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Algorithm for reconstruction of Riemann surface with given boundary in complex projective space

A. D. Agaltsov¹, G. M. Henkin²

Abstract

In this paper we propose a numerically realizable algorithm for reconstruction of a complex curve with known boundary and without compact components in complex projective space.

Key words: Riemann surface, reconstruction algorithm, Burgers equation, Cauchy type formulas.

1 Introduction

Let us denote by \mathbb{CP}^2 the complex projective space with homogeneous coordinates $(w_0 : w_1 : w_2)$. Let a real closed rectifiable, oriented curve γ in \mathbb{CP}^2 be the boundary of a complex curve $X \subset \mathbb{CP}^2$ with notation $\gamma = bX$. Without restriction of generality we suppose that the following conditions of general position hold:

$$(0 : 1 : 0) \notin X, \quad w_0|_\gamma \neq 0.$$

Put $\mathbb{C}^2 = \{w \in \mathbb{CP}^2 : w_0 \neq 0\}$ with coordinates $z_1 = \frac{w_1}{w_0}$, $z_2 = \frac{w_2}{w_0}$. For almost all $\xi = (\xi_0, \xi_1) \in (\mathbb{C}^2)^*$ the points of intersection of X with complex line $\mathbb{C}_\xi^1 = \{z \in \mathbb{C}^2 : \xi_0 + \xi_1 z_1 + z_2 = 0\}$ form a finite set of points

$$(z_1^{(j)}(\xi), z_2^{(j)}(\xi)) = (h_j(\xi_0, \xi_1), -\xi_0 - \xi_1 h_j(\xi_0, \xi_1)), \quad j = 1, \dots, N_+(\xi).$$

The following Cauchy type formula from [DH2] plays the essential role in reconstruction of X through γ :

$$G_m(\xi_0, \xi_1) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_\gamma z_1^m (\xi_0 + \xi_1 z_1 + z_2)^{-1} d(\xi_0 + \xi_1 z_1 + z_2) = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi_0, \xi_1) + P_m(\xi_0, \xi_1), \quad m = 0, 1, \dots, \quad (1)$$

where $N_+(\xi) = N_+(\xi_0, \xi_1)$ is the number of points of intersection (multiplicities taken into account) of X with complex line \mathbb{C}_ξ^1 , $P_m(\xi_0, \xi_1)$ is a polynomial of

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degree at most m with respect to ξ_0 . In addition, $P_0(\xi_0, \xi_1) = -N_-$, where N_- is the number of points of intersection of X with infinity $\{w \in \mathbb{CP}^2: w_0 = 0\}$,

$$P_1(\xi_0, \xi_1) = \sum_{k=1}^{N_-(\xi)} \frac{a_k \xi_0 - b_k}{a_k \xi_1 + 1}, \quad (2)$$

$a_k = w_2(q_k)$, $b_k = \frac{dw_2}{dw_0}(q_k)$, where q_k , $k = 1, \dots, N_-$, are the points of intersection of X with infinity $\{w \in \mathbb{CP}^2: w_0 = 0\}$. In particular, the following corollary of (1) holds:

$$G_0(\xi_0, \xi_1) = \frac{1}{2\pi i} \int_{\gamma} \frac{d(\xi_0 + \xi_1 z_1 + z_2)}{\xi_0 + \xi_1 z_1 + z_2} = N_+(\xi) - N_-. \quad (3)$$

Let further $\xi_1 = 0$ and let $\pi_2: \mathbb{C}^2 \rightarrow \mathbb{C}$ be the projection on second factor: $\pi_2(z_1, z_2) = z_2$. We have $\pi_2\gamma \subset \mathbb{C}$, $\mathbb{C} \setminus \pi_2\gamma = \cup_{l=0}^L \Omega_l$, where $\{\Omega_l\}$ are the connected components of $\mathbb{C} \setminus \pi_2\gamma$. For every component Ω_l the number of points of intersection of X with line $z_2 = -\xi_0$, $\xi_0 \in \Omega_l$, multiplicities taken into account, will be denoted by $\mu_l = N_+(\xi_0, 0)$. Let Ω_0 denote the unbounded component of set $\mathbb{C} \setminus \pi_2\gamma$. From definition of N_{\pm} it follows that

$$\mu_0 = N_+(\xi_0, 0) = N_-, \quad \xi_0 \in \Omega_0. \quad (4)$$

Assume that complex curve X does not contain compact components, or equivalently, satisfies the following condition of minimality from the work [HM]:

For arbitrary complex curve $\tilde{X} \subset \mathbb{CP}^2$ with condition $b\tilde{X} = bX = \gamma$ and for almost all $\xi \in (\mathbb{C}^2)^$ the number of points of intersection $\tilde{N}_+(\xi)$ of \tilde{X} with line \mathbb{C}_{ξ}^1 , multiplicities taken into account, is not less than the number $N_+(\xi)$ for curve X .* (*)

Condition of minimality (*) is a condition of general position and is fulfilled for X if, for example, every irreducible component of X is a transcendental complex curve. Note that from theorems of Chow [C] and Harvey, Shiffman [HS] it follows that an arbitrary complex curve $\tilde{X} \subset \mathbb{CP}^2$ with condition $b\tilde{X} = bX$ admits the unique representation $\tilde{X} = X \cup V$, where X is a curve with condition of minimality (*), and V is a compact algebraic curve, possibly with multiple components.

In this work in development of [DH2] we have obtained a numerically realizable algorithm for reconstruction of complex curve $X \subseteq \mathbb{CP}^2$ with known boundary and with condition of minimality. This algorithm permits, in particular, to make applicable the result of [HM] about principal possibility to reconstruct topology and conformal structure of a two-dimensional surface X in \mathbb{R}^3 with constant scalar conductivity from measurements on bX of electric current densities, being created by three potentials in general position.

Our algorithm depends on parameter $\mu_0 = N_{\pm}(\xi_0, 0)$, $\xi_0 \in \Omega_0$. It was tested on many examples and was rigorously justified for $\mu_0 = 0, 1, 2$. Despite the

difficulty of justification of the algorithm for $\mu_0 \geq 3$, there are no principal obstacles for the realization of our algorithm for any μ_0 . Moreover, in Theorem 3.2 we propose a method for finding parameter μ_0 in terms of γ . This makes the algorithm much more applicable.

This paper is an extended version of our work [AH]. We have added new results (Theorem 3.2 and Proposition 3.1) and we have improved Section 4 (Visualization).

2 Cauchy type formulas and Riemann-Burgers equations

Let us give at first a new proof of the Cauchy type formula (1) from [DH2], permitting to obtain explicit expressions for functions $P_m(\xi_0, \xi_1)$.

Theorem 2.1. *Let $X \subset \mathbb{CP}^2 \setminus [0 : 1 : 0]$ be a complex curve without compact components, $\gamma = bX \subset \mathbb{C}^2$ be a real rectifiable oriented curve. Suppose that for almost all $\xi \in (\mathbb{C}^2)^*$ all the points of intersection of X with \mathbb{CP}_ξ^1 have multiplicity at most one. Then the following equalities are fulfilled:*

$$G_m(\xi_0, \xi_1) = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi_0, \xi_1) + P_m(\xi_0, \xi_1), \quad \xi = (\xi_0, \xi_1) \in (\mathbb{C}^2)^*, \quad m \geq 1, \quad (5)$$

where $P_m(\xi_0, \xi_1)$ is a polynomial of degree at most m with respect to ξ_0 of the following form:

$$P_m(\xi_0, \xi_1) = \sum_{s=1}^{\mu_0} \sum_{k=0}^{m-1} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s) \frac{d^{m-k}}{dw_0^{m-k}} \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s} - \sum_{s=1}^{\mu_0} \sum_{i_1+\dots+i_m=m} \frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s).$$

In particular, if $\mu_0 = 0$ then $P_m \equiv 0$.

Remark 2.1. *In the exceptional case, when $[0 : 1 : 0] \in X$, the term $P_m(\xi_0, \xi_1)$ in (5) need not to be a polynomial in general.*

Proof. Put $\tilde{g} = \xi_0 w_0 + \xi_1 w_1 + w_2$ and $g = \frac{\tilde{g}}{w_0} = \xi_0 + \xi_1 z_1 + z_2$. Consider differential forms

$$\omega_m \stackrel{\text{def}}{=} z_1^m \frac{dg}{g} = \frac{w_1^m w_0}{w_0^m \tilde{g}} d\left(\frac{\tilde{g}}{w_0}\right) = \frac{w_1^m d\tilde{g}}{w_0^m \tilde{g}} - \frac{w_1^m}{w_0^{m+1}} dw_0, \quad m = 0, 1, \dots$$

Then $G_m(\xi) = \frac{1}{2\pi i} \int_\gamma \omega_m$. Let us compute this integral explicitly. Denote by p_j , $j = 1, \dots, N_+(\xi)$ the points of intersection of X with \mathbb{CP}_ξ^1 , and by q_s ,

$s = 1, \dots, \mu_0$ the points of intersection of X with infinity $\{w \in \mathbb{CP}^2: w_0 = 0\}$. Denote by B_j^ε the intersection of X with the ball of radius ε in \mathbb{CP}^2 centered at p_j and by D_s^ε the intersection of X with the ball of radius ε centered at q_s . The restriction of form ω_m on X is meromorphic with poles at points p_j and q_s . Thus the following equality is valid:

$$G_m(\xi) = \frac{1}{2\pi i} \int_{\gamma} \omega_m = \sum_{j=1}^{N_+(\xi)} \frac{1}{2\pi i} \int_{bB_j^\varepsilon} \omega_m + \sum_{s=1}^{\mu_0} \frac{1}{2\pi i} \int_{bD_s^\varepsilon} \omega_m.$$

If $\mu_0 = 0$, then the second group of terms is absent. The integral $\int_{bB_j^\varepsilon} \omega_m$ can be calculated as a residue at the first order pole:

$$\int_{bB_j^\varepsilon} \omega_m = \int_{bB_j^\varepsilon} z_1^m \frac{d\tilde{g}}{\tilde{g}} - \int_{bB_j^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = \int_{bB_j^\varepsilon} z_1^m \frac{d\tilde{g}}{\tilde{g}} = 2\pi i h_j^m(\xi).$$

Let $\mu_0 > 0$. Computation of integral $\int_{bD_s^\varepsilon} \omega_1$ will be done in two steps. Let us calculate first $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0$. Consider the expansion of $w_1(w_0)$ into power series in w_0 in the neighborhood of point q_s :

$$w_1(w_0) = w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2w_1}{dw_0^2}(q_s)w_0^2 + \dots$$

Note further that

$$w_1^m(w_0) = \sum_{k=0}^{\infty} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \dots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^k.$$

The coefficient near w_0^m can be presented in the form

$$\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = 2\pi i \sum_{i_1+\dots+i_m=m} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \dots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s).$$

Now we can calculate the integral $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{\tilde{g}}$. Using relation $d\tilde{g} = \frac{d\tilde{g}}{dw_0} dw_0$ and expansion of $w_1(w_0)$ into power series in w_0 we obtain:

$$\begin{aligned} \int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{\tilde{g}} &= \int_{bD_s^\varepsilon} \frac{1}{w_0^m} \left(w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2w_1}{dw_0^2}(q_s)w_0^2 + \dots \right)^m \frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} dw_0 = \\ &= \sum_{k=0}^{\infty} \int_{bD_s^\varepsilon} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \dots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} dw_0 = \\ &= \sum_{k=0}^{m-1} \int_{bD_s^\varepsilon} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \dots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} dw_0 = \\ &= \sum_{k=0}^{m-1} \frac{2\pi i}{(m-k-1)!} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \dots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) \lim_{w_0 \rightarrow 0} \frac{d^{m-k-1}}{dw_0^{m-k-1}} \left(\frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} \right). \end{aligned}$$

From here, taking into account the relation $\frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} = \frac{d \ln \tilde{g}}{dw_0}$, we obtain, finally

$$\int_{bD_\xi} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{\tilde{g}} = 2\pi i \sum_{k=0}^{m-1} \sum_{i_1+\dots+i_m=k} \frac{\frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)!} \frac{d^{m-k}}{dw_0^{m-k}} \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s}.$$

It is a polynomial of degree at most m with respect to ξ_0 . \square

Before considering the algorithm for reconstruction of complex curve we need to prove one more theorem, which connects among themselves different functions $h_j(\xi_0, \xi_1)$, $j = 1, \dots, N_+(\xi_0, \xi_1)$. Denote by $\Omega_l^{(k)}$ the infinitesimal neighborhood of order k of the set Ω_l .

Theorem 2.2. *Let $X \subset \mathbb{CP}^2 \setminus [0 : 1 : 0]$ be a complex curve without compact components, $\gamma = bX \subset \mathbb{C}^2$. Fix $l \in \{0, \dots, L\}$. Suppose that functions \hat{h}_j , $j = 1, \dots, \mu_l$, are mutually distinct and analytic in $\Omega_l^{(1)}$ and satisfy the Riemann–Burgers equation in $\xi \in \Omega_l^{(1)}$:*

$$\frac{\partial \hat{h}_j}{\partial \xi_1}(\xi) = \hat{h}_j(\xi) \frac{\partial \hat{h}_j}{\partial \xi_0}(\xi), \quad j = 1, \dots, \mu_l. \quad (6)$$

Then the functions \hat{h}_j , $j = 1, \dots, \mu_l$, satisfy the system

$$G_m(\xi) = \sum_{j=1}^{\mu_l} \hat{h}_j^m(\xi) + \hat{P}_m(\xi), \quad \xi \in \Omega_l^{(1)}, \quad m = 1, 2, \dots, \quad (7)$$

where \hat{P}_m , $m = 1, 2, \dots$, are some analytic functions in $\Omega_l^{(1)}$, being polynomials of degree at most m with respect to ξ_0 , if and only if the functions \hat{h}_j , $j = 1, \dots, \mu_l$, satisfy the equation

$$0 = \frac{\partial^2}{\partial \xi_0^2} \left(G_1(\xi) - \sum_{j=1}^{\mu_l} \hat{h}_j(\xi) \right), \quad \xi \in \Omega_l^{(1)}. \quad (8)$$

Moreover, for minimal $\{\mu_l\}$ with properties (7)–(8) there exists the unique set of functions \hat{h}_j , $j = 1, \dots, \mu_l$, satisfying the equivalent conditions (7)–(8) and the unique set of functions \hat{P}_m , $m = 1, 2, \dots$, from condition (7). Furthermore, $\hat{h}_j = h_j$, $\hat{P}_m = P_m$ for $j = 1, \dots, \mu_l$, $m = 1, \dots$, where functions h_j and P_m are defined in Theorem (2.1).

Proof. Necessity. From (7) it follows that

$$G_1(\xi) - \sum_{j=1}^{\mu_l} \hat{h}_j(\xi) = \hat{P}_1(\xi), \quad \xi \in \Omega_l^{(1)}.$$

Differentiating the latter equality two times with respect to ξ_0 and taking into account that \hat{P}_1 is a polynomial in ξ_0 of degree at most 1 we obtain (8).

Sufficiency. Suppose that mutually distinct functions $\{\widehat{h}_j(\xi)\}$ on $\Omega_l^{(1)}$, $l = 0, 1, \dots, L$, are holomorphic and satisfy the equations (6), (8). In particular, for any $\xi_0 \in \Omega_l^{(0)}$ we have

$$\frac{\partial \widehat{h}_j}{\partial \xi_1}(\xi_0, 0) = \widehat{h}_j(\xi_0, 0) \frac{\partial \widehat{h}_j}{\partial \xi_0}(\xi_0, 0).$$

By Cauchy–Kowalewski theorem in a neighborhood of arbitrary $\xi^* \in \Omega_l^{(1)}$ there exist unique holomorphic functions $\{\widetilde{h}_j(\xi_0, \xi_1)\}$, satisfying the Riemann–Burgers equation (6) and such that $\widetilde{h}_j|_{\Omega_l^{(1)}} = \widehat{h}_j$.

From here and from Proposition 3.3.3 of [DH2] we obtain existence and uniqueness of holomorphic functions $\{\widehat{P}_m(\xi)\}$, being polynomials of degree at most m in ξ_0 , such that $\{\widehat{h}_j(\xi)\}$ and $\{\widehat{P}_m(\xi)\}$ satisfy the system (7) for $m = 1, 2, \dots, j = 1, \dots, \mu_l$, $\xi \in \Omega_l^{(1)}$.

Existence and uniqueness. Existence of functions $\{\widehat{h}_j\}$ and $\{\widehat{P}_m\}$ with necessary properties follows from Theorem 2.1. More precisely, $\widehat{h}_j = h_j$ and $\widehat{P}_m = P_m$, $j = 1, \dots, \mu_l$, $m \geq 1$. Uniqueness of functions $\{\widehat{h}_j\}$ for minimal $\{\mu_l\}$ with properties (7)–(8) follows from Theorem II of [DH2] and from Theorem 3 of [HM]. Uniqueness of polynomials $\{\widehat{P}_m\}$ for minimal $\{\mu_l\}$ with properties (7)–(8) follows from the proof of sufficiency. \square

3 Reconstruction algorithm

Consider now the reconstruction algorithm of complex curve $X \subseteq \mathbb{CP}^2$ with given boundary bX and with condition of minimality (*). Let us consider the cases $\mu_0 = 0, 1, 2$.

The reconstruction algorithm is based on formulas (5) with polynomials P_m , $m = 0, 1, \dots$. The next theorem permits to calculate these polynomials.

Theorem 3.1. *Let $X \subseteq \mathbb{CP}^2$ be a complex curve without algebraic subdomains, $\gamma \subseteq \mathbb{C}^2$ be its boundary. Let mutually distinct holomorphic in $\xi \in \Omega_l^{(1)}$, $l = 0, 1, \dots, L$, functions $\{h_j(\xi)\}$ and holomorphic in $\xi \in \Omega_l^{(1)}$, $l = 0, \dots, L$, functions $P_m(\xi_0, \xi_1)$, being polynomials of degree at most m in ξ_0 , satisfy the system (5) for $\xi \in \Omega_l^{(1)}$, $j = 1, \dots, N_+(\xi)$ with minimal $N_+(\xi)$ (existence and uniqueness of such functions follow from Theorem 2.2). Then the following statements are valid:*

1. *If $\mu_0 = 0$, then $P_m(\xi_0, 0) \equiv 0$ for all m . Besides, $G_1(\xi_0, \xi_1) = 0$, if $|\xi_0| \geq \text{const}(X)(1 + |\xi_1|)$.*
2. *If $\mu_0 = 1$, then $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$, where constants c_{11} and c_{12} satisfy*

identity in $\xi_0 \in \Omega_0$:

$$\begin{aligned} c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) + c_{12} \left(\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) + G_1(\xi_0, 0) \right) &= \\ &= G_1(\xi_0, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0). \end{aligned} \quad (9)$$

3. If $\mu_0 = 2$, then $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$, $P_2(\xi_0, 0) = c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2$, where constants c_{11} , c_{12} , c_{21} , c_{22} , c_{23} satisfy identity in $\xi_0 \in \Omega_0$:

$$\begin{aligned} 2(G_1 - c_{11} - c_{12}\xi_0) \frac{\partial G_1}{\partial \xi_1} - G_1(c_{22} + 2c_{23}\xi_0) - \frac{\partial G_2}{\partial \xi_1} - 3c_{12}^2 \mathfrak{a} + \mathfrak{a}c_{23} &= \\ = \left((G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \frac{\partial G_1}{\partial \xi_0} + & \quad (10) \\ + \left(G_1^2 - 2G_1c_{11} - 2G_1c_{12}\xi_0 - G_2 \right) c_{12}, \quad \text{where } \mathfrak{a} = \frac{1}{2\pi i} \int_{\gamma} z_1 dz_2. \end{aligned}$$

Proof. By Theorem 2.2 functions P_m from condition of this theorem are defined in Theorem 2.1.

1. By Theorem 2.1 $P_m \equiv 0$, if $\mu_0 = 0$.

2. For $\xi \in \Omega_l^{(1)}$ we have equality $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$. We need to find constants $c_{11} = C_{11}(0)$ and $c_{12} = C_{12}(0)$.

Differentiate the equation (5) with respect to ξ_0 and rescript this equation and its differentiated version to $\xi \in \Omega_l^{(0)}$:

$$\begin{aligned} h_1(\xi_0, 0) &= G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0, \\ \frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0, \\ \frac{\partial h_1}{\partial \xi_0}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12}, \end{aligned}$$

where $\xi_0 \in \Omega_0$. By Theorem 2.2 for $\xi_0 \in \Omega_0$ function $h_1(\xi_0, 0)$ satisfies the equality $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) = h_1(\xi_0, 0) \frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$. If we substitute in this equality the expressions for $h_1(\xi_0, 0)$, $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0)$ and $\frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$, we will obtain the equation

$$\begin{aligned} \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 &= \\ = \left(G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0 \right) \left(\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12} \right). \end{aligned} \quad (11)$$

This equation is valid for $\xi_0 \in \Omega_0$. Let us divide it into ξ_0 and tend $\xi_0 \rightarrow \infty$. We obtain equality $\dot{C}_{12}(0) = -c_{12}^2$. Taking into account this equality, we can rewrite the equation (11) in the form

$$\begin{aligned} \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) &= \\ = \left(G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0 \right) \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \left(G_1(\xi_0, 0) - c_{11} \right) c_{12}. \end{aligned} \quad (12)$$

Taking into account that $\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) \rightarrow 0$ as $\xi_0 \rightarrow \infty$ and passing $\xi_0 \rightarrow \infty$ in (12), we obtain the equality $\dot{C}_{11}(0) = -c_{11}c_{12}$. Due to the just obtained equality the equation (12) takes the desired form.

3. By Theorem 2.2 functions $h_1(\xi)$ and $h_2(\xi)$ satisfy the Riemann–Burgers equation for $\xi \in \Omega_l^{(1)}$. So, the following equalities are valid:

$$\frac{\partial(h_1 h_2)}{\partial \xi_1} = h_1 \frac{\partial h_2}{\partial \xi_1} + \frac{\partial h_1}{\partial \xi_1} h_2 = h_1 h_2 \frac{\partial(h_1 + h_2)}{\partial \xi_0}, \quad (13)$$

$$\frac{\partial(h_1^2 + h_2^2)}{\partial \xi_0} = 2h_1 \frac{\partial h_1}{\partial \xi_0} + 2h_2 \frac{\partial h_2}{\partial \xi_0} = 2 \frac{\partial(h_1 + h_2)}{\partial \xi_1}. \quad (14)$$

Note that $h_1 h_2 = \frac{1}{2}(h_1 + h_2)^2 - \frac{1}{2}(h_1^2 + h_2^2)$. Therefore the system (13)–(14) is equivalent to the system

$$\frac{\partial(h_1 + h_2)^2}{\partial \xi_1} - \frac{\partial(h_1^2 + h_2^2)}{\partial \xi_1} = \left((h_1 + h_2)^2 - (h_1^2 + h_2^2) \right) \frac{\partial(h_1 + h_2)}{\partial \xi_0}, \quad (15)$$

$$\frac{\partial(h_1^2 + h_2^2)}{\partial \xi_0} = 2 \frac{\partial(h_1 + h_2)}{\partial \xi_1}. \quad (16)$$

We substitute into this system $h_1^2 + h_2^2$ and $h_1 + h_2$ from equation (5), using notations $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$, $P_2(\xi_0, \xi_1) = C_{21}(\xi_1) + C_{22}(\xi_1)\xi_0 + C_{23}(\xi_1)\xi_0^2$. Equation (16) restricted to $\Omega_l^{(0)}$ takes the form

$$\frac{\partial G_2}{\partial \xi_0}(\xi_0, 0) - c_{22} - 2c_{23}\xi_0 = 2 \left(\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 \right). \quad (17)$$

Divide this equation into ξ_0 and tend $\xi_0 \rightarrow \infty$. We obtain the equality $\dot{C}_{12}(0) = c_{23}$. Taking this equality into account and passing $\xi_0 \rightarrow \infty$ in (17) we obtain the equality $\dot{C}_{11}(0) = \frac{1}{2}c_{22}$.

Now substitute the expressions for $h_1^2 + h_2^2$ and $h_1 + h_2$ into (15) and restrict the obtained formula to $\Omega_l^{(0)}$. We obtain the equality

$$\begin{aligned} & 2 \left(G_1 - c_{11} - c_{12}\xi_0 \right) \left(\frac{\partial G_1}{\partial \xi_1} - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 \right) - \frac{\partial G_2}{\partial \xi_1} + \dot{C}_{21}(0) + \dot{C}_{22}(0)\xi_0 + \dot{C}_{23}(0)\xi_0^2 = \\ & = \left((G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \left(\frac{\partial G_1}{\partial \xi_0} + c_{12} \right). \end{aligned} \quad (18)$$

Divide this equation into ξ_0^2 and pass $\xi_0 \rightarrow \infty$. This leads to the equality

$$2c_{12}\dot{C}_{12}(0) + \dot{C}_{23}(0) = (c_{12}^2 + c_{23})c_{12}.$$

Using the latter equality, divide (18) into ξ_0 and pass $\xi_0 \rightarrow \infty$ to obtain the equality

$$2c_{11}\dot{C}_{12}(0) + 2c_{12}\dot{C}_{11}(0) + \dot{C}_{22}(0) = (-2c_{11} + c_{22})c_{12}.$$

Taking into account the obtained equalities one can rewrite (18) in the form

$$\begin{aligned}
& 2(G_1 - c_{11} - c_{12}\xi_0) \frac{\partial G_1}{\partial \xi_1} - 2G_1(\dot{C}_{11}(0) + \dot{C}_{12}(0)\xi_0) + 2c_{11}\dot{C}_{11}(0) - \frac{\partial G_2}{\partial \xi_1} + \dot{C}_{21}(0) = \\
& = \left((G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \frac{\partial G_1}{\partial \xi_0} + \\
& \quad + \left((G_1 - c_{11})^2 - 2G_1c_{12}\xi_0 - G_2 + c_{21} \right) c_{12}. \quad (19)
\end{aligned}$$

Pass $\xi_0 \rightarrow \infty$ in this equality and note that the following relations are valid

$$\begin{aligned}
\lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{\partial G_1}{\partial \xi_1} &= \lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_1}{\xi_0 + z_2} = \frac{1}{2\pi i} \int_{\gamma} z_1 dz_1 = \langle b\gamma, \frac{1}{4\pi i} z_1^2 \rangle = 0, \\
\lim_{\xi_0 \rightarrow \infty} \xi_0 G_1 &= \lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{\xi_0 + z_2} = \frac{1}{2\pi i} \int_{\gamma} z_1 dz_2, \\
\lim_{\xi_0 \rightarrow \infty} \xi_0^2 \frac{\partial G_1}{\partial \xi_0} &= - \lim_{\xi_0 \rightarrow \infty} \xi_0^2 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{(\xi_0 + z_2)^2} = - \frac{1}{2\pi i} \int_{\gamma} z_1 dz_2.
\end{aligned}$$

We obtain

$$-2\mathfrak{a}\dot{C}_{12}(0) + 2c_{11}\dot{C}_{11}(0) + \dot{C}_{21}(0) = -(c_{12}^2 + c_{23})\mathfrak{a} + (c_{11}^2 - 2c_{12}\mathfrak{a} + c_{21})c_{12}.$$

Express constants $\dot{C}_{ij}(0)$ through c_{ij} in obtained equations:

$$\begin{aligned}
\dot{C}_{11}(0) &= \frac{1}{2}c_{22}, \\
\dot{C}_{12}(0) &= c_{23}, \\
\dot{C}_{23}(0) &= c_{12}^3 - c_{12}c_{23}, \\
\dot{C}_{22}(0) &= -2c_{11}(c_{12} + c_{23}), \\
\dot{C}_{21}(0) &= (c_{11}^2 - 3c_{12}\mathfrak{a} + c_{21})c_{12} + \mathfrak{a}c_{23} - c_{11}c_{22}.
\end{aligned} \quad (20)$$

Substituting these constants to (19), we obtain the third statement of Theorem 3.1. \square

The next theorem permits to find μ_0 through γ .

Theorem 3.2. *Let $X \subset \mathbb{CP}^2 \setminus [0 : 1 : 0]$ be a complex curve without algebraic subdomains, $\gamma = bX \subset \mathbb{C}^2$ be the boundary of X . Let functions $G_m(\xi_0, \xi_1)$, $m \geq 1$, be defined by formula (1) and number μ_0 defined by formula (4). Then the following statements are valid:*

1. *If $G_1(\xi_0, \xi_1) = 0$ for $|\xi_0| \geq \text{const}(X)(1 + |\xi_1|)$, then $\mu_0 = 0$.*

2. If there exist such complex constants c_{11} , c_{12} that for any $\xi \in \Omega_0^{(1)}$ the following equality is valid:

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi) + c_{12} \left(\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi) + G_1(\xi) \right) = G_1(\xi) \frac{\partial G_1}{\partial \xi_0}(\xi) - \frac{\partial G_1}{\partial \xi_1}(\xi), \quad (21)$$

then $\mu_0 \leq 1$.

3. If there exist such complex constants c_{11} , c_{12} , c_{21} , c_{22} , c_{23} that the following equality is valid:

$$\begin{aligned} & 2(G_1(\xi) - c_{11} - c_{12}\xi_0) \frac{\partial G_1}{\partial \xi_1}(\xi) - G_1(\xi)(c_{22} + 2c_{23}\xi_0) - \\ & - \frac{\partial G_2}{\partial \xi_1}(\xi) - 3c_{12}^2 \mathfrak{a} + \mathfrak{a} c_{23} = \left((G_1(\xi) - c_{11} - c_{12}\xi_0)^2 - G_2(\xi) + \right. \\ & \left. + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \frac{\partial G_1}{\partial \xi_0}(\xi) + \left(G_1^2(\xi) - 2G_1(\xi)c_{11} - \right. \\ & \left. - 2G_1(\xi)c_{12}\xi_0 - G_2(\xi) \right) c_{12}, \quad \text{where } \mathfrak{a} = \frac{1}{2\pi i} \int_{\gamma} z_1 dz_2, \quad \xi \in \Omega_0^{(1)}, \end{aligned} \quad (22)$$

then $\mu_0 \leq 2$.

Proof. 1. Equality $G_1(\xi_0, \xi_1) = 0$ for $|\xi_0| \geq \text{const}(X)(1 + |\xi_1|)$ imply according to [DH2] the moment condition

$$\int_{\gamma} z_1^{k_1} z_2^{k_2} dz_2 = 0 \quad \text{for all } k_1, k_2 \in \mathbb{N}.$$

From here according to [W] and [HL] it follows that for an appropriate choice of orientation γ is the boundary of a complex curve in \mathbb{C}^2 . Hence either $\mu_0 = 0$ or X is a domain on algebraic curve in \mathbb{CP}^2 . But X can't be a domain on algebraic curve in \mathbb{CP}^2 because X doesn't contain algebraic subdomains.

2. Let conditions of general position be fulfilled. Put

$$\begin{aligned} h(\xi_0, \xi_1) &= G_1(\xi_0, \xi_1) - C_{11}(\xi_1) - C_{12}(\xi_1)\xi_0, \\ C_{11}(\xi_1) &= c_{11} - c_{11}c_{12}\xi_1, \\ C_{12}(\xi_1) &= c_{12} - c_{11}^2\xi_1, \end{aligned} \quad (23)$$

where $\xi = (\xi_0, \xi_1) \in \Omega_0^{(1)}$.

Taking into account (23), we can rewrite the equality (21) in the form

$$\frac{\partial h}{\partial \xi_1}(\xi_0, \xi_1) = h(\xi_0, \xi_1) \frac{\partial h}{\partial \xi_0}(\xi_0, \xi_1), \quad \xi = (\xi_0, \xi_1) \in \Omega_0^{(1)}. \quad (24)$$

From definition (23) we obtain the following equality:

$$\frac{\partial^2}{\partial \xi_0^2} (G_1(\xi_0, \xi_1) - h(\xi_0, \xi_1)) = 0, \quad \xi = (\xi_0, \xi_1) \in \Omega_0^{(1)}. \quad (25)$$

From equalities (24), (25) due to Theorems 2.1, 2.2 and Theorem 3 from [HM] we obtain

$$X \cap \{z_2 = -\xi_0\} = \{(h(\xi_0, 0), -\xi_0)\}, \quad \xi_0 \in \Omega_0. \quad (26)$$

From here it follows that $\mu_0 \leq 1$.

3. Let conditions of general position be fulfilled. Let us define functions h_1 and h_2 by the following relations:

$$\begin{aligned} h_1(\xi_0, \xi_1) + h_2(\xi_0, \xi_1) &= G_1(\xi_0, \xi_1) - C_{11}(\xi_1) - C_{12}(\xi_1)\xi_0, \\ h_1^2(\xi_0, \xi_1) + h_2^2(\xi_0, \xi_1) &= G_2(\xi_0, \xi_1) - C_{21}(\xi_1) - C_{22}(\xi_1)\xi_0 - C_{23}(\xi_1)\xi_0^2, \end{aligned} \quad (27)$$

$$C_{ij}(\xi_1) = c_{ij} + \dot{C}_{ij}(0)\xi_1, \quad i = 1, j = 1, 2 \text{ and } i = 2, j = 1, 2, 3, \quad (28)$$

where $\xi = (\xi_0, \xi_1) \in \Omega_0^{(1)}$ and constants $\dot{C}_{ij}(0)$ are defined by formulas (20).

Taking into account definitions (28) the identity (22) is equivalent to the identity (18), where $\xi = (\xi_0, \xi_1) \in \Omega_0^{(1)}$.

Taking into account definitions (27) the identity (18) for $\xi = (\xi_0, \xi_1) \in \Omega_0^{(1)}$ is equivalent to the identity (15) for $\xi \in \Omega_0^{(1)}$.

By Lemma 3.2.1 from paper [DH1] for $\xi = (\xi_0, \xi_1) \in \Omega_0^{(1)}$ the following equality is valid:

$$\frac{\partial G_2}{\partial \xi_0}(\xi_0, \xi_1) = 2 \frac{\partial G_1}{\partial \xi_1}(\xi_0, \xi_1). \quad (29)$$

From definitions (27), (28) and from equality (29) we obtain the equality (16) for $\xi \in \Omega_0^{(1)}$.

Equalities (15), (16) mean that functions h_1 and h_2 satisfy the Riemann–Burgers equations:

$$\frac{\partial h_j}{\partial \xi_1}(\xi_0, \xi_1) = h_j(\xi_0, \xi_1) \frac{\partial h_j}{\partial \xi_0}(\xi_0, \xi_1), \quad \xi = (\xi_0, \xi_1) \in \Omega_0^{(1)}. \quad (30)$$

Further, because of definition (27) the following equality is valid:

$$\frac{\partial^2}{\partial \xi_0^2} (G_1(\xi_0, \xi_1) - h_1(\xi_0, \xi_1) - h_2(\xi_0, \xi_1)) = 0. \quad (31)$$

From equalities (30), (31) and using Theorems 2.1, 2.2 and Theorem 3 of the paper [HM] we obtain:

$$X \cap \{z_2 = -\xi_0\} = \{(h_j(\xi_0, 0), -\xi_0) \mid j = 1, 2\}, \quad \xi_0 \in \Omega_0.$$

From here it follows that $\mu_0 \leq 2$. □

Let us describe an effective method of finding of constants $\{c_{ij}\}$ from conditions of Theorem (3.1).

Let $X \subset \mathbb{CP}^2$ be a complex curve without compact components, $\gamma = bX \subset \mathbb{C}^2$ be its compact boundary. Put

$$\mathfrak{a}_{\delta,\rho}^{\alpha,\beta} = \frac{1}{2\pi i} \int_{\gamma} z_1^\alpha z_2^\beta dz_1 + z_1^\delta z_2^\rho dz_2,$$

where $\alpha, \beta, \gamma, \delta$ are nonnegative integers. Put by definition $\alpha_{\gamma,\delta}^{\alpha,\beta} = 0$, if at least one number of $\alpha, \beta, \gamma, \delta$ is negative.

Let us define constants

$$\begin{aligned} a_{1,k}^1 &= k\mathfrak{a}_{1,k-1}^{00}, \\ a_{2,k}^1 &= -k\mathfrak{a}_{1,k}^{00}, \\ b_k^1 &= \frac{k}{2} \sum_{i+j=k-2} \mathfrak{a}_{1,i}^{00}\mathfrak{a}_{1,j}^{00} + k\mathfrak{a}_{2,k-1}^{00} + \mathfrak{a}_{00}^{1,k}, \end{aligned}$$

and constants

$$\begin{aligned} a_{1,k}^2 &= (k-1) \sum_{i+j=k-3} \mathfrak{a}_{1,i}^{00}\mathfrak{a}_{1,j}^{00} + 2(k-1)\mathfrak{a}_{2,k-2}^{00} + 2\mathfrak{a}_{00}^{1,k-1}, \\ a_{2,k}^2 &= (k+1) \sum_{i+j=k-2} \mathfrak{a}_{1,i}^{00}\mathfrak{a}_{1,j}^{00} + (2k+1)\mathfrak{a}_{2,k-1}^{00} + 2\mathfrak{a}_{00}^{1,k}, \\ a_{3,k}^2 &= (k-1)\mathfrak{a}_{1,k-2}^{00}, \\ a_{4,k}^2 &= -(k-1)\mathfrak{a}_{1,k-1}^{00}, \\ a_{5,k}^2 &= (k-1)\mathfrak{a}_{1,k}^{00}, \\ a_{6,k}^2 &= -2(k+1)\mathfrak{a}_{1,k-1}^{00}, \\ a_{7,k}^2 &= (k+3)\mathfrak{a}_{1,k}^{00}, \\ b_k^2 &= \frac{k-1}{3} \sum_{i+j+l=k-4} \mathfrak{a}_{1,i}^{00}\mathfrak{a}_{1,j}^{00}\mathfrak{a}_{1,l}^{00} + 2 \sum_{i+j=k-2} \mathfrak{a}_{1,i}^{00}\mathfrak{a}_{00}^{1,j} + \\ &\quad + \sum_{i=0}^{k-3} (k+i)\mathfrak{a}_{2,i}^{00}\mathfrak{a}_{1,k-3-i}^{00} + (k-1)\mathfrak{a}_{3,k-2}^{00} + \mathfrak{a}_{00}^{2,k-1}, \end{aligned}$$

where $k = 1, 2, \dots$

For arbitrary collection of integers k_1, \dots, k_s , where $1 \leq k_1 < \dots < k_s$, $s \geq 2$, let us define

$$A^j(k_1, \dots, k_s) = \begin{bmatrix} a_{1,k_1}^j & \cdots & a_{s,k_1}^j \\ \vdots & \ddots & \vdots \\ a_{1,k_s}^j & \cdots & a_{s,k_s}^j \end{bmatrix}, \quad b^j(k_1, \dots, k_s) = \begin{bmatrix} b_{k_1}^j \\ \vdots \\ b_{k_s}^j \end{bmatrix}, \quad (32)$$

where $j = 1, 2$.

Proposition 3.1. *Let $X \subset \mathbb{C}P^2 \setminus [0 : 1 : 0]$ be a complex curve without algebraic subdomains, $\gamma = bX \subset \mathbb{C}^2$ be its compact boundary, where μ_0 is defined by formula (4) and matrices $A^k(k_1, \dots, k_s)$ and vectors $b^j(k_1, \dots, k_s)$ are defined by formula (32). Then the following statements are valid:*

1. *Let $\mu_0 = 1$ and numbers c_{11}, c_{12} be defined in the statement 2 of Theorem 3.1. Suppose that for some collection of integers k_1, k_2 , where $1 \leq k_1 < k_2$, the inequality $\det A^1(k_1, k_2) \neq 0$ is valid. Let us define numbers x_1, x_2 by formula $(x_1, x_2) = (A^1(k_1, k_2))^{-1} b^1(k_1, k_2)$. Then $c_{11} = x_1, c_{12} = x_2$.*
2. *Let $\mu_0 = 2$ and numbers $c_{11}, c_{12}, c_{21}, c_{22}, c_{23}$ be defined in the statement 3 of Theorem 3.1. Suppose that for some collection of integers k_1, \dots, k_7 , where $1 \leq k_1 < \dots < k_7$, the inequality $\det A^2(k_1, \dots, k_7) \neq 0$ is valid. Let us define integers x_1, \dots, x_7 by formula $(x_1, \dots, x_7) = (A^2(k_1, \dots, k_7))^{-1} b^2(k_1, \dots, k_7)$. Then the following equalities are valid:*

$$c_{11} = x_1, c_{12} = x_2, c_{21} = x_3 - x_1^2, c_{22} = x_4, c_{23} = x_5.$$

Proof. Statements 1, 2 follow from decompositions of identities (9) and (10) into power series by degrees of ξ_0 and by equalizations of coefficients near the terms of equal degree in ξ_0 . \square

Let us describe the algorithm of reconstruction of a complex curve X in $\mathbb{C}P^2$ without compact components (satisfying minimality condition (*)). As above, $\gamma = bX$ is a compact real curve.

The algorithm for reconstruction of curve X permits to find a curve coinciding with original curve in given finite number of points and obtained by interpolation in other points. Let $\{\xi^k\}_{k=1}^N, \xi^k \in \mathbb{C}$ be an arbitrary grid on \mathbb{C} , $\xi^i \neq \xi^j, i \neq j$, and $\xi^k \notin \pi_2\gamma, k = 1, \dots, N$. Complex curve X intersects complex line $\{z_2 = \xi^k\}$ in $N_+(\xi^k, 0)$ points. Algorithm allows to find these points.

The algorithm takes as input points $\{\xi^k\}_{k=1}^N$ and a curve γ (for example, represented as a finite number of points belonging to γ). On the output of the algorithm we obtain a set of points $(h_s(\xi^k, 0), \xi^k), k = 1, \dots, N; s = 1, \dots, N_+(\xi^k, 0)$, belonging to the complex curve X .

The case of $\mu_0 = 0$

1. *Calculation of μ_l .* By formula (3) for every domain $\Omega_l, l = 1, \dots, L$, the number μ_l is equal to the winding number of curve $\pi_2\gamma$ with respect to $\xi_0 \in \Omega_l$:

$$\mu_l \equiv N_+(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz_2}{z_2 - \xi_0} \equiv \frac{1}{2\pi i} \int_{\pi_2\gamma} \frac{dz}{z - \xi_0}, \quad \xi_0 \in \Omega_l.$$

2. *Computation of power sums.* In the case of $\mu_0 = 0$ for every point $\xi^k \in \Omega_l$, $l = 1, \dots, L$, by Theorem 3.1 we have equalities $P_m(\xi^k, 0) \equiv 0$. By formula (5) we have the following formulas for the power sums:

$$s_m(\xi^k) \equiv h_1^m(\xi^k, 0) + \dots + h_{\mu_l}^m(\xi^k, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^m dz_2}{z_2 - \xi^k}, \quad m = \overline{1, \mu_l}.$$

By Theorem 2.1 the points $(h_s(\xi^k, 0), \xi^k)$, $s = 1, \dots, N_+(\xi^k, 0)$; $k = 1, \dots, N$, are the desired points of X .

3. *Computation of symmetric functions.* For every point $\xi^k \in \Omega_l$, $l = 1, \dots, L$, the Newton identities

$$k\sigma_k(\xi^k) = \sum_{i=1}^k (-1)^{i-1} \sigma_{k-i}(\xi^k) s_i(\xi^k), \quad k = 1, \dots, N_+(\xi^k).$$

allow to reconstruct the elementary symmetric functions:

$$\begin{aligned} \sigma_1(\xi^k) &= h_1(\xi^k, 0) + \dots + h_{\mu_l}(\xi^k, 0), \\ &\dots = \dots \\ \sigma_{\mu_l}(\xi^k) &= h_1(\xi^k, 0) \times \dots \times h_{\mu_l}(\xi^k, 0). \end{aligned}$$

4. *Desymmetrisation.* For every point $\xi^k \in \Omega_l$ using Vieta formulas one can find complex numbers $h_1(\xi^k, 0), \dots, h_{\mu_l}(\xi^k, 0)$. The points $(h_s(\xi^k, 0), \xi^k)$, $s = 1, \dots, N_+(\xi^k, 0)$; $k = 1, \dots, N$, are the required points of complex curve X .

The cases of $\mu_0 = 1, 2$

These cases are reduced to the case $\mu_0 = 0$ in the following way. Since $\pi_2\gamma \subset \mathbb{C}$ is a compact real curve, there exists such $R > 0$, that the set $B_R^c(0) = \{z \in \mathbb{C} \mid |z| \geq R\}$ belongs to Ω_0 . Without restriction of generality, one can suppose that $|\xi^k| < R$ for all $k = 1, \dots, N$. Otherwise one can increase R .

Let us define the auxiliary complex curve $X_R = \{(z_1, z_2) \in X \mid |z_2| \leq R\}$. Its boundary γ_R consists in two disjoint parts (possibly, multiconnected): the first part is γ and the second is a real curve obtained by lifting of the circle $S_R = \{z \in \mathbb{C} \mid |z| = R\}$ on surface X by inversion of projection $\pi_2: X \rightarrow \mathbb{C}$. Complex curve X_R does not intersect infinity and, as a consequence $\mu_0(X_R) = 0$. Moreover, every point $(z_1(\xi^k), \xi^k)$, $k = 1, \dots, N$, belongs to X if and only if it belongs to X_R . Therefore, in order to reconstruct the complex curve X_R it is sufficient to reconstruct the real curve obtained by lifting of S_R on X and to solve the reconstruction problem for surface X_R , being in conditions of the case of $\mu_0 = 0$. Finally, we come to the following algorithm:

1. *New boundary.* Choose a sufficiently large constant R , so that the exterior of the disk of radius R centered at origin belongs to Ω_0 and all ξ^k belong

to this disk. Denote the boundary of this disk by S_R . In the case of $\mu_0 = 1$ by virtue of formulas (5) the points $\xi_0 \in S_R$ satisfy the equality

$$h_1(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{z_2 - \xi_0} + P_1(\xi_0, 0).$$

In the case of $\mu_0 = 2$ we have two equalities:

$$h_1(\xi_0, 0) + h_2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{z_2 - \xi_0} + P_1(\xi_0, 0),$$

$$h_1^2(\xi_0, 0) + h_2^2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^2 dz_2}{z_2 - \xi_0} + P_2(\xi_0, 0),$$

where the polynomials can be found using Theorem 3.1. In the case of $\mu_0 = 1$ by lifting the curve S_R on X we obtain at once the real curve of the form $\{(h_1(\xi_0, 0), \xi_0) \mid \xi_0 \in S_R\}$. In the case of $\mu_0 = 2$ we have to apply Newton identities and Vieta formulas in order to obtain h_1 and h_2 from functions $h_1 + h_2$ and $h_1^2 + h_2^2$.

2. *Reduction.* In order to find the complex curve X_R with boundary $bX_R = \gamma_R$ we apply the algorithm of reconstruction for the case of $\mu_0 = 0$. The discussion before the description of the algorithm shows that we will obtain the desired points.

4 Visualization

Let us describe in few words the algorithm of visualisation of complex curves that we have used in our examples. Denote by $\pi_1: \mathbb{C}^2 \rightarrow \mathbb{C}$ the projection into the first factor: $\pi_1(z_1, z_2) = z_1$. Suppose that X is a complex curve in \mathbb{C}^2 such that the covering $\pi_1: X \setminus \{\text{ramification points}\} \rightarrow \mathbb{C}$ has multiplicity L . Consider, for simplicity, a rectangular grid Λ in \mathbb{C} :

$$\Lambda = \left\{ z_1^{ij} : \operatorname{Re} z_1^{ij} = \frac{i}{N}, \operatorname{Im} z_1^{ij} = \frac{j}{N}, i, j = 0, \dots, N \right\},$$

where N is a natural number. Suppose now that we are given the set $X_{\Lambda} = \pi_1^{-1}(\Lambda) \cap X$ and we need to visualize the part of X lying above the rectangle $0 \leq \operatorname{Re} z_1 \leq 1, 0 \leq \operatorname{Im} z_1 \leq 1$.

Let us introduce some terminology. We define a path in Λ as a map $\gamma: \{1, \dots, M\} \rightarrow \Lambda$ such that $|\gamma(k+1) - \gamma(k)| = \frac{1}{N}$ for all admissible k , where M is some natural number.

Let $\gamma: \{1, \dots, M\} \rightarrow \Lambda$ be a path in Λ and let $i: \{1, \dots, M\} \rightarrow [1, M]$ be the inclusion map. Define the function $i_*\gamma: [1, M] \rightarrow \mathbb{C}$ such that $i_*\gamma(k) = \gamma(k)$ for integer k and $i_*\gamma|_{[k, k+1]}$ is linear for all admissible k . It is clear that $i_*\gamma$ is a continuous function and hence it can be lifted to X by the map π_1 .

We define a path in X_Λ as a map $\Gamma: \{1, \dots, M\} \rightarrow X_\Lambda$ such that $\gamma = \pi_1 \circ \Gamma$ is a path in Λ and $\Gamma = i^*L(i_*\gamma)$, where i^* is the pullback map with respect to i and $L(i_*\gamma)$ is some lift of $i_*\gamma$ to X by π_1 , i. e. $L(i_*\gamma)$ is a continuous map from $[1, M]$ to X such that $\pi_1 \circ L(i_*\gamma) = i_*\gamma$. We also say that Γ is obtained by lifting of γ .

We will call subsets of Λ and X_Λ path-connected if every two points of these sets can be connected by a path in Λ and X_Λ , respectively.

Let us describe the practical way to lift paths in Λ to paths in X_Λ . Suppose that N is sufficiently large. Let $\gamma: \{1, \dots, M\} \rightarrow \Lambda$ be a path in Λ and let $\Gamma(1) \in \pi^{-1}(\gamma(1)) \cap X$ be an arbitrary point. We select $\Gamma(k) \in \pi^{-1}(\gamma(k)) \cap X$ in such a way that

$$|\Gamma(k) - \Gamma(k-1)| = \min\{|z - \Gamma(k-1)|: z \in \pi^{-1}(\gamma(k)) \cap X\}, \quad k = 2, \dots, M.$$

Then Γ is a path in X_Λ obtained by lifting of γ . All possible lifts of γ may be obtained by varying $\Gamma(1)$. Note that if γ is closed, i. e. $\gamma(1) = \gamma(M)$, Γ need not to be closed.

Finding of ramification points and making branch cuts. The first step in visualization procedure consists in finding of ramification points of X with respect to projection π_1 . Since we have only a finite number of points on X we can find ramification points only approximately. More precisely, we will localize them in small circles.

Without restriction of generality we suppose that all ramification points are projected by π_1 into interior points of Λ . Take any interior point $z_1 \in \Lambda$ and select a small closed path $\gamma: \{1, \dots, M\} \rightarrow \Lambda$ around z_1 so that there is at most one ramification point inside the polygon $\overline{\gamma(1) \dots \gamma(M)}$. For example, one can take as γ the following path:

$$z_1 + \frac{1}{N} \rightarrow z_1 + \frac{1+i}{N} \rightarrow z_1 + \frac{i}{N} \rightarrow \dots \rightarrow z_1 + \frac{1-i}{N} \rightarrow z_1 + \frac{1}{N},$$

where i is the imaginary unit.

Now consider different lifts of γ to X_Λ . If at least one lift is not closed, mark z_1 as a possible ramification point (meaning that it is situated near the projection of some ramification point of X). Now vary z_1 and mark all possible ramification points. The resulting set will consist of several path-connected components each of which localizes the position of one ramification point of X with respect to π_1 .

Now connect each of the obtained connected components of possible ramification points by path with boundary of the grid Λ in such a way that different paths don't intersect. Denote the union of the set of possible ramification points with images of these paths by Λ_c . An important observation is that every closed path in $\Lambda \setminus \Lambda_c$ always lifts to a closed path in X_Λ since it doesn't contain π_1 -projections of ramification points inside.

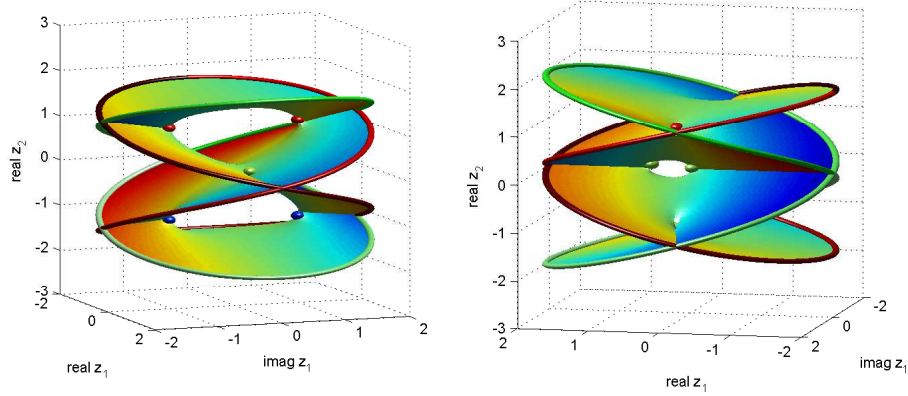


Figure 1: Riemann surface of function $f(z) = \sqrt{\exp\left(\frac{z}{4}\right) + \sqrt{z^2 + 1}}$, $|z| \leq 2$, obtained by the visualization algorithm. Red and green curves represent two connected components of the surface boundary, colored small balls represent ramification points

Visualization. Now denote $\Lambda \setminus \Lambda_c = \cup_{s=1}^S \Lambda_s$, where Λ_s are different path-connected components. Take any $z_1^s \in \Lambda_s$ and $z_2^s \in \pi_1^{-1}(z_1^s) \cap X$. Now take other $z_1 \in \Lambda_s$ and connect z_1^s with z_1 by some path γ . Then γ lifts to a path Γ with $\Gamma(1) = (z_1^s, z_2^s)$ and $\Gamma(2) = (z_1, z_2)$ for some $z_2 \in \pi_1^{-1}(z_1) \cap X$ and z_2 doesn't depend on γ . Varying z_1 we thus obtain the map $\Sigma(z_1^s, z_2^s): \Lambda_s \rightarrow X_\Lambda$ which allows us to visualize the part of X .

Varying $z_2^s \in \pi_1^{-1}(z_1^s) \cap X$ (the latter is the finite set, namely, it consists of L elements) we obtain the other maps $\Sigma(z_1^s, z_2^s)$ which allow us to visualize other parts of X . Clearly, the set of obtained maps doesn't depend on the choice of $z_1^s \in \Lambda_s$. Hence we can denote the obtained maps by Σ_s^l , $l = 1, \dots, L$. It is clear that $\cup_{l=1}^L \Sigma_s^l(\Lambda_s) = \pi^{-1}(\Lambda_s) \cap X$. Now vary s to visualize

$$\cup_{s=1}^S \cup_{l=1}^L \Sigma_s^l(\Lambda_s) = \pi^{-1}(\cup_{s=1}^S \Lambda_s) \cap X = X_\Lambda \setminus \pi^{-1}(\Lambda_c).$$

The part $\pi^{-1}(\Lambda_c) \cap X_\Lambda$ consists of cuts and preimages of possible ramification points. The cuts can be visualized as the already visualized part of the surface. The only problem is the visualization of π_1 -preimages of possible ramification points. But the latter take a little part of the surface when N is large and one can just forget about their visualization. On the other hand, in our examples they were visualized using low-level graphics approach.

Examples of application of this algorithm are given in fig. 1 and 2. The visualization algorithm can be easily generalized to the case of general grids. For instance, in our examples we have used a modification with periodic grid.

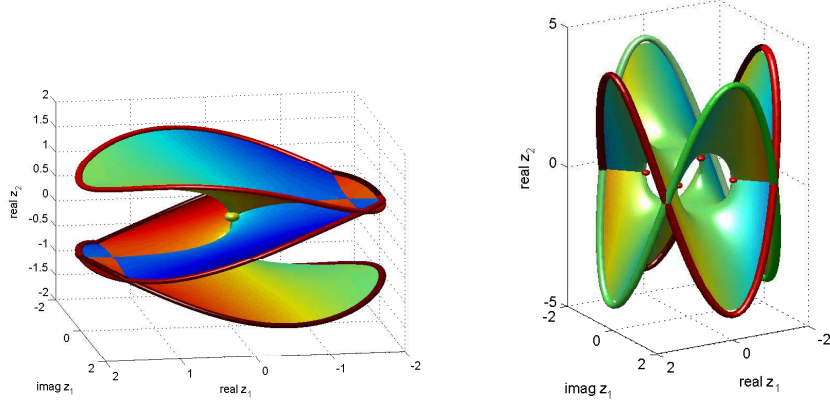


Figure 2: Riemann surfaces of functions $f(z) = \sqrt{\sin(z)}$, $|z| \leq 2$ (left) and $f(z) = \sqrt{z^4 + 1}$, $|z| \leq 2$ (right) obtained by the visualization algorithm.

5 Examples

5.1 The case of $\mu_0 = 1$

Consider an example of reconstruction of a Riemann surface with given boundary. The simplest case is the case of $\mu_0 = 0$ but it follows from discussion of the reconstruction algorithm that this case is directly included in any other case. Therefore, we begin with the next simplest case, namely, the case of $\mu_0 = 1$.

Let us reconstruct the surface

$$X_1 = \{(z_1, z_2) \in \mathbb{C}^2 \mid (z_1 - 1)z_2 = \exp(z_1^2), \quad |z_1| \leq 2\}.$$

We suppose that the boundary $\gamma_1 = bX_1$ is given in the form of discrete number of points (see further fig. 3).

Note that if point $(z_1, z_2) \in X_1$ is such that z_1 approaches to 1, then z_2 approaches infinity. Choose R large enough, e. g. $R = 60$, and reconstruct the real curve $\Gamma = \{(z_1, z_2) \in X_1 \mid |z_2| = R\}$. Compute, first, for two different ξ_0^1, ξ_0^2 , $|\xi_0^1| = |\xi_0^2| = R$, the values of functions

$$\begin{aligned} G_1(\xi_0, 0) &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{z_1 dz_2}{z_2 + \xi_0}, \\ \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) &= -\frac{1}{2\pi i} \int_{\gamma_2} \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2}, \\ \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) &= \frac{1}{2\pi i} \int_{\gamma_2} \left(\frac{z_1 dz_1}{z_2 + \xi_0} - \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2} \right) \end{aligned}$$

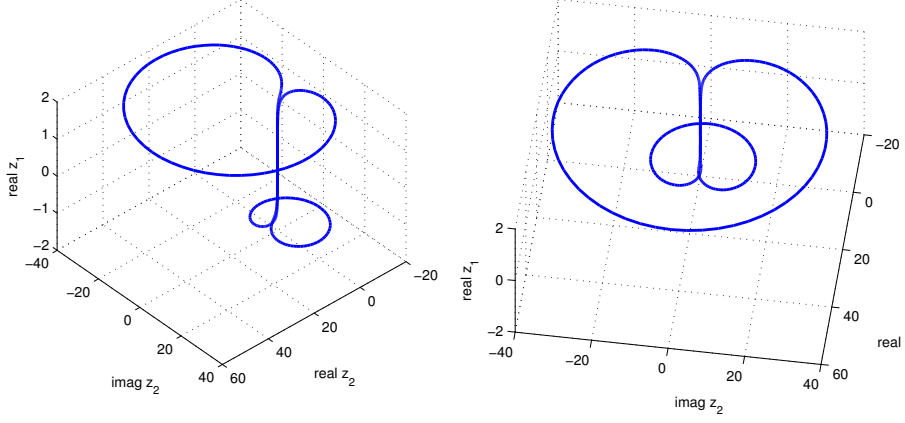


Figure 3: Boundary $\gamma_1 = bX_1$ of the surface X_1 .

and find from linear system for c_{11} and c_{12}

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) + c_{12} \left(\xi_0^1 \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) + G_1(\xi_0^1, 0) \right) = G_1(\xi_0^1, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0^1, 0),$$

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) + c_{12} \left(\xi_0^2 \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) + G_1(\xi_0^2, 0) \right) = G_1(\xi_0^2, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0^2, 0)$$

values $c_{11} = 1$, $c_{12} = 0$. Now calculate the values of functions $G_1(\xi_0, 0)$ on the circle $|\xi_0| = R$ and find function $h_1(\xi_0) = G_1(\xi_0, 0) + c_{11} + c_{12}\xi_0$, $|\xi_0| = R$. So we reconstruct a real curve $\Gamma = \{(h_1(\xi_0), -\xi_0) \mid |\xi_0| = R\} \subseteq X_1$ (see fig. 4).

We apply further the reconstruction algorithm for the case of $\mu_0 = 0$ to the surface $X_1^R = \{z \in X_1 \mid |z_2| \leq 60\}$ with boundary $bX_1^R = \gamma_1 + \Gamma$. We can compute the values of function

$$\sigma_0(\xi_0) = \frac{1}{2\pi i} \int_{\gamma_1 + \Gamma} \frac{dz_2}{z_2 + \xi_0}.$$

The value of function $\sigma_0(\xi_0)$ at point ξ_0 is equal to number $N_+(\xi_0)$ of points of surface X_1^R , projected into the point $-\xi_0$ under projection $(z_1, z_2) \mapsto z_2$. Further, for every point with $N_+(\xi_0) > 0$ we compute functions

$$s_k(\xi_0) = \frac{1}{2\pi i} \int_{\gamma_1 + \Gamma} \frac{z_1^k dz_2}{z_2 + \xi_0}, \quad k = 1, \dots, N_+(\xi_0).$$

From functions $s_k(\xi_0)$ we can find functions $\sigma_k(\xi_0)$ using Newton identities:

$$k\sigma_k(\xi_0) = \sum_{i=1}^k (-1)^{i-1} \sigma_{k-i}(\xi_0) s_i(\xi_0), \quad k = 1, \dots, N_+(\xi_0).$$

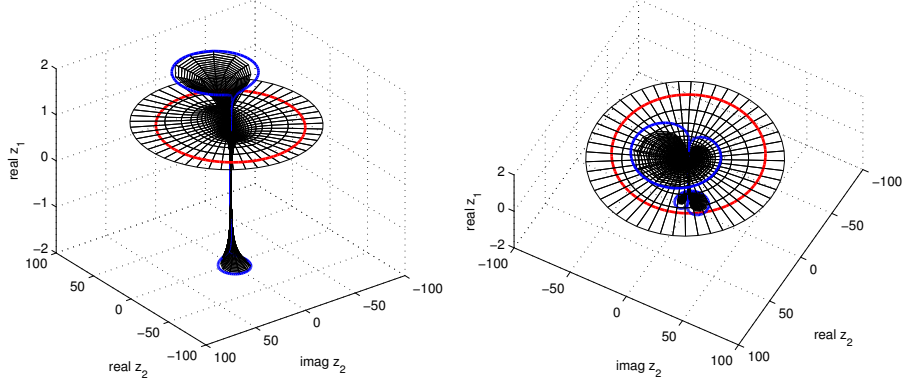


Figure 4: The contour of surface X_1 (black), boundary γ_1 of X_1 (blue), reconstructed curve Γ , belonging to X_1 (red).

After, we find roots $h_1(\xi_0), \dots, h_{\sigma_0(\xi_0)}(\xi_0)$ of polynomial

$$t^{N+(\xi_0)} - \sigma_1(\xi_0)t^{N+(\xi_0)-1} + \dots + (-1)^{N+(\xi_0)}\sigma_{N+(\xi_0)}(\xi_0) = 0.$$

The points $\{(h_k(\xi_0), -\xi_0) \mid k = 1, \dots, N_+(\xi_0)\}$ represent the set of all points of X_1^R , projected into $-\xi_0$ by projection $(z_1, z_2) \rightarrow z_2$. Visualization of the obtained set of points $\{(h_k(\xi_0), -\xi_0)\}$ corresponding to varying ξ_0 can be realized by the visualization algorithm described in the previous section. Reconstructed surface is represented in fig. 5.

5.2 The case of $\mu_0 = 2$

Consider an example of reconstruction of Riemann surface for the case of $\mu_0 = 2$. We are going to reconstruct the surface

$$X_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2(z_1^2 - 1) = z_1 \exp(z_1^2), \quad |z_1| \leq 2\}$$

given its boundary γ_2 represented as an array of finite number of equidistributed points on γ_2 (see fig. 6).

Choose R large enough, e. g. $R = 60$, and consider the circle C_R of radius

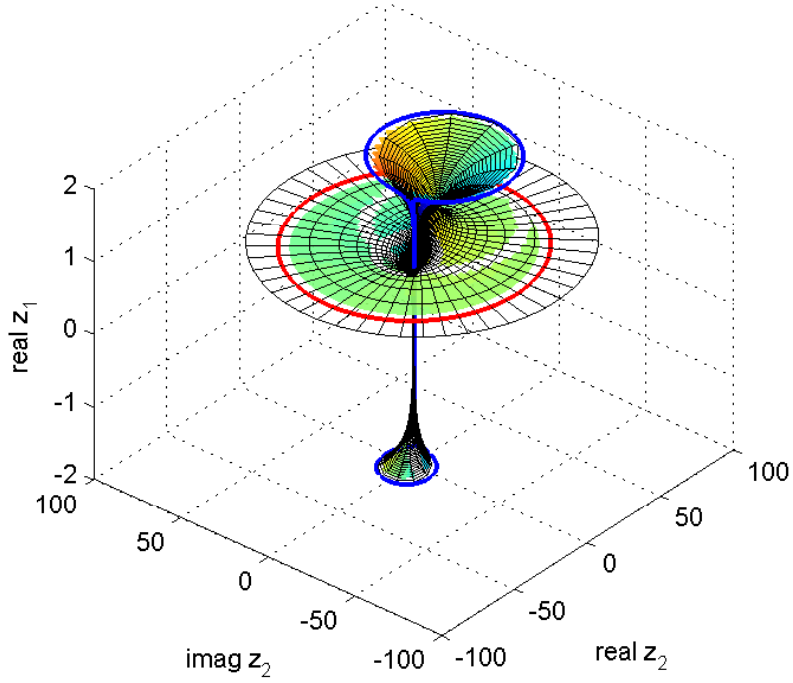


Figure 5: The contour of surface X_1 (black), boundary γ_1 of X_1 (blue), reconstructed curve Γ , belonging to X_1 (red), colored domains represent the reconstructed leaves of surface X_1 .

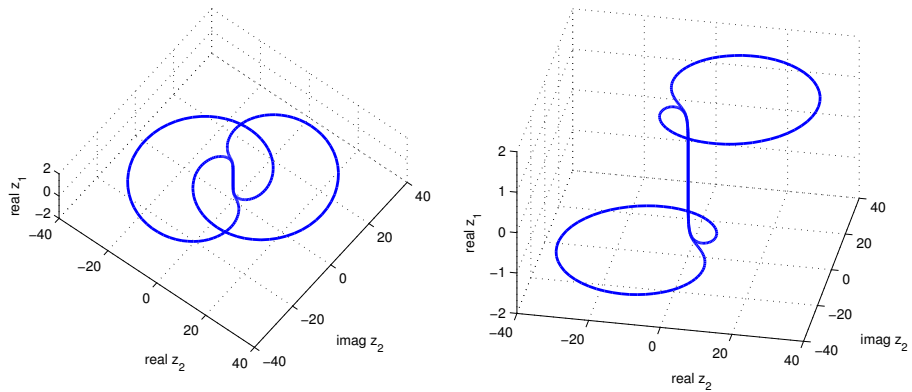


Figure 6: Boundary γ_2 of the surface X_2 .

R in z_2 -plane centered at origin. Compute for $-\xi_0 \in C_R$ the values of functions

$$\begin{aligned} G_1(\xi_0, 0) &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{z_1 dz_2}{z_2 + \xi_0}, \\ G_2(\xi_0, 0) &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{z_1^2 dz_2}{z_2 + \xi_0}, \\ \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) &= -\frac{1}{2\pi i} \int_{\gamma_2} \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2}, \\ \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) &= \frac{1}{2\pi i} \int_{\gamma_2} \left(\frac{z_1 dz_1}{z_2 + \xi_0} - \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2} \right), \\ \frac{\partial G_2}{\partial \xi_1}(\xi_0, 0) &= \frac{1}{2\pi i} \int_{\gamma_2} \left(\frac{z_1^2 dz_1}{z_2 + \xi_0} - \frac{z_1^3 dz_2}{(z_2 + \xi_0)^2} \right) \end{aligned}$$

and the value of constant $\mathfrak{a} = \frac{1}{2\pi i} \int_{\gamma_2} z_1 dz_2$, for example, using the method of rectangles.

In order to find constants c_{11} , c_{12} , c_{21} , c_{22} , c_{23} we solve numerically the problem of minimization of $L_2(C_R)$ -norm of function

$$\begin{aligned} f_{\text{error}}(\xi_0) &= 2(G_1 - c_{11} - c_{12}\xi_0) \frac{\partial G_1}{\partial \xi_1} - G_1(c_{22} + 2c_{23}\xi_0) - \frac{\partial G_2}{\partial \xi_1} - 3c_{12}^2 \mathfrak{a} + \mathfrak{a}c_{23} + \\ &+ (-1) \left((G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \frac{\partial G_1}{\partial \xi_0} + \\ &+ (-1) \left(G_1^2 - 2G_1c_{11} - 2G_1c_{12}\xi_0 - G_2 \right) c_{12}, \end{aligned}$$

in variables c_{11} , c_{12} , c_{21} , c_{22} , c_{23} . As a result of solving of this minimization problem we find $c_{11} = 0$, $c_{12} = 0$, $c_{21} = 2$, $c_{22} = 0$, $c_{23} = 0$ (see fig. 7).

Let us compute symmetric functions $\sigma_1 = G_1 + c_{11} + c_{12}\xi_0$, $\sigma_2 = G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2$ and σ_0 on the circle C_R and desymmetrize them using Newton identities. The value of $\sigma_0(\xi_0)$ is a sum of $\mu_0 = 2$ and of the winding number of γ_2 with respect to $-\xi_0$, which by definition is equal to $G_0(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{dz_2}{z_2 + \xi_0}$. As a result of desymmetrization we obtain functions $h_1(\xi_0)$ and $h_2(\xi_0)$ on the circle C_R , which perform lifting of the circle C_R to the surface X_2 . Denote by $\Gamma_{1,2} = \{(h_{1,2}(z_2), -z_2) \mid z_2 \in C_R\}$ the curves, obtained by corresponding lifting of C_R to X_2 (see fig. 8).

Now we consider the curve $\gamma_2 + \Gamma_1 + \Gamma_2$ as a new initial curve and we reconstruct the surface $X_2^R = \{z \in X_2 \mid |z_2| \leq R\}$. Further, our considerations are similar to those for the case of $\mu_0 = 1$. At first, we compute functions s_k . Then, we find symmetric functions σ_k . Further, we solve the algebraic equation (numerically) and find functions $h_k(\xi_0)$, $k = 1, \dots, N_+(\xi_0)$. The reconstructed surface is given by fig. 9.

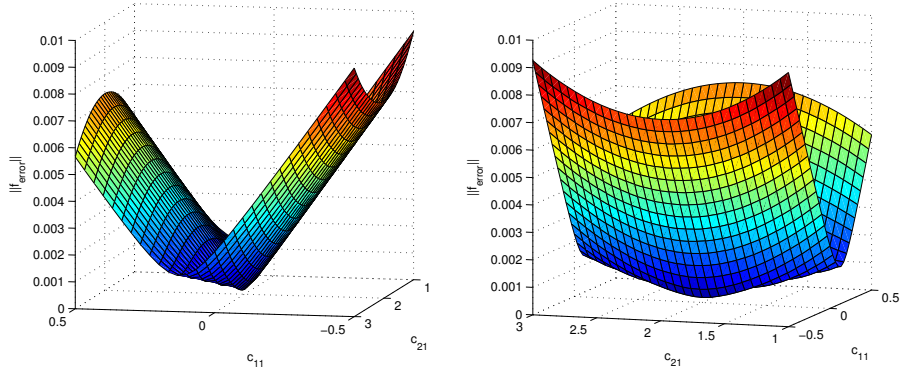


Figure 7: Graph of function $\|f_{\text{error}}\|(c_{11}, c_{21})$ in a neighborhood of the point of global minimum.

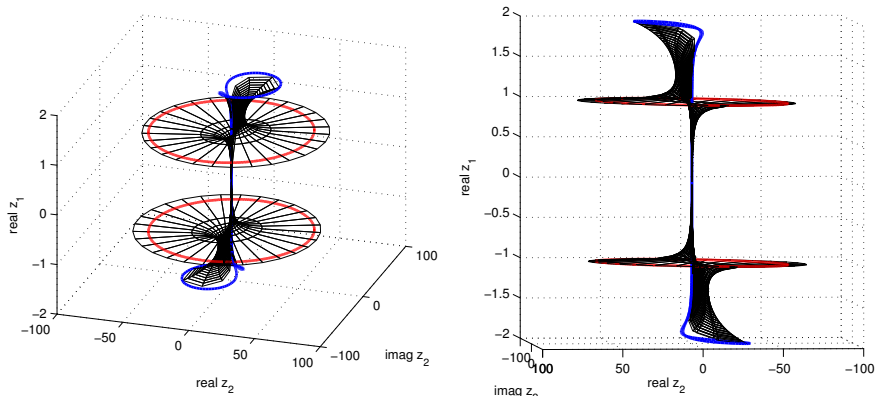


Figure 8: The contour of surface X_2 (black), boundary γ_2 of X_2 (blue), reconstructed curves $\Gamma_{1,2}$, belonging to X_2 (red).

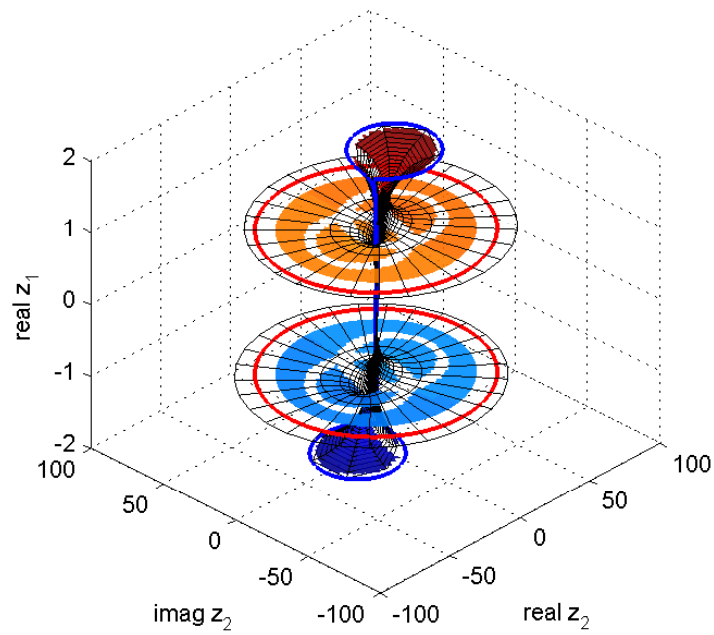


Figure 9: The contour of surface X_2 (black), boundary γ_2 of X_2 (blue), reconstructed curves $\Gamma_{1,2}$, belonging to X_2 (red), reconstructed leaves of the surface are represented by dark-blue, orange, red and blue domains.

References

- [AH] A. Agaltsov, G. Henkin, Algorithm for reconstruction of Riemann surface with given boundary in complex projective space, hal-00912925, version 1, 03 Dec. 2013
- [C] W. L. Chow, On Compact Complex Analytic Varieties, Amer. J. of Math. **71** (4), 1949, pp. 893–914
- [DH1] P. Dolbeault, G. Henkin, Surfaces de Riemann de Bord Donné Dans $\mathbb{C}P^n$, Contributions to Complex Analysis and Analytic Geometry, Aspects of Mathematics **E 26**, 1994, pp. 163–187
- [DH2] P. Dolbeault, G. Henkin, Chaines Holomorphes de Bord Donné Dans $\mathbb{C}P^n$, Bull. Soc. Math. de France **125**, 1997, pp. 383–445
- [HL] F. R. Harvey, H. Lawson, Boundaries of Complex Analytic Varieties, Ann. of Math., **102** (2), 1975, pp. 223–290
- [HS] F. R. Harvey, B. Shiffman, A Characterization of Holomorphic Chaines, Ann. of Math. **99** (3), 1974, pp. 553–587
- [HM] G. Henkin, V. Michel, On the Explicit Reconstruction of a Riemann Surface From its Dirichlet–Neumann Operator, Geom. Funct. Anal. **17** (1), 2007, pp. 116–155
- [W] J. Wermer, The Hull of a Curve in \mathbb{C}^n , Ann. of Math. **58** (3), 1958, pp. 550–561