon}} k(v) := g((x-y)/\varepsilon), \quad (4.37)$$

and one can find similar bounds for the two others. Note that by our assumption on k , g is an integrable function. Using the inequality $ab \leq a^2/2 + b^2/2$ together with the symmetry in x and y for the exponential term in (4.34) and combining it with the bound we have just obtained for the covariance we obtain

$$\begin{aligned} \mathbb{E}[(M_{\varepsilon',s}^\gamma - M_{\varepsilon,s}^\gamma)^2 \mid \mathcal{F}_\varepsilon] &\leq C \int_{[0,1]^2} e^{2\gamma X_{\varepsilon,s}(x) - \gamma^2 \mathbb{E}[X_{\varepsilon,s}^2(x)]} \\ &\quad \times \left(2g\left(\frac{x-y}{\varepsilon}\right) + g\left(\frac{x-y-s}{\varepsilon}\right) + g\left(\frac{x-y+s}{\varepsilon}\right) \right) dx dy. \end{aligned} \quad (4.38)$$

Now we can extend the above integral to $y \in \mathbb{R}$ to obtain an upper bound (and then three terms in the second line above give the same contribution by translation invariance), and make the change of variable $z = \frac{y-x}{\varepsilon}$ to obtain

$$\begin{aligned} \mathbb{E}[(M_{\varepsilon',s}^\gamma - M_{\varepsilon,s}^\gamma)^2 \mid \mathcal{F}_\varepsilon] &\leq C\varepsilon \left(\int_0^1 e^{2\gamma X_{\varepsilon,s}(x) - \gamma^2 \mathbb{E}[X_{\varepsilon,s}^2(x)]} dx \right) \left(\int_{-\infty}^{\infty} g(z) dz \right) \\ &\leq C'\varepsilon \left(\int_0^1 e^{2\gamma X_{\varepsilon,s}(x) - \gamma^2 \mathbb{E}[X_{\varepsilon,s}^2(x)]} dx \right). \end{aligned} \quad (4.39)$$

Plugging this inequality into (4.33), we have

$$\mathbb{E} \left[|M_{\varepsilon',s}^\gamma - M_{\varepsilon,s}^\gamma|^q \right] \leq C\varepsilon^{q/2} \mathbb{E} \left[\left(\int_0^1 e^{2\gamma X_{\varepsilon,s}(x) - \gamma^2 \mathbb{E}[X_{\varepsilon,s}^2(x)]} dx \right)^{q/2} \right]. \quad (4.40)$$

Now we split the interval $[0, 1]$ into $K = \lceil \varepsilon^{-1} \rceil$ disjoint intervals of length $1/K$ setting

$$I_\varepsilon^j := [(j-1)/K, j/K], \quad j = 1..K$$

Using the inequality $(\sum a_i)^\theta \leq \sum a_i^\theta$ for $\theta < 1$ we have

$$\mathbb{E} \left[\left(\int_0^1 e^{2\gamma X_{\varepsilon,s}(x) - \gamma^2 \mathbb{E}[X_{\varepsilon,s}^2(x)]} dx \right)^{q/2} \right] = K \mathbb{E} \left[\left(\int_{I_\varepsilon^0} e^{2\gamma X_{\varepsilon,s}(x) - \gamma^2 \mathbb{E}[X_{\varepsilon,s}^2(x)]} dx \right)^{q/2} \right]. \quad (4.41)$$

Using the Hölder inequality we have, for any $p > 1$ (let $p' = p/(p-1)$ denote its Hölder conjugate),

$$\begin{aligned} &\mathbb{E} \left[\left(\int_{I_\varepsilon^0} e^{2\gamma X_{\varepsilon,s}(x) - \gamma^2 \mathbb{E}[X_{\varepsilon,s}^2(x)]} dx \right)^{q/2} \right] \\ &\leq (1/K)^{q/2} \mathbb{E} \left[e^{\gamma p q X_{\varepsilon,s}(0) - \frac{p q \gamma^2}{2} \mathbb{E}[X_{\varepsilon,s}^2(x)]} \right]^{1/p} \mathbb{E} \left[\left(K \int_{I_\varepsilon^0} e^{2\gamma(X_{\varepsilon,s}(x) - X_{\varepsilon,s}(0))} dx \right)^{p' q/2} \right]^{1/p'}. \end{aligned} \quad (4.42)$$

By Jensen's inequality, if $p'q/2 > 1$ (which holds whenever $p \leq 2$ as $q > 1$) the last factor is smaller than

$$\mathbb{E} \left[K \int_{I_\varepsilon^0} e^{\gamma p' q (X_{\varepsilon,s}(x) - X_{\varepsilon,s}(0))} dx \right]^{1/p'} \leq \exp \left(\frac{\gamma^2 p' q^2}{2} \max_{x \in [0, 1/K]} \mathbb{E}[(X_{\varepsilon,s}(x) - X_{\varepsilon,s}(0))^2] \right).$$

which bounded above by a constant that does not depend on ε . On the other hand the first term is equal to

$$e^{\frac{\gamma^2 q(pq-1)}{2}} \mathbb{E}[X_{\varepsilon,s}^2(x)] \leq \varepsilon^{-2\gamma^2 q(pq-1)}$$

because $\mathbb{E}[X_{\varepsilon,s}(x)^2] \leq 4\mathbb{E}[X_\varepsilon(x)^2] = 4|\ln \varepsilon|$. In the end, combining (4.40), (4.41) and (4.42) (recall that $K \leq 2\varepsilon^{-1}$), we get that

$$\mathbb{E} \left[|M_{\varepsilon',s}^\gamma - M_{\varepsilon,s}^\gamma|^q \right] \leq C\varepsilon^{q-1-2\gamma^2 q(pq-1)}. \quad (4.43)$$

Then the result is proved provided one can find $p > 1$ such that $q - 1 > 2\gamma^2 q(pq - 1)$, which is possible whenever $q < 1/(2\gamma^2)$. \square

4.4 Proof of Proposition 4.6

Recall (4.15). The aim of this Section is to show that in probability

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^{\beta^2-1/2} \int_0^1 \frac{2dr}{G_\varepsilon(r)^{\beta^2+1/2}} |\ln \varepsilon|^{1/2} \hat{M}_{\varepsilon,r}^{1/\sqrt{2}} \\ = \left(\lim_{\varepsilon \rightarrow 0} \varepsilon^{\beta^2-1/2} \int_0^1 \frac{2dr}{G_\varepsilon(r)^{\beta^2+1/2}} \right) \sqrt{\frac{2}{\pi}} M'[0,1] = \sigma^2(1/2 + \beta^2) \sqrt{\frac{2}{\pi}} M'[0,1]. \end{aligned} \quad (4.44)$$

The second equality is in fact (4.6) and has already been proved. In fact, we can replace $\hat{M}_{\varepsilon,r}^{1/\sqrt{2}}$ by $M_{\varepsilon,r}^{1/\sqrt{2}}$ in the l.h.s. as

$$\int_0^1 \varepsilon^{\beta^2-1/2} \frac{2dr}{G_\varepsilon(r)^{\beta^2+1/2}} |\ln \varepsilon|^{1/2} \mathbb{E}[|\hat{M}_{\varepsilon,r}^{1/\sqrt{2}} - M_{\varepsilon,r}^{1/\sqrt{2}}|] = \int_0^1 \varepsilon^{\beta^2-1/2} \frac{2rdr}{G_\varepsilon(r)^{\beta^2+1/2}} |\ln \varepsilon|^{1/2}. \quad (4.45)$$

which tends to 0 thanks to Lemma 4.7. Hence what remains to prove is that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\beta^2-1/2} \left| \int_0^1 \frac{dr}{G_\varepsilon(r)^{\beta^2+1/2}} \left(|\ln \varepsilon|^{1/2} M_{\varepsilon,r}^{1/\sqrt{2}} - \sqrt{\frac{2}{\pi}} M'[0,1] \right) \right| = 0 \quad (4.46)$$

or with a change of variable $r \rightarrow s = r/\varepsilon$

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^{\varepsilon^{-1}} \frac{ds}{(G_\varepsilon(\varepsilon s)/\varepsilon)^{\beta^2+1/2}} \left(|\ln \varepsilon|^{1/2} M_{\varepsilon,\varepsilon s}^{1/\sqrt{2}} - \sqrt{\frac{2}{\pi}} M'[0,1] \right) \right| = 0. \quad (4.47)$$

We will split the proof of (4.47) in two lemmas: one taking care of the contribution of the small s in the integral (which is the most important step) and another one controlling the contribution of the larger s :

Lemma 4.11. *We have in probability*

$$\lim_{A \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \left| \int_A^{\varepsilon^{-1}} \frac{ds}{(G_\varepsilon(\varepsilon s)/\varepsilon)^{\beta^2+1/2}} \left(|\ln \varepsilon|^{1/2} M_{\varepsilon,\varepsilon s}^{1/\sqrt{2}} - \sqrt{\frac{2}{\pi}} M'[0,1] \right) \right| = 0 \quad (4.48)$$

Lemma 4.12. *For all A , we have in probability*

$$\lim_{\varepsilon \rightarrow 0} \left| \int_0^A \frac{ds}{(G_\varepsilon(\varepsilon s)/\varepsilon)^{\beta^2+1/2}} \left(|\ln \varepsilon|^{1/2} M_{\varepsilon,\varepsilon s}^{1/\sqrt{2}} - \sqrt{\frac{2}{\pi}} M'[0,1] \right) \right| = 0 \quad (4.49)$$

The limit (4.47) results from a combination of (4.48) and (4.49). Before proceeding with the proofs of these lemmas, let us introduce some notations. In what follows, for some $\kappa > 0$, we will consider the set

$$\begin{aligned} B_\kappa &= \left\{ \sup_{x \in [0,1]} \sup_{u \in [0,1]} X_u(x) + \sqrt{2} \ln u \leq \kappa \right\}, \\ B_{\kappa, \varepsilon} &= \left\{ \sup_{x \in [0,1]} \sup_{u \in [\varepsilon,1]} X_u(x) + \sqrt{2} \ln u \leq \kappa \right\}. \end{aligned} \quad (4.50)$$

Obviously $B_\kappa \subset B_{\kappa, \varepsilon}$ for all $\varepsilon > 0$, and it is proved in [22, Proposition 19] that

$$\mathbb{P}(B_\kappa) \rightarrow 1 \quad \text{as } \kappa \rightarrow \infty.$$

For $x \in [0, 1]$ and $\kappa > 0$, we also introduce the stopping time

$$\tau_x^\kappa = \sup\{u \in]0, 1]; X_u(x) + \sqrt{2} \ln u > \kappa\}.$$

It is readily seen that, on B_κ , we have $\tau_x^\kappa = 0$ for all $x \in [0, 1]$.

Proof of Lemma 4.11. The reader can check using Lemma 4.7 that

$$\lim_{A \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \int_A^{\varepsilon^{-1}} \frac{ds}{(G_\varepsilon(\varepsilon s)/\varepsilon)^{\beta^2+1/2}} = 0.$$

Hence what we have to prove is

$$\lim_{A \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \int_A^{\varepsilon^{-1}} \frac{ds}{(G_\varepsilon(\varepsilon s)/\varepsilon)^{\beta^2+1/2}} |\ln \varepsilon|^{1/2} M_{\varepsilon, \varepsilon s}^{1/\sqrt{2}} = 0 \quad \text{in probability.} \quad (4.51)$$

To do so, it is enough to show that the expectation of the above expression restricted to the event B_κ vanishes, for all κ , as $A \rightarrow \infty$ and $\varepsilon \rightarrow 0$. The key point is to prove that there exists a constant C such that for all t :

$$\mathbb{E}[M_{\varepsilon, \varepsilon t}^{1/\sqrt{2}} \mathbf{1}_{B_\kappa}] \leq \sqrt{\frac{2}{\pi}} \frac{\kappa + \sqrt{2}/2(1 + \ln(Ct))}{|\ln \varepsilon|^{1/2}}. \quad (4.52)$$

Indeed, this implies that for all $\kappa > 0$

$$\begin{aligned} & \lim_{A \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\mathbf{1}_{B_\kappa} \int_A^{\varepsilon^{-1}} \frac{ds}{(G_\varepsilon(\varepsilon s)/\varepsilon)^{\beta^2+1/2}} |\ln \varepsilon|^{1/2} M_{\varepsilon, \varepsilon s}^{1/\sqrt{2}} \right] \\ & \leq \lim_{A \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \sqrt{2/\pi} \int_A^{\varepsilon^{-1}} \frac{\kappa + \sqrt{2}/2(1 + \ln(Ct))}{(G_\varepsilon(\varepsilon s)/\varepsilon)^{\beta^2+1/2}} \\ & = 0. \end{aligned}$$

Let us now prove (4.52).

$$\begin{aligned} \mathbb{E}[M_{\varepsilon, \varepsilon t}^{1/\sqrt{2}} \mathbf{1}_{B_\kappa}] &= \int_0^1 \mathbb{E} \left[\mathbf{1}_{B_\kappa} e^{1/\sqrt{2} X_{\varepsilon, \varepsilon t}(z) - \frac{1}{4} \mathbb{E}[X_{\varepsilon, \varepsilon t}(z)^2]} \right] dz \\ &\leq \int_0^1 \mathbb{E} \left[\mathbf{1}_{\forall u \in [\varepsilon, 1] X_u(z) + \sqrt{2} \ln u \leq \kappa} e^{\sqrt{2} X_{\varepsilon, \varepsilon t}(z) - \frac{1}{4} \mathbb{E}[X_{\varepsilon, \varepsilon t}(z)^2]} \right] dz. \end{aligned}$$

By the Girsanov formula, the expectation in the r.h.s. of the above expectation is equal to (recall that K_ε denotes the covariance kernel of X_ε):

$$\mathbb{P}\left(\sup_{u \in [\varepsilon, 1]} X_u(z) + \sqrt{2}/2 \ln u + \sqrt{2}/2 K_u(\varepsilon t) \leq \kappa\right). \quad (4.53)$$

Because of Assumption (A), we may find a constant C such that $|k(0) - k(x)| \leq C|x|$ for all $x \in \mathbb{R}$. Thus we have for $u \in [\varepsilon, 1]$:

$$\begin{aligned} \ln \frac{1}{u} - K_u(\varepsilon t) &= \int_1^{u^{-1}} \frac{1 - k(vt\varepsilon)}{v} dv \leq \int_\varepsilon^1 \frac{1 - k(vt)}{v} dv \leq \int_\varepsilon^{(Ct)^{-1}} Ct dv + \int_{(Ct)^{-1}}^1 \frac{dv}{v} \\ &\leq 1 + \ln(Ct). \end{aligned}$$

For some given z , the process $X_u(z)_{u \in [0, 1]}$ has the same law of a time changed standard Brownian motion $(B_{\sqrt{-\ln u}})_{u \in [0, 1]}$ and hence (4.53) is smaller than

$$\mathbb{P}\left(\sup_{s \in [0, |\ln \varepsilon|]} B_s \leq \kappa + \sqrt{2}/2(1 + \ln(Ct))\right) = \sqrt{\frac{2}{\pi}} \frac{\kappa + \sqrt{2}/2(1 + \ln(Ct))}{|\ln \varepsilon|^{1/2}},$$

which is enough to conclude. \square

Proof of Lemma 4.12. We set $\varepsilon'' = \sqrt{\varepsilon}$, $\varepsilon' = \varepsilon \ln(\varepsilon^{-1})$. From [23], we know that $|\ln \varepsilon|^{1/2} \varepsilon M_{\varepsilon'}^{\sqrt{2}, 0}[0, 1]$ converges towards $\sqrt{2/\pi} M'[0, 1]$ (note that $\ln \varepsilon' / \ln \varepsilon \rightarrow 1$) when ε tends to zero and thus it is enough to prove (4.49) with $\sqrt{2/\pi} M'[0, 1]$ replaced by $|\ln \varepsilon|^{1/2} \varepsilon M_{\varepsilon'}^{\sqrt{2}, 0}[0, 1]$.

We introduce $\tilde{X}_{\varepsilon, \varepsilon', s}$ which is an interpolation between $X_{\varepsilon, s}$ and $X_{\varepsilon'}$ (when $\varepsilon \leq \varepsilon' \leq 1$):

$$\tilde{X}_{\varepsilon, \varepsilon', s} := 2X_{\varepsilon'} + (X_{\varepsilon, s} - X_{\varepsilon', s}).$$

We then use the following decomposition

$$\begin{aligned} &\left| \int_0^A \frac{|\ln \varepsilon|^{1/2} ds}{(G_\varepsilon(\varepsilon s)/\varepsilon)^{\beta^2 + 1/2}} \left(M_{\varepsilon, \varepsilon s}^{1/\sqrt{2}} - \varepsilon M_{\varepsilon'}^{\sqrt{2}, 0} \right) \right| \\ &\leq \left| \int_0^A \frac{|\ln \varepsilon|^{1/2} ds}{(G_\varepsilon(\varepsilon s)/\varepsilon)^{\beta^2 + 1/2}} \left(\int_0^1 e^{(1/\sqrt{2})X_{\varepsilon, \varepsilon t}(z) - \frac{1}{4}\mathbb{E}[X_{\varepsilon, \varepsilon t}(z)^2]} - e^{(1/\sqrt{2})\tilde{X}_{\varepsilon, \varepsilon'', \varepsilon t}(z) - \frac{1}{4}\mathbb{E}[\tilde{X}_{\varepsilon, \varepsilon'', \varepsilon t}(z)^2]} dz \right) \right| \\ &+ \left| \int_0^A \frac{|\ln \varepsilon|^{1/2} ds}{(G_\varepsilon(\varepsilon s)/\varepsilon)^{\beta^2 + 1/2}} \left(\int_0^1 e^{(1/\sqrt{2})\tilde{X}_{\varepsilon, \varepsilon'', \varepsilon t}(z) - \frac{1}{4}\mathbb{E}[\tilde{X}_{\varepsilon, \varepsilon'', \varepsilon t}(z)^2]} - e^{(1/\sqrt{2})\tilde{X}_{\varepsilon, \varepsilon', \varepsilon t}(z) - \frac{1}{4}\mathbb{E}[\tilde{X}_{\varepsilon, \varepsilon', \varepsilon t}(z)^2]} dz \right) \right| \\ &+ \left| \int_0^A \frac{|\ln \varepsilon|^{1/2} ds}{(G_\varepsilon(\varepsilon s)/\varepsilon)^{\beta^2 + 1/2}} \left(\int_0^1 e^{(1/\sqrt{2})\tilde{X}_{\varepsilon, \varepsilon', \varepsilon t}(z) - \frac{1}{4}\mathbb{E}[\tilde{X}_{\varepsilon, \varepsilon', \varepsilon t}(z)^2]} - e^{\sqrt{2}X_\varepsilon(z) - \mathbb{E}[X_\varepsilon(z)^2]} dz \right) \right| \end{aligned}$$

and show that each of the three terms converges to zero in probability.

The first term converges in \mathbb{L}_1 norm. Indeed, from Lemma 4.9 we have

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^A \frac{|\ln \varepsilon|^{1/2} dt}{(G_\varepsilon(\varepsilon t)/\varepsilon)^{\beta^2+1/2}} \left(\int_0^1 e^{(1/\sqrt{2})X_{\varepsilon,\varepsilon t}(z) - \frac{1}{4}\mathbb{E}[X_{\varepsilon,\varepsilon t}(z)^2]} - e^{(1/\sqrt{2})\tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t}(z) - \frac{1}{4}\mathbb{E}[\tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t}(z)^2]} dz \right) \right| \right] \\ & \leq \int_0^A \frac{|\ln \varepsilon|^{1/2} dt}{(G_\varepsilon(\varepsilon t)/\varepsilon)^{\beta^2+1/2}} \int_0^1 \mathbb{E} \left[\left| e^{(1/\sqrt{2})X_{\varepsilon,\varepsilon t}(z) - \frac{1}{4}\mathbb{E}[X_{\varepsilon,\varepsilon t}(z)^2]} - e^{(1/\sqrt{2})\tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t}(z) - \frac{1}{4}\mathbb{E}[\tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t}(z)^2]} \right| \right] dz \\ & \leq C \int_0^A \frac{|\ln \varepsilon|^{1/2} dt}{(G_\varepsilon(\varepsilon t)/\varepsilon)^{\beta^2+1/2}} \sqrt{\mathbb{E}[(X_{\varepsilon,\varepsilon t}(z) - \tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t}(z))^2]} \leq C' |\ln \varepsilon|^{1/2} \varepsilon^{1/4}. \end{aligned}$$

The second term converges to zero in probability. This is a bit more tricky to show because we do not have \mathbb{L}_1 convergence but we can manage to obtain it by restricting ourselves to the event $B_{\kappa,\varepsilon''}$ with say $\kappa = \ln \ln |\ln \varepsilon|$ (whose probability tends to one). By Jensen's inequality, the expectation of the second term on the event $B_{\kappa,\varepsilon''}$ is smaller than

$$\int_0^A \frac{|\ln \varepsilon|^{1/2} dt}{(G_\varepsilon(\varepsilon t)/\varepsilon)^{\beta^2+1/2}} \int_0^1 \mathbb{E} \left[\mathbf{1}_{B_{\kappa,\varepsilon''}} \left| e^{(1/\sqrt{2})\tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t}(z) - \frac{1}{4}\mathbb{E}[\tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t}(z)^2]} - e^{(1/\sqrt{2})\tilde{X}_{\varepsilon,\varepsilon',\varepsilon t}(z) - \frac{1}{4}\mathbb{E}[\tilde{X}_{\varepsilon,\varepsilon',\varepsilon t}(z)^2]} \right| \right] dz. \quad (4.54)$$

Then, by independence of the increments of $(X_{e^{-t}})_t \geq 0$, we have

$$\begin{aligned} & \mathbb{E} \left[\mathbf{1}_{B_{\kappa,\varepsilon''}} \left| \int_0^1 e^{(1/\sqrt{2})\tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t}(z) - \frac{1}{4}\mathbb{E}[\tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t}(z)^2]} - e^{(1/\sqrt{2})\tilde{X}_{\varepsilon,\varepsilon',\varepsilon t}(z) - \frac{1}{4}\mathbb{E}[\tilde{X}_{\varepsilon,\varepsilon',\varepsilon t}(z)^2]} dz \right| \right] \quad (4.55) \\ & = \int_0^1 \mathbb{E} \left[\mathbf{1}_{B_{\kappa,\varepsilon''}} e^{\sqrt{2}X_{\varepsilon''}(z) - \mathbb{E}[X_{\varepsilon''}(z)^2]} \mathbb{E} \left[\left| e^{(1/\sqrt{2})(\tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t} - 2X_{\varepsilon''})(z) - \frac{1}{4}\mathbb{E}[(\tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t} - 2X_{\varepsilon''})^2](z)} \right. \right. \right. \\ & \quad \left. \left. \left. - e^{(1/\sqrt{2})(\tilde{X}_{\varepsilon,\varepsilon',\varepsilon t} - 2X_{\varepsilon''})(z) - \frac{1}{4}\mathbb{E}[(\tilde{X}_{\varepsilon,\varepsilon',\varepsilon t} - 2X_{\varepsilon''})^2](z)} \right| \right] \right] dz. \end{aligned}$$

By a Girsanov transform one sees that the first factor in the r.h.s. of (4.55) is equal to

$$\mathbb{P} \left[\max_{u \in (\varepsilon'', 1]} X_u(z) \leq \kappa \right] = \mathbb{P} \left[|X_{\varepsilon''}(z)| \leq \kappa \right] = \sqrt{\frac{2}{\pi}} \frac{\kappa}{(\ln \varepsilon)^{1/2}}. \quad (4.56)$$

Using Lemma 4.9 one sees that the second factor is smaller than

$$C \sqrt{\mathbb{E}[(\tilde{X}_{\varepsilon,\varepsilon'',\varepsilon t} - \tilde{X}_{\varepsilon,\varepsilon',\varepsilon t})^2]} \leq C \sqrt{A(\ln |\ln \varepsilon|)^{-1}},$$

Hence (4.54) is smaller than

$$C \sqrt{A} \int_0^A \frac{(\ln |\ln \varepsilon|)^{-1/2} dt}{(G_\varepsilon(\varepsilon t)/\varepsilon)^{\beta^2+1/2}} \leq C \sqrt{A} \kappa (\ln |\ln \varepsilon|)^{-1/2}. \quad (4.57)$$

Finally we show that

$$\left| \int_0^A \frac{|\ln \varepsilon|^{1/2}}{(G_\varepsilon(t\varepsilon)/\varepsilon)^{\beta^2+1/2}} \left(\int_0^1 e^{(1/\sqrt{2})\tilde{X}_{\varepsilon,\varepsilon',\varepsilon t}(z) - \frac{1}{4}\mathbb{E}[\tilde{X}_{\varepsilon,\varepsilon',\varepsilon t}(z)^2]} - e^{\sqrt{2}X_{\varepsilon'}(z) - \mathbb{E}[X_{\varepsilon'}(z)^2]} dz \right) dt \right| \quad (4.58)$$

tends to zero in probability. To do so we prove that for some $q < 1$ the q -th moment converges to zero. The content of $|\cdot|$ can be rewritten as

$$|\ln \varepsilon|^{1/2} \int_0^1 e^{\sqrt{2}X_{\varepsilon'}(z) - \mathbb{E}[X_{\varepsilon'}(z)^2]} \xi(z) dz, \quad (4.59)$$

where

$$\xi(z) := \int_0^A \frac{dt}{(G_\varepsilon(\varepsilon t)/\varepsilon)^{\beta^2+1/2}} \left(e^{(1/\sqrt{2})(X_{\varepsilon,\varepsilon t} - X_{\varepsilon',\varepsilon t})(z) - \frac{1}{4}\mathbb{E}[(X_{\varepsilon,\varepsilon t} - X_{\varepsilon',\varepsilon t})(z)]^2)} - 1 \right).$$

Then using conditional Jensen's inequality, we obtain that the q -th moment is smaller than

$$(-\ln \varepsilon)^{q/2} \mathbb{E} \left[\left(\int_{[0,1]^2} e^{\sqrt{2}(X_{\varepsilon'}(z) + X_{\varepsilon'}(z')) - 2\mathbb{E}[X_{\varepsilon'}(z)^2]} \mathbb{E}[\xi(z)\xi(z')] dz dz' \right)^{q/2} \right]. \quad (4.60)$$

then we need to have an upper bound on $\mathbb{E}[\xi(z)\xi(z')]$ to conclude.

Lemma 4.13. *We stick to the notation of (4.37). For any z and z' we have*

$$\mathbb{E}[\xi(z)\xi(z')] \leq C \left(\frac{\varepsilon'}{\varepsilon} \right)^2 \ln(\varepsilon'/\varepsilon) g \left(\frac{|z - z'| - \varepsilon A}{\varepsilon'} \right).$$

Proof. Expanding the two integrals we get

$$\begin{aligned} \mathbb{E}[\xi(z)\xi(z')] &= \int_0^A \int_0^A \frac{1}{(G_\varepsilon(\varepsilon t)/\varepsilon)^{\beta^2+1/2} (G_\varepsilon(\varepsilon t')/\varepsilon)^{\beta^2+1/2}} \\ &\quad \times \left(e^{\mathbb{E}[(X_{\varepsilon,\varepsilon t} - X_{\varepsilon',\varepsilon t})(z)(X_{\varepsilon,\varepsilon t'} - X_{\varepsilon',\varepsilon t'})(z')]/2} - 1 \right) dt dt'. \end{aligned} \quad (4.61)$$

Using the inequality $e^x - 1 \leq e^K x$ for $x \leq K$ and the fact that

$$\mathbb{E}[(X_{\varepsilon,\varepsilon t} - X_{\varepsilon',\varepsilon t})(z)(X_{\varepsilon,\varepsilon t'} - X_{\varepsilon',\varepsilon t'})(z')] \leq 4\mathbb{E}[(X_\varepsilon - X_{\varepsilon'})^2(0)].$$

we get

$$e^{\mathbb{E}[(X_{\varepsilon,\varepsilon t} - X_{\varepsilon',\varepsilon t})(z)(X_{\varepsilon,\varepsilon t'} - X_{\varepsilon',\varepsilon t'})(z')]} - 1 \leq \left(\frac{\varepsilon'}{\varepsilon} \right)^2 \mathbb{E}[(X_{\varepsilon,\varepsilon t} - X_{\varepsilon',\varepsilon t})(z)(X_{\varepsilon,\varepsilon t'} - X_{\varepsilon',\varepsilon t'})(z')]/2$$

To get a bound on (4.61), we need a bound on $\mathbb{E}[(X_{\varepsilon,\varepsilon t} - X_{\varepsilon',\varepsilon t})(z)(X_{\varepsilon,\varepsilon t'} - X_{\varepsilon',\varepsilon t'})(z')]$ that does not depend on t nor t' . We do so by noticing that for any values of t, t' in $[0, A]$, we have

$$\min(|z - z'|, |z - z' + \varepsilon t|, |z - z' - \varepsilon t'|, |z - z' + \varepsilon t - \varepsilon t'|) \geq |z - z'| - \varepsilon A. \quad (4.62)$$

Hence, using the notation introduced in (4.37), we have

$$\mathbb{E}[(X_{\varepsilon,\varepsilon t} - X_{\varepsilon',\varepsilon t})(z)(X_{\varepsilon,\varepsilon t'} - X_{\varepsilon',\varepsilon t'})(z')] \leq C \ln(\varepsilon'/\varepsilon) g \left(\frac{(|z - z'| - \varepsilon A)_+}{\varepsilon'} \right). \quad (4.63)$$

The result follows from the combination of the above inequalities and the fact that $\int_0^A \frac{dt}{(G_\varepsilon(\varepsilon t)/\varepsilon)^{\beta^2+1/2}}$ is bounded uniformly in A and ε from Lemma 4.7. \square

From Lemma 4.13 and (4.60) we have that the q -th moment of (4.58) is smaller than

$$C |\ln \varepsilon|^{q/2} \left(\frac{\varepsilon}{\varepsilon'} \right)^2 \ln(\varepsilon/\varepsilon') \times \mathbb{E} \left[\left(\int_{[0,1]^2} e^{\sqrt{2}(X_{\varepsilon'}(z)+X_{\varepsilon'}(z'))-2\mathbb{E}[X_{\varepsilon'}(z)^2]} g \left(\frac{(|z-z'|-\varepsilon A)_+}{\varepsilon'} \right) dz dz' \right)^{q/2} \right]. \quad (4.64)$$

In the above expectation, by using the relation

$$e^{\sqrt{2}(X_{\varepsilon'}(z)+X_{\varepsilon'}(z'))-2\mathbb{E}[X_{\varepsilon'}(z)^2]} \leq \frac{1}{2} e^{2\sqrt{2}X_{\varepsilon'}(z)-2\mathbb{E}[X_{\varepsilon'}(z)^2]} + \frac{1}{2} e^{2\sqrt{2}X_{\varepsilon'}(z')-2\mathbb{E}[X_{\varepsilon'}(z')^2]},$$

we get the following upper-bound by symmetrization:

$$\begin{aligned} & \int_{[0,1]^2} e^{\sqrt{2}(X_{\varepsilon'}(z)+X_{\varepsilon'}(z'))-2\mathbb{E}[X_{\varepsilon'}(z)^2]} g \left(\frac{(|z-z'|-\varepsilon A)_+}{\varepsilon'} \right) dz dz' \\ & \leq \int_{[0,1]^2} e^{2\sqrt{2}X_{\varepsilon'}(z)-2\mathbb{E}[X_{\varepsilon'}(z)^2]} g \left(\frac{(|z-z'|-\varepsilon A)_+}{\varepsilon'} \right) dz dz' \\ & \leq \varepsilon' \int_0^1 e^{2\sqrt{2}X_{\varepsilon'}(z)-2\mathbb{E}[X_{\varepsilon'}(z)^2]} dz \int_{\mathbb{R}} g \left((|z''| - (\varepsilon A)/\varepsilon')_+ \right) dz'' \end{aligned}$$

where the last line was obtained by a change of variables and expanding the integral over \mathbb{R} . The function in the second integral is smaller than $g(|z''| - 1)_+$, which is integrable. And hence it remains to show that

$$|\ln \varepsilon|^{q/2} \left(\frac{\varepsilon'}{\varepsilon} \right)^2 \ln(\varepsilon'/\varepsilon) \mathbb{E} \left[\left(\varepsilon' \int_0^1 e^{2\sqrt{2}X_{\varepsilon'}(z)-2\mathbb{E}[X_{\varepsilon'}(z)^2]} dz \right)^{q/2} \right], \quad (4.65)$$

tends to zero.

Now, we use Lemma 3.14 with $\gamma = 2\sqrt{2}$ and $\alpha = q/2 < 1/2$, and we obtain the following bound

$$E \left[\left(\varepsilon' \int_0^1 e^{2\sqrt{2}X_{\varepsilon'}(z)-2\mathbb{E}[X_{\varepsilon'}(z)^2]} dz \right)^{q/2} \right] \leq C |\ln \varepsilon'|^{-\frac{3q}{2}},$$

so that the expression (4.65) is smaller than

$$C |\ln \varepsilon|^{-q} \left(\frac{\varepsilon'}{\varepsilon} \right)^2 \ln(\varepsilon'/\varepsilon),$$

which tends to zero according to our definition of ε' . \square

4.5 Proof of Theorem 4.2 and 4.3

Now, we conclude the proofs of Theorem 4.2 and 4.3. For simplicity, we consider the case $d = 1$. We only treat the proof of Theorem 4.2 since Theorem 4.3 can be dealt with similarly. We consider l disjoint intervals A_1, \dots, A_l . We fix k vectors u_1, \dots, u_l in \mathbb{R}^2 . We denote by $\langle u, x \rangle$ the Euclidean scalar product on \mathbb{R}^2 . We also introduce a sequence $(f_j)_{j \geq 1}$ of continuous and bounded functions which is dense in the space of continuous functions with compact support. We want to show that

$$Z_{\varepsilon}^{u_1, \dots, u_l} := \sum_{i=1}^l \varepsilon^{\gamma^2 - \frac{1}{2}} \langle u_i, M_{\varepsilon}^{\gamma, \beta}(A_i) \rangle$$

converges in law to $Z^{u_1, \dots, u_l} := \sum_{i=1}^l W_{\sigma^2 M^{2\gamma, 0}}(A_i)$ as ε goes to 0. If this was not the case, we could find an index j_0 and a subsequence ε_n going to 0 such that $\mathbb{E}[f_{j_0}(Z_{\varepsilon_n}^{u_1, \dots, u_l})]$ does not converge to $\mathbb{E}[f_{j_0}(Z^{u_1, \dots, u_l})]$. By applying a diagonal extraction argument to proposition A.1, we can find a subsequence $(n_p)_{p \geq 1}$ such that for all $k \geq 1$ we get the following almost sure convergence (with respect to X):

$$\mathbb{E}[(Z_{\varepsilon_{n_p}}^{u_1, \dots, u_l})^k | X] \xrightarrow{p \rightarrow \infty} \mathbb{E}[(Z^{u_1, \dots, u_l})^k | X]$$

By the method of moments, we deduce that almost surely we have:

$$\mathbb{E}[f_{j_0}(Z_{\varepsilon_{n_p}}^{u_1, \dots, u_l}) | X] \xrightarrow{p \rightarrow \infty} \mathbb{E}[f_{j_0}(Z^{u_1, \dots, u_l}) | X]$$

Hence by dominated convergence, we get that $\mathbb{E}[f_{j_0}(Z_{\varepsilon_{n_p}}^{u_1, \dots, u_l})] \xrightarrow{p \rightarrow \infty} \mathbb{E}[f_{j_0}(Z^{u_1, \dots, u_l})]$ which contradicts our assumption.

5 Conjectures

5.1 Reminder: conjectures on $\beta = 0$

In the case $\beta = 0$, there is still some open questions about the renormalization of the measures $(M_\varepsilon^{\gamma, 0})_\varepsilon$. Some conjectures are stated in [6, 22] that we recall here:

Conjecture 5.1. *Assume $\gamma > \sqrt{2d}$ and set $\alpha = \frac{\sqrt{2d}}{\gamma}$. Then*

$$(-\ln \varepsilon)^{\frac{3\gamma}{2\sqrt{2d}}} \varepsilon^{\gamma\sqrt{2d}-d} M_\varepsilon^{\gamma, 0}(dx) \xrightarrow{\text{law}} c_\gamma N_\alpha(dx), \quad \text{as } \varepsilon \rightarrow 0 \quad (5.1)$$

where c_γ is a positive constant depending on γ and the law of N_α is given, conditioned on the derivative martingale M' , by an independently scattered random measure the law of which is characterized by

$$\forall A \in \mathcal{B}(\mathbb{R}^d), \forall q \geq 0, \quad \mathbb{E}[e^{-qN_\alpha(A)} | M'] = e^{-q^\alpha M'(A)}.$$

5.2 Conjectures on the inner phase II

We state here conjectures on this phase that we should be able to prove in the case of discrete cascades thanks to the exact study of the extremal process combined with our argument to establish convergence in law towards a complex Gaussian random measure conditionally on X . In the context of Gaussian multiplicative chaos, we have to rely on the conjecture 5.1 to state:

Conjecture 5.2. *Let $\beta > 0$ and $\gamma > \sqrt{\frac{d}{2}}$ such that $\gamma + \beta > \sqrt{2d}$. Then we get the following convergence in law:*

$$\left(\left(\ln \frac{1}{\varepsilon} \right)^{\frac{3\gamma}{2\sqrt{2d}}} \varepsilon^{\gamma\sqrt{2d}-d} M_\varepsilon^{\gamma, \beta}(A) \right)_{A \subset \mathbb{R}^d} \Rightarrow \left(W_{\sigma^2 N_{M'}^\alpha}(A) \right)_{A \subset \mathbb{R}^d}.$$

where, conditionally on $N_{M'}^\alpha$, $W_{\sigma^2 N_{M'}^\alpha}$ is a complex Gaussian random measure with intensity $N_{M'}^\alpha$, and $N_{M'}^\alpha$ is a α -stable random measure with intensity M' and $\alpha = \sqrt{\frac{d}{2}} \frac{1}{\gamma}$, namely an independently scattered random measure whose law is characterized by $\mathbb{E}[e^{-qN_{M'}^\alpha(A)}] = e^{-q^\alpha M'(A)}$ for all $u \geq 0$ and A bounded Borelian set. The constant σ^2 depends on (γ, β) .

5.3 Triple point

Concerning the triple point, the situation is a bit more delicate. When looking at the proof of subsection 4.4, it is natural to expect:

Conjecture 5.3. For $\beta = \gamma = \sqrt{\frac{d}{2}}$, the following convergence in law holds:

$$\left(|\ln \varepsilon|^{-\frac{1}{4}} M_\varepsilon^{\gamma, \beta}(A) \right)_{A \subset \mathbb{R}^d} \xrightarrow{\varepsilon \rightarrow 0} (W_{\sigma^2 M'}(A))_{A \subset \mathbb{R}^d}.$$

where, conditionally on M' , $W_{\sigma^2 M'}(\cdot)$ is a complex Gaussian random measure with intensity $\sigma^2 M'$, and σ^2 is a constant.

5.4 Continuity in dimension 1 of the limiting process in the frontier I/II

Looking at the proof of lemma 3.13, one can write the following heuristics for $k \leq n$:

$$\begin{aligned} \mathbb{E} \left[(2^{-n})^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} M_{\frac{1}{2^n}}^{\gamma, \beta} [0, 2^{-k}]^2 \mid \mathcal{F}^X \right] &= \int_{[0, \frac{1}{2^k}] \times [0, \frac{1}{2^k}]} (2^{-n})^{\gamma^2} \frac{M_{\frac{1}{2^n}}^{\gamma, 0}(dx) M_{\frac{1}{2^n}}^{\gamma, 0}(dy)}{G_{\frac{1}{2^n}}(y-x)(\sqrt{2}-\gamma)^2} \\ &\approx \sum_{j=k}^n 2^{j\beta^2} \sum_{l=1}^{2^{j-k}} \frac{1}{2^{j\gamma^2}} M_{\frac{1}{2^j}}^{\gamma, 0} \left[\frac{l-1}{2^j}, \frac{l}{2^j} \right]^2 \\ &\approx \sum_{j=k}^n \sum_{l=1}^{2^{j-k}} e^{2\gamma(X_{2^{-j}(\frac{l}{2^j})} - \sqrt{2} \ln 2^j)} \end{aligned}$$

Recall that it is conjectured that $\sum_{l=1}^{2^{j-k}} e^{2\gamma(X_{2^{-j}(\frac{l}{2^j})} - \sqrt{2} \ln 2^j)}$ converges in law to some atomic random measure ν (see (5.1)). Hence, the limit $M^{\gamma, \beta}$ should satisfy the bound $\mathbb{E}[M^{\gamma, \beta}([0, \frac{1}{2^k}])^2 \mid \mathcal{F}^X] \leq \nu([0, \frac{1}{2^k}]) (k^{\frac{3\gamma}{\sqrt{2}} - 1})^{-1}$ and more generally:

$$\mathbb{E}[M^{\gamma, \beta}([s, t])^2 \mid \mathcal{F}^X] \leq \frac{\nu([s, t])}{(\ln_+ \frac{1}{|t-s|})^{\frac{3\gamma}{\sqrt{2}} - 1}}$$

It is therefore natural to conjecture that in dimension 1, we can reinforce the above result by an almost sure convergence in the space of continuous functions. Indeed, as soon as one can show that the limiting measure $M^{\gamma, \beta}$ is cadlag, the above estimates entail continuity.

6 Gaussian Free Fields

The Gaussian Free field with mass $m \geq 0$ on a set $D \subset \mathbb{R}^2$ (for simplicity we can say that D is either a planar bounded domain or the whole plane) and Dirichlet boundary condition is the Gaussian field whose covariance function is given by the Green function of the problem

$$\Delta u - 2mu = -2\pi f \text{ on } D, \quad u|_{\partial D} = 0.$$

Notice the unusual normalization factor 2π in order to get correlations of the form (1.2). When $D = \mathbb{R}^2$ we have to consider $m > 0$: otherwise the Green function is infinite everywhere. In the case of a bounded domain D , we are mostly interested in the case $m = 0$.

The Green function can be written as

$$g(x, y) = \pi \int_0^\infty e^{-rm} p(r, x, y) dr.$$

where $p(t, x, y)$ will denote the transition densities of the Brownian motion on D killed upon touching ∂D . A formal way to define the complex Gaussian field $X + iY$ (with X and Y independent GFF) is to consider two independent white noises W^X, W^Y on $\mathbb{R}_+ \times D$ and define

$$\begin{aligned} X(x) &= \sqrt{\pi} \int_0^\infty \int_D e^{-mr/2} p(r/2, x, y) W^X(dr, dy), \\ Y(x) &= \sqrt{\pi} \int_0^\infty \int_D e^{-mr/2} p(r/2, x, y) W^Y(dr, dy). \end{aligned} \tag{6.1}$$

To define the exponential of the field $X + iY$, we need to use a cut-off procedure. So we define the approximations X_ε and Y_ε (respectively of the fields X and Y) by integrating over $(\varepsilon^2, \infty) \times D$ in (6.1) instead of $(0, \infty) \times D$. The covariance function for these approximations is given by

$$\mathbb{E}[X_\varepsilon(x)X_{\varepsilon'}(y)] = \mathbb{E}[Y_\varepsilon(x)Y_{\varepsilon'}(y)] = \pi \int_{\varepsilon^2 \vee \varepsilon'^2}^\infty e^{-rm} p(r, x, y) dr. \tag{6.2}$$

6.1 Massive Gaussian Free Field in the plane

Observe that, in the case of massive Gaussian Free Field on \mathbb{R}^2 (see subsection 2.1), the kernel p is translation invariant and has a simple expression

$$p(t, x, y) = \frac{1}{2\pi t} e^{-\frac{|x-y|^2}{2t}}.$$

The whole plane massive Green function then takes the form

$$\forall x, y \in \mathbb{R}^2, \quad G_m(x, y) = \int_0^\infty e^{-mu - \frac{|x-y|^2}{2u}} \frac{du}{2u}. \tag{6.3}$$

and can be rewritten as

$$G_m(x, y) = \int_1^{+\infty} \frac{k_m(u(x-y))}{u} du. \tag{6.4}$$

for some continuous covariance kernel $k_m = \frac{1}{2} \int_0^\infty e^{-\frac{m}{v}|z|^2 - \frac{v}{2}} dv$. Therefore the whole plane massive free field strictly enters the framework of the first part of our paper.

6.2 Gaussian Free Field on planar bounded domains

The case of the GFF on a planar bounded domain is a bit more delicate (but nothing too serious) as $p(t, x, y)$ is not translation invariant this time. We use the change of variables $r \rightarrow r^{-2}$ in the integral in the r.h.s. of (6.2) to find something closer to the setup that we have worked with in the previous sections and obtain:

$$\mathbb{E}[X_\varepsilon(x)X_{\varepsilon'}(y)] = \mathbb{E}[Y_\varepsilon(x)Y_{\varepsilon'}(y)] = 2\pi \int_0^{\varepsilon^{-1} \wedge \varepsilon'^{-1}} e^{-m/r^2} r^{-3} p(r^{-2}, x, y) dr. \tag{6.5}$$

The kernel $2\pi e^{-m/r^2} r^{-3} p(r^{-2}, x, y)$ will have to play the role of $\frac{k(u(x-y))}{u}$.

We have

$$2\pi r^{-3}p(r^{-2}, x, y) \leq \frac{e^{-\frac{(r|x-y|)^2}{2}}}{r}, \quad \forall(x, y, r). \quad (6.6)$$

Furthermore the two kernels are asymptotically equivalent in the interior of D in the sense that for any compact $K \subset D$:

$$\lim_{r \rightarrow \infty} \sup_{x, y \in K} |2\pi r^{-2}p(r^{-2}, x, y)e^{\frac{(r|x-y|)^2}{2}} - 1| = 0. \quad (6.7)$$

We will also consider the following decomposition of the covariance function

$$\begin{aligned} g_\varepsilon(x, y) &:= \mathbb{E}[X_\varepsilon(x)X_\varepsilon(y)] \\ &= \int_1^{\varepsilon^{-1}} 2\pi r^{-3}p(r^{-2}, x, y) dr + \int_0^1 2\pi r^{-3}p(r^{-2}, x, y) dr \\ &=: \tilde{g}_\varepsilon(x, y) + g'(x, y), \end{aligned}$$

This corresponds to writing

$$X_\varepsilon = \tilde{X}_\varepsilon + X' \quad (6.8)$$

where X' and \tilde{X}_ε have respective covariance functions $\tilde{g}_\varepsilon(x, y)$ and $g'(x, y)$.

The conformal radius $C(x, D)$ of a point x in the planar bounded domain D is defined by

$$C(x, D) := \frac{1}{|\varphi'(x)|} \quad (6.9)$$

where φ is any conformal mapping of D to the unit disc such that $\varphi(x) = 0$. In fact, we will use the following definition which is more useful for our purpose. Let φ be any conformal map from D to the upper half plane \mathbb{H} . Then we have the following expression for the conformal radius:

$$C(x, D) = \frac{2\text{Im}(\varphi(x))}{|\varphi'(x)|}. \quad (6.10)$$

Set

$$G_\varepsilon(x, y) := e^{-g_\varepsilon(x, y)}.$$

The following claim is proved in the Appendix.

Lemma 6.1. *For all $x \in D$, we set $C_\varepsilon(x, D) = \varepsilon/G_\varepsilon(x, x)$. For all $x \in D$, we have:*

$$\lim_{\varepsilon \rightarrow 0} C_\varepsilon(x, D) = C(x, D) \quad (6.11)$$

uniformly on the compact subsets of D .

We define for $(\gamma, \beta) \in \mathbb{R}_+^2$ and ε the following operator:

$$M_\varepsilon^{\gamma, \beta}(\varphi) = \int_D \varphi(x) e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x) - \frac{\gamma^2 - \beta^2}{2} G_\varepsilon(x, x)} C(x, D)^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} dx.$$

where $\varphi(x)$ is a bounded measurable function on D . Notice that the renormalization term $\frac{\gamma^2 - \beta^2}{2} G_\varepsilon(x, x)$ in the exponential is chosen such that $M_\varepsilon^{\gamma, \beta}(\varphi)$ is a martingale in ε .

Given another planar domain \tilde{D} and a conformal map $\psi : \tilde{D} \rightarrow D$, we will denote by $(X_\varepsilon^\psi)_{\varepsilon \in]0,1]}$ and $(Y_\varepsilon^\psi)_{\varepsilon \in]0,1]}$ the random fields defined by

$$X_\varepsilon^\psi(x) = X_\varepsilon(\psi(x)) \quad \text{and} \quad Y_\varepsilon^\psi(x) = Y_\varepsilon(\psi(x)).$$

These two family form two independent white noise approximating sequences of the GFFs $X \circ \psi$ and $Y \circ \psi$ defined on \tilde{D} . Then we define for φ defined on \tilde{D} :

$$M_\varepsilon^{\gamma,\beta,\psi}(\varphi) = \int_{\tilde{D}} \varphi(x) e^{\gamma X_\varepsilon^\psi(x) + i\beta Y_\varepsilon^\psi(x) - \frac{\gamma^2 - \beta^2}{2} G_\varepsilon(\psi(x), \psi(x))} C(x, \tilde{D})^{\frac{\gamma^2 - \beta^2}{2}} |\psi'(x)|^{2\gamma} dx.$$

When ψ is the identity we simply write $M_\varepsilon^{\gamma,\beta}$. This allows us to define simultaneously the GFF on every planar bounded domain conformally equivalent to D . We also mention the rule, for $x \in \tilde{D}$, $|C(\psi(x), D)| = |\psi'(x)| |C(x, \tilde{D})|$ where $|\psi'(x)|^2$ is the Jacobian of the mapping $\psi : \tilde{D} \rightarrow D$ (this follows right away from the definition of the conformal radius (6.9)).

Phase I and frontier I/II

Consider a couple $(\gamma, \beta) \in \mathbb{R}_+ \times \mathbb{R}$ and define

$$\zeta(p) = \left(2 + \frac{\gamma^2}{2} - \frac{\beta^2}{2}\right)p - \frac{\gamma^2}{2}p^2.$$

We have the following behavior inside phase I:

Theorem 6.2. *Consider a couple $(\gamma, \beta) \in \mathbb{R}_+ \times \mathbb{R}$ in phase I or in the frontier I/II (excluding the extremal points). Consider $p \in]1, 2]$ such that $\zeta(p) > 2$ in the inner phase I or $p \in [1, \frac{2}{\gamma}[$ on the frontier I/II. Then:*

1. For every bounded planar bounded domain \tilde{D} and conformal map $\psi : \tilde{D} \rightarrow D$, for all bounded function $\varphi \subset \tilde{D}$, the martingale:

$$(M_\varepsilon^{\gamma,\beta,\psi}(\varphi))_\varepsilon$$

is uniformly bounded in \mathbb{L}_p .

2. Almost surely, the convergence of $M_\varepsilon^{\gamma,\beta,\psi}(\cdot)$ holds in the sense of distributions of order 2. The operator norm in the space of distributions of order 2 of the limiting distribution, denoted by $M^{\gamma,\beta,\psi}(\cdot)$, is \mathbb{L}_p -integrable.
3. For all $q \in [0, p]$ and all function $\varphi \in C_c^2(\tilde{D})$:

$$\mathbb{E}[|M^{\gamma,\beta,\psi}(\varphi(\cdot/r))|^p] \underset{r \rightarrow 0}{\sim} C_x r^{\zeta(p)}$$

for all $x \in \tilde{D}$ and some constant $C_x > 0$, which is continuous with respect to x on \tilde{D} .

Proof. The proofs of items 1,2,3 are exactly the same as in Section 3: when computing the \mathbb{L}_p norm, thanks to (6.6), we can use Proposition 2.2 to compare the capacity with the one of the stationary case $k(x) = e^{-x^2/2}$. \square

Remark 6.3. As a consequence of the above Theorem we have for any $\varphi \in C_c^2(\tilde{D})$,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_D \varphi(x) e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx = M_\varepsilon^{\gamma, \beta}(\varphi),$$

in the \mathbb{L}_p sense.

To see this, the reader can check that with our definitions

$$\begin{aligned} & \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_D \varphi(x) e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx - M_\varepsilon^{\gamma, \beta}(\varphi) \\ &= \int_D (C_\varepsilon(x, D) - C(x, D))^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \varphi(x) e^{\gamma X_\varepsilon^\psi(x) + i\beta Y_\varepsilon^\psi(x) - \frac{\gamma^2 - \beta^2}{2} G_\varepsilon(x, x)} dx. \end{aligned} \quad (6.12)$$

As $C_\varepsilon(x, D)$ converges uniformly (Lemma 6.1), Theorem 3.1 (Item 1) shows that the moment of order p of the above quantity tends to zero.

Gaussian Free Field on planar bounded domains: another approach in phase I and frontier I/II

Circle averages. In this subsection, we extend the framework of [25] to the complex case. Let X and Y be two independent GFFs on a domain D . Without loss of generality, we suppose that D contains the square $[0, 1]^2$ (otherwise we could consider a smaller square inside D). We introduce the circle averages $(X_\varepsilon)_{\varepsilon \in [0, 1]}$ and $(Y_\varepsilon)_{\varepsilon \in [0, 1]}$ of radius ε , i.e. $X_\varepsilon(x)$ (resp. $Y_\varepsilon(x)$) stands for the mean value of X (resp. Y) on the circle centered at x with radius ε (cf. [25] for further details). We then consider the operator:

$$\varphi \mapsto M_\varepsilon^{\gamma, \beta}(\varphi) = \int_D \varphi(x) e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x) - (\gamma^2/2 - \beta^2/2) E[X_\varepsilon(x)^2]} dx.$$

We set $G_{\varepsilon, \varepsilon'}(x, y) = \mathbb{E}[X_\varepsilon(x) X_{\varepsilon'}(y)]$. We can now state the following theorem:

Theorem 6.4. *In the inner phase I, we consider $p \in]1, 2]$ such that $\zeta(p) > 2$ and on the frontier I/II (excluding the extremal points), we consider $p \in]1, \frac{2}{\gamma}]$. For all bounded measurable function φ with compact support in D , the family $(M_\varepsilon^{\gamma, \beta}(\varphi))_\varepsilon$ converges in \mathbb{L}_p towards a variable $M^{\gamma, \beta}(\varphi)$.*

Proof. Here $M_\varepsilon^{\gamma, \beta}$ is not a martingale so we cannot content ourselves with proving boundedness in \mathbb{L}_p : we must show that the sequence is Cauchy. If $\varepsilon, \varepsilon' > 0$, we have the following bounds:

$$\begin{aligned} & \mathbb{E}[|M_\varepsilon^{\gamma, \beta}([0, 1]^2) - M_{\varepsilon'}^{\gamma, \beta}([0, 1]^2)|^p] \\ & \leq \mathbb{E}[\mathbb{E}[|M_\varepsilon^{\gamma, \beta}([0, 1]^2) - M_{\varepsilon'}^{\gamma, \beta}([0, 1]^2)|^2 | X]^{p/2}] \\ & = \mathbb{E}[|A(\varepsilon, \varepsilon) + A(\varepsilon', \varepsilon') - 2A(\varepsilon, \varepsilon')|^{p/2}] \end{aligned} \quad (6.13)$$

where we have set:

$$A(\varepsilon, \varepsilon') = \int_{([0, 1]^2)^2} e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2} \mathbb{E}[X_\varepsilon(x)^2]} e^{\gamma X_{\varepsilon'}(y) - \frac{\gamma^2}{2} \mathbb{E}[X_{\varepsilon'}(y)^2]} e^{\beta^2 G_{\varepsilon, \varepsilon'}(x, y)} dx dy.$$

If $\delta > 0$, we further define

$$\begin{aligned} A(\varepsilon, \varepsilon', \delta) &= \int_{([0, 1]^2)^2, |x-y| \leq \delta} e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2} \mathbb{E}[X_\varepsilon(x)^2]} e^{\gamma X_{\varepsilon'}(y) - \frac{\gamma^2}{2} \mathbb{E}[X_{\varepsilon'}(y)^2]} e^{\beta^2 G_{\varepsilon, \varepsilon'}(x, y)} dx dy \\ C(\varepsilon, \varepsilon', \delta) &= \int_{([0, 1]^2)^2, |x-y| > \delta} e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2} \mathbb{E}[X_\varepsilon(x)^2]} e^{\gamma X_{\varepsilon'}(y) - \frac{\gamma^2}{2} \mathbb{E}[X_{\varepsilon'}(y)^2]} e^{\beta^2 G_{\varepsilon, \varepsilon'}(x, y)} dx dy. \end{aligned}$$

The main idea of what follows is the following: we split the integrals appearing in (6.13) in two regions $|x - y| \leq \delta$ and $|x - y| > \delta$ for some $\delta > 0$. On the set $|x - y| > \delta$, the singularity $e^{\beta^2 G_{\varepsilon, \varepsilon'}(x, y)}$ is bounded by a constant (eventually depending on δ). Therefore, the convergence of the term

$$\mathbb{E}[|C(\varepsilon, \varepsilon, \delta) + C(\varepsilon', \varepsilon', \delta) - 2C(\varepsilon, \varepsilon', \delta)|^{p/2}]$$

towards 0 boils down to establishing the convergence of the family $(M_\varepsilon^{\gamma, 0})$ in \mathbb{L}^p . This is "almost" proved in [25]: actually, the authors in [25] only prove almost sure convergence. On the other hand, it is plain to check (using Proposition 2.2 get a comparison with a stationary field) that this family is uniformly bounded in \mathbb{L}_q for some $q > p$. The claim of convergence in \mathbb{L}_p follows. We deduce:

$$\begin{aligned} & \limsup_{\varepsilon, \varepsilon' \rightarrow 0} \mathbb{E}[|M_\varepsilon^{\gamma, \beta}([0, 1]^2) - M_{\varepsilon'}^{\gamma, \beta}([0, 1]^2)|^p] \\ & \leq \limsup_{\varepsilon, \varepsilon' \rightarrow 0} \mathbb{E}[|A(\varepsilon, \varepsilon, \delta)|^{p/2}] + \mathbb{E}[|A(\varepsilon', \varepsilon', \delta)|^{p/2}] + 2\mathbb{E}[|A(\varepsilon, \varepsilon', \delta)|^{p/2}]. \end{aligned}$$

By the capacity lemmas 3.10 or 3.13 (depending if we are in the inner phase I or the frontier I/II), the above quantity goes to 0 as δ goes to 0; therefore $(M_\varepsilon^{\gamma, \beta}([0, 1]^2))_\varepsilon$ is a Cauchy sequence in \mathbb{L}_p . □

Orthonormal basis expansion of the GFF. As a preliminary, the reader is referred to [53, 25] for further background about the decomposition of the GFF along orthonormal basis. Let us denote by $H(D)$ the Hilbert space closure of the space $C_c^\infty(D)$ with respect to the inner product

$$(f, g)_\nabla = \frac{1}{2\pi} \int_D \nabla f(x) \cdot \nabla g(x) dx.$$

Now we want to expand the GFF along a given orthonormal basis of $H(D)$ to produce another way of defining the limiting random variable $M^{\gamma, \beta}$. We will also show that the limit obtained with this procedure does not depend on the choice of the orthonormal basis.

So we consider an orthonormal basis $(f_k)_{k \geq 1}$ of $H(D)$ made up of continuous functions. We consider the projections of X and Y onto this orthonormal basis, namely we define the sequence of i.i.d. Gaussian random variables:

$$\varepsilon_k = \frac{1}{2\pi} \int_D \nabla X(x) \nabla f_k(x) dx, \quad \text{and} \quad \varepsilon'_k = \frac{1}{2\pi} \int_D \nabla Y(x) \nabla f_k(x) dx.$$

The projections of X and Y onto the span of $\{f_1, \dots, f_n\}$ are given by:

$$X_n(x) = \sum_{k=1}^n \varepsilon_k f_k(x) \quad \text{and} \quad Y_n(x) = \sum_{k=1}^n \varepsilon'_k f_k(x).$$

In this context, we set:

$$M_n^{\gamma, \beta}(A) = \int_A e^{\gamma X_n(x) + i\beta Y_n(x) - (\gamma^2/2 - \beta^2/2)E[X_n(x)^2]} dx. \tag{6.14}$$

We have the following result:

Theorem 6.5. *In the inner phase I, we consider $p \in]1, 2]$ such that $\zeta(p) > 2$ and on the frontier I/II (excluding the extremal points), we consider $p \in]1, \frac{2}{\gamma}[$. For all bounded measurable function φ with compact support in D , the sequence $(M_n^{\gamma, \beta}(\varphi))_n$ converges almost surely and in \mathbb{L}_p to $M^{\gamma, \beta}(\varphi)$, i.e. the same limit as the circle average approximations of Theorem 6.4.*

Proof. First observe that the sequence $(M_n^{\gamma, \beta}(\varphi))_n$ is a martingale uniformly bounded in \mathbb{L}_p (because of Lemma 3.10 or Lemma 3.13 depending if we are in the inner phase I or the frontier I/II). Thus it converges almost surely and in \mathbb{L}_p .

Let $\varepsilon > 0$. For $n \geq 1$, we denote by $X_{n, \varepsilon}(x)$ (resp. $Y_{n, \varepsilon}(x)$) the circle average of $X_n(x)$ (resp. $Y_n(x)$), i.e. the mean value of X_n (resp. Y_n) along the circle centered at x with radius ε . For all $n \leq m$, we have the following:

$$\begin{aligned} & \mathbb{E}\left[\int_D \varphi(x) e^{\gamma X_{m, \varepsilon}(x) + i\beta Y_{m, \varepsilon}(x) - (\gamma^2/2 - \beta^2/2)E[X_{m, \varepsilon}(x)^2]} dx \mid (\varepsilon_k, \varepsilon'_k)_{k \leq n}\right] \\ &= \int_D \varphi(x) e^{\gamma X_{n, \varepsilon}(x) + i\beta Y_{n, \varepsilon}(x) - (\gamma^2/2 - \beta^2/2)E[X_{n, \varepsilon}(x)^2]} dx \end{aligned}$$

Now, we take the limit as $m \rightarrow \infty$ and get that:

$$\mathbb{E}[M_\varepsilon^{\gamma, \beta}(\varphi) \mid (\varepsilon_k, \varepsilon'_k)_{k \leq n}] = \int_D \varphi(x) e^{\gamma X_{n, \varepsilon}(x) + i\beta Y_{n, \varepsilon}(x) - (\gamma^2/2 - \beta^2/2)E[X_{n, \varepsilon}(x)^2]} dx.$$

Since the variable $M_\varepsilon^{\gamma, \beta}$ converges in \mathbb{L}_p , we can take the limit in the above identity as $\varepsilon \rightarrow 0$ hence getting:

$$\mathbb{E}[M^{\gamma, \beta}(\varphi) \mid (\varepsilon_k, \varepsilon'_k)_{k \leq n}] = M_n^{\gamma, \beta}(\varphi).$$

Now, we conclude that $M_n^{\gamma, \beta}(\varphi)$ is a martingale bounded in \mathbb{L}_p which converges to $M^{\gamma, \beta}(\varphi)$. \square

Remark 6.6. *The almost sure limit of Theorem 6.2 is the same in distribution as the one defined in Theorem 6.4 or 6.5.*

Phases II and III, frontier I/III and II/III

Now let us discuss how to adapt the proofs of Theorems 4.2 and (4.3) to the case of the GFF with Dirichlet boundary condition.

Concerning the corresponding statements, we have to specify what the value of σ and the intensity measure are. It appears through the computations that the natural thing to do is to renormalize $e^{\gamma X_\varepsilon + i\beta Y_\varepsilon}$ by a power of ε (i.e. by considering Wick ordering). So we consider here

$$M_\varepsilon^{\gamma, \beta}(A) = \varepsilon^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} \int_A e^{\gamma X_\varepsilon(x) + i\beta Y_\varepsilon(x)} dx$$

for all measurable bounded set $A \subset D$.

Theorem 6.7. \bullet *When $\gamma \in [0, 1[$ and $\beta^2 + \gamma^2 > 2$, we have*

$$\left(\varepsilon^{\gamma^2 - 1} M_\varepsilon^{\gamma, \beta}(A)\right)_{A \subset \mathbb{R}^2} \Rightarrow \left(W_{\sigma^2 \widetilde{M}^{2\gamma, 0}}(A)\right)_{A \subset \mathbb{R}^d}. \quad (6.15)$$

with

$$\sigma^2 = \sigma^2(\beta^2 + \gamma^2) := 2\pi \int_0^\infty \exp\left(-(\gamma^2 + \beta^2) \int_0^1 \frac{1 - e^{-(ur)^2/2}}{u} du\right) r dr,$$

and

$$\begin{aligned}\widetilde{M}^{2\gamma,0}(\mathrm{d}x) &:= \lim_{\varepsilon \rightarrow 0} \varepsilon^{2\gamma^2} e^{2\gamma X_\varepsilon(x)} \mathrm{d}x \\ &= C(x, D)^{2\gamma^2} \lim_{\varepsilon \rightarrow 0} e^{2\gamma X_\varepsilon(x) - 2\gamma^2 \mathbb{E}[X_\varepsilon^2(x)]} \mathrm{d}x =: C(x, D)^{2\gamma^2} M^{2\gamma,0}(\mathrm{d}x) : \quad (6.16)\end{aligned}$$

and W is a standard complex Gaussian measure on \mathbb{R}^d with intensity $\sigma^2 M^{2\gamma,0}$. The above convergence holds in the sense of convergence in law of the finite dimensional marginals.

- When $\gamma \in [0, 1[$ and $\beta^2 + \gamma^2 = 2$, we have

$$\left(\varepsilon^{\gamma^2-1} |\ln \varepsilon|^{-1/2} M_\varepsilon^{\gamma,\beta}(A) \right)_{A \subset \mathbb{R}^d} \Rightarrow \left(W_{\sigma^2 \widetilde{M}^{2\gamma,0}}(A) \right)_{A \subset \mathbb{R}^d}. \quad (6.17)$$

with

$$\sigma^2 = \sigma^2(2) := 2\pi \exp \left(\int_0^\infty \frac{e^{-u^2/2} - \mathbb{1}_{[0,1]}(u)}{u} \mathrm{d}u \right).$$

and W is a standard complex Gaussian measure on \mathbb{R}^d with intensity $\sigma^2 M^{2\gamma,0}$. The above convergence holds in the sense of convergence in law of the finite dimensional distributions.

Theorem 6.8. When $\gamma = 1$ and $\beta^2 + \gamma^2 > d$, we have

$$\left((-\ln \varepsilon)^{1/4} M_\varepsilon^{\gamma,\beta}(A) \right)_{A \subset \mathbb{R}^d} \Rightarrow \left(W_{\sigma^2 \widetilde{M}'}(A) \right)_{A \subset \mathbb{R}^d}. \quad (6.18)$$

with

$$\sigma^2 = \sqrt{\frac{2}{\pi}} \sigma^2(\beta^2 + 1).$$

Convergence holds in the sense of convergence in law of the finite dimensional distributions and the law of $W_{\sigma^2 \widetilde{M}'(\cdot)}$ is that of a complex Gaussian random measure with intensity $\sigma^2 M'$ where

$$\begin{aligned}\widetilde{M}'(\mathrm{d}x) &:= \lim_{\varepsilon \rightarrow 0} \sqrt{\frac{\pi}{2}} \varepsilon^2 |\ln \varepsilon|^{1/2} e^{2X_\varepsilon(x)} \mathrm{d}x \\ &= C(x, D)^2 \lim_{\varepsilon \rightarrow 0} [2\mathbb{E}[X_\varepsilon^2(x)] - X_\varepsilon(x)] e^{2X_\varepsilon(x) - 2\mathbb{E}[X_\varepsilon^2(x)]} \mathrm{d}x =: C(x, D)^2 M'(\mathrm{d}x). \quad (6.19)\end{aligned}$$

Remark 6.9. Note that the expression that we find for σ is that of Theorem 4.2 where k is taken equal to $e^{-u^2/2}$.

Computations of the constant σ^2 . When adapting the proof, there is only some work to be done in the computation of the second moment (Proposition 4.5 and 4.6). In particular there is some challenge in computing the value of σ^2 . For simplicity we compute the second moment for A a square included in D (and which does not touch the boundary).

Repeating the computations of Section 4.2 leading to (4.13) we obtain after having made the change of variables $(x, y) \in A^2 \rightarrow (x, z)$ with $z = y - x$:

$$\begin{aligned}\mathbb{E} \left[(\varepsilon^{\gamma^2-1} M_\varepsilon^{\gamma,\beta}(A))^2 \mid \mathcal{F}^X \right] &= \varepsilon^{\gamma^2+\beta^2-1} \int_{A-A} \left(\int_{A \cap (A-z)} \frac{(C_\varepsilon(x, D) C_\varepsilon(x+z, D))^{\frac{\gamma^2-\beta^2}{2}}}{G_\varepsilon(x, x+z)^{\gamma^2+\beta^2}} e^{\gamma X_{\varepsilon,z}(x) - \frac{\gamma^2}{2} \mathbb{E}[X_{\varepsilon,z}^2(x)]} \mathrm{d}x \right) \mathrm{d}z, \\ &:= \varepsilon^{\gamma^2+\beta^2-1} \int_{A-A} \frac{\widehat{M}_{\varepsilon,z}^\gamma}{(G_\varepsilon^0(z))^{\gamma^2+\beta^2}} \mathrm{d}z. \quad (6.20)\end{aligned}$$

where

$$\begin{aligned}
X_{\varepsilon,z}(x) &:= X_\varepsilon(x) + X_\varepsilon(x+z) \\
G_\varepsilon^0(z) &:= \exp\left(-\int_1^{\varepsilon^{-1}} \frac{k_0(uz)du}{u}\right) \quad \text{with } k_0(z) = e^{-|z^2|/2}, \\
\hat{M}_{\varepsilon,z}^\gamma &= \int_{A \cap (A-z)} \frac{(C_\varepsilon(x,D)C_\varepsilon(x+z,D))^{\frac{\gamma^2-\beta^2}{2}} G_\varepsilon^0(z)^{\gamma^2+\beta^2}}{G_\varepsilon(x,x+z)^{\gamma^2+\beta^2}} e^{\gamma X_{\varepsilon,z}(x) - \frac{\gamma^2}{2} \mathbb{E}[X_{\varepsilon,z}^2(x)]} dx \\
M_{\varepsilon,z}^\gamma &= \int_A C(x,D)^{2\gamma^2} e^{\gamma X_{\varepsilon,z}(x) - \frac{\gamma^2}{2} \mathbb{E}[X_{\varepsilon,z}^2(x)]} dx.
\end{aligned} \tag{6.21}$$

Also, we have similar expressions for the frontiers I/III or II/III.

We need to prove the following equivalent of Lemma 4.8

Lemma 6.10. *For $\gamma < 1$ we have the following convergences in \mathbb{L}_1*

$$\lim_{\varepsilon, z \rightarrow 0} \hat{M}_{\varepsilon,z}^\gamma = \lim_{\varepsilon, z \rightarrow 0} M_{\varepsilon,z}^\gamma = \tilde{M}^{2\gamma}(A). \tag{6.22}$$

Proof. The proof of the convergence of $M_{\varepsilon,z}^\gamma$ is very similar to what we have done in the translation invariant case and we wish not to repeat it. What there is to do is to prove that $M_{\varepsilon,z}^\gamma - \hat{M}_{\varepsilon,z}^\gamma(A)$ converges to zero. First we notice that

$$C_\varepsilon(x,D)/C(x,D) \text{ and } C_\varepsilon(x+z,D)/C(x,D)$$

converges to one uniformly in A when ε and z tend to zero so that the C_ε in \hat{M} can be replaced by $C(x,D)$. The second thing is to check that

$$\lim_{z, \varepsilon \rightarrow 0} G_\varepsilon^0(z)/G_\varepsilon(x,x+z) = C(x,D) \tag{6.23}$$

uniformly in A . The $-\ln$ of the quotient above is equal to

$$\begin{aligned}
&\int_1^{\varepsilon^{-1}} \frac{e^{-(r|z|)^2/2}}{r} - 2\pi r^{-3} p(r^{-2}, x, x+z) dr \\
&= \int_1^{\varepsilon^{-1}} \frac{e^{-(r|z|)^2/2}}{r} (1 - 2\pi r^{-2} p(r^{-2}, x, x)) dr + 2\pi \int_1^{\varepsilon^{-1}} r^{-3} \left(e^{-(|z|r)^2/2} p(r^{-2}, x, x) - p(r^{-2}, x, x+z) \right) dr.
\end{aligned} \tag{6.24}$$

By the monotone convergence Theorem, the first term converges to

$$\lim_{\varepsilon \rightarrow \infty} \int_1^{\varepsilon^{-1}} \frac{1}{r} (1 - 2\pi r^{-2} p(r^{-2}, x, x)) dr = \int_1^\infty \frac{1}{r} (1 - 2\pi r^{-2} p(r^{-2}, x, x)) dr$$

which according to Lemma 6.1 is equal to $-\ln C(x,D)$.

We have to show that the second term goes to zero. Let $P_{x,y}^t$ denote the law of the standard Brownian bridge $(Y_s)_{s \in [0,t]}$ with lifetime t starting from x and ending at y . Then using the definition of p , we have

$$p(t, x, y) = \frac{1}{2\pi t} e^{-\frac{(x-y)^2}{2t}} P_{x,y}^t((Y_s)_{s \in [0,t]} \text{ stays in } D).$$

Hence the second term in (6.24) is equal in absolute value to

$$\begin{aligned} & \left| \int_1^{\varepsilon^{-1}} r^{-1} e^{-(|z|r)^2/2} \left(P_{x,x}^{r^{-2}}(Y \text{ exits } D) - P_{x,x+z}^{r^{-2}}(Y \text{ exits } D) \right) dr \right| \\ & \leq \int_1^{\infty} r^{-1} |P_{x,x}^{r^{-2}}(Y \text{ exits } D) - P_{x,x+z}^{r^{-2}}(Y \text{ exits } D)| dt. \end{aligned} \quad (6.25)$$

A classic fact for Brownian Motion is that if x and $x+z$ are at a positive distance d of the boundary of D , then there exists constants c and C such that

$$P_{x,x+z}^{r^{-2}}(Y \text{ exits } D) \leq C(d) e^{-c(d)r^{-2}}.$$

As for fixed r

$$\lim_{z \rightarrow 0} |P_{x,x}^{r^{-2}}(Y \text{ exits } D) - P_{x,x+z}^{r^{-2}}(Y \text{ exits } D)| = 0$$

we can apply the dominate convergence Theorem to show that the r.h.s. of (6.25) tends to zero.

The third step is to change the domain of integration from $A \cap (A-z)$ to A but this is easy to check that the contribution to the integral of $A \setminus (A-z)$ is negligible. \square

Other GFFs

They are other possible choices of the underlying GFF. One may for instance consider Neumann boundary conditions instead of Dirichlet's, or consider a GFF with vanishing mean on the sphere or the torus. We will only give a few details here in the case of the torus. This straightforwardly adapts to the case of the sphere. But we do not treat the case of the GFF with Neumann boundary conditions: we believe but have not checked in details that everything works the same.

Consider a GFF on the torus with vanishing mean $(X(x))_{x \in \mathbb{T}}$ where \mathbb{T} is the two-dimensional torus. All the theorems stated for the GFF with Dirichlet boundary applies. The difference here concerns the computation of the constant σ^2 . Let us expand the corresponding Green function on the torus G along the eigenvalues of the Laplacian on \mathbb{T} . More precisely, it is the Green function associated to the problem: for $f \in L^2(\mathbb{T})$ and $\int_{\mathbb{T}} f = 0$, find a solution u to the problem:

$$\Delta u = -2\pi f, \quad u \in L^2(\mathbb{T}) \quad \text{and} \quad \int_{\mathbb{T}} u = 0.$$

The eigenvalues of the Laplacian are given for $x = (x_1, x_2) \in \mathbb{T}$ by:

$$e_{p,q}(x) = \sqrt{2} \cos(2\pi p x_1 + 2\pi q x_2) \quad f_{p,q}(x) = \sqrt{2} \sin(2\pi p x_1 + 2\pi q x_2)$$

with associated eigenvalue $-\lambda_{p,q} = 2\pi(p^2 + q^2)$ for $(p, q) \in \mathbb{N}^* \times \mathbb{Z} \cup \{0\} \times \mathbb{N}^* \stackrel{\text{def}}{=} E_n$. For $n \in \mathbb{N}^*$. We define the truncated Green function

$$G_n(x) = \frac{1}{\pi} \sum_{(p,q) \in E_n} \frac{1}{p^2 + q^2} \cos(2\pi p x_1 + 2\pi q x_2).$$

In this context, the equivalent of Lemma 4.7 reads:

Lemma 6.11. *We have the following convergence for all $x \in \mathbb{R}^2$:*

$$G_n\left(\frac{x}{n}\right) = \ln n - F(x) + g_n(x)$$

where $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by:

$$F(x) = -\kappa + \frac{1}{2\pi} \int_{[-1,1]^2} \frac{1 - \cos(2\pi(x_1u + x_2v))}{u^2 + v^2} dudv$$

and g_n satisfies $\sup_{|x/n| \leq 1} |g_n(x)| \leq C$ and $g_n(x) \rightarrow 0$ as $n \rightarrow \infty$. The constant κ is determined by

$$\frac{1}{\pi} \sum_{\substack{(p,q) \in E_n \\ (p,q) \neq 0}} \frac{1}{p^2 + q^2} = \ln n + \kappa + o(1) \quad \text{as } n \rightarrow \infty$$

and, for all $\gamma^2 + \beta^2 > 2$, we have $\int_{\mathbb{R}^2} e^{-(\gamma^2 + \beta^2)F(x)} dx < \infty$.

Proof. Because of the definition of κ , it suffices to estimate the quantity

$$\frac{1}{2\pi} \sum_{\substack{|p|, |q| \leq n \\ (p,q) \neq 0}} \frac{1}{p^2 + q^2} \left(\cos(2\pi p \frac{x_1}{n} + 2\pi q \frac{x_2}{n}) - 1 \right)$$

It can be rewritten as

$$\frac{1}{n^2} \sum_{\substack{|p|, |q| \leq n \\ (p,q) \neq 0}} \frac{1}{2\pi} \frac{1}{(\frac{p}{n})^2 + (\frac{q}{n})^2} \left(\cos(2\pi p \frac{x_1}{n} + 2\pi q \frac{x_2}{n}) - 1 \right).$$

This is a sum of Riemann type. It converges as n goes to ∞ towards

$$\frac{1}{2\pi} \int_{[-1,1]^2} \frac{\cos(2\pi x_1 u + 2\pi x_2 v) - 1}{u^2 + v^2} dudv.$$

The function g_n is the remainder in this convergence and its properties are easily established via standard techniques of Riemann approximations. This proves the claim about the structure of F . Let us prove that F satisfies the announced integrability condition. Obviously, F is continuous and therefore locally integrable. We just have to study its behaviour close to ∞ . By making a change of variables, we get:

$$F(x) = -\kappa + \frac{1}{2\pi} \int_{[-|x|, |x|]^2} \frac{1 - \cos(2\pi(\frac{x_1}{|x|}u + \frac{x_2}{|x|}v))}{u^2 + v^2} dudv.$$

This quantity is then easily seen to be equivalent to $\frac{1}{2\pi} \int_{[-|x|, |x|]^2 \setminus [-1,1]^2} \frac{1}{u^2 + v^2} dudv$, which is in turn equivalent to $\ln |x|$ as $x \rightarrow \infty$. \square

One can thus apply Theorems 6.7 and 6.8 with the constant

$$\sigma^2(\gamma^2 + \beta^2) = \begin{cases} \int_{\mathbb{R}^2} \exp\left(-(\gamma^2 + \beta^2)F(x)\right) dx & \text{if } \gamma^2 + \beta^2 > 2, \\ 2\pi e^{2\kappa - 2\tilde{\kappa}} & \text{if } \gamma^2 + \beta^2 = 2. \end{cases}$$

where

$$\tilde{\kappa} = \int_{[-1,1]^2 \setminus B(0,1)} \frac{1}{|u|^2} du + \lim_{t \rightarrow \infty} \int_0^t \frac{\mathbf{1}_{r \in]0,1[} - (2\pi)^{-1} \int_0^{2\pi} \cos(2\pi r \cos \theta) d\theta}{r} dr$$

7 Applications in 2D-string theory

7.1 Introduction

Euclidian quantum gravity is an attempt to quantize general relativity based on Feynman's functional integral and on the Einstein-Hilbert action principle. One integrates over all Riemannian metrics on a d -dimensional manifold Σ . The Lorentzian signature is then hopefully recovered via Wick rotation.

General relativity is a reparametrization invariant theory which can be formulated with no reference to coordinates at all and this diffeomorphism invariance is a central issue in quantum theory. The main motivation for considering 2D (for two-dimensional) quantum gravity comes from the fact that the Einstein-Hilbert action becomes trivial in 2D as it reduces to a topological term and the cosmological constant coupled to the volume of space-time. All the non trivial dynamics of the two-dimensional theory thus come from gauge fixing the diffeomorphisms while keeping the geometry exactly fixed. This is the famous representation of the functional integral over geometries as a Liouville field theory by Polyakov [47] (see also [47, 40, 12]).

More precisely, one couples a Conformal Field Theory (CFT) (or more generally a matter field or quantum field theory) to gravity via any reparametrization invariant action for conformal matter fields with central charge c . A famous example is the coupling of c free scalar matter fields to gravity, which can also be interpreted as an embedding of Σ in a c -dimensional Euclidian space, thus leading to an interpretation of such a specific theory of 2D-Liouville Quantum Gravity as a bosonic string theory in c dimensions [47].

The following discussion will focus on the coupling of one free scalar matter field to gravity: the CFT is then said to have a central charge $c = 1$ and this corresponds to the bosonic string in 1 dimension. For a central charge $c = 1$, it is shown in [47, 40, 12]) that the reparametrization invariant action of the CFT (here called the Polyakov action) factorizes as a tensor product: the fluctuations of the metric are independent of the CFT. More precisely, the random metric roughly takes on the form [47, 40, 12] (we consider an Euclidean background metric for simplicity):

$$g(z) = e^{bX(z)} dz^2, \tag{7.1}$$

where the fluctuations of the field X are governed by the Liouville action (with $b = 2$ for a central charge $c = 1$) and the CFT becomes an independent free field, say Y . When the cosmological constant is set to 0, one talks about **critical 2D-Liouville Quantum Gravity** and the Liouville action turns the field X in (7.1) into a Free Field, with appropriate boundary conditions. For an excellent review on 2D string theory, we refer to Klebanov's lecture notes [38]. As expressed by Klebanov in [38]: "Two-dimensional string theory is the kind of toy model which possesses a remarkably simple structure but at the same time incorporates some of the physics of string theories embedded in higher dimensions". The reader is also referred to [13, 12, 25, 16, 17, 28, 29, 30, 31, 38, 40, 45, 47] for more insights on 2D-Liouville quantum gravity.

Therefore, in the following section, we review the basic notions of CFT with central charge $c = 1$ and then recall the basic notions of CFT with central charge $c = 1$ coupled to gravity, i.e. two-dimensional string theory. In particular, we show that our work enables to define mathematically the so-called **Tachyon fields**.

7.2 Conformal Field theory with central charge $c = 1$

We consider a domain D and a GFF Y on D with Dirichlet boundary conditions. In the physics literature, one considers the conformally invariant action:

$$S(Y) = \frac{1}{4\pi} \int_D |\nabla Y(x)|^2 dx \quad (7.2)$$

and all averages of functionals $F(Y)$ are denoted formally as:

$$\mathbb{E}[F(Y)] = \int F(Y) e^{-S(Y)} dY.$$

Note that the normalization in the definition of S ensures that:

$$\mathbb{E}[Y(y)Y(x)] \Big|_{|y-x| \rightarrow 0} \sim \ln \frac{1}{|y-x|}.$$

In this context, Conformal Field Theory (CFT) with central charge $c = 1$ involves defining and studying operators (or fields) formally constructed as functions of the GFF: see the mathematically oriented article [36] for more on this. Since the GFF is a distribution (generalized function), this is often not straightforward mathematically. Of particular importance are the so-called vertex operators denoted $V_\alpha = e^{\alpha Y}$ ($\alpha \in \mathbb{C}$) by physicists that we will rather denote formally in the following way:

$$V_\alpha(Y(x), x) = C(x, D)^{\frac{\alpha^2}{2}} e^{\alpha Y(x) - \frac{\alpha^2}{2} E[Y(x)^2]} \quad (7.3)$$

where $C(z, D)$ is the conformal radius. This formal definition is more accurate to denote what physicists of CFT or Quantum gravity call normal or Wick ordering of $e^{\alpha Y}$ (in other fields, in the Wick ordering of $e^{\alpha Y}$, the conformal radius does not appear in expression 7.3). Under this form, we recognize Gaussian multiplicative chaos if α is a nonnegative real less than 2. If α is complex with $|\alpha| < \sqrt{2}$, one can define V_α as a random distribution (see [36]). The construction is obvious because $|\alpha| < \sqrt{2}$ ensure the fields are \mathbb{L}_2 -integrable.

In this paper and in the following discussion, we consider the case $\alpha = i\beta$ where β is real. The conformal dimension of $V_{i\beta}(Y, z)$ measures how the field changes when one switches to another parametrization. More precisely, let $\psi : \tilde{D} \rightarrow D$ be a conformal map and set $\tilde{Y}(\tilde{x}) = (Y \circ \psi)(\tilde{x})$. Since the action (7.2) maps $Y \rightarrow \tilde{Y}$ under ψ , the conformal dimension $\Delta_{i\beta}$ of $V_{i\beta}$ is defined by

$$V_{i\beta}(Y(\psi(\tilde{x})), \psi(\tilde{x})) = |\psi'(\tilde{x})|^{-2\Delta_{i\beta}} V_{i\beta}(\tilde{Y}(\tilde{x}), \tilde{x}). \quad (7.4)$$

In fact, this corresponds to the definition of conformal dimension for spinless operators. By the rule $|C(\psi(\tilde{x}), D)| = |\psi'(\tilde{x})| |C(\tilde{x}, \tilde{D})|$, we get that $\Delta_{i\beta} = \frac{\beta^2}{4}$. In particular, we get the following rigorous relation by integrating the formal relation (7.4) for some compact set \tilde{K} ($K = \psi(\tilde{K})$):

$$\begin{aligned} \int_{\tilde{K}} V_{i\beta}(\tilde{Y}(\tilde{x}), \tilde{x}) d\tilde{x} &= \int_{\tilde{K}} V_{i\beta}(Y(\psi(\tilde{x})), \psi(\tilde{x})) |\psi'(\tilde{x})|^{2\Delta_{i\beta}} d\tilde{x} \\ &= \int_K V_{i\beta}(Y(x), x) |\psi'(\psi^{-1}(x))|^{2\Delta_{i\beta}-2} dx \end{aligned}$$

where K is some compact set.

7.3 CFT with central charge $c = 1$ coupled to gravity

In the special case $c = 1$ (hence the case of a GFF Y with Dirichlet boundary conditions on some domain D), we have $b = 2$ and the Polyakov action can be written under the following tensor form:

$$S(X, Y) = \frac{1}{4\pi} \int_D |\nabla Y(x)|^2 d^2x + \frac{1}{4\pi} \int_D |\nabla X(x)|^2 + QR(x)X(x)d^2x \quad (7.5)$$

where the first term is the classical GFF action (CFT with $c = 1$) and the second is the classical Liouville action (where we have set the cosmological constant to 0, R is the curvature and $Q = \frac{2}{b} + \frac{b}{2} = 2$).

Following the physics literature (see [17, 30, 45]), we consider the following equivalence class of random surfaces: if $\psi : \tilde{D} \rightarrow D$ is a conformal map then we get the following rule for the fields (X, Y) :

$$(X, Y) \rightarrow (X \circ \psi + 2 \ln |\psi'|, Y \circ \psi)$$

This equivalence class is a generalization of [25] in the sense that now we incorporate the matter field Y . The reparametrization rule for Y is a just a consequence of the conformal invariance of the Free Field, i.e. the action $\int_D |\nabla Y(x)|^2 d^2x$ is conformally invariant. Since Liouville Quantum Gravity is a conformal field theory, the relevant operators are the ones which are invariant under the above rule; this ensures that the operators are stable under reparametrization, i.e. are independent of the underlying background metric which is used to define the theory. For consistency reasons (no conformal anomaly), one can only consider conformally invariant dressed operators within Liouville Quantum Gravity. The simplest of such operators are the so-called **tachyon fields** (see the reviews [38, 45]). More precisely, the tachyon fields are the CFT vertex operators $e^{i\beta Y}$ with gravitational dressing of the form $e^{\gamma X}$ that are conformally invariant under the action (7.5). Hence they are formally of the form $e^{\gamma X(x) + i\beta Y(x)}$. Here, we stress the fact that $e^{\gamma X(x) + i\beta Y(x)}$ is a function of the two GFFs X, Y in order to determine the way it changes under reparametrization. In what follows, we will thus consider the Wick ordering of the field $e^{\gamma X(x) + i\beta Y(x)}$ and see if it properly defines a conformally invariant operator under the action (7.5).

The special point $\gamma = 2, \beta = 0$

At the special point ($\gamma = 2, \beta = 0$), we recover the special tachyon field

$$M_{X,Y}^{\gamma,\beta}(A) = \int_A C(z, D)^2 M'(dz)$$

where M' is the derivative martingale defined in [22, 23]:

$$M'(dz) = (2\mathbb{E}[X(z)^2] - X) e^{2X(z) - 2\mathbb{E}[X(z)^2]} dz.$$

Tachyons within phase I and frontier I/II

It is natural to first look for other tachyons fields in phase I together with the frontier of phases I/II (excluding the extremal points) because the renormalization is standard in this area. The Wick ordering of the field $e^{\gamma X(x) + i\beta Y(x)}$ then corresponds to (see subsection 6.2):

$$M_{X,Y}^{\gamma,\beta}(dx) = e^{\gamma X(x) + i\beta Y(x) - \frac{\gamma^2}{2} \mathbb{E}[X(x)^2] + \frac{\beta^2}{2} \mathbb{E}[Y(x)^2]} C(x, D)^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} dx.$$

We are thus looking for the couples (γ, β) satisfying

$$M_{X,Y}^{\gamma,\beta}(\varphi \circ \psi^{-1}) = M_{X \circ \psi + 2 \ln |\psi'|, Y \circ \psi}^{\gamma,\beta}(\varphi)$$

for every function $\varphi \in C_c^2(\tilde{D})$. Now, we get that:

$$\begin{aligned}
& M_{X \circ \psi + 2 \ln |\psi'|, Y \circ \psi}^{\beta, \gamma}(\varphi) \\
&= \int_{\tilde{D}} \varphi(\tilde{x}) e^{\gamma(X(\psi(\tilde{x})) + 2 \ln |\psi'(\tilde{x})|) + i\beta Y(\psi(\tilde{x})) - \frac{\gamma^2}{2} E[X(\psi(\tilde{x}))^2] + \frac{\beta^2}{2} E[Y(\psi(\tilde{x}))^2]} |C(\tilde{x}, \tilde{D})|^{\gamma^2/2 - \beta^2/2} d^2 \tilde{x} \\
&= \int_{\tilde{D}} \varphi(\tilde{x}) e^{\gamma X(\psi(\tilde{x})) + i\beta Y(\psi(\tilde{x})) - \frac{\gamma^2}{2} E[X(\psi(\tilde{x}))^2] + \frac{\beta^2}{2} E[Y(\psi(\tilde{x}))^2]} |C(\psi(\tilde{x}), D)|^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} |\psi'(\tilde{x})|^{2\gamma - \frac{\gamma^2}{2} + \frac{\beta^2}{2}} d^2 \tilde{x} \\
&= \int_D \varphi(\psi^{-1}(x)) e^{\gamma X(x) + i\beta Y(x) - \frac{\gamma^2}{2} E[X(x)^2] + \frac{\beta^2}{2} E[Y(x)^2]} |C(x, D)|^{\frac{\gamma^2}{2} - \frac{\beta^2}{2}} |\psi'(\psi^{-1}(x))|^{2\gamma - \frac{\gamma^2}{2} + \frac{\beta^2}{2} - 2} d^2 x.
\end{aligned}$$

Hence, for the field to be conformally invariant, one must solve the equation $2\gamma = \gamma^2/2 - \beta^2/2 + 2$, yielding the points of the frontier of phases I/II: $\gamma \pm \beta = 2$ for $\gamma \in]1, 2[$). This partially confirms predictions of physicists (see [38] and references therein) in the sense that it was claimed that we get tachyons on the segment $\gamma \pm \beta = 2$ for $\gamma \in [0, 2[$). Yet, we argue that one cannot find tachyons for the values $\gamma \pm \beta = 2$ and $\gamma \in [0, 1]$ by this Wick ordering procedure: we enter phase III and the triple point, which involves non standard renormalization that highly perturbs the behaviour of the exponent of the conformal radius, yielding operators that are not conformally invariant. In order to get tachyon fields, it might be necessary to study the second order which corresponds to analytic continuation in the physics literature (see [26]). For instance, in the case $\gamma = 0$, the paper [26] predicts as $\varepsilon \rightarrow 0$

$$\varepsilon^{-\frac{\beta^2}{2}} e^{i\beta Y_\varepsilon(x)} \varphi(x) dx \simeq \varepsilon^{1 - \frac{\beta^2}{2}} W(\varphi) + T(\varphi) \quad (7.6)$$

where W corresponds to the white noise and T is a distribution independent of W , which should correspond to the tachyon field.

KPZ formula for the tachyon fields

Consider the vertex operators $V_{i\beta} = e^{i\alpha Y}$, with $\beta \in [0, 1)$, of a Conformal Field Theory (CFT) with central charge $c = 1$, i.e. Y is a GFF. We have seen that the conformal dimension $\Delta_{i\beta}^0$ of $V_{i\beta}$ is $\frac{\beta^2}{4}$. The quantum dimension $\Delta_{i\beta}^q$ of the operator $V_{i\beta}$ is defined as the value such that the operator

$$e^{b(1 - \Delta_{i\beta}^q)X} V_{i\beta}$$

is conformally invariant within the theory, i.e. becomes a tachyon field (recall that $b = 2$). We have seen that we must choose

$$b(1 - \Delta_{i\beta}^q) + \beta = 2$$

in order for this field to be conformally invariant, yielding

$$\Delta_{i\beta}^q = \frac{\beta}{2}.$$

We thus recover the celebrated KPZ formula (see [40])

$$\Delta_{i\beta}^0 = \Delta_{i\beta}^q + \frac{b^2}{4} \Delta_{i\beta}^q (\Delta_{i\beta}^q - 1)$$

in its original derivation for (critical) Liouville quantum gravity with a $c = 1$ ($b = 2$) central charge.

Further comments on the special point $\gamma = 2, \beta = 0$

We would like to make some further comments about the physics literature at $c = 1$. As pointed out in [23], this literature suggests (at least) three possible interpretations of the tachyon field e^{2X} , corresponding to the point $(\gamma = 2, \beta = 0)$ in the diagram of Figure 2. The cornerstone of the discussion is that this point is a wild discontinuity point. The first interpretation is the derivative martingale M' (formally Xe^{2X}) constructed in [22], which turns out to coincide with e^{2X} via the Seneta-Heyde scaling (see [23]). However, this atypical tachyon field e^{2X} in Liouville quantum Gravity has been associated to another, non-standard, form of critical $c = 1, b = 2$ random surface models. Indeed, the introduction of higher trace terms in the action of the $c = 1$ matrix model of two-dimensional quantum gravity is believed to generate a new critical behavior of the random surface [32, 37, 39, 55], with an enhanced *critical* proliferation of spherical bubbles connected one to another by microscopic “wormholes”. The mathematical meaning of this physics poetry is to add atoms on top of the measure M' , meaning considering a random distribution $N_{M'}^1$ where, conditionally on the derivative martingale M' , $N_{M'}^1$ is an independently scattered random measure distributed on \mathbb{R}^2 and characterized by

$$\mathbb{E}[e^{iqN_{M'}^1(A)}] = e^{-c|q|M'(A)},$$

for some constant c . We further stress that the law $N_{M'}^1$ is exactly the same as the law (of the real part) of $W_{N_{M'}^{1/2}}$ (with the notation of Figure 2) that one gets along the half-line $\gamma = 2, \beta > 0$ of Figure 2. We believe that the introduction of these higher trace terms, though finely tuned so as to reach the point $(\gamma = 2, \beta = 0)$ at the scaling limit, make you reach this point via a path living strictly inside the phase II. Therefore, the scaling limit that you get is indeed a $2D$ -Cauchy process on top of the derivative martingale (i.e. $W_{N_{M'}^{1/2}}$ or equivalently $N_{M'}^1$) but does not correspond to the tachyon field because of the discontinuity at the point $(\gamma = 2, \beta = 0)$.

Anyway, this remains a striking fact from the angle of (still conjectural) analogy between random surfaces models and $2D$ -Liouville Quantum Gravity (see [14] for recent advances): all the random surface models developed to approximate $c = 1$ Liouville Quantum Gravity correspond to all the possible limits that you get when approaching the discontinuity point $(\gamma = 2, \beta = 0)$ in the diagram of Figure 2.

7.4 Chodos-Thorn/Feigin-Fuks Theory: Gaussian Free Field with a background charge

We discuss here applications of our results to Gaussian conformal field theory in the presence of a background charge, also known as Chodos-Thorn/Feigin-Fuks Theory (CTFF) [11, 19, 20, 30, 36]. By inserting an imaginary background charge to the free field action, this theory allows one to shift the $c = 1$ central charge of the standard free field action to a $c < 1$ central charge, with $c = 1 - 6\chi^2$ for some $\chi > 0$. In a way, this could be interpreted as a random geometry with imaginary curvature (see [30, 54]).

More precisely, we still consider the vertex operators $V_\alpha = e^{\alpha Y}$ ($\alpha \in \mathbb{C}$) defined by

$$V_\alpha(Y(x), x) = C(x, D)^{\frac{\alpha^2}{2}} e^{\alpha Y(x) - \frac{\alpha^2}{2} \mathbb{E}[Y(x)^2]} \tag{7.7}$$

where Y is a GFF on a planar domain D , say with Dirichlet boundary conditions. Once again, we can make sense of these operators via martingale techniques in the \mathbb{L}_2 phase, i.e. for $|\alpha|^2 < 2$. We continue this discussion while assuming now that α is purely imaginary, i.e. $\alpha = i\beta$. Our work

establishes that the only relevant values of β are such that $|\beta|^2 < 2$ since we have seen that other values of β enter the phase III (or its II/III boundary) and yield a limiting white noise.

One can still measure how the field changes under conformal maps provided that we fix a reparameterization rule. We assume from now on that D is the complex half-plane \mathbb{H} , i.e. $\mathbb{H} = \{z \in \mathbb{C}; \Im(z) > 0\}$. Given another simply connected planar domain \tilde{D} with a marked boundary point $q \in \partial\tilde{D}$ and a conformal map $\psi : \tilde{D} \rightarrow \mathbb{H}$ sending q to ∞ , we will denote by Y^ψ the centered Gaussian Free Field on \tilde{D} defined by

$$Y^\psi(x) = Y(\psi(x)).$$

We consider the following equivalence class of random surfaces (also called imaginary surface in [44, 54]). If $\psi : (\tilde{D}, q) \rightarrow (\mathbb{H}, \infty)$ is a conformal map then we consider the following rule for the field Y :

$$Y \rightarrow \tilde{Y}(\tilde{x}) = Y \circ \psi(\tilde{x}) - \chi \arg \psi'(\tilde{x}).$$

For test functions φ on D , we have

$$\begin{aligned} & \int_{\tilde{D}} \varphi(\psi(\tilde{x})) V_{i\beta}(\tilde{Y}(\tilde{x}), \tilde{x}) d\tilde{x} \\ &= \int_{\tilde{D}} \varphi(\psi(\tilde{x})) e^{i\beta(Y(\psi(\tilde{x})) - \chi \arg \psi'(\tilde{x})) + \frac{\beta^2}{2} E[Y(\psi(\tilde{x}))^2]} |C(\tilde{x}, \tilde{D})|^{-\beta^2/2} d^2\tilde{x} \\ &= \int_D \varphi(x) e^{i\beta Y(x) + \frac{\beta^2}{2} E[Y(x)^2]} |C(x, D)|^{-\frac{\beta^2}{2}} |\psi'(\psi^{-1}(x))|^{\frac{\beta^2}{2} - 2} e^{-i\beta \chi \arg(\psi'(\psi^{-1}(x)))} d^2x \\ &= \int_D \varphi(x) V(Y(x), x) \psi'(\psi^{-1}(x))^{\frac{\beta^2}{4} - \frac{\beta\chi}{2} - 1} \bar{\psi}'(\psi^{-1}(x))^{\frac{\beta^2}{4} + \frac{\beta\chi}{2} - 1} d^2x, \end{aligned}$$

where $\bar{\psi}'$ denotes the complex conjugate of ψ' . This operator is not spinless in the sense that the exponent of ψ' differs from that of $\bar{\psi}'$ in the above right-hand side. The conformal dimensions of this operator are then given by $(\Delta_{i\beta}, \bar{\Delta}_{i\beta}) = (\frac{\beta^2}{4} - \frac{\beta\chi}{2}, \frac{\beta^2}{4} + \frac{\beta\chi}{2})$. The quantity $\Delta_{i\beta} - \bar{\Delta}_{i\beta} = -\chi\beta$ is called the *conformal spin*.

Let us make some further comments. If the conformal spin of the vertex operator $V_{i\beta}$, that is formally $V_{i\beta} = e^{iY/\chi}$ (to be understood as the limit $\lim_{\epsilon \rightarrow 0} \epsilon^{-\beta^2/2} e^{iY_\epsilon(x)} dx$ as defined in this paper), is -1 then the direction of the field transforms as the direction of a vector field under conformal maps in such a way that the flow lines $\dot{v} = V_{i\chi^{-1}}(v)$ of the vertex operator $V_{i\chi^{-1}}$ are conformally invariant curves, which are studied in [44, 54], leading to a SLE based treatment of the CTFF theory. Let us mention the correspondence $b = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ with the standard κ parameter of SLE theory (see [36, 54]). One may also consult [21] for applications of vertex operators to the study of conformally invariant curves.

Our paper naturally raises the following question. As a phase transition occurs when renormalizing the field $V_{i\chi^{-1}}$ (we get a limiting white noise for $|\chi|^{-2} \geq 2$), how can this be interpreted in the SLE framework? For instance, does this mean that we cannot approximate the flow lines in [44, 54] for $|\chi|^{-2} \geq 2$ by discretizing the GFF? As in the case of tachyons, it might be necessary to consider the approximation (7.6) and to consider the flow line of the limiting object T , discarding the effect of the white noise.

Keeping up with this train of thoughts, a possible generalization could be the following. One may for instance consider the flow lines of the operator $e^{\gamma X + i\beta Y}$ where X, Y are two independent GFF on D . Such operators occur for instance when coupling the CTFF theory with central charge $c = 1 - 6\chi^2$ to $2D$ -gravity, in which case the string susceptibility matches $b = \sqrt{4 + \chi^2} - \chi$. The

reparametrization rule for the field (X, Y) is then given by

$$(X, Y) \mapsto \tilde{Y}(\tilde{x}) = (X \circ \psi(\tilde{x}) + \sqrt{4 + \chi^2} \ln |\psi'(\tilde{x})|, Y \circ \psi(\tilde{x}) - \chi \arg \psi'(\tilde{x}))$$

The conformal spin of the operator $e^{\gamma X + i\beta Y}$ remains equal to $-\beta\chi$ so we still impose $\beta = 1/\chi$ in order to get conformally invariant flow lines. But there is no need to make tedious computations: it is straightforward to check that the flow lines \tilde{v} of $e^{\gamma X + iY/\chi}$ are just the flow lines v of $e^{iY/\chi}$ time changed by $F^{-1}(t)$ where $\dot{F}(t) = e^{-\gamma X(v_t)}$, i.e. $u = v \circ F^{-1}$. If one knows a bit of information about the capacity properties of the occupation measure of the path v , then one can apply Kahane's theory of Gaussian multiplicative chaos [35] to define the change of time F . For instance, this strategy has been applied in [29] to define the natural quantum parameterization of Brownian motion. Similar questions may be addressed in phase II : do we need to consider higher order terms in order to get rid of the stable measure with random intensity?

A Control of moments of order $2k$

In this appendix, we gather technical estimates on the convergence of the higher order moments in phase III and its frontiers with the other phases. The main purpose of this appendix is to prove proposition A.1 below. The following results have straightforward analogs in all dimensions: for the sake of clarity, we state and prove them in dimension 1.

Proposition A.1. *Let k and k' be natural integers. Then for any interval J , in the phase III, I/III and II/III we have the following convergence in probability:*

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E} \left[(M_\varepsilon^{\gamma, \beta}(J))^k (\bar{M}_\varepsilon^{\gamma, \beta}(J))^{k'} \mid \mathcal{F}^X \right]}{\mathbb{E} \left[M_\varepsilon^{\gamma, \beta}(J) \bar{M}_\varepsilon^{\gamma, \beta}(J) \mid \mathcal{F}^X \right]^{k+k'/2}} = k! \mathbf{1}_{k=k'} \quad (\text{A.1})$$

For all $l \geq 2$ and all $2l$ -tuple of natural integers (k_1, \dots, k_{2l}) , for any collection of disjoint intervals J_1, \dots, J_l , we have the following convergence in probability:

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathbb{E} \left[\prod_{1 \leq i \leq l} (M_\varepsilon^{\gamma, \beta}(J_i))^{k_{2i-1}} (\bar{M}_\varepsilon^{\gamma, \beta}(J_i))^{k_{2i}} \mid \mathcal{F}^X \right]}{\prod_{1 \leq i \leq l} \mathbb{E} \left[M_\varepsilon^{\gamma, \beta}(J_i) \bar{M}_\varepsilon^{\gamma, \beta}(J_i) \mid \mathcal{F}^X \right]^{(k_{2i-1} + k_{2i})/2}} = \prod_{1 \leq i \leq l} k_{2i-1}! \mathbf{1}_{k_{2i-1} = k_{2i}}. \quad (\text{A.2})$$

A.1 Optimal matching between two finite sets in \mathbb{R}^d

Before starting the proof, we introduce a matching procedure. This algorithm was introduced by Gale and Shapley [27] to provide a solution to the stable marriage problem.

Given $k \leq k'$, let (x_1, \dots, x_k) and $(y_1, \dots, y_{k'})$ be two sets of points in \mathbb{R}^d such that all the pairwise distances $|x_i - y_j|$ are distinct. The **optimal matching** of \mathbf{x} with \mathbf{y} is an injective application:

$$\sigma(\mathbf{x}, \mathbf{y}) : \{1, \dots, k\} \rightarrow \{1, \dots, k'\}$$

obtained by the following procedure:

- (i) If x_i and y_j are mutually closest, i.e. if:

$$\forall i', j' \quad |x_i - y_j| < |x_{i'} - y_{j'}| \text{ and } |x_i - y_j| < |x_i - y_{j'}|$$

then we set $\sigma(i) = j$.

- (ii) We delete the points that have been matched in step (i).
- (iii) We iterate the procedure until all the x_i 's have been matched.

A.2 Proof of proposition A.1 for matching indices $k = k'$

We prove (A.1) for $J = [0, 1]$. The general case can be proved along the same lines. We introduce the following notation:

$$M_\varepsilon^{\gamma, \beta}(\mathbf{dx}) := \prod_{j=1}^k M_\varepsilon^{\gamma, \beta}(dx_j)$$

The moment $\mathbb{E} \left[\left| M_\varepsilon^{\gamma, \beta}([0, 1]) \right|^{2k} \mid \mathcal{F}^X \right]$ is given by

$$\varepsilon^{k\beta^2} \int_{[0,1]^{2k}} \frac{M_\varepsilon^{\gamma, 0}(\mathbf{dxdy}) \left(\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k} G_\varepsilon(y_i - y_j) \right)^{\beta^2}}{\prod_{i,j=1}^k G_\varepsilon(x_i - y_j)^{\beta^2}}. \quad (\text{A.3})$$

Hence (A.1) for $k = k'$ corresponds to proving

$$\lim_{\varepsilon \rightarrow 0} \frac{\int_{[0,1]^{2k}} \frac{M_\varepsilon^{\gamma, 0}(\mathbf{dxdy}) \left(\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k} G_\varepsilon(y_i - y_j) \right)^{\beta^2}}{\prod_{i,j=1}^k G_\varepsilon(x_i - y_j)^{\beta^2}}}{\left(\int_{[0,1]^2} M_\varepsilon^{\gamma, 0}(\mathbf{dxdy}) G_\varepsilon(x - y)^{-\beta^2} \right)^k} = k!. \quad (\text{A.4})$$

Let $\sigma(\mathbf{x}, \mathbf{y})$ be the permutation obtained by the optimal matching procedure described in the previous section (which is Lebesgue almost-everywhere well defined) and set

$$B := \{(\mathbf{x}, \mathbf{y}) \mid \sigma(\mathbf{x}, \mathbf{y}) = \mathbb{1}\},$$

where $\mathbb{1}$ denotes the identity. By symmetry of the indices, we can rewrite (A.3) (divided by $\varepsilon^{k\beta^2}$) as

$$k! \int_B \frac{M_\varepsilon^{\gamma, 0}(\mathbf{dxdy}) \left(\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k} G_\varepsilon(y_i - y_j) \right)^{\beta^2}}{\prod_{i,j=1}^k G_\varepsilon(x_i - y_j)^{\beta^2}}. \quad (\text{A.5})$$

A reformulation of (A.4) is that one can find a δ that tends to zero with ε which is such that for all k with high probability

$$\begin{aligned} & (1 - \delta) \left(\int_{[0,1]^2} G_\varepsilon(x - y)^{-\beta^2} M_\varepsilon^{\gamma, 0}(\mathbf{dxdy}) \right)^k \\ & \leq \int_B \frac{M_\varepsilon^{\gamma, 0}(\mathbf{dxdy}) \left(\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k} G_\varepsilon(y_i - y_j) \right)^{\beta^2}}{\prod_{i,j=1}^k G_\varepsilon(x_i - y_j)^{\beta^2}} \\ & \leq (1 + \delta) \left(\int_{[0,1]^2} G_\varepsilon(x - y)^{-\beta^2} M_\varepsilon^{\gamma, 0}(\mathbf{dxdy}) \right)^k. \quad (\text{A.6}) \end{aligned}$$

Let us consider functions $a(\varepsilon)$ and $b(\varepsilon)$ such that $\varepsilon \ll a(\varepsilon) \ll b(\varepsilon) \ll 1$. We set

$$A := \{(\mathbf{x}, \mathbf{y}) \in [0, 1]^{2k} \mid \forall i, |x_i - y_i| \leq a(\varepsilon), \forall i \neq j, |x_i - x_j| \geq b(\varepsilon), |y_i - y_j| \geq b(\varepsilon)\}. \quad (\text{A.7})$$

Note that $A \subset B$ (because when $(\mathbf{x}, \mathbf{y}) \in A$ all the pairs (x_i, y_i) are mutually closest).

Lemma A.2. *When $(\mathbf{x}, \mathbf{y}) \in A$ we have*

$$\begin{aligned} (1 - \delta) \prod_{i=1}^k G_\varepsilon(x_i - y_i)^{-\beta^2} &\leq \frac{\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j)^{\beta^2} \prod_{1 \leq i < j \leq k} G_\varepsilon(y_i - y_j)^{\beta^2}}{\prod_{i,j=1}^k G_\varepsilon(x_i - y_j)^{\beta^2}} \\ &\leq (1 + \delta) \prod_{i=1}^k G_\varepsilon(x_i - y_i)^{-\beta^2} \end{aligned} \quad (\text{A.8})$$

where $\delta = \delta(\varepsilon)$ tends to zero when ε does.

For $(\mathbf{x}, \mathbf{y}) \in B$ we have a general upper bound

$$\frac{\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j)^{\beta^2} \prod_{1 \leq i < j \leq k} G_\varepsilon(y_i - y_j)^{\beta^2}}{\prod_{i,j=1}^k G_\varepsilon(x_i - y_j)^{\beta^2}} \leq C(k, \beta) \prod_{i=1}^k G_\varepsilon(x_i - y_i)^{-\beta^2}. \quad (\text{A.9})$$

Proof. We prove (A.9) by induction on k . We can assume that $x_1 = y_1$ are mutually closest (there is at least one pair of mutually closest vertices and we exchange the indices if needed). Then we have for all $j \in \{2, \dots, k\}$

$$\begin{aligned} |y_1 - y_j| &\leq |x_1 - y_1| + |x_1 - y_j| \leq 2|x_1 - y_j| \\ |x_1 - x_j| &\leq |x_1 - y_1| + |y_1 - x_j| \leq 2|y_1 - x_j|. \end{aligned} \quad (\text{A.10})$$

We have from Lemma 4.7

$$\sup_{s \in [0, 1], t \leq 2s, \varepsilon \leq 1} \frac{G_\varepsilon(t)}{G_\varepsilon(s)} = C_1 < \infty,$$

and hence, using the inequalities (A.10) we obtain that

$$\begin{aligned} \frac{\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k} G_\varepsilon(y_i - y_j)}{\prod_{i,j=1}^k G_\varepsilon(x_i - y_j)} &\leq C_1^{2(k-1)} \frac{1}{G_\varepsilon(x_1 - y_1)} \frac{\prod_{2 \leq i < j \leq k} G_\varepsilon(y_i - y_j)}{\prod_{i,j=2}^k G_\varepsilon(x_i - y_j)}. \end{aligned} \quad (\text{A.11})$$

Then using the induction hypothesis, (note that the identity is still the optimal matching once (x_1, y_1) have been deleted) we obtain (A.9) with $C(k, \beta) = C_1^{\beta^2 k(k-1)}$.

The inequality (A.8) is obtained by noticing that on the set A for all $i < j$

$$\begin{aligned} ||y_i - y_j| - |y_i - x_j|| &\leq |x_j - y_j| \leq a(\varepsilon) \\ ||x_i - x_j| - |x_i - y_j|| &\leq |x_j - y_j| \leq a(\varepsilon) \end{aligned} \quad (\text{A.12})$$

We have from Lemma 4.7

$$\lim_{\varepsilon \rightarrow 0} \sup_{s \geq b(\varepsilon), t \in (s-a(\varepsilon), s+a(\varepsilon))} \left| \frac{G_\varepsilon(s)}{G_\varepsilon(t)} - 1 \right| = 0.$$

Let $\delta_1(\varepsilon)$ be the quantity in the limit. Using (A.12) we obtain (A.8) with $(1 \pm \delta_1(\varepsilon))^{\beta^2 k(k-1)}$ instead of $(1 \pm \delta)$. \square

Now, we state the following lemma whose proof is postponed to the next subsection.

Lemma A.3. *For all k , we have the following convergence in probability*

$$\lim_{\varepsilon \rightarrow 0} \frac{\int_{[0,1]^{2k} \setminus A} M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) \prod_{i=1}^k G_\varepsilon(x_i - y_i)^{-\beta^2}}{\left(\int_{[0,1]^2} M_\varepsilon^{\gamma,0}(\mathbf{d}x\mathbf{d}y) G_\varepsilon(x - y)^{-\beta^2} \right)^k} = 0. \quad (\text{A.13})$$

With this lemma and Lemma A.2, we can conclude the proof of (A.1). Let us first prove the lower bound in (A.6). First we replace the domain integration B by A which is smaller. Then we use (A.8) and obtain that

$$\begin{aligned} & \int_B \frac{M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) \left(\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k} G_\varepsilon(y_i - y_j) \right)^{\beta^2}}{\prod_{i,j=1}^k G_\varepsilon(x_i - y_j)^{\beta^2}} \\ & \geq (1 - \delta) \int_A M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) \prod_{i=1}^k G_\varepsilon(x_i - y_i)^{-\beta^2} \\ & \geq (1 - \delta') \left(\int_{[0,1]^2} M_\varepsilon^{\gamma,0}(\mathbf{d}x\mathbf{d}y) G_\varepsilon(x - y)^{-\beta^2} \right)^k. \end{aligned}$$

where the last line holds with high probability according to Lemma A.3.

For the upper bound in (A.6), we remark that from (A.8) we have:

$$\begin{aligned} & \int_A \frac{M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) \left(\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k} G_\varepsilon(y_i - y_j) \right)^{\beta^2}}{\prod_{i,j=1}^k G_\varepsilon(x_i - y_j)^{\beta^2}} \\ & \leq (1 + \delta) \int_A M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) \prod_{i=1}^k G_\varepsilon(x_i - y_i)^{-\beta^2}. \quad (\text{A.14}) \end{aligned}$$

Thus it is sufficient to control the contribution of $B \setminus A$ to conclude. From (A.9) we have:

$$\begin{aligned} & \int_{B \setminus A} \frac{M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) \left(\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k} G_\varepsilon(y_i - y_j) \right)^{\beta^2}}{\prod_{i,j=1}^k G_\varepsilon(x_i - y_j)^{\beta^2}} \\ & \leq C(k, \beta^2) \int_{[0,1]^{2k} \setminus A} M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) \prod_j G_\varepsilon(x_j - y_j)^{-\beta^2}. \quad (\text{A.15}) \end{aligned}$$

According to (A.13), the r.h.s. is smaller than $\delta \left(\int_{[0,1]^2} M_\varepsilon^{\gamma,0}(\mathbf{d}x\mathbf{d}y) G_\varepsilon(x - y)^{-\beta^2} \right)^k$ provided ε is chosen sufficiently small.

A.3 Proof of Lemma A.3

We decompose the set $[0, 1]^{2k} \setminus A$ as a union of non-disjoint events as follows

$$\begin{aligned}
[0, 1]^{2k} \setminus A &:= \bigcup_{i=1}^k \{(\mathbf{x}, \mathbf{y}) \in [0, 1]^{2k} \mid |x_i - y_i| > a(\varepsilon)\} \\
&\quad \cup \bigcup_{1 \leq i < j \leq k} \{(\mathbf{x}, \mathbf{y}) \in [0, 1]^{2k} \mid |x_i - x_j| < b(\varepsilon)\} \\
&\quad \cup \bigcup_{1 \leq i < j \leq k} \{(\mathbf{x}, \mathbf{y}) \in [0, 1]^{2k} \mid |y_i - y_j| < b(\varepsilon)\} \\
&=: \bigcup_{i=1}^k \bar{A}_i \cup \bigcup_{1 \leq i < j \leq k} \bar{A}_{i,j} \cup \bigcup_{1 \leq i < j \leq k} \bar{A}'_{i,j}
\end{aligned}$$

Then by permutation of the indices and symmetry in x, y we have:

$$\begin{aligned}
\int_{[0,1]^{2k} \setminus A} M_\varepsilon^{\gamma,0}(\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}) \prod_{i=1}^k G_\varepsilon(x_i - y_i)^{-\beta^2} &\leq k \int_{\bar{A}_1} M_\varepsilon^{\gamma,0}(\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}) \prod_{i=1}^k G_\varepsilon(x_i - y_i)^{-\beta^2} \\
&\quad + k(k-1) \int_{\bar{A}_{1,2}} M_\varepsilon^{\gamma,0}(\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}) \prod_{i=1}^k G_\varepsilon(x_i - y_i)^{-\beta^2}.
\end{aligned}$$

Hence it is sufficient to show (A.13) with $[0, 1]^{2k} \setminus A$ replaced by \bar{A}_1 and $\bar{A}_{1,2}$ in the numerator's integrand. After simplification it amounts to showing two things (see the next lemma). One sets

$$\begin{aligned}
D_2 &:= \{(x, y) \in [0, 1]^2 \mid |x - y| > a(\varepsilon)\} \\
D_4 &:= \{(\mathbf{x}, \mathbf{y}) \in [0, 1]^4 \mid |x_1 - x_2| < b(\varepsilon)\}.
\end{aligned} \tag{A.16}$$

Lemma A.4. *The two following convergences hold in probability:*

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \frac{\int_{D_2} G_\varepsilon(x - y)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y})}{\int_{[0,1]^2} G_\varepsilon(x - y)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y})} &= 0, \\
\lim_{\varepsilon \rightarrow 0} \frac{\int_{D_4} \prod_{i=1}^2 G_\varepsilon(x_i - y_i)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y})}{\left(\int_{[0,1]^2} G_\varepsilon(x - y)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y})\right)^2} &= 0.
\end{aligned} \tag{A.17}$$

Proof. Using the results of section 4.2 concerning convergence of the second moment, one can make the following replacement in the denominator of (A.17)

$$\int_{[0,1]^2} G_\varepsilon(x - y)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}) \approx \begin{cases} \varepsilon^{1-2\gamma^2-\beta^2} & \text{if } \gamma < 1/\sqrt{2} \text{ and } \gamma^2 + \beta^2 > 1, \\ \varepsilon^{-\gamma^2} |\ln(\varepsilon)| & \text{if } \gamma < 1/\sqrt{2} \text{ and } \gamma^2 + \beta^2 = 1, \\ \varepsilon^{1-2\gamma^2-\beta^2} |\ln(\varepsilon)|^{-1/2} & \text{if } \gamma = 1/\sqrt{2} \text{ and } \gamma^2 + \beta^2 > 1, \end{cases} \tag{A.18}$$

in the sense that the above ratios converge in probability towards a positive random variable. Then the first line of (A.17) comes from an easy \mathbb{L}_1 computation

$$\mathbb{E} \left[\int_{D_2} G_\varepsilon(x - y)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}) \right] \approx \begin{cases} \varepsilon^{-\gamma^2} a(\varepsilon)^{1-\gamma^2-\beta^2} & \text{if } \gamma \leq 1/\sqrt{2} \text{ and } \gamma^2 + \beta^2 > 1, \\ \varepsilon^{-\gamma^2} |\ln(a(\varepsilon))| & \text{if } \gamma < 1/\sqrt{2} \text{ and } \gamma^2 + \beta^2 = 1. \end{cases} \tag{A.19}$$

which is good enough as $a(\varepsilon) > 0$ (we only need to use the Markov inequality). When $\gamma = 1/\sqrt{2}$ we require $(a(\varepsilon)/\varepsilon)^{\gamma^2+\beta^2-1} \gg |\ln \varepsilon|^{1/2}$ to make things work.

A similar computation works for $\gamma < \frac{1}{\sqrt{2}}$ for the second line of (A.17) but miserably fails in the other cases. Set $B(\varepsilon) = b(\varepsilon)^{-1}$ (and we assume that b is defined so that B is an integer). For $j = 1, \dots, B-1$, we define $I_j := [b(j-1), b(j+1)]$.

We have

$$\int_{D_4} \prod_{i=1}^2 G_\varepsilon(x_i - y_i)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) \leq \sum_{j=1}^{B-1} \left(\int_{I_j \times [0,1]} G_\varepsilon(x-y)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) \right)^2. \quad (\text{A.20})$$

Now we treat the case $\gamma < 1/\sqrt{2}$ and $\gamma^2 + \beta^2 > 1$, the others can be dealt with similarly. We set

$$\widetilde{M}_\varepsilon^{2\gamma,0}(I) = \varepsilon^{2\gamma^2+\beta^2-1} \int_{I \times [0,1]} G_\varepsilon(x-y)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y})$$

and therefore get:

$$(\varepsilon^{2\gamma^2+\beta^2-1})^2 \int_{D_4} \prod_{i=1}^2 G_\varepsilon(x_i - y_i)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) \leq \sup_{1 \leq j \leq B} (\widetilde{M}_\varepsilon^{2\gamma,0}[I_j]) \widetilde{M}_\varepsilon^{2\gamma,0}([0,1]) \quad (\text{A.21})$$

By the results of section 4.2, the random measures $\widetilde{M}_\varepsilon^{2\gamma,0}$ converge in probability in the space of Radon measures to the measure $M^{2\gamma,0}$. By extracting a subsequence, we can assume that almost sure convergence holds. Let $b > 0$ be fixed. We have:

$$\overline{\lim}_{\varepsilon \rightarrow 0} \sup_{1 \leq j \leq B} (\widetilde{M}_\varepsilon^{2\gamma,0}[I_j]) \widetilde{M}_\varepsilon^{2\gamma,0}([0,1]) \leq \sup_{1 \leq j \leq b} (M^{2\gamma,0}(I_j)) M^{2\gamma,0}([0,1])$$

Now one can conclude by letting b go to 0 in the above inequality and using the fact that $M^{2\gamma,0}$ has no atoms.

Hence we have proved that:

$$\lim_{\varepsilon \rightarrow 0} \frac{\int_{D_4} \prod_{i=1}^2 G_\varepsilon(x_i - y_i)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y})}{\left(\int_{[0,1]^2} G_\varepsilon(x-y)^{-\beta^2} M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) \right)^2} = 0. \quad (\text{A.22})$$

□

A.4 The case $k < k'$

This case is easier and only uses the tools developed for the $k = k'$ case.

An easy computation shows that (A.1) for $k < k'$ corresponds to proving

$$\lim_{\varepsilon \rightarrow 0} \frac{\int_{[0,1]^{k+k'}} \frac{M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) (\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k'} G_\varepsilon(y_i - y_j))^{\beta^2}}{\prod_{i=1}^k \prod_{j=1}^{k'} G_\varepsilon(x_i - y_j)^{\beta^2}}}{\left(\int_{[0,1]^2} M_\varepsilon^{\gamma,0}(\mathbf{d}\mathbf{x}\mathbf{d}\mathbf{y}) G_\varepsilon(x-y)^{-\beta^2} \right)^{(k+k')/2}} = 0. \quad (\text{A.23})$$

Let σ denote the function obtained from the matching procedure of $\mathbf{x} \in \mathbb{R}^k$, $\mathbf{y} \in \mathbb{R}^{k'}$. Set

$$B := \{(\mathbf{x}, \mathbf{y}) \in [0,1]^{k+k'} \mid \sigma(\mathbf{x}, \mathbf{y})(i) = i, \forall i \in \{1, \dots, k\}\}.$$

We have by invariance under permutation of the indices that the numerator above is equal to

$$\frac{k'!}{(k' - k)!} \int_B \frac{M_\varepsilon^{\gamma,0}(\mathbf{dxdy}) \left(\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k'} G_\varepsilon(y_i - y_j) \right)^{\beta^2}}{\prod_{i=1}^k \prod_{j=1}^{k'} G_\varepsilon(x_i - y_j)^{\beta^2}}. \quad (\text{A.24})$$

Now we can adapt the proof of (A.9) in Lemma A.2 and show that for all $(\mathbf{x}, \mathbf{y}) \in B$

$$\frac{\left(\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k'} G_\varepsilon(y_i - y_j) \right)^{\beta^2}}{\prod_{i=1}^k \prod_{j=1}^{k'} G_\varepsilon(x_i - y_j)^{\beta^2}} \leq C_1^{\beta^2 k(k'-1)} \frac{\prod_{k \leq i < j \leq k'} G_\varepsilon(y_i - y_j)^{\beta^2}}{\prod_{i=1}^k G_\varepsilon(x_i - y_i)^{\beta^2}}. \quad (\text{A.25})$$

Then using Lemma 4.7, we see that the numerator of the r.h.s. above is bounded by a constant. Hence there is a constant C such that

$$\begin{aligned} & \int_B \frac{M_\varepsilon^{\gamma,0}(\mathbf{dxdy}) \left(\prod_{1 \leq i < j \leq k} G_\varepsilon(x_i - x_j) \prod_{1 \leq i < j \leq k'} G_\varepsilon(y_i - y_j) \right)^{\beta^2}}{\prod_{i=1}^k \prod_{j=1}^{k'} G_\varepsilon(x_i - y_j)^{\beta^2}} \\ & \leq C \int_{[0,1]^{k+k'}} \frac{M_\varepsilon^{\gamma,0}(\mathbf{dxdy})}{\prod_{i=1}^k G_\varepsilon(x_i - y_i)^{\beta^2}} \\ & = \left(\int_{[0,1]^2} M_\varepsilon^{\gamma,0}(\mathbf{dxdy}) G_\varepsilon(x - y)^{-\beta^2} \right)^k (M_\varepsilon^{\gamma,0}([0,1]))^{k'-k}. \end{aligned}$$

Hence to prove (A.23), it is sufficient to prove that

$$\lim_{\varepsilon \rightarrow 0} \frac{M_\varepsilon^{\gamma,0}([0,1])}{\sqrt{\int_{[0,1]^2} M_\varepsilon^{\gamma,0}(\mathbf{dxdy}) G_\varepsilon(x - y)^{-\beta^2}}} = 0. \quad (\text{A.26})$$

This is a simple consequence of the results of Section 4.2.

A.5 Proof of (A.2)

We treat only the case of two intervals, more intervals meaning only more notational problems. We further assume that J_1 and J_2 are at positives distance from one another for simplicity.

If this is not the case, as when $J_1 = [-1, 0]$ and $J_2 = [0, 1]$, we can split J_2 into two intervals $J_2' = [0, \delta]$ and $J_2'' = [\delta, 1]$ and expand the factor $M(J_2) = M(J_2') + M(J_2'')$ in the product. Then using the Hlder inequality together with (A.1) we prove that all the term where J_2' appear have a negligible contribution when δ goes to zero.

The moment in the numerator is equal to

$$\begin{aligned} & \varepsilon^{k\beta^2} \int_{\mathbf{x}^i \in I^{k_i}} M_\varepsilon^{\gamma,0}(\mathbf{dx}) \frac{\left(\prod_{l=1}^4 \prod_{1 \leq i < j \leq k_l} G_\varepsilon(x_i^l - x_j^l) \right)^{\beta^2}}{\left(\prod_{(l,m) \in \{(1,2), (3,4)\}} \prod_{i=1}^{k_l} \prod_{j=1}^{k_m} G_\varepsilon(x_i^l - x_j^m) \right)^{\beta^2}} \\ & \quad \times \frac{\left(\prod_{i=1}^{k_1} \prod_{j=1}^{k_3} G_\varepsilon(x_i^1 - x_j^3) \right)^{\beta^2} \left(\prod_{i=1}^{k_2} \prod_{j=1}^{k_4} G_\varepsilon(x_i^2 - x_j^4) \right)^{\beta^2}}{\left(\prod_{(l,m) \in \{(1,4), (2,3)\}} \prod_{i=1}^{k_l} \prod_{j=1}^{k_m} G_\varepsilon(x_i^l - x_j^m) \right)^{\beta^2}} \quad (\text{A.27}) \end{aligned}$$

Now if I and J are at a positive distance, the term appearing on the second line is uniformly bounded, and hence we obtain the result for free when either $k_1 \neq k_2$ or $k_3 \neq k_4$. In the case $k_1 = k_2$ or $k_3 = k_4$ we have to show that the cross term

$$\frac{\left(\prod_{i=1}^{k_1} \prod_{j=1}^{k_3} G_\varepsilon(x_i^1 - x_j^3)\right)^{\beta^2} \left(\prod_{i=1}^{k_2} \prod_{j=1}^{k_4} G_\varepsilon(x_i^2 - x_j^4)\right)^{\beta^2}}{\left(\prod_{(l,m) \in \{(1,4), (2,3)\}} \prod_{i=1}^{k_l} \prod_{j=1}^{k_m} G_\varepsilon(x_i^l - x_j^m)\right)^{\beta^2}}$$

is roughly equal to one on the subset of $J_1^{2k_1} \times J_2^{2k_3}$ that really matters. First we remark that multiplying by $k_1!k_3!$ we can restrict to the set

$$B_1 \times B_2 := \{(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4) \in J_1^{2k_1} \times J_2^{2k_3} \mid \sigma(\mathbf{x}^1, \mathbf{x}^2) = \mathbf{1}, \sigma(\mathbf{x}^3, \mathbf{x}^4) = \mathbf{1}\}.$$

Then we show that on the set $A_1 \times A_2$ where A_1 and A_2 are defined similarly to A in (A.7) we show that the cross term is in the interval $(1 - \delta, 1 + \delta)$. Then one can conclude by using Lemma A.3.

B Differentiability of Gaussian processes

In this section, we state a few results about Gaussian processes, mainly about differentiability and convergence in law in the sense of distributions. These results are certainly not new but we have not found any proper reference.

In what follows, D stands for a compact subset D of \mathbb{R}^d and for X be a \mathbb{R}^m -valued stochastically continuous stochastic process defined on D .

First we recall the classical

Proposition B.1. *If, for some $\beta, \alpha, C > 0$:*

$$\forall x, z \in D, \quad \mathbb{E}[|X_x - X_z|^q] \leq C|x - z|^{d+\beta}.$$

For all $\gamma \in]0, \frac{\beta}{q}[$, we set $L = \sup_{x \neq z} \frac{|X_x - X_z|}{|x - z|^\gamma}$. Then, for all $p < q$, $\mathbb{E}[L^\beta] \leq 1 + \frac{Cp^{2\beta - q\gamma}}{(q-p)(2\beta - q\gamma - 1)}$.

Now we claim the following:

Proposition B.2. *Assume that X is a centered Gaussian process with covariance kernel K . Then:*

- 1) *if for some $\alpha > 0$ and $\forall x, y \in D$, $K(x, x) + K(z, z) - 2K(x, z) \leq C|x - z|^\alpha$ then X admits a γ -Hölder modification for all $\gamma \in]0, \alpha/2[$.*
- 2) *if, for some $k \in \mathbb{N}^*$ and $\alpha > 0$, K is of class C^{2k} and $\forall x, y \in D$:*

$$\partial_x^k \partial_z^k K(x, x) + \partial_x^k \partial_z^k K(z, z) - 2\partial_x^k \partial_z^k K(x, z) \leq C|x - z|^\alpha, \quad (\text{B.1})$$

then X is of class C^k . Furthermore the k -th derivative is γ -Hölder for all $\gamma \in]0, \alpha/2[$ and the Hölder constant is in L^p for all $p > 0$.

Proof. The first claim is just an application of Proposition B.1. We outline the second claim. We consider a mollifying sequence $(\rho_n)_n$ and we define the Gaussian process by convolution $X_n = X \star \rho_n$. For each n , this process is infinitely differentiable and the covariance structure is readily seen to be given, after integration by parts, by:

$$\mathbb{E}[\partial_x^k X_n(x) \partial_x^k X_n(z)] = \int \int \rho_n(y) \rho_n(y') \partial_x^k \partial_z^k K(x - y, z - y') dy dy' \stackrel{\text{def}}{=} K_{k,n}(x, z).$$

Because of (B.1), it is plain to see that there exists a constant C such that, for all n and $x, y \in D$

$$\mathbb{E}[|\partial_{x_i}^k X_n(x) - \partial_{x_i}^k X_n(z)|^2] \leq C|x - y|.$$

The same argument holds for all the derivatives of order k' for $k' = 0, \dots, k$. The Kolmogorov criterion (Prop. B.1) ensures that the sequence $(X_n)_n$ is tight in C^k equipped with the topology of uniform convergence of all derivatives of order k' for $k' = 0, \dots, k$. Since $(X_n)_n$ converges almost surely in $C(D)$ towards X , we deduce that X is of class C^k . \square

We deduce:

Corollary B.3. *Consider a sequence $(X_n)_n$ of centered Gaussian processes with respective covariance kernel K_n of class C^{2k} such that all the derivatives up to order $2k$ uniformly converges on D towards a covariance kernel K . Then the sequence $(X_n, \partial_x X_n, \dots, \partial_x^k X_n)$ converges in law in $C(D)$ towards $(X, \partial_x X, \dots, \partial_x^k X)$, where X is a centered Gaussian process on D .*

C Auxiliary results of section 6

Proof of Lemma 6.1. We suppose that $B(x, \delta) \subset D$. We set G_D to be the Green function in D . We have for all function $F \geq 0$:

$$\begin{aligned} \int_D F(y) \left(\int_{\varepsilon^2}^{\infty} p_D(t, x, y) dt \right) dy &= \mathbb{E}^x \left[\int_{\varepsilon^2}^{\infty} F(B_t) \mathbf{1}_{\{\tau_D > t\}} dt \right] = \mathbb{E}^x [\mathbb{E}^{B_{\varepsilon^2}} \left[\int_0^{\infty} F(B_t) \mathbf{1}_{\tau_D > t} dt \right] \mathbf{1}_{\{\tau_D > \varepsilon^2\}}] \\ &= \mathbb{E}^x \left[\int_D G_D(B_{\varepsilon^2}, y) F(y) dy \mathbf{1}_{\{\tau_D > \varepsilon^2\}} \right] = \int_D F(y) \mathbb{E}^x [G_D(B_{\varepsilon^2}, y) \mathbf{1}_{\{\tau_D > \varepsilon^2\}}] dy. \end{aligned}$$

Hence we have $\int_{\varepsilon^2}^{\infty} p_D(t, x, y) dt = \mathbb{E}^x [G_D(B_{\varepsilon^2}, y) \mathbf{1}_{\{\tau_D > \varepsilon^2\}}]$. Now, we extend $G_D(x, y)$ to all $\mathbb{R}^2 \times \mathbb{R}^2$ by setting it equal to 0 as soon as x or y are not in the domain D . Recall that there exists some constant $C > 0$ such that for all x, y in \mathbb{R}^2 , $G_D(x, y) \leq \ln \frac{1}{|y-x|} + C$. Now, we have:

$$\begin{aligned} |\mathbb{E}^x [G_D(B_{\varepsilon^2}, x)] - \mathbb{E}^x [G_D(B_{\varepsilon^2}, x) \mathbf{1}_{\{\tau_D > \varepsilon^2\}}]| &= \mathbb{E}^x [G_D(B_{\varepsilon^2}, x) \mathbf{1}_{\{\tau_D \leq \varepsilon^2\}}] \\ &\leq \mathbb{E}^x \left[\left(\ln \frac{1}{|B_{\varepsilon^2} - x|} + C \right) \mathbf{1}_{\{\sup_{u \leq \varepsilon^2} |B_u - x| \geq \delta\}} \right] \\ &\leq \mathbb{E}^x \left[\left(\ln \frac{1}{|B_{\varepsilon^2} - x|} + C \right)^2 \right]^{1/2} \mathbb{P} \left(\sup_{u \leq \varepsilon^2} |B_u - x| \geq \delta \right)^{1/2} \\ &\leq C \left(\ln \frac{1}{\varepsilon} \right)^2 e^{-C\delta^2/\varepsilon^2}. \end{aligned}$$

Hence, we can replace $\mathbb{E}^x [G_D(B_{\varepsilon^2}, x) \mathbf{1}_{\tau_D > \varepsilon^2}]$ by $\mathbb{E}^x [G_D(B_{\varepsilon^2}, x)]$. Similarly, we can replace $\mathbb{E}^x [G_D(B_{\varepsilon^2}, x)]$ by $\mathbb{E}^x [G_D(B_{\varepsilon^2}, x) \mathbf{1}_{\{|B_{\varepsilon^2} - x| \leq \delta\}}]$. Therefore we get

$$\begin{aligned} \mathbb{E}^x [G_D(B_{\varepsilon^2}, x) \mathbf{1}_{\{\tau_D > \varepsilon^2\}}] &= \mathbb{E}^x [G_D(B_{\varepsilon^2}, x) \mathbf{1}_{\{|B_{\varepsilon^2} - x| \leq \delta\}}] + o(1) \\ &= \int_{|u| \leq \frac{\delta}{\varepsilon}} \frac{e^{-|u|^2/2}}{2\pi} G_D(x + \varepsilon u, x) du + o(1). \end{aligned}$$

Now, by conformal invariance, we get that:

$$\begin{aligned} \int_{|u| \leq \frac{\delta}{\varepsilon}} \frac{e^{-|u|^2/2}}{2\pi} G_D(x + \varepsilon u, x) du &= \int_{|u| \leq \frac{\delta}{\varepsilon}} \frac{e^{-|u|^2/2}}{2\pi} G_{\mathbb{H}}(\varphi(x + \varepsilon u), \varphi(x)) du \\ &\approx \int_{|u| \leq \frac{\delta}{\varepsilon}} \frac{e^{-|u|^2/2}}{2\pi} G_{\mathbb{H}}(\varphi(x) + \varepsilon \varphi'(x)u, \varphi(x)) du \end{aligned}$$

where the last line can be made rigorous by using the explicit expression of $G_{\mathbb{H}}$:

$$G_{\mathbb{H}}(x, y) = \ln \frac{1}{|x - y|} - \ln \frac{1}{|x - \bar{y}|}.$$

Therefore we get:

$$\begin{aligned} \int_{\varepsilon^2}^{\infty} p_D(t, x, y) dt &= \int_{|u| \leq \frac{\delta}{\varepsilon}} \frac{e^{-|u|^2/2}}{2\pi} G_{\mathbb{H}}(\varphi(x) + \varepsilon\varphi'(x)u, \varphi(x)) du + o(1) \\ &= \int_{|u| \leq \frac{\delta}{\varepsilon}} \frac{e^{-|u|^2/2}}{2\pi} \ln \frac{1}{|\varepsilon\varphi'(x)u|} du \\ &\quad - \int_{|u| \leq \frac{\delta}{\varepsilon}} \frac{e^{-|u|^2/2}}{2\pi} \ln \frac{1}{|\varphi(x) - \bar{\varphi}(x) + \varepsilon\varphi'(x)u|} du + o(1) \\ &= \ln \frac{1}{\varepsilon} + \ln \frac{1}{|\varphi'(x)|} + \ln 2\text{Im}(\varphi(x)) + o(1). \end{aligned}$$

The result follows via (6.10). □

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