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# Algorithm for reconstruction of Riemann surface with given boundary in complex projective space

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**Abstract** In this paper we propose the numerically realizable algorithm for reconstruction of complex curve with known boundary and without compact components in complex projective space.

**Key words:** Riemann surface, reconstruction algorithm, Burgers equation, Cauchy type formulas

## 1 Introduction.

Let us denote by  $\mathbb{CP}^2$  complex projective space with homogeneous coordinates  $(w_0 : w_1 : w_2)$ . Let real closed rectifiable, oriented curve  $\gamma$  in  $\mathbb{CP}^2$  be boundary of complex curve  $X \subset \mathbb{CP}^2$  with notation  $\gamma = bX$ . Without restriction of generality we suppose that the following conditions of general position be realized:

$$(0 : 1 : 0) \notin X, \quad w_l|_\gamma \neq 0, \quad l = 0, 1, 2. \quad (\#)$$

Put  $\mathbb{C}^2 = \{w \in \mathbb{CP}^2 : w_0 \neq 0\}$  with coordinates  $z_1 = \frac{w_1}{w_0}$ ,  $z_2 = \frac{w_2}{w_0}$ . For almost all  $\xi = (\xi_0, \xi_1) \in (\mathbb{C}^2)^*$  the points of intersection  $X$  with complex line  $\mathbb{C}_\xi^1 = \{z \in \mathbb{C}^2 : \xi_0 + \xi_1 z_1 + z_2 = 0\}$  form finite set of points

$$(z_1^{(j)}(\xi), z_2^{(j)}(\xi)) = (h_j(\xi_0, \xi_1), -\xi_0 - \xi_1 h_j(\xi_0, \xi_1)), \quad j = 1, \dots, N_+(\xi).$$

The essential role in the reconstruction of  $X$  through  $\gamma$  plays the following formula of Cauchy type from [2]:

$$G_m(\xi_0, \xi_1) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_\gamma z_1^m (\xi_0 + \xi_1 z_1 + z_2)^{-1} d(\xi_0 + \xi_1 z_1 + z_2) = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi_0, \xi_1) + P_m(\xi_0, \xi_1), \quad m = 0, 1, \dots, \quad (1)$$

where  $N_+(\xi) = N_+(\xi_0, \xi_1)$  be the number of points of intersection (multiplicity taking into account) of  $X$  with complex line  $\mathbb{C}_\xi^1$ ,  $P_m(\xi_0, \xi_1)$  be polynomial of degree  $m$  with respect to  $\xi_0$ . In addition,  $P_0(\xi_0, \xi_1) = -N_-$ , where  $N_-$  is the number of points of intersection of  $X$  with infinity  $\{w \in \mathbb{CP}^2 : w_0 = 0\}$ ,

$$P_1(\xi_0, \xi_1) = \sum_{k=1}^{N_-(\xi)} \frac{a_k \xi_0 - b_k}{a_k \xi_1 + 1}, \quad (2)$$

$a_k = w_2(q_k)$ ,  $b_k = \frac{dw_2}{dw_0}(q_k)$ , where  $q_k$ ,  $k = 1, \dots, N_-$ , — be points of intersection of  $X$  with infinity  $\{w : w_0 = 0\}$ . In particular, the following corollary of (1) takes place:

$$G_0(\xi_0, \xi_1) = \frac{1}{2\pi i} \int_{\gamma} \frac{d(\xi_0 + \xi_1 z_1 + z_2)}{\xi_0 + \xi_1 z_1 + z_2} = N_+(\xi) - N_-. \quad (3)$$

Let further  $\xi_1 = 0$  and let  $\pi_2: \mathbb{C}^2 \rightarrow \mathbb{C}$  — be projection:  $\pi_2(z_1, z_2) = z_2$ . We have  $\pi_2\gamma \subset \mathbb{C}$ ,  $\mathbb{C} \setminus \pi_2\gamma = \cup_{l=0}^L \Omega_l$ , where  $\{\Omega_l\}$  — be connected component of  $\mathbb{C} \setminus \pi_2\gamma$ . For every component  $\Omega_l$  the number of points of intersection of  $X$  with line  $z_2 = \xi_0$ ,  $\xi_0 \in \Omega_l$ , multiplicity taking into account will be denoted by  $\mu_l = N_+(\xi_0, 0)$ . Let  $\Omega_0$  denotes unbounded component of set  $\mathbb{C} \setminus \pi_2\gamma$ . From definition of  $N_{\pm}$  it follows the formula

$$\mu_0 = N_+(\xi_0, 0) = N_-, \quad \xi_0 \in \Omega_0.$$

Assume that complex curve  $X$  does not contain compact components, or equivalently, satisfies the following condition of minimality from the work [4]:

*For arbitrary complex curve  $\tilde{X} \subset \mathbb{CP}^2$  with condition  $b\tilde{X} = bX = \gamma$  and for almost all  $\xi \in (\mathbb{C}^2)^*$  the number of points of intersection  $\tilde{N}_+(\xi)$  of  $\tilde{X}$  with line  $\mathbb{C}_{\xi}^1$ , multiplicity taking into account, is not less than the number  $N_+(\xi)$  for curve  $X$ .* (\*)

Condition of minimality (\*) is condition of general position and is fulfilled for  $X$ , if, for example, every irreducible component of  $X$  is transcendental complex curve. Note, that from theorems of Chow [1], [2] and Harvey, Shiffman [3], it follows that arbitrary complex curve  $X \subset \mathbb{CP}^2$  with condition  $b\tilde{X} = bX$  admits unique representation  $\tilde{X} = X \cup V$ , where  $X$  — is curve with condition of minimality (\*), and  $V$  — is compact algebraic curve, possibly with multiple components.

In this work in the development of [2] we obtained the numerically realizable algorithm for reconstruction of complex curve  $X \subseteq \mathbb{CP}^2$  with known boundary and with condition of minimality. This algorithm permits, in particular, to make applicable the result of [4] about principal possibility to reconstruct topology and conformal structure of two-dimensional surface  $X$  in  $\mathbb{R}^3$  with constant scalar conductivity from measurements on  $bX$  of electric current densities, being created by three potentials in general position.

Our algorithm depends of parameter  $\mu_0 = N_{\pm}(\xi_0, 0)$ ,  $\xi_0 \in \Omega_0$ , verified for many examples and rigorously justified for  $\mu_0 = 0, 1, 2$ . The difficulty of justification increases with increasing of  $\mu_0$ .

## 2 Cauchy type formulas and Riemann-Burgers equations.

Let us give at first a new proof of Cauchy type formula (1) from [2], permitting to obtain explicit expressions for functions  $P_m(\xi_0, \xi_1)$ .

**Theorem 2.1.** *Let  $X \subset \mathbb{CP}^2$  be complex curve without compact components,  $\gamma = bX$  be real rectifiable oriented curve. Let conditions of general position (#) be fulfilled and all points of intersection of  $X$  and  $\mathbb{CP}_\xi^1$ , for almost all  $\xi \in \mathbb{C}^2$ , have multiplicity one and their coordinate  $w_2$  is not equal to zero. Then following equalities are fulfilled*

$$G_m(\xi_0, \xi_1) = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi_0, \xi_1) + P_m(\xi_0, \xi_1), \quad (4)$$

where  $P_m(\xi_0, \xi_1)$  is polynomial of degree at most  $m$  with respect to  $\xi_0$ , having the form

$$\begin{aligned} P_m(\xi_0, \xi_1) = & \sum_{s=1}^{\mu_0} \sum_{k=0}^{m-1} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s) \frac{d^{m-k}}{dw_0^{m-k}} \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s} - \\ & \sum_{s=1}^{\mu_0} \sum_{i_1+\dots+i_m=m} \frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s). \end{aligned}$$

In particular, if  $\mu_0 = 0$ , then  $P_m \equiv 0$ .

*Proof.* Put  $\tilde{g} = \xi_0 w_0 + \xi_1 w_1 + w_2$  and  $g = \frac{\tilde{g}}{w_0} = \xi_0 + \xi_1 z_1 + z_2$ . Consider differential forms

$$\omega_m \stackrel{\text{def}}{=} z_1^m \frac{dg}{g} = \frac{w_1^m w_0}{w_0^m \tilde{g}} d\left(\frac{\tilde{g}}{w_0}\right) = \frac{w_1^m d\tilde{g}}{w_0^m \tilde{g}} - \frac{w_1^m}{w_0^{m+1}} dw_0, \quad m = 0, 1, \dots$$

Then  $G_m(\xi) = \frac{1}{2\pi i} \int_\gamma \omega_m$ . Let us calculate this integral explicitly. Denote by  $p_j$ ,  $j = 1, \dots, N_+(\xi)$  the points of intersection of  $X$  and  $\mathbb{CP}_\xi^1$ , and by  $q_s$ ,  $s = 1, \dots, \mu_0$  the points of intersection of  $X$  with infinity  $\{w_0 = 0\}$ . Denote by  $B_j^\varepsilon$  intersections of  $X$  with ball of radius  $\varepsilon$  in  $\mathbb{CP}^2$  with center in  $p_j$  and by  $D_s^\varepsilon$  intersections of  $X$  with ball of radius  $\varepsilon$  with center in  $q_s$ . The restriction of form  $\omega_m$  on  $X$  is meromorphic form with poles in points  $p_j$  and  $q_s$ . Thus the following equality is valid

$$G_m(\xi) = \frac{1}{2\pi i} \int_\gamma \omega_m = \sum_{j=1}^{N_+(\xi)} \frac{1}{2\pi i} \int_{bB_j^\varepsilon} \omega_m + \sum_{s=1}^{\mu_0} \frac{1}{2\pi i} \int_{bD_s^\varepsilon} \omega_m.$$

If  $\mu_0 = 0$ , then the second group of terms is absent. The integral  $\int_{bB_j^\varepsilon} \omega_m$  can be calculated as residue of the first order pole:

$$\int_{bB_j^\varepsilon} \omega_m = \int_{bB_j^\varepsilon} z_1^m \frac{d\tilde{g}}{\tilde{g}} - \int_{bB_j^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = \int_{bB_j^\varepsilon} z_1^m \frac{d\tilde{g}}{\tilde{g}} = 2\pi i h_j^m(\xi).$$

Let  $\mu_0 > 0$ . Calculation of integral  $\int_{bD_s^\varepsilon} \omega_1$  will be done in two steps. Let us calculate first  $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0$ . Consider expansion of  $w_1(w_0)$  into power series in  $w_0$  in the neighborhood of point  $q_s$ :

$$w_1(w_0) = w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2w_1}{dw_0^2}(q_s)w_0^2 + \dots$$

Note further that

$$w_1^m(w_0) = \sum_{k=0}^{\infty} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)w_0^k.$$

The coefficient near  $w_0^m$  can be presented in the form

$$\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = 2\pi i \sum_{i_1+\dots+i_m=m} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s).$$

Now we can calculate the integral  $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{g}$ . Using relation  $d\tilde{g} = \frac{d\tilde{g}}{dw_0} dw_0$  and expansion of  $w_1(w_0)$  into power series in  $w_0$  we obtain:

$$\begin{aligned} \int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{g} &= \int_{bD_s^\varepsilon} \frac{1}{w_0^m} \left( w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2w_1}{dw_0^2}(q_s)w_0^2 + \dots \right)^m \frac{d\tilde{g}}{dw_0} \frac{1}{g} dw_0 = \\ &= \sum_{k=0}^{\infty} \int_{bD_s^\varepsilon} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\tilde{g}}{dw_0} \frac{1}{g} dw_0 = \\ &= \sum_{k=0}^{m-1} \int_{bD_s^\varepsilon} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\tilde{g}}{dw_0} \frac{1}{g} dw_0 = \\ &= \sum_{k=0}^{m-1} \frac{2\pi i}{(m-k-1)!} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) \lim_{w_0 \rightarrow 0} \frac{d^{m-k-1}}{dw_0^{m-k-1}} \left( \frac{d\tilde{g}}{dw_0} \frac{1}{g} \right). \end{aligned}$$

From here, taking into account relation  $\frac{d\tilde{g}}{dw_0} \frac{1}{g} = \frac{d \ln \tilde{g}}{dw_0}$ , we obtain finally

$$\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{g} = 2\pi i \sum_{k=0}^{m-1} \sum_{i_1+\dots+i_m=k} \frac{\frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)!} \frac{d^{m-k}}{dw_0^{m-k}} \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s}.$$

It is polynomial of degree at most  $m$  with respect to  $\xi_0$ .  $\square$

Before considering the algorithm for reconstruction of complex curve we need to prove one more theorem, which connects among themselves different functions  $h_j(\xi_0, \xi_1)$ ,  $j = 1, \dots, N_+(\xi_0, \xi_1)$ . Denote by  $\Omega_l^{(k)}$  the infinitesimal neighborhood of order  $k$  of the set  $\Omega_l$ .

**Theorem 2.2.** *Let  $X \subset \mathbb{CP}^2$  be complex curve without compact components,  $\gamma = bX$ , and between points of intersection of  $X$  with infinity  $\{w_0 = 0\}$  there is no point  $(0 : 1 : 0)$ .*

*Then for existence and uniqueness of mutually distinct holomorphic in  $\xi \in \Omega_l^{(1)}$  functions  $\hat{h}_j(\xi)$  and holomorphic functions  $\hat{P}_m(\xi)$ , being polynomials of degree at most  $m$  in  $\xi_0$ , satisfying to system*

$$G_m(\xi) = \sum_{j=1}^{N_+(\xi)} \hat{h}_j^m(\xi) + \hat{P}_m(\xi), \quad m, j = 1, \dots, N_+(\xi), \quad l = 0, 1, \dots, L, \quad (5)$$

*it is necessary and sufficient the existence and uniqueness of mutually distinct holomorphic in  $\xi \in \Omega_l^{(1)}$  functions  $\tilde{h}_j(\xi)$ , satisfying to equation  $\frac{\partial^2}{\partial \xi_0^2}(G_1(\xi) - \sum_{j=1}^{N_+(\xi)} \tilde{h}_j(\xi)) = 0$  and to the system of Riemann-Burgers equations:*

$$\frac{\partial \tilde{h}_j}{\partial \xi_1} = \tilde{h}_j(\xi) \frac{\partial \tilde{h}_j}{\partial \xi_0}, \quad j = 1, \dots, N_+(\xi), \quad \xi \in \Omega_l^{(1)}, \quad l = 0, 1, \dots, L.$$

*Moreover, functions  $\{\tilde{h}_j\}$ ,  $\{\hat{h}_j\}$ ,  $\{\hat{P}_m\}$ , satisfying these systems, exist and  $\tilde{h}_j = \hat{h}_j$ ,  $j = 1, \dots, N_+(\xi)$ .*

*Proof. Proof. Necessity.* Let there exist and unique up to order mutually distinct holomorphic in  $\xi \in \Omega_l^{(1)}$  functions  $\{\hat{h}_j(\xi)\}$ ,  $j = 1, \dots, N_+(\xi_0, 0)$ ,  $l = 0, 1, \dots, L$ , and holomorphic functions  $\{\hat{P}_m(\xi)\}$ , being polynomials of degree at most  $m$  in  $\xi_0$ , and satisfying system (5),  $m = 1, \dots, N_+(\xi)$ . By theorem II from [2] and theorem 3 from [4] for almost all  $\xi^* = (\xi_0^*, 0)$  in the neighborhood of  $\xi^*$  there exist and unique up to order mutually distinct holomorphic solutions  $\{\tilde{h}_j(\xi)\}$  of Riemann-Burgers equation:

$$\frac{\partial \tilde{h}_j(\xi)}{\partial \xi_1} = \tilde{h}_j(\xi) \frac{\partial \tilde{h}_j(\xi)}{\partial \xi_0},$$

such that  $\frac{\partial^2}{\partial \xi_0^2}(G_1(\xi) - \sum_j \tilde{h}_j(\xi)) = 0$ ,  $j = 1, \dots, N_+(\xi^*)$ . Proposition 3.3.3 from [2] ensure that functions  $\{\tilde{h}_j\}$  satisfy to system (5). From uniqueness of families of functions  $\{\hat{h}_j(\xi)\}$  and  $\{\tilde{h}_j(\xi)\}$  it follows that these functions satisfy simultaneously to Riemann-Burgers equation.

*Sufficiency.* Suppose, that there exist and unique up to order mutually distinct holomorphic functions  $\{h_j(\xi)\}$  on  $\Omega_l^{(1)}$ ,  $l = 0, 1, \dots, L$ , satisfying to Riemann-Burgers equation and also to the following equation

$$\frac{\partial^2}{\partial \xi_0^2}(G_1(\xi) - \sum_j h_j(\xi)) = 0, \quad \xi \in \Omega_l^{(1)}.$$

In particular, for any  $\xi_0 \in \Omega_l^{(0)}$  we have relations

$$\frac{\partial h_j}{\partial \xi_1}(\xi_0, 0) = h_j(\xi_0, 0) \frac{\partial h_j}{\partial \xi_0}(\xi_0, 0).$$

By Cauchy-Kowalewski theorem for any  $l = 0, 1, \dots, L$  in a neighborhood of arbitrary  $\xi^* \in \Omega_l^{(1)}$  there exist holomorphic functions  $\{\tilde{h}_j(\xi_0, \xi_1)\}$ , satisfying to Riemann-Burgers equation and such that  $\tilde{h}_j|_{\Omega_l^{(1)}} = h_j$ .

From here and Proposition 3.3.3 from [2] we obtain existence and uniqueness of holomorphic functions  $\{P_m(\xi)\}$ , being polynomials of degree at most  $m$  in  $\xi_0$ , such that  $\{h_j(\xi)\}$  and  $\{P_m(\xi)\}$  satisfy to system (5) for  $m, j = 1, \dots, N_+(\xi)$ ,  $l = 0, 1, \dots, L$ ,  $\xi \in \Omega_l^{(1)}$ .

Existence and uniqueness of functions  $\{\tilde{h}_j\}$  with necessary properties follow from [2], theorem II, and from [4], theorem 3. From here and from already proved sufficiency it follows the existence and uniqueness of functions  $\{\tilde{h}_j\}$  and  $\{\hat{P}_m(\xi)\}$  with property (5). Equalities  $\hat{h}_j = \tilde{h}_j$ ,  $j = 1, \dots, N_+(\xi)$ , follow from property of functions  $\{\tilde{h}_j\}$  to satisfy (5) and from uniqueness of functions satisfying to (5).  $\square$

### 3 Reconstruction algorithm.

Consider now the reconstruction algorithm of complex curve  $X \subseteq \mathbb{CP}^2$  with given boundary  $bX$  and with condition of minimality (\*). Let us consider the cases  $\mu_0 = 0, 1, 2$ .

The reconstruction algorithm is based on formulas (4) with polynomials  $P_m$ ,  $m = 0, 1, \dots$ . The next theorem permits to calculate these polynomials.

**Theorem 3.1.** *Let  $\gamma = bX$ ,  $X \subseteq \mathbb{CP}^2$  be complex curves without compact components,  $\gamma \subseteq \mathbb{C}^2$  be its boundary. Let mutually distinct holomorphic in  $\xi \in \Omega_l^{(1)}$ ,  $l = 0, 1, \dots, L$ , functions  $\{h_j(\xi)\}$  and polynomials of degree at most  $m$  in  $\xi_0$   $P_m(\xi_0, \xi_1)$  with holomorphic coefficients satisfy to system (4) for  $\xi \in \Omega_l^{(1)}$ ,  $j = 1, \dots, N_+(\xi)$  (existence and uniqueness of such functions follow from theorem of 2.2). Then the following statements are valid :*

1. *If  $\mu_0 = 0$ , then  $P_m(\xi_0, 0) \equiv 0$  for all  $m$ ,*
2. *If  $\mu_0 = 1$ , then  $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$ , where constants  $c_{11}$  and  $c_{12}$  can be found from identity in  $\xi_0 \in \Omega_0$ :*

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) + c_{12} \left( \xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) + G_1(\xi_0, 0) \right) = G_1(\xi_0, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0).$$

3. *If  $\mu_0 = 2$ , then  $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$ ,  $P_2(\xi_0, 0) = c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2$ , where constants  $c_{11}$ ,  $c_{12}$ ,  $c_{21}$ ,  $c_{22}$ ,  $c_{23}$  can be found from identity in*

$\xi_0 \in \Omega_0$ :

$$\begin{aligned} & 2(G_1 - c_{11} - c_{12}\xi_0) \frac{\partial G_1}{\partial \xi_1} - G_1(c_{22} + 2c_{23}\xi_0) - \frac{\partial G_2}{\partial \xi_1} - 3c_{12}^2 \mathfrak{a} + \mathfrak{a}c_{23} = \\ & = \left( (G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \frac{\partial G_1}{\partial \xi_0} + \\ & \quad + \left( G_1^2 - 2G_1c_{11} - 2G_1c_{12}\xi_0 - G_2 \right) c_{12}, \end{aligned}$$

where  $\mathfrak{a} = \frac{1}{2\pi i} \int_{\gamma} z_1 dz_2$ .

*Proof.* By theorem 2.2 functions  $P_m$  from condition of theorem coincide with functions  $P_m$  from theorem 2.1.

1. By theorem 2.1  $P_m \equiv 0$ , if  $\mu_0 = 0$ .

2. For  $\xi \in \Omega_l^{(0)}$  we have equality  $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$ . We need to find constants  $c_{11} = C_{11}(0)$  and  $c_{12} = C_{12}(0)$ . Let us write out the restriction of equation (4) for  $\xi \in \Omega_l^{(1)}$  and equation derived by differentiation of (4) in  $\xi_0$  and by restriction on  $\xi \in \Omega_l^{(0)}$ :

$$\begin{aligned} h_1(\xi_0, 0) &= G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0, \\ \frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0, \\ \frac{\partial h_1}{\partial \xi_0}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12}, \end{aligned}$$

where  $\xi_0 \in \Omega_0$ . By theorem 2.2 for  $\xi_0 \in \Omega_0$  function  $h_1(\xi_0, 0)$  satisfies to equality  $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) = h_1(\xi_0, 0) \frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$ . If we substitute in this equality expressions for  $h_1(\xi_0, 0)$ ,  $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0)$  and  $\frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$ , we obtain equation

$$\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 = \left( G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0 \right) \left( \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12} \right). \quad (6)$$

This equation is valid for  $\xi_0 \in \Omega_0$ . Let us divide it into  $\xi_0$  and tend  $\xi_0 \rightarrow \infty$ . We obtain equality  $\dot{C}_{12}(0) = -c_{12}^2$ . Taking into account this equality, the equation (6) may be rewritten in the form

$$\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) = \left( G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0 \right) \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \left( G_1(\xi_0, 0) - c_{11} \right) c_{12}. \quad (7)$$

Taking into account that  $\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) \rightarrow 0$  for  $\xi_0 \rightarrow \infty$ , passage to the limit in (7) when  $\xi_0 \rightarrow \infty$  gives equality  $\dot{C}_{11}(0) = -c_{11}c_{12}$ . Using just obtained equality, equation (7) takes the required form.

3. By theorem 2.2 functions  $h_1(\xi)$  and  $h_2(\xi)$  satisfy to Riemann-Burgers



equation for  $\xi \in \Omega_i^{(1)}$ . So, the following equalities are valid.

$$\frac{\partial(h_1 h_2)}{\partial \xi_1} = h_1 \frac{\partial h_2}{\partial \xi_1} + \frac{\partial h_1}{\partial \xi_1} h_2 = h_1 h_2 \frac{\partial(h_1 + h_2)}{\partial \xi_0}, \quad (8)$$

$$\frac{\partial(h_1^2 + h_2^2)}{\partial \xi_0} = 2h_1 \frac{\partial h_1}{\partial \xi_0} + 2h_2 \frac{\partial h_2}{\partial \xi_0} = 2 \frac{\partial(h_1 + h_2)}{\partial \xi_1}. \quad (9)$$

Note that  $h_1 h_2 = \frac{1}{2}(h_1 + h_2)^2 - \frac{1}{2}(h_1^2 + h_2^2)$ . Therefore the system (8)–(9) is equivalent to the system

$$\frac{\partial(h_1 + h_2)^2}{\partial \xi_1} - \frac{\partial(h_1^2 + h_2^2)}{\partial \xi_1} = \left( (h_1 + h_2)^2 - (h_1^2 + h_2^2) \right) \frac{\partial(h_1 + h_2)}{\partial \xi_0}, \quad (10)$$

$$\frac{\partial(h_1^2 + h_2^2)}{\partial \xi_0} = 2 \frac{\partial(h_1 + h_2)}{\partial \xi_1}. \quad (11)$$

We substitute into this system  $h_1^2 + h_2^2$  and  $h_1 + h_2$  from equation (4), using notations  $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$ ,  $P_2(\xi_0, \xi_1) = C_{21}(\xi_1) + C_{22}(\xi_1)\xi_0 + C_{23}(\xi_1)\xi_0^2$ . Equation (11) restricted on  $\Omega_i^{(0)}$  takes the form

$$\frac{\partial G_2}{\partial \xi_0}(\xi_0, 0) - c_{22} - 2c_{23}\xi_0 = 2 \left( \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 \right).$$

Divide this equation into  $\xi_0$  and tend  $\xi_0 \rightarrow \infty$ . We obtain the equality  $\dot{C}_{12}(0) = c_{23}$ . In this equation, using last equality, in the passage to the limit for  $\xi_0 \rightarrow \infty$  we obtain the equality  $\dot{C}_{11}(0) = \frac{1}{2}c_{22}$ .

Now substitute into (10) expression for  $h_1^2 + h_2^2$  and  $h_1 + h_2$  and restrict obtained formula to  $\Omega_i^{(0)}$ . We obtain the equality

$$\begin{aligned} & 2 \left( G_1 - c_{11} - c_{12}\xi_0 \right) \left( \frac{\partial G_1}{\partial \xi_1} - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 \right) - \frac{\partial G_2}{\partial \xi_1} + \dot{C}_{21}(0) + \dot{C}_{22}(0)\xi_0 + \dot{C}_{23}(0)\xi_0^2 = \\ & = \left( (G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \left( \frac{\partial G_1}{\partial \xi_0} + c_{12} \right). \quad (12) \end{aligned}$$

Divide this equation into  $\xi_0^2$  and in passage to the limit for  $\xi_0 \rightarrow \infty$  we obtain the equality

$$2c_{12}\dot{C}_{12}(0) + \dot{C}_{23}(0) = (c_{12}^2 + c_{23})c_{12}.$$

Using last equality, divide (12) into  $\xi_0$  and in passage to the limit for  $\xi_0 \rightarrow \infty$  we obtain the equality

$$2c_{11}\dot{C}_{12}(0) + 2c_{12}\dot{C}_{11}(0) + \dot{C}_{22}(0) = (-2c_{11} + c_{22})c_{12}.$$

Taking into account obtained equalities one can rewrite (12) in the form

$$\begin{aligned}
& 2(G_1 - c_{11} - c_{12}\xi_0) \frac{\partial G_1}{\partial \xi_1} - 2G_1 (\dot{C}_{11}(0) + \dot{C}_{12}(0)\xi_0) + 2c_{11}\dot{C}_{11}(0) - \frac{\partial G_2}{\partial \xi_1} + \dot{C}_{21}(0) = \\
& = \left( (G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \frac{\partial G_1}{\partial \xi_0} + \\
& \quad + \left( (G_1 - c_{11})^2 - 2G_1 c_{12}\xi_0 - G_2 + c_{21} \right) c_{12}. \quad (13)
\end{aligned}$$

Take  $\xi_0 \rightarrow \infty$  in this equality and note that the following relations are valid

$$\begin{aligned}
\lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{\partial G_1}{\partial \xi_1} &= \lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_1}{\xi_0 + z_2} = \frac{1}{2\pi i} \int_{\gamma} z_1 dz_1 = \langle b\gamma, \frac{1}{4\pi i} z_1^2 \rangle = 0, \\
\lim_{\xi_0 \rightarrow \infty} \xi_0 G_1 &= \lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{\xi_0 + z_2} = \frac{1}{2\pi i} \int_{\gamma} z_1 dz_2, \\
\lim_{\xi_0 \rightarrow \infty} \xi_0^2 \frac{\partial G_1}{\partial \xi_0} &= - \lim_{\xi_0 \rightarrow \infty} \xi_0^2 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{(\xi_0 + z_2)^2} = - \frac{1}{2\pi i} \int_{\gamma} z_1 dz_2.
\end{aligned}$$

We obtain

$$-2\mathfrak{a}\dot{C}_{12}(0) + 2c_{11}\dot{C}_{11}(0) + \dot{C}_{21}(0) = -(c_{12}^2 + c_{23})\mathfrak{a} + (c_{11}^2 - 2c_{12}\mathfrak{a} + c_{21})c_{12}.$$

In obtained equations we express constants  $\dot{C}_{ij}(0)$  through  $c_{ij}$ :

$$\begin{aligned}
\dot{C}_{11}(0) &= \frac{1}{2}c_{22}, \\
\dot{C}_{12}(0) &= c_{23}, \\
\dot{C}_{23}(0) &= c_{12}^3 - c_{12}c_{23}, \\
\dot{C}_{22}(0) &= -2c_{11}(c_{12} + c_{23}), \\
\dot{C}_{21}(0) &= (c_{11}^2 - 3c_{12}\mathfrak{a} + c_{21})c_{12} + \mathfrak{a}c_{23} - c_{11}c_{22}.
\end{aligned}$$

Substituting these constants to (13), we obtain the third statement of theorem 3.1.  $\square$

Let us describe algorithm of reconstruction of complex curve  $X$  in  $\mathbb{C}P^2$  without compact components (satisfying minimality condition (\*)). As above,  $\gamma = bX$  is compact real curve.

Algorithm for reconstruction of curve  $X$  permits, instead of original curves, to find curve coinciding with original curve in the finite number of points and obtained by interpolation in other points. Let  $\{\xi^k\}_{k=1}^N$ ,  $\xi^k \in \mathbb{C}$  be arbitrary frame on  $\mathbb{C}$ ,  $\xi^i \neq \xi^j$ ,  $i \neq j$ , and  $\xi^k \notin \pi_2\gamma$ ,  $k = 1, \dots, N$ . Complex curve  $X$  intersects complex line  $\{z_2 = \xi^k\}$  in  $N_+(\xi_k, 0)$  points. Algorithm allows to find these points.

For all algorithms in input we have points  $\{\xi^k\}_{k=1}^N$  and curve  $\gamma$  (for example, as finite number of points belonging to  $\gamma$ ). In output we have a set of points  $(h_s(\xi^k, 0), \xi^k)$ ,  $k = 1, \dots, N$ ;  $s = 1, \dots, N_+(\xi^k, 0)$ , belonging to complex curve  $X$ .

The case of  $\mu_0 = 0$ .

1. *Calculation of  $\mu_l$ .* By formula (3) for every domain  $\Omega_l$ ,  $l = 1, \dots, L$ , number  $\mu_l$  is equal to index of curve  $\pi_2\gamma$  with respect to  $\xi_0 \in \Omega_l$ :

$$\mu_l \equiv N_+(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz_2}{z_2 - \xi_0} \equiv \frac{1}{2\pi i} \int_{\pi_2\gamma} \frac{dz}{z - \xi_0}, \quad \xi_0 \in \Omega_l.$$

2. *Calculation of power sums.* For the case  $\mu_0 = 0$  for every point  $\xi^k \in \Omega_l$ ,  $l = 1, \dots, L$ , by theorem 3.1 we have equalities  $P_m \equiv 0$ . By formula (4) we have the following formulas for power sums:

$$s_m(\xi^k) \equiv h_1^m(\xi^k, 0) + \dots + h_{\mu_l}^m(\xi^k, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^m dz_2}{z_2 - \xi^k}, \quad m = \overline{1, \mu_l}.$$

By theorem 2.1 the points  $(h_s(\xi^k, 0), \xi^k)$ ,  $s = 1, \dots, N_+(\xi^k, 0)$ ;  $k = 1, \dots, N$ , are required points of  $X$ .

3. *Calculation of symmetric functions.* For every point  $\xi^k \in \Omega_l$ ,  $l = 1, \dots, L$ , the Newton formulas

$$k\sigma_k(\xi^k) = \sum_{i=1}^k (-1)^{i-1} \sigma_{k-i}(\xi^k) s_i(\xi^k), \quad k = 1, \dots, N_+(\xi^k).$$

allow to reconstruct elementary symmetric functions:

$$\begin{aligned} \sigma_1(\xi^k) &= h_1(\xi^k, 0) + \dots + h_{\mu_l}(\xi^k, 0), \\ &\dots = \dots \\ \sigma_{\mu_l}(\xi^k) &= h_1(\xi^k, 0) \times \dots \times h_{\mu_l}(\xi^k, 0). \end{aligned}$$

4. *Desymmetrisation.* For any point  $\xi^k \in \Omega_l$  using Vieta formulas one can find numbers  $h_1(\xi^k, 0), \dots, h_{\mu_l}(\xi^k, 0)$ . Required points  $(h_s(\xi^k, 0), \xi^k)$ ,  $s = 1, \dots, N_+(\xi^k, 0)$ ;  $k = 1, \dots, N$ , of complex curve  $X$  are obtained.

The cases for  $\mu_0 = 1, 2$ . These cases are reduced to the case  $\mu_0 = 0$  by the following way. Since  $\pi_2\gamma \subset \mathbb{C}$  is compact real curve, there exists such  $R > 0$ , that the set  $B_R^c(0) = \{z \in \mathbb{C} \mid |z| \geq R\}$  belongs to  $\Omega_0$ . Without restriction of generality, one can suppose that  $|\xi^k| < R$  for all  $k = 1, \dots, N$ . Otherwise one can increase  $R$ .

Let us define auxiliary complex curve  $X_R = \{(z_1, z_2) \in X \mid |z_2| \leq R\}$ . Its boundary  $\gamma_R$  consists in two disjoint parts (possibly, multiconnected):  $\gamma$  and

real curve, obtained by lifting of the circle  $S_R = \{z \in \mathbb{C} \mid |z| = R\}$  on surface  $X$ , by inversion of projection  $\pi_2: X \rightarrow \mathbb{C}$ . Complex curve  $X_R$  does not intersect infinity and, as a consequence  $\mu_0(X_R) = 0$ . Moreover, any point  $(z_1(\xi^k), \xi^k)$ ,  $k = 1, \dots, N$ , belongs to  $X$  iff it belongs to  $X_R$ . Therefore, for reconstruction of complex curve  $X_R$  it is sufficient to reconstruct, at first, real curve obtained by lifting of  $S_R$  on  $X$  and after to solve reconstruction problem, as in the case  $\mu_0 = 0$ . Finally, we come to the following algorithm.

1. *New boundary.* We choose constant  $R$  so large, that the exterior of the disk of radius  $R$  with center in zero belong to  $\Omega_0$  and all  $\xi^k$  belong to this disk. Denote boundary of this disk by  $S_R$ . In the case  $\mu_0 = 1$  for points  $\xi_0 \in S_R$  by formula (4) we have equality

$$h_1(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{z_2 - \xi_0} + P_1(\xi_0, 0).$$

In the case  $\mu_0 = 2$  we have two equalities:

$$h_1(\xi_0, 0) + h_2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{z_2 - \xi_0} + P_1(\xi_0, 0),$$

$$h_1^2(\xi_0, 0) + h_2^2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^2 dz_2}{z_2 - \xi_0} + P_2(\xi_0, 0),$$

where polynomials can be found from theorem 3.1. In the case  $\mu_0 = 1$  by lifting the curve  $S_R$  on  $X$  we obtain at once the real curve of the form  $\{(h_1(\xi_0, 0), \xi_0) \mid \xi_0 \in S_R\}$ . In the case  $\mu_0 = 2$  we have to apply, at first, Newton formula and after Vieta formula in order from functions  $h_1 + h_2$  and  $h_1^2 + h_2^2$  to obtain  $h_1$  and  $h_2$ .

2. *Reduction.* For finding of complex curve  $X_R$  with boundary  $bX_R = \gamma_R$  we apply algorithm of reconstruction for the case  $\mu_0 = 0$ . The points obtained in such a way will be those, which we need due to explanation before algorithm.

## 4 Visualization.

Let us write a little bit about algorithm of visualization of Riemann surface by selection of appropriate finite number of points. For illustration of this algorithm we consider visualization of Riemann surface of the function

$$f(z) = \sqrt{\exp\left(\frac{z}{4}\right) + \sqrt{z^2 + 1}}, \quad |z| \leq 2.$$

Note, firstly, that Riemann surface of this function has six algebraic points of ramification of index 2 in the domain  $\{|z| \leq 2\}$ , two of which correspond to

points  $z = i$  and  $z = -i$  of nulling of argument of interior radical, and four others correspond to points of nulling of argument of exterior radical.

We assume that we have grid  $\{z_1^{ij}\}$  in variable  $z_1$  and a set of points  $\{(z_1^{ij}, z_2^{ij})\}$  on Riemann surface  $X = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = f(z_1), |z_1| \leq 2\}$  of function  $f(z)$ , where  $1 \leq i \leq N_1$ ,  $1 \leq j \leq N_2$ , and to neighbor values of  $i, j$  correspond neighbor points of grid. Suppose, that for each point  $z_1^{ij}$  the number of corresponding points  $(z_1^{ij}, z_2^{ij})$  is equal to number of leafs of Riemann surface  $X$ . Moreover, one can accept that some point  $z_1^{ij}$  may be a projection of some point of ramification on  $X$ , permitting also multiple points  $(z_1^{ij}, z_2^{ij})$ . Suppose also, that points of ramification can be only interior points of the grid  $\{z_1^{ij}\}$ . We will solve further the problem of visualization in `Matlab`.

1. **Finding of ramification points.** The first step of visualization algorithm consists in finding of ramification points. Choose an arbitrary point  $(z_1^{i_0 j_0}, z_2^{i_0 j_0})$  such, that point  $z_1^{i_0 j_0}$  is interior point of the grid. Let us go around the point  $z_1^{i_0 j_0}$  along discrete circle of sufficiently small radius with center in  $z_1^{i_0 j_0}$  (such that inside of the circle would not be more than one projection of ramification point of  $X$ ). Choose an arbitrary point from the grid  $\{z_1^{i_1 j_1}, z_2^{i_1 j_1}\}$ , corresponding to the first point  $z_1^{i_1 j_1}$  of this circuit and let us in the process of circuit associate with current point  $z_1^{i_k j_k}$  the point  $z_2^{i_k j_k}$  such, that the distance between points  $(z_1^{i_{k-1} j_{k-1}}, z_2^{i_{k-1} j_{k-1}})$  and  $(z_1^{i_k j_k}, z_2^{i_k j_k})$  be minimal. We associate the initial point of circuit  $(z_1^{i_1 j_1}, z_2^{i_1 j_1})$  with the final point  $(z_1^{i_N j_N}, z_2^{i_N j_N})$  of circuit. If in this case  $z_2^{i_1 j_1} \neq z_2^{i_N j_N}$ , we mark the disk with center in  $z_1^{i_0 j_0}$  with the same radius as for circuit of circle by the marked disk containing projection of some ramification point on  $X$ . This method of obtaining from discret circle on the grid  $\{z_1^{ij}\}$  of the set of points  $\{(z_1^{ij}, z_2^{ij})\}$  will be called by lifting of this circle .
2. **Realization of slits.** We joint the disks marked in the first item with boundary of the grid by some discrete curves on the grid  $\{z_1^{ij}\}$ , in such a way that different discrete curves don't intersect each other and call these curves by slits. It is impossible now to go around any point  $z_1^{ij}$  on discrete grid by closed discrete curve without intersections with these slits and the marked disks such that for describing lifting of this discrete curve into the set of points  $\{(z_1^{ij}, z_2^{ij})\}$  the initial and final points would be lifted into different points of this set of points.
3. **Representation of leafs.** Due to previous item the discrete grid  $\{z_1^{ij}\}$  without marked disks and without slits could be uniquely lifted into the set  $\{(z_1^{ij}, z_2^{ij})\}$  by fixing of lifting of some arbitrary point for each connected component on the set  $\{z_1^{ij}\}$ . These lifted sets of points we call leafs. These leafs one can visualize as usual surfaces.
4. **Lifting of slits.** By definition of slits  $\{z_1^{ij}\}$  every slit admits unique lifting into the set  $\{(z_1^{ij}, z_2^{ij})\}$  if for any slit to fix lifting of one of its points.

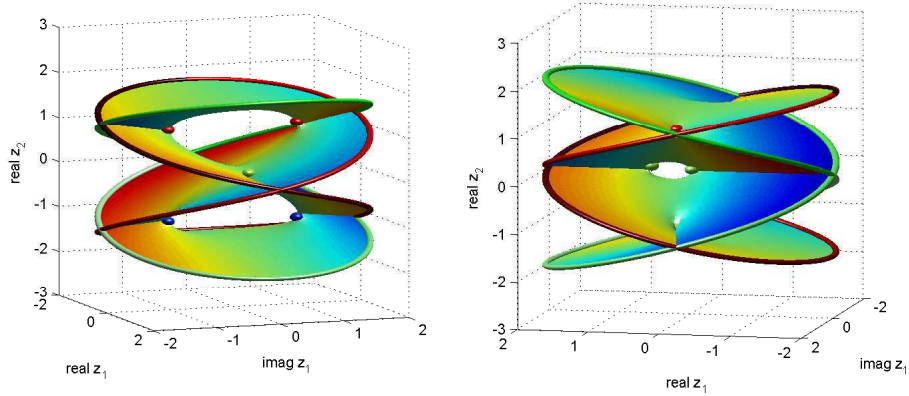


Figure 1: Riemann surface for function  $f(z) = \sqrt{\exp\left(\frac{z}{4}\right) + \sqrt{z^2 + 1}}$ ,  $|z| \leq 2$ , obtained by visualization algorithm. Red and green curves— two connected components of surface boundary, colored small balls — points of ramification.

This item is similar to previous one.

## 5 Examples.

### 5.1 The case $\mu_0 = 1$ .

Consider example of reconstruction of Riemann surface for the case  $\mu_0 = 1$ . The case  $\mu_0 = 2$  will be considered further. The case  $\mu_0 = 0$  is directly included in considered cases because of specific character of reconstruction algorithm. Let us reconstruct surface

$$X_1 = \{(z_1, z_2) \in \mathbb{C}^2 \mid (z_1 - 1)z_2 = \exp(z_1^2), \quad |z_1| \leq 2\}.$$

The boundary  $\gamma_1 = bX_1$  is given in the form of discrete number of points ( see further fig. 3).

Note, that if point  $(z_1, z_2) \in X_1$  is such that  $z_1$  approaches to 1, then  $z_2$  increases indefinitely. Choose  $R$  large enough, for example  $R = 60$ , and reconstruct real curve  $\Gamma = \{(z_1, z_2) \in X_1 \mid |z_2| = R\}$ . Compute, first, for two

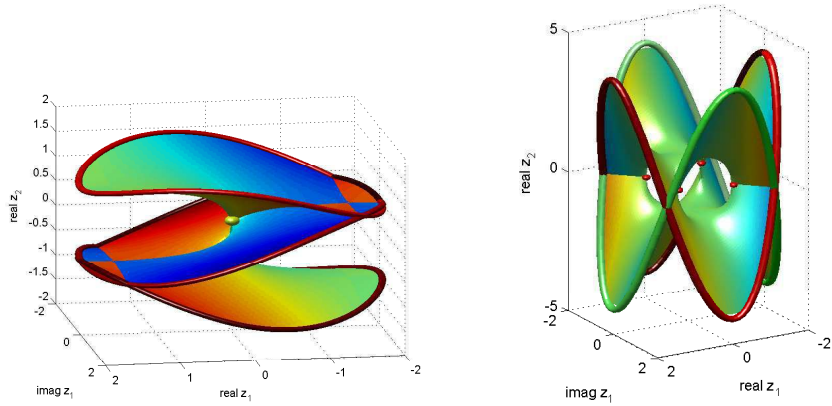


Figure 2: From the left Riemann surface of function  $f(z) = \sqrt{\sin(z)}$ ,  $|z| \leq 2$ , from the right — functions  $f(z) = \sqrt{z^4 + 1}$ ,  $|z| \leq 2$ , obtained by visualization algorithm.

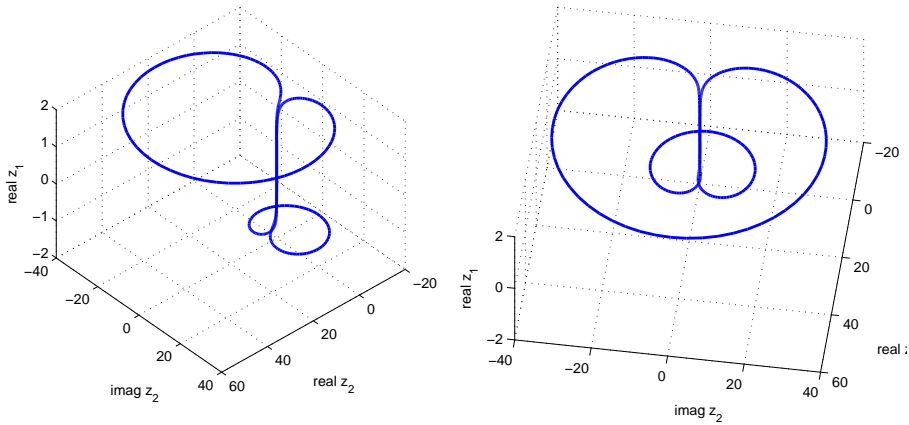


Figure 3: Boundary  $\gamma_1 = bX_1$  of the surface  $X_1$  by which the surface  $X_1$  is to reconstruct.

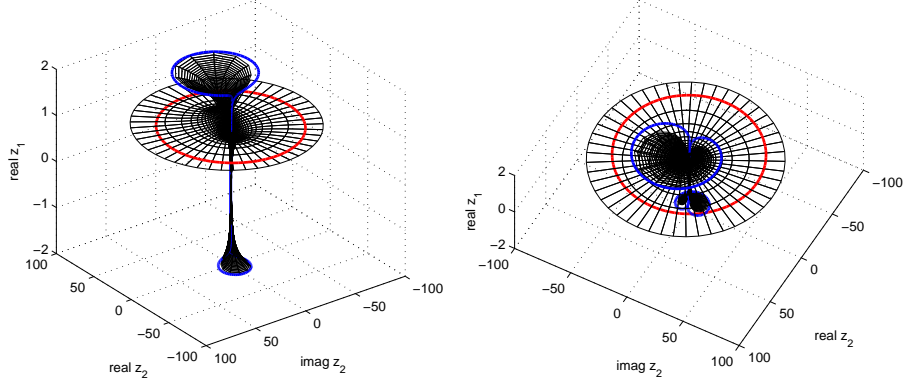


Figure 4: By black is represented contour of surface  $X_1$ , by blue —  $\gamma_1$ , by red — reconstructed curve  $\Gamma$ , belonging to  $X_1$ .

different  $\xi_0^1, \xi_0^2, |\xi_0^1| = |\xi_0^2| = R$ , the values of functions

$$G_1(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{z_1 dz_2}{z_2 + \xi_0},$$

$$\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) = -\frac{1}{2\pi i} \int_{\gamma_2} \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2},$$

$$\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma_2} \left( \frac{z_1 dz_1}{z_2 + \xi_0} - \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2} \right)$$

and find from linear system for  $c_{11}$  and  $c_{12}$

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) + c_{12} \left( \xi_0^1 \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) + G_1(\xi_0^1, 0) \right) = G_1(\xi_0^1, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0^1, 0),$$

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) + c_{12} \left( \xi_0^2 \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) + G_1(\xi_0^2, 0) \right) = G_1(\xi_0^2, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0^2, 0)$$

values  $c_{11} = 1, c_{12} = 0$ . Now calculate the values of functions  $G_1(\xi_0, 0)$  on the circle  $|\xi_0| = R$  and find function  $h_1(\xi_0) = G_1(\xi_0, 0) + c_{11} + c_{12}\xi_0, |\xi_0| = R$ . So we reconstruct a real curve  $\Gamma = \{(h_1(\xi_0), -\xi_0) \mid |\xi_0| = R\} \subseteq X_1$  (see fig. 4).

We can apply further the reconstruction algorithm of Riemann surface for the case  $\mu_0 = 0$  to the surface  $X_1^R = \{z \in X_1 \mid |z_2| \leq 60\}$  with boundary  $bX_1^R = \gamma_1 + \Gamma$ . We can calculate the values of function

$$\sigma_0(\xi_0) = \frac{1}{2\pi i} \int_{\gamma_1 + \Gamma} \frac{dz_2}{z_2 + \xi_0}.$$



The value of function  $\sigma_0(\xi_0)$  in point  $\xi_0$  is equal to number  $N_+(\xi_0)$  of points of surface  $X_1^R$ , projected into the point  $-\xi_0$  under projection  $(z_1, z_2) \mapsto z_2$ . Further for every point with  $N_+(\xi_0) > 0$  we calculate functions

$$s_k(\xi_0) = \frac{1}{2\pi i} \int_{\gamma_1 + \Gamma} \frac{z_1^k dz_2}{z_2 + \xi_0}, \quad k = 1, \dots, N_+(\xi_0).$$

From functions  $s_k(\xi_0)$  we can find functions  $\sigma_k(\xi_0)$ , using Newton formulas:

$$k\sigma_k(\xi_0) = \sum_{i=1}^k (-1)^{i-1} \sigma_{k-i}(\xi_0) s_i(\xi_0), \quad k = 1, \dots, N_+(\xi_0).$$

After we find roots  $h_1(\xi_0), \dots, h_{\sigma_0(\xi_0)}(\xi_0)$  of polynomial

$$t^{N_+(\xi_0)} - \sigma_1(\xi_0)t^{N_+(\xi_0)-1} + \dots + (-1)^{N_+(\xi_0)}\sigma_{N_+(\xi_0)}(\xi_0) = 0.$$

The points  $\{(h_k(\xi_0), -\xi_0) \mid k = 1, \dots, N_+(\xi_0)\}$  represent the set of all points of  $X_1^R$ , projected into  $-\xi_0$  by projection  $(z_1, z_2) \rightarrow z_2$ . Visualization, obtained for all  $\xi_0$ , of set of points can be realized by visualization algorithm described in previous section. Reconstructed surface is represented in fig. 5.

## 5.2 The case $\mu_0 = 2$ .

Consider example of reconstruction of Riemann surface for the case  $\mu_0 = 2$ . As surface for reconstruction we choose the reconstruction

$$X_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2(z_1^2 - 1) = z_1 \exp(z_1^2), \quad |z_1| \leq 2\}.$$

From definition of  $X_2$  one can see that if  $(z_1, z_2) \in X_2$  and  $z_1$  approaches to 1 or to  $-1$ , then  $z_2$  increases indefinitely. From given boundary  $\gamma_2 = bX_2$  (see fig. 6), we need to reconstruct surface  $X_2$  as a finite number of points.

Choose  $R$  large enough, for example,  $R = 60$ , and consider circle  $C_R$  of radius  $R$  in the plane of variable  $z_2$  with center in zero. Calculate for  $-\xi_0 \in C_R$  the values of functions

$$\begin{aligned} G_1(\xi_0, 0) &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{z_1 dz_2}{z_2 + \xi_0}, \\ G_2(\xi_0, 0) &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{z_1^2 dz_2}{z_2 + \xi_0}, \\ \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) &= -\frac{1}{2\pi i} \int_{\gamma_2} \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2}, \\ \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) &= \frac{1}{2\pi i} \int_{\gamma_2} \left( \frac{z_1 dz_1}{z_2 + \xi_0} - \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2} \right), \\ \frac{\partial G_2}{\partial \xi_1}(\xi_0, 0) &= \frac{1}{2\pi i} \int_{\gamma_2} \left( \frac{z_1^2 dz_1}{z_2 + \xi_0} - \frac{z_1^3 dz_2}{(z_2 + \xi_0)^2} \right) \end{aligned}$$

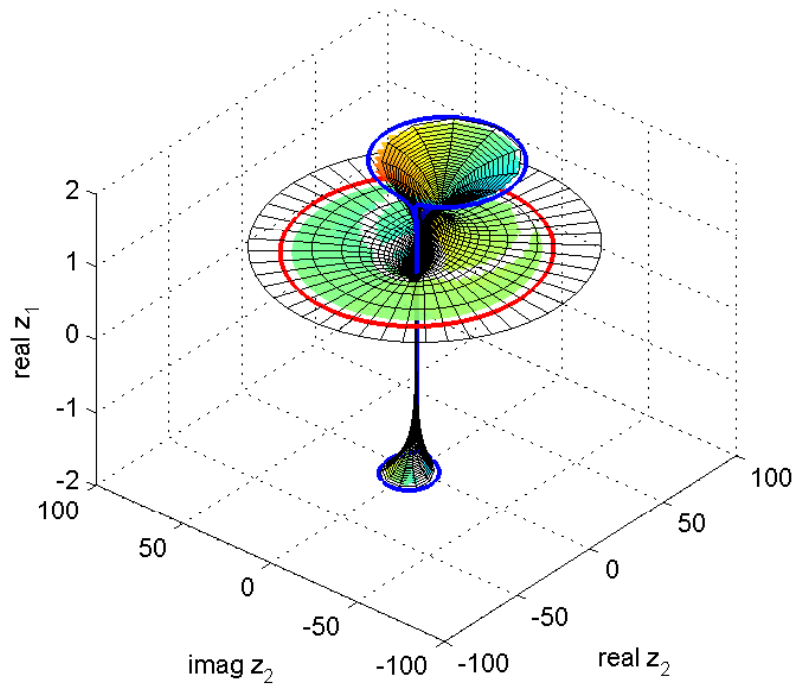


Figure 5: By black is represented contour of surface  $X_1$ , by blue —  $\gamma_1$ , by red — reconstructed curve  $\Gamma$ , belonging to  $X_1$ , colored domains — reconstructed leafs of surface  $X_1$ .

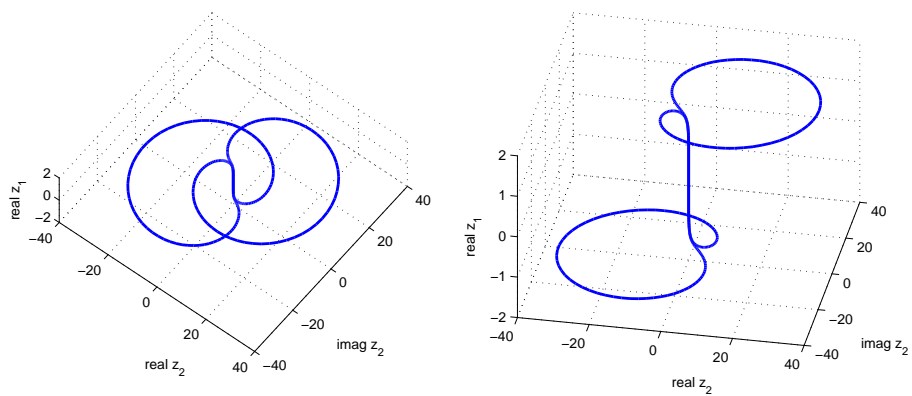


Figure 6: Boundary  $bX_2$  of surface  $X_2$  by which surface  $X_2$  is to reconstruct.

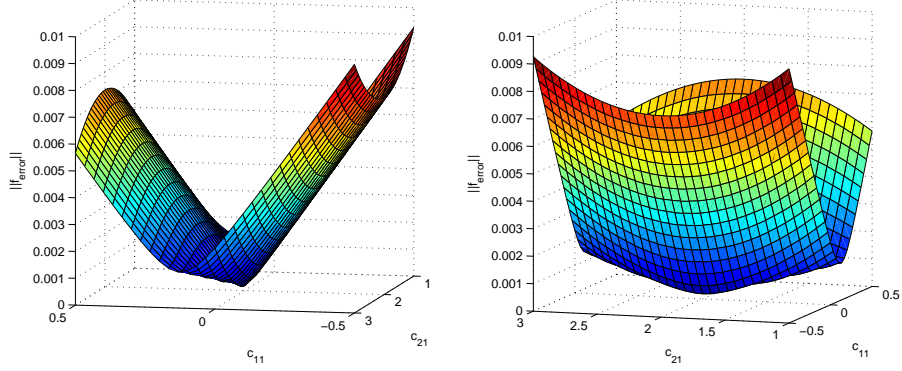


Figure 7: Graph of function  $\|f_{error}\|(c_{11}, c_{21})$  in the neighborhood of point of global minimum.

and the value of constant  $\varkappa = \frac{1}{2\pi i} \int_{\gamma_2} z_1 dz_2$ , for example, using the method of rectangles. We will solve numerically the problem of minimization of  $L_2(C_R)$ -norm of function

$$\begin{aligned}
f_{error}(\xi_0) = & 2(G_1 - c_{11} - c_{12}\xi_0) \frac{\partial G_1}{\partial \xi_1} - G_1(c_{22} + 2c_{23}\xi_0) - \frac{\partial G_2}{\partial \xi_1} - 3c_{12}^2\varkappa + \varkappa c_{23} + \\
& + (-1) \left( (G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \frac{\partial G_1}{\partial \xi_0} + \\
& + (-1) \left( G_1^2 - 2G_1c_{11} - 2G_1c_{12}\xi_0 - G_2 \right) c_{12},
\end{aligned}$$

in variables  $c_{11}, c_{12}, c_{21}, c_{22}, c_{23}$ . As a result of solving of minimization problem we find  $c_{11} = 0, c_{12} = 0, c_{21} = 2, c_{22} = 0, c_{23} = 0$  (see fig 7).

Let us calculate symmetric functions  $\sigma_1 = G_1 + c_{11} + c_{12}\xi_0, \sigma_2 = G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2$  and  $\sigma_0$  on the circle  $C_R$  and we do its desymmetrization, using Newton formulas. Value of  $\sigma_0(\xi_0)$  is a sum of  $\mu_0 = 2$  and index of point  $-\xi_0$  with respect to curve  $\gamma_2$ , which by definition is equal to  $G_0(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{dz_2}{z_2 + \xi_0}$ . As a result of desymmetrization we obtain functions  $h_1(\xi_0)$  and  $h_2(\xi_0)$  on the circle  $C_R$ , which realize lifting of circle  $C_R$  on the surface  $X_2$ . Denote by  $\Gamma_{1,2} = \{(h_{1,2}(z_2), -z_2) \mid z_2 \in C_R\}$  curves, obtained by corresponding lifting  $C_R$  on  $X_2$  (see fig. 8).

Now we consider the curve  $\gamma_2 + \Gamma_1 + \Gamma_2$  as a new initial curve and reconstruct the surface  $X_2^R = \{z \in X_2 \mid |z_2| \leq R\}$ . Further, our considerations are similar to those for the case  $\mu_0 = 1$ . We calculate, first, functions  $s_k$ , then symmetric functions  $\sigma_k$ , then we solve algebraic equation (numerically) and find functions  $h_k(\xi_0), k = 1, \dots, N_+(\xi_0)$ . Reconstructed surface is given by fig. 9.

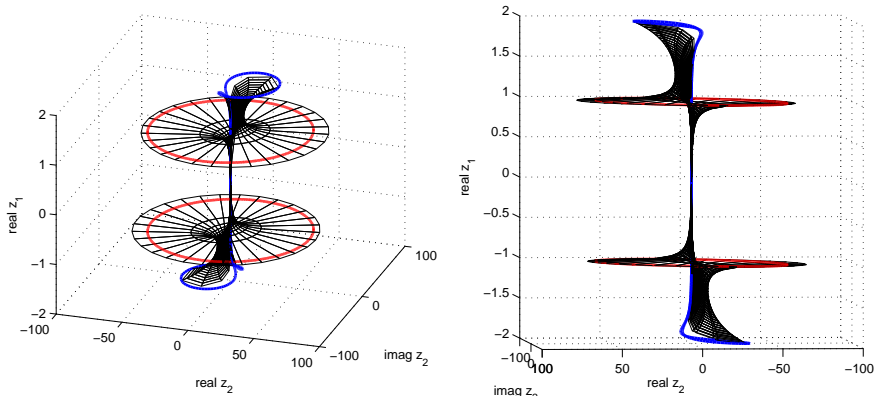


Figure 8: By black is represented contour of surface  $X_2$ , by blue —  $\gamma_2$ , by red — reconstructed curves  $\Gamma_{1,2}$ , belonging to  $X_2$ .

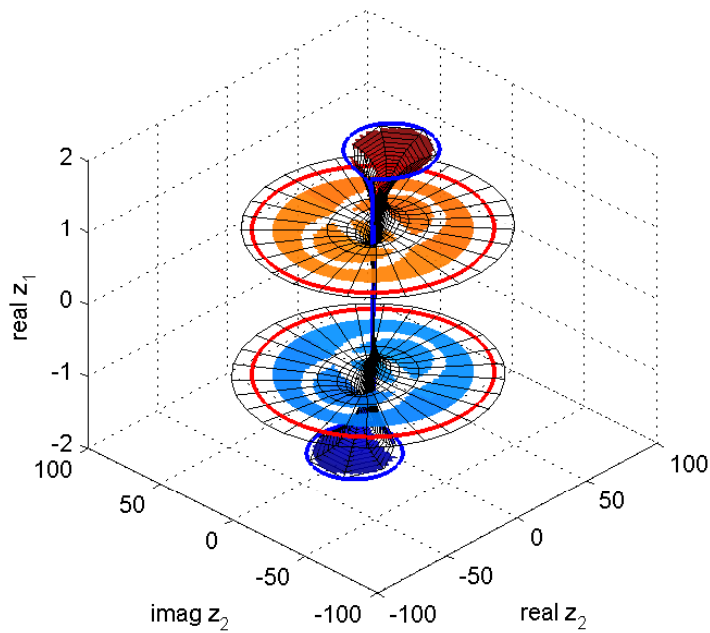


Figure 9: By black is represented contour of surface  $X_2$ , by blue —  $\gamma_2$ , by red — reconstructed curves  $\Gamma_{1,2}$ , belonging to  $X_2$ , colored by dark-blue, orange, red and blue domains — reconstructed leaves of surface.

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