

# A Random Field Model and its Application in Industrial Production.

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We describe now three theoretical examples used in the article. These dummy problems have been designed to meet the following requirements. First, it is defined for an arbitrary factor space dimension  $D$ . In each of these examples, the space  $X$  is a hyper-rectangle in  $\mathbb{R}^D$  of the form  $X = \prod_{i=1}^D [X_{\min,i}, X_{\max,i}]$ . It is equipped with a probability measure  $P$ . Secondly, a theoretical risk, denoted  $r_t$ , can be expressed in closed-form. Let  $m \in \mathbb{R}$ , the out-of-specifications space is defined by  $A := [m, +\infty)$ . As a consequence, the real failure set is  $X_A := \{x \in X \mid y(x) > m\}$ . The true failure risk is given by  $r_t = P(X_A)$ . If the distribution  $P$  is uniform,  $r_t$  simply equals the volume of  $X_A$ . All numerical calculations are performed with Matlab (2012.a).

## 1 Quadric example

Fix  $m > 0$ ,  $a_1, a_2, \dots, a_D > m^2$ ,  $X := [-1, 1]^D$  and for  $\mathbf{x} := (x_1, x_2, \dots, x_D) \in X$ :

$$y(\mathbf{x}) := \sqrt{\sum_{i=1}^D a_i x_i^2}.$$

This is a degenerate quadric. In particular, for  $D = 2$ , it's a cone. The boundary of failure set  $X_A$  is a generalization of an ellipse in dimension  $D$ . The failure set volume is:

$$\frac{1}{\text{vol}(X)} \int_X 1_{[y(\mathbf{x}) > m]} d\mathbf{x} = 1 - \frac{1}{2^D} \frac{\pi^{\frac{D}{2}} m^D}{\Gamma(\frac{D}{2} + 1)} \prod_{i=1}^D a_i^{-\frac{1}{2}}.$$

The particular example that we consider in the article is defined as follow.

Set  $D$  to 5, ( $a_1 = 4.0, a_2 = 4.2, a_3 = 4.4, a_4 = 4.6, a_5 = 4.8$ ) and  $m$  to 1.9. The theoretical risk is then  $r_t = 1.008 \times 10^{-1}$ .

## 2 Sine example

Fix  $a_1, a_2, \dots, a_D$  non zero integers,  $X := [0, 1]^D$  and for  $\mathbf{x} := (x_1, x_2, \dots, x_D) \in X$ :

$$y(\mathbf{x}) := \sin\left(2\pi \sum_{i=1}^D a_i x_i\right).$$

It can be proved that for all  $m \in [-1, 1]$ :

$$\int_X 1_{[y(\mathbf{x}) > m]} d\mathbf{x} = \int_0^1 1_{[\sin(2\pi t) > m]} dt = \frac{1}{2} - \frac{\arcsin(m)}{\pi}.$$

Here is the particular example that we consider. Set  $D$  to 3,  $(a_1 = 1, a_2 = -1, a_3 = 2)$  and  $m$  to 0.8. The theoretical risk is  $r_t = 2.048 \times 10^{-1}$ .

### 3 Bell-shaped example

Let  $R$  be a positive integer,  $X = [0, 1]^D$  and for  $\mathbf{x} \in X$ :

$$y(\mathbf{x}) := \max_{1 \leq i \leq R} f_i(\mathbf{x}, \boldsymbol{\mu}_i, \sigma_i)$$

where for all  $1 \leq i \leq R$ ,  $\boldsymbol{\mu}_i \in \mathbb{R}^D$ ,  $\sigma_i \in \mathbb{R}$ ,  $\sigma_i > 0$  and

$$f_i(\mathbf{x}, \boldsymbol{\mu}_i, \sigma_i) = \frac{1}{(\sqrt{2\pi}\sigma_i)^D} \exp\left(-\frac{1}{2\sigma_i^2} \|\mathbf{x} - \boldsymbol{\mu}_i\|^2\right)$$

such that for a fixed threshold  $m$ , the failure areas associated to each  $f_i$  do not overlap each other. When this condition is satisfied, it follows that:

$$\int_X 1_{[y(\mathbf{x}) > m]} d\mathbf{x} = \sum_{i=1}^R \int_X 1_{[f_i(\mathbf{x}, \boldsymbol{\mu}_i, \sigma_i) > m]} d\mathbf{x}.$$

More precisely, in order to get a simple closed-form expression for  $r_t$ , we consider the case defined by the statements below, involving the quantities:

$$r_i := \sqrt{-2\sigma_i^2 \log\left(m(2\pi)^{\frac{D}{2}} \sigma_i^D\right)}, \quad \forall 1 \leq i \leq R$$

1. Failure subsets are not empty:

$$0 < \sigma_i < \frac{1}{\sqrt{2\pi}m^{\frac{1}{D}}}, \quad \forall 1 \leq i \leq R$$

2. Failure subsets are entirely included in the space  $X$ :

$$0 \leq \mu_{ij} \pm r_i \leq 1, \quad \forall 1 \leq i \leq R, \forall 1 \leq j \leq D$$

3. Failure subsets are disjoint:

$$\|\boldsymbol{\mu}_i - \boldsymbol{\mu}_j\| > r_i + r_j, \quad \forall j \neq i.$$

Then, for each  $1 \leq i \leq R$ , the elementary failure area is a  $D$ -sphere centered in  $\boldsymbol{\mu}_i$  and with the radius  $r_i$ . Its volume is:

$$\frac{\pi^{\frac{D}{2}} r_i^D}{\Gamma\left(\frac{D}{2} + 1\right)}.$$

Finally, the failure risk is simply the sum of the volumes of  $R$  disjoint  $D$ -spheres:

$$\int_X 1_{[y(\mathbf{x}) > m]} d\mathbf{x} = \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2} + 1\right)} \sum_{i=1}^R r_i^D.$$

Here is the particular example presented in the article. Set  $D$  to 5,  $R$  to 10 and  $m$  to 0.1. The bell-shaped function  $y$  consists of 10 multi-dimensional Gaussian functions defined in Table 1.

The theoretical risk is about  $r_t = 4.341 \times 10^{-3}$ .

	$\mu_{i1}$	$\mu_{i2}$	$\mu_{i3}$	$\mu_{i4}$	$\mu_{i5}$	$\sigma_i$
$f_1$	$4.889 \times 10^{-1}$	$6.241 \times 10^{-1}$	$6.791 \times 10^{-1}$	$3.955 \times 10^{-1}$	$3.674 \times 10^{-1}$	$6.259 \times 10^{-1}$
$f_2$	$5.870 \times 10^{-1}$	$4.139 \times 10^{-1}$	$3.091 \times 10^{-1}$	$2.638 \times 10^{-1}$	$7.588 \times 10^{-1}$	$6.297 \times 10^{-1}$
$f_3$	$7.637 \times 10^{-1}$	$5.588 \times 10^{-1}$	$1.838 \times 10^{-1}$	$4.980 \times 10^{-1}$	$5.179 \times 10^{-1}$	$6.292 \times 10^{-1}$
$f_4$	$2.318 \times 10^{-1}$	$3.963 \times 10^{-1}$	$7.051 \times 10^{-1}$	$5.586 \times 10^{-1}$	$7.566 \times 10^{-1}$	$6.299 \times 10^{-1}$
$f_5$	$5.806 \times 10^{-1}$	$2.989 \times 10^{-1}$	$7.137 \times 10^{-1}$	$3.605 \times 10^{-1}$	$7.185 \times 10^{-1}$	$6.298 \times 10^{-1}$
$f_6$	$3.805 \times 10^{-1}$	$3.022 \times 10^{-1}$	$2.945 \times 10^{-1}$	$2.598 \times 10^{-1}$	$4.635 \times 10^{-1}$	$6.258 \times 10^{-1}$
$f_7$	$1.684 \times 10^{-1}$	$7.401 \times 10^{-1}$	$8.728 \times 10^{-1}$	$3.555 \times 10^{-1}$	$1.788 \times 10^{-1}$	$6.304 \times 10^{-1}$
$f_8$	$7.885 \times 10^{-1}$	$5.891 \times 10^{-1}$	$4.363 \times 10^{-1}$	$5.497 \times 10^{-1}$	$3.530 \times 10^{-1}$	$6.300 \times 10^{-1}$
$f_9$	$8.601 \times 10^{-1}$	$3.204 \times 10^{-1}$	$6.636 \times 10^{-1}$	$9.056 \times 10^{-1}$	$6.854 \times 10^{-1}$	$6.318 \times 10^{-1}$
$f_{10}$	$2.995 \times 10^{-1}$	$8.727 \times 10^{-1}$	$3.751 \times 10^{-1}$	$7.984 \times 10^{-1}$	$8.168 \times 10^{-1}$	$6.318 \times 10^{-1}$

Table 1: Parameters values of the bell-shaped function.