

A Random Field Model and its Application in Industrial Production.

Julie OGER^{*†}, Emmanuel LESIGNE^{*}, Philippe LEDUC[†]

^{*} Université François-Rabelais, CNRS, LMPT UMR7350, Tours, France

[†] STMicroelectronics, Tours, France

We describe now three theoretical examples used in the article. These dummy problems have been designed to meet the following requirements. First, it is defined for an arbitrary factor space dimension D . In each of these examples, the space X is a hyper-rectangle in \mathbb{R}^D of the form $X = \prod_{i=1}^D [X_{\min,i}, X_{\max,i}]$. It is equipped with a probability measure P . Secondly, a theoretical risk, denoted r_t , can be expressed in closed-form. Let $m \in \mathbb{R}$, the out of specifications space is defined by $A := [m, +\infty)$. As a consequence, the real failure set is $X_A := \{x \in X \mid y(x) > m\}$. The true failure risk is given by $r_t = P(X_A)$. If the distribution P is uniform, r_t simply equals the volume of X_A . All numerical calculations are performed with Matlab (2012.a).

1 Quadric example

Fix $m > 0$, $a_1, a_2, \dots, a_D > m^2$, $X := [-1, 1]^D$ and for $\mathbf{x} := (x_1, x_2, \dots, x_D) \in X$:

$$y(\mathbf{x}) := \sqrt{\sum_{i=1}^D a_i x_i^2}.$$

This is a degenerate quadric. In particular, for $D = 2$, it's a cone. The boundary of failure set X_A is a generalization of an ellipse in dimension D . The failure set volume is:

$$\frac{1}{\text{vol}(X)} \int_X 1_{[y(\mathbf{x}) > m]} \, d\mathbf{x} = 1 - \frac{1}{2^D} \frac{\pi^{\frac{D}{2}} m^D}{\Gamma(\frac{D}{2} + 1)} \prod_{i=1}^D a_i^{-\frac{1}{2}}.$$

The particular example that we consider in the article is defined as follow.

Set D to 5, ($a_1 = 4.0, a_2 = 4.2, a_3 = 4.4, a_4 = 4.6, a_5 = 4.8$) and m to 1.9. The theoretical risk is then

$$r_t = 1.008 \times 10^{-1}.$$

2 Sine example

Fix a_1, a_2, \dots, a_D non zero integers, $X := [0, 1]^D$ and for $\mathbf{x} := (x_1, x_2, \dots, x_D) \in X$:

$$y(\mathbf{x}) := \sin\left(2\pi \sum_{i=1}^D a_i x_i\right).$$

It can be proved that for all $m \in [-1, 1]$:

$$\int_X 1_{[y(\mathbf{x}) > m]} d\mathbf{x} = \int_0^1 1_{[\sin(2\pi t) > m]} dt = \frac{1}{2} - \frac{\arcsin(m)}{\pi}.$$

Here is the particular example that we consider. Set D to 3, ($a_1 = 1, a_2 = -1, a_3 = 2$) and m to 0.8. The theoretical risk is $r_t = 2.048 \times 10^{-1}$.

3 Bell-shaped example

Let R be a positive integer, $X = [0, 1]^D$ and for $\mathbf{x} \in X$:

$$y(\mathbf{x}) := \max_{1 \leq i \leq R} f_i(\mathbf{x}, \boldsymbol{\mu}_i, \sigma_i)$$

where for all $1 \leq i \leq R$, $\boldsymbol{\mu}_i \in \mathbb{R}^D$, $\sigma_i \in \mathbb{R}$, $\sigma_i > 0$ and

$$f_i(\mathbf{x}, \boldsymbol{\mu}_i, \sigma_i) = \frac{1}{(\sqrt{2\pi}\sigma_i)^D} \exp\left(-\frac{1}{2\sigma_i^2} \|\mathbf{x} - \boldsymbol{\mu}_i\|^2\right)$$

such that for a fixed threshold m , the failure areas associated to each f_i do not overlap each other. When this condition is satisfied, it follows that:

$$\int_X 1_{[y(\mathbf{x}) > m]} d\mathbf{x} = \sum_{i=1}^R \int_X 1_{[f_i(\mathbf{x}, \boldsymbol{\mu}_i, \sigma_i) > m]} d\mathbf{x}.$$

More precisely, in order to get a simple closed-form expression for r_t , we consider the case defined by the statements below, involving the quantities:

$$r_i := \sqrt{-2\sigma_i^2 \log\left(m(2\pi)^{\frac{D}{2}} \sigma_i^D\right)}, \quad \forall 1 \leq i \leq R$$

1. Failure subsets are not empty:

$$0 < \sigma_i < \frac{1}{\sqrt{2\pi m^{\frac{1}{D}}}}, \quad \forall 1 \leq i \leq R$$

2. Failure subsets are entirely included in the space X :

$$0 \leq \mu_{ij} \pm r_i \leq 1, \quad \forall 1 \leq i \leq R, \forall 1 \leq j \leq D$$

3. Failure subsets are disjoint:

$$\|\boldsymbol{\mu}_i - \boldsymbol{\mu}_j\| > r_i + r_j, \quad \forall j \neq i.$$

Then, for each $1 \leq i \leq R$, the elementary failure area is a D -sphere centered in $\boldsymbol{\mu}_i$ and with the radius r_i .

Its volume is:

$$\frac{\pi^{\frac{D}{2}} r_i^D}{\Gamma\left(\frac{D}{2} + 1\right)}.$$

Finally, the failure risk is simply the sum of the volumes of R disjoint D -spheres:

$$\int_X 1_{[y(\mathbf{x}) > m]} d\mathbf{x} = \frac{\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2} + 1\right)} \sum_{i=1}^R r_i^D.$$

Here is the particular example presented in the article. Set D to 5, R to 10 and m to 0.1. The bell-shaped function y consists of 10 multi-dimensional Gaussian functions defined in Table 1.

The theoretical risk is about $r_t = 4.341 \times 10^{-3}$.

	μ_{i1}	μ_{i2}	μ_{i3}	μ_{i4}	μ_{i5}	σ_i
f_1	4.889×10^{-1}	6.241×10^{-1}	6.791×10^{-1}	3.955×10^{-1}	3.674×10^{-1}	6.259×10^{-1}
f_2	5.870×10^{-1}	4.139×10^{-1}	3.091×10^{-1}	2.638×10^{-1}	7.588×10^{-1}	6.297×10^{-1}
f_3	7.637×10^{-1}	5.588×10^{-1}	1.838×10^{-1}	4.980×10^{-1}	5.179×10^{-1}	6.292×10^{-1}
f_4	2.318×10^{-1}	3.963×10^{-1}	7.051×10^{-1}	5.586×10^{-1}	7.566×10^{-1}	6.299×10^{-1}
f_5	5.806×10^{-1}	2.989×10^{-1}	7.137×10^{-1}	3.605×10^{-1}	7.185×10^{-1}	6.298×10^{-1}
f_6	3.805×10^{-1}	3.022×10^{-1}	2.945×10^{-1}	2.598×10^{-1}	4.635×10^{-1}	6.258×10^{-1}
f_7	1.684×10^{-1}	7.401×10^{-1}	8.728×10^{-1}	3.555×10^{-1}	1.788×10^{-1}	6.304×10^{-1}
f_8	7.885×10^{-1}	5.891×10^{-1}	4.363×10^{-1}	5.497×10^{-1}	3.530×10^{-1}	6.300×10^{-1}
f_9	8.601×10^{-1}	3.204×10^{-1}	6.636×10^{-1}	9.056×10^{-1}	6.854×10^{-1}	6.318×10^{-1}
f_{10}	2.995×10^{-1}	8.727×10^{-1}	3.751×10^{-1}	7.984×10^{-1}	8.168×10^{-1}	6.318×10^{-1}

Table 1: Parameters values of the bell-shaped function.