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Abstract In this paper we propose the numerically realizable algorithm for reconstruction of complex curve with known boundary and without compact components in complex projective space.

Key words: Riemann surface, reconstruction algorithm, Burgers equation, Cauchy type formulas

1 Introduction.

Let us denote by $\mathbb{C}\mathrm{P}^2$ complex projective space with homogeneous coordinates $(w_0 : w_1 : w_2)$. Let real closed rectifiable, oriented curve γ in $\mathbb{C}\mathrm{P}^2$ be boundary of complex curve $X \subset \mathbb{C}\mathrm{P}^2$ with notation $\gamma = bX$. Without restriction of generality we suppose that the following conditions of general position be realized:

$$(0:1:0) \notin X, \quad w_l|_\gamma \neq 0, \quad l = 0, 1, 2. \tag{♯}$$

Put $\mathbb{C}^2 = \{w \in \mathbb{C}\mathrm{P}^2 \colon w_0 \neq 0\}$ with coordinates $z_1 = \frac{w_1}{w_0}$, $z_2 = \frac{w_2}{w_0}$. For almost all $\xi = (\xi_0, \xi_1) \in (\mathbb{C}^2)^*$ the points of intersection X with complex line $\mathbb{C}^1_\xi = \{z \in \mathbb{C}^2 \colon \xi_0 + \xi_1 z_1 + z_2 = 0\}$ form finite set of points

$$\big(z_1^{(j)}(\xi), z_2^{(j)}(\xi)\big) = \big(h_j(\xi_0, \xi_1), -\xi_0 - \xi_1 h_j(\xi_0, \xi_1)\big), \quad j = 1, \ldots, N_+(\xi).$$

The essential role in the reconstruction of X through γ plaies the following formula of Cauchy type from [2]:

$$G_m(\xi_0, \xi_1) \overset{\text{def}}{=\!=} \frac{1}{2\pi i} \int\limits_\gamma z_1^m (\xi_0 + \xi_1 z_1 + z_2)^{-1} d(\xi_0 + \xi_1 z_1 + z_2) = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi_0, \xi_1) + P_m(\xi_0, \xi_1), \quad m = 0, 1, \ldots, \tag{1}$$

where $N_+(\xi) = N_+(\xi_0, \xi_1)$ be the number of points of intersection (multiplicity taking into account) of X with complex line $\mathbb{C}^1_\xi$, $P_m(\xi_0, \xi_1)$ be polynomial of degree m with respect to ξ_0. In addition, $P_0(\xi_0, \xi_1) = -N_-$, where N_- is the number of points of intersection of X with infinity $\{w \in \mathbb{C}\mathrm{P}^2 \colon w_0 = 0\}$,

$$P_1(\xi_0, \xi_1) = \sum_{k=1}^{N_-(\xi)} \frac{a_k \xi_0 - b_k}{a_k \xi_1 + 1}, \tag{2}$$

$a_k = w_2(q_k)$, $b_k = \frac{dw_2}{dw_0}(q_k)$, where q_k, $k = 1, \ldots, N_-$, — be points of intersection of X with infinity $\{w : w_0 = 0\}$. In particular, the following corollary of (1) takes place:

$$G_0(\xi_0, \xi_1) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{d(\xi_0 + \xi_1 z_1 + z_2)}{\xi_0 + \xi_1 z_1 + z_2} = N_+(\xi) - N_-. \tag{3}$$

Let further $\xi_1 = 0$ and let $\pi_2 \colon \mathbb{C}^2 \to \mathbb{C}$ — be projection: $\pi_2(z_1, z_2) = z_2$. We have $\pi_2\gamma \subset \mathbb{C}$, $\mathbb{C} \setminus \pi_2\gamma = \cup_{l=0}^{L} \Omega_l$, where $\{\Omega_l\}$ — be connected component of $\mathbb{C} \setminus \pi_2\gamma$. For every component Ω_l the number of points of intersection of X with line $z_2 = \xi_0$, $\xi_0 \in \Omega_l$, multiplicity taking into account will be denoted by $\mu_l = N_+(\xi_0, 0)$. Let Ω_0 denotes unbounded component of set $\mathbb{C} \setminus \pi_2\gamma$. From definition of $N_\pm$ it follows the formula

$$\mu_0 = N_+(\xi_0, 0) = N_-, \quad \xi_0 \in \Omega_0.$$

Assume that complex curve X does not contain compact components , or equivalently, satisfies the following condition of minimality from the work [4]:

For arbitrary complex curve $\widetilde{X} \subset \mathbb{C}\mathrm{P}^2$ with condition $b\widetilde{X} = bX = \gamma$ and for almost all $\xi \in (\mathbb{C}^2)^*$ the number of points of intersection $\widetilde{N}_+(\xi)$ of $\widetilde{X}$ with line $\mathbb{C}^1_\xi$, multiplicity taking into account, is not less than the number $N_+(\xi)$ for curve X. (∗)

Condition of minimality (∗) is condition of general position and is fulfilled for X, if, for example, every irreducible component of X is transcendental complex curve. Note, that from theorems of Chow [1], [2] and Harvey, Shiffman [3], it follows that arbitrary complex curve $X \subset \mathbb{C}\mathrm{P}^2$ with condition $b\widetilde{X} = bX$ admits unique representation $\widetilde{X} = X \cup V$, where X — is curve with condition of minimality (∗), and V — is compact algebraic curve, possibly with multiple components.

In this work in the development of [2] we obtained the numerically realizable algorithm for reconstruction of complex curve $X \subseteq \mathbb{C}\mathrm{P}^2$ with known boundary and with condition of minimality. This algorithm permits, in particular, to make applicable the result of [4] about principal possibility to reconstruct topology and conformal structure of two-dimensional surface X in $\mathbb{R}^3$ with constant scalar conductivity from measurements on bX of electric current densities, being created by three potentials in general position.

Our algorithm depends of parameter $\mu_0 = N_\pm(\xi_0, 0)$, $\xi_0 \in \Omega_0$, verified for many examples and rigorously justified for $\mu_0 = 0, 1, 2$. The difficulty of justification increases with increasing of μ_0.

2 Cauchy type formulas and Riemann-Burgers equations.

Let us give at first a new proof of Cauchy type formula (1) from [2], permitting to obtain explicit expressions for functions $P_m(\xi_0, \xi_1)$.

Theorem 2.1. *Let $X \subset \mathbb{C}\mathrm{P}^2$ be complex curve without compact components , $\gamma = bX$ be real rectifiable oriented curve. Let conditions of general position ($\sharp$) be fulfilled and all points of intersection of X and $\mathbb{C}\mathrm{P}^1_\xi$, for almost all $\xi \in \mathbb{C}^2$, have multiplicity one and their coordinate w_2 is not equal to zero. Then following equalities are fulfilled*

$$G_m(\xi_0, \xi_1) = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi_0, \xi_1) + P_m(\xi_0, \xi_1), \tag{4}$$

where $P_m(\xi_0, \xi_1)$ is polynomial of degree at most m with respect to ξ_0, having the form

$$P_m(\xi_0, \xi_1) =$$
$$\sum_{s=1}^{\mu_0} \sum_{k=0}^{m-1} \sum_{i_1+\cdots+i_m=k} \frac{\frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)!} \frac{d^{m-k}}{dw_0^{m-k}} \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s} -$$
$$\sum_{s=1}^{\mu_0} \sum_{i_1+\cdots+i_m=m} \frac{d^{i_1} w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m} w_1}{dw_0^{i_m}}(q_s).$$

In particular, if $\mu_0 = 0$, then $P_m \equiv 0$.

Proof. Put $\widetilde{g} = \xi_0 w_0 + \xi_1 w_1 + w_2$ and $g = \frac{\widetilde{g}}{w_0} = \xi_0 + \xi_1 z_1 + z_2$. Consider differential forms

$$\omega_m \stackrel{\text{def}}{=\!=} z_1^m \frac{dg}{g} = \frac{w_1^m}{w_0^m} \frac{w_0}{\widetilde{g}} d\left(\frac{\widetilde{g}}{w_0}\right) = \frac{w_1^m}{w_0^m} \frac{d\widetilde{g}}{\widetilde{g}} - \frac{w_1^m}{w_0^{m+1}} dw_0, \quad m = 0, 1, \ldots$$

Then $G_m(\xi) = \frac{1}{2\pi i} \int_\gamma \omega_m$. Let us calculate this integral explicitly. Denote by p_j, $j = 1, \ldots, N_+(\xi)$ the points of intersection of X and $\mathbb{C}\mathrm{P}^1_\xi$, and by q_s, $s = 1, \ldots, \mu_0$ the points of intersection of X with infinity $\{w_0 = 0\}$. Denote by B_j^ε intersections of X with ball of radius ε in $\mathbb{C}\mathrm{P}^2$ with center in p_j and by D_s^ε intersections of X with ball of radius ε with center in q_s. The restriction of form ω_m on X is meromorphic form with poles in points p_j and q_s. Thus the following equality is valid

$$G_m(\xi) = \frac{1}{2\pi i} \int_\gamma \omega_m = \sum_{j=1}^{N_+(\xi)} \frac{1}{2\pi i} \int_{bB_j^\varepsilon} \omega_m + \sum_{s=1}^{\mu_0} \frac{1}{2\pi i} \int_{bD_s^\varepsilon} \omega_m.$$

If $\mu_0 = 0$, then the second group of terms is absent. The integral $\int_{bB_j^\varepsilon} \omega_m$ can be calculated as residue of the first order pole:

$$\int_{bB_j^\varepsilon} \omega_m = \int_{bB_j^\varepsilon} z_1^m \frac{d\widetilde{g}}{\widetilde{g}} - \int_{bB_j^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = \int_{bB_j^\varepsilon} z_1^m \frac{d\widetilde{g}}{\widetilde{g}} = 2\pi i \, h_j^m(\xi).$$

Let $\mu_0 > 0$. Calculation of integral $\int_{bD_s^\varepsilon} \omega_1$ will be done in two steps. Let us calculate first $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0$. Consider expansion of $w_1(w_0)$ into power series in w_0 in the neighborhood of point q_s:

$$w_1(w_0) = w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2w_1}{dw_0^2}(q_s)w_0^2 + \cdots .$$

Note further that

$$w_1^m(w_0) = \sum_{k=0}^{\infty} \sum_{i_1+\cdots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^k.$$

The coefficient near w_0^m can be presented in the form

$$\int\limits_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = 2\pi i \sum_{i_1+\cdots+i_m=m} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s).$$

Now we can calculate the integral $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\widetilde{g}}{\widetilde{g}}$. Using relation $d\widetilde{g} = \frac{d\widetilde{g}}{dw_0} dw_0$ and expension of $w_1(w_0)$ into power series in w_0 we obtain:

$$\int\limits_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\widetilde{g}}{\widetilde{g}} = \int_{bD_s^\varepsilon} \frac{1}{w_0^m} \left(w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2w_1}{dw_0^2}(q_s)w_0^2 + \cdots \right)^m \frac{d\widetilde{g}}{dw_0} \frac{1}{\widetilde{g}} dw_0 =$$

$$\sum_{k=0}^{\infty} \int_{bD_s^\varepsilon} \sum_{i_1+\cdots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\widetilde{g}}{dw_0} \frac{1}{\widetilde{g}} dw_0 =$$

$$\sum_{k=0}^{m-1} \int_{bD_s^\varepsilon} \sum_{i_1+\cdots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\widetilde{g}}{dw_0} \frac{1}{\widetilde{g}} dw_0 =$$

$$\sum_{k=0}^{m-1} \frac{2\pi i}{(m-k-1)!} \sum_{i_1+\cdots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) \lim_{w_0 \to 0} \frac{d^{m-k-1}}{dw_0^{m-k-1}} \left(\frac{d\widetilde{g}}{dw_0} \frac{1}{\widetilde{g}} \right).$$

From here, taking into account relation $\frac{d\widetilde{g}}{dw_0} \frac{1}{\widetilde{g}} = \frac{d \ln \widetilde{g}}{dw_0}$, we obtain finally

$$\int\limits_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\widetilde{g}}{\widetilde{g}} == 2\pi i \sum_{k=0}^{m-1} \sum_{i_1+\cdots+i_m=k} \frac{\frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)!} \frac{d^{m-k}}{dw_0^{m-k}} \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s}.$$

It is polynomial of degree at most m with respect to ξ_0. □

Before considering the algorithm for reconstruction of complex curve we need to prove one more theorem, which connects among themselves different functions $h_j(\xi_0, \xi_1)$, $j = 1, \ldots, N_+(\xi_0, \xi_1)$. Denote by $\Omega_l^{(k)}$ the infinitesimal neighborhood of order k of the set Ω_l.

Theorem 2.2. *Let $X \subset \mathbb{C}\mathrm{P}^2$ be complex curve without compact components, $\gamma = bX$, and between points of intersection of X with infinity $\{w_0 = 0\}$ there is no point $(0 : 1 : 0)$.*

Then for existence and uniqueness of mutually distinct holomorphic in $\xi \in \Omega_l^{(1)}$ functions $\widehat{h}_j(\xi)$ and holomorphic functions $\widehat{P}_m(\xi)$, being polynomials of degree at most m in ξ_0, satisfying to system

$$G_m(\xi) = \sum_{j=1}^{N_+(\xi)} \widehat{h}_j^m(\xi) + \widehat{P}_m(\xi), \quad m, j = 1, \ldots, N_+(\xi),\ l = 0, 1, \ldots, L, \qquad (5)$$

it is necessary and sufficient the existence and uniqueness of mutually distinct holomorphic in $\xi \in \Omega_l^{(1)}$ functions $\widetilde{h}_j(\xi)$, satisfying to equation $\frac{\partial^2}{\partial \xi_0^2}\big(G_1(\xi) - \sum_{j=1}^{N_+(\xi)} \widetilde{h}_j(\xi)\big) = 0$ and to the system of Riemann-Burgers equations:

$$\frac{\partial \widetilde{h}_j}{\partial \xi_1} = \widetilde{h}_j(\xi) \frac{\partial \widetilde{h}_j}{\partial \xi_0}, \quad j = 1, \ldots, N_+(\xi), \quad \xi \in \Omega_l^{(1)}, \quad l = 0, 1, \ldots, L.$$

Moreover, functions $\{\widetilde{h}_j\}$, $\{\widehat{h}_j\}$, $\{\widehat{P}_m\}$, satisfying these systems, exist and $\widetilde{h}_j = \widehat{h}_j$, $j = 1, \ldots, N_+(\xi)$.

Proof. Proof. Necessity. Let there exist and unique up to order mutually distinct holomorphic in $\xi \in \Omega_l^{(1)}$ functions $\{\widehat{h}_j(\xi)\}$, $j = 1, \ldots, N_+(\xi_0, 0)$, $l = 0, 1, \ldots, L$, and holomorphic functions $\{\widehat{P}_m(\xi)\}$, being polynomials of degree at most m in ξ_0, and satisfying system (5), $m = 1, \ldots, N_+(\xi)$. By theorem II from [2] and theorem 3 from [4] for almost all $\xi^* = (\xi_0^*, 0)$ in the neighborhood of ξ^* there exist and unique up to order mutually distinct holomorphic solutions $\{\widetilde{h}_j(\xi)\}$ of Riemann-Burgers equation:

$$\frac{\partial \widetilde{h}_j(\xi)}{\partial \xi_1} = \widetilde{h}_j(\xi) \frac{\partial \widetilde{h}_j(\xi)}{\partial \xi_0},$$

such that $\frac{\partial^2}{\partial \xi_0^2}\big(G_1(\xi) - \sum_j \widetilde{h}_j(\xi)\big) = 0$, $j = 1, \ldots, N_+(\xi^*)$. Proposition 3.3.3 from [2] ensure that functions $\{\widetilde{h}_j\}$ satisfy to system (5). From uniqueness of families of functions $\{\widehat{h}_j(\xi)\}$ and $\{\widetilde{h}_j(\xi)\}$ it follows that these functions satisfy simultaneously to Riemann-Burgers equation.

Sufficiency. Suppose, that there exist and unique up to order mutually distinct holomorphic functions $\{h_j(\xi)\}$ on $\Omega_l^{(1)}$, $l = 0, 1, \ldots, L$, satisfying to Riemann-Burgers equation and also to the following equation

$$\frac{\partial^2}{\partial \xi_0^2}\big(G_1(\xi) - \sum_j h_j(\xi)\big) = 0, \quad \xi \in \Omega_l^{(1)}.$$

In particular, for any $\xi_0 \in \Omega_l^{(0)}$ we have relations

$$\frac{\partial h_j}{\partial \xi_1}(\xi_0, 0) = h_j(\xi_0, 0) \frac{\partial h_j}{\partial \xi_0}(\xi_0, 0).$$

By Cauchy-Kowalewski theorem for any $l = 0, 1, \ldots, L$ in a neighborhood of arbitrary $\xi^* \in \Omega_l^{(1)}$ there exist holomorphic functions $\{\widetilde{h}_j(\xi_0, \xi_1)\}$, satisfying to Riemann-Burgers equation and such that $\tilde{h}_j|_{\Omega_l^{(1)}} = h_j$.

From here and Proposition 3.3.3 from [2] we obtain existence and uniqueness of holomorphic functions $\{P_m(\xi)\}$, being polynomials of degree at most m in ξ_0, such that $\{h_j(\xi)\}$ and $\{P_m(\xi)\}$ satisfy to system (5) for $m, j = 1, \ldots, N_+(\xi)$, $l = 0, 1, \ldots, L$, $\xi \in \Omega_l^{(1)}$.

Existence and uniqueness of functions $\{\widetilde{h}_j\}$ with necessary properties follow from [2], theorem II, and from [4], theorem 3. From here and from already proved sufficiency it follows the existence and uniqueness of functions $\{\widehat{h}_j\}$ and $\{\widehat{P}_m(\xi)\}$ with property (5). Equalities $\widehat{h}_j = \widetilde{h}_j$, $j = 1, \ldots, N_+(\xi)$, follow from property of functions $\{\widetilde{h}_j\}$ to satisfy (5) and from uniqueness of functions satisfying to (5). □

3 Reconstruction algorithm.

Consider now the reconstruction algorithm of complex curve $X \subseteq \mathbb{C}\mathrm{P}^2$ with given boundary bX and with condition of minimality $(*)$. Let us consider the cases $\mu_0 = 0, 1, 2$.

The reconstruction algorithm is based on formulas (4) with polynomials P_m, $m = 0, 1, \ldots$. The next theorem permits to calculate these polynomials.

Theorem 3.1. *Let $\gamma = bX$, $X \subseteq \mathbb{C}\mathrm{P}^2$ be complex curves without compact components, $\gamma \subseteq \mathbb{C}^2$ be its boundary. Let mutually distinct holomorphic in $\xi \in \Omega_l^{(1)}$, $l = 0, 1, \ldots, L$, functions $\{h_j(\xi)\}$ and polynomials of degree at most m in ξ_0 $P_m(\xi_0, \xi_1)$ with holomorphic coefficients satisfy to system (4) for $\xi \in \Omega_l^{(1)}$, $j = 1, \ldots, N_+(\xi)$ (existence and uniqueness of such functions follow from theorem of 2.2). Then the following statements are valid :*

1. *If $\mu_0 = 0$, then $P_m(\xi_0, 0) \equiv 0$ for all m,*
2. *If $\mu_0 = 1$, then $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$, where constants c_{11} and c_{12} can be found from identity in $\xi_0 \in \Omega_0$:*

$$c_{11}\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) + c_{12}\left(\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) + G_1(\xi_0, 0)\right) = G_1(\xi_0, 0)\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0).$$

3. *If $\mu_0 = 2$, then $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$, $P_2(\xi_0, 0) = c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2$, where constants c_{11}, c_{12}, c_{21}, c_{22}, c_{23} can be found from identity in*

$\xi_0 \in \Omega_0$:

$$2\big(G_1 - c_{11} - c_{12}\xi_0\big)\frac{\partial G_1}{\partial \xi_1} - G_1\big(c_{22} + 2c_{23}\xi_0\big) - \frac{\partial G_2}{\partial \xi_1} - 3c_{12}^2 æ + æ c_{23} =$$
$$= \bigg(\big(G_1 - c_{11} - c_{12}\xi_0\big)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2\bigg)\frac{\partial G_1}{\partial \xi_0} +$$
$$+ \bigg(G_1^2 - 2G_1c_{11} - 2G_1c_{12}\xi_0 - G_2\bigg)c_{12},$$

where $æ = \frac{1}{2\pi i}\int_\gamma z_1\, dz_2$.

Proof. By theorem 2.2 functions P_m from condition of theorem coincide with functions P_m from theorem 2.1.

1. By theorem 2.1 $P_m \equiv 0$, if $\mu_0 = 0$.

2. For $\xi \in \Omega_l^{(0)}$ we have equality $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$. We need to find constants $c_{11} = C_{11}(0)$ and $c_{12} = C_{12}(0)$. Let us write out the restriction of equation (4) for $\xi \in \Omega_l^{(1)}$ and equation derived by differentiation of (4) in ξ_0 and by restriction on $\xi \in \Omega_l^{(0)}$:

$$\begin{aligned} h_1(\xi_0, 0) &= G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0, \\ \frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0, \\ \frac{\partial h_1}{\partial \xi_0}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12}, \end{aligned}$$

where $\xi_0 \in \Omega_0$. By theorem 2.2 for $\xi_0 \in \Omega_0$ function $h_1(\xi_0, 0)$ satisfies to equality $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) = h_1(\xi_0, 0)\frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$. If we substitute in this equality expressions for $h_1(\xi_0, 0)$, $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0)$ and $\frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$, we obtain equation

$$\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 = \bigg(G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0\bigg)\bigg(\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12}\bigg). \tag{6}$$

This equation is valid for $\xi_0 \in \Omega_0$. Let us divide it into ξ_0 and tend $\xi_0 \to \infty$. We obtain equality $\dot{C}_{12}(0) = -c_{12}^2$. Taking into account this equality, the equation (6) may be rewritten in the form

$$\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) = \bigg(G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0\bigg)\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \bigg(G_1(\xi_0, 0) - c_{11}\bigg)c_{12}. \tag{7}$$

Taking into account that $\xi_0\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) \to 0$ for $\xi_0 \to \infty$, passage to the limit in (7) when $\xi_0 \to \infty$ gives equality $\dot{C}_{11}(0) = -c_{11}c_{12}$. Using just obtained equality, equation (7) takes the required form.

3. By theorem 2.2 functions $h_1(\xi)$ and $h_2(\xi)$ satisfy to Riemann-Burgers

equation for $\xi \in \Omega_l^{(1)}$. So, the following equalities are valid.

$$\frac{\partial(h_1h_2)}{\partial\xi_1} = h_1\frac{\partial h_2}{\partial\xi_1} + \frac{\partial h_1}{\partial\xi_1}h_2 = h_1h_2\frac{\partial(h_1+h_2)}{\partial\xi_0}, \tag{8}$$

$$\frac{\partial(h_1^2+h_2^2)}{\partial\xi_0} = 2h_1\frac{\partial h_1}{\partial\xi_0} + 2h_2\frac{\partial h_2}{\partial\xi_0} = 2\frac{\partial(h_1+h_2)}{\partial\xi_1}. \tag{9}$$

Note that $h_1h_2 = \frac{1}{2}\big(h_1+h_2\big)^2 - \frac{1}{2}\big(h_1^2+h_2^2\big)$. Therefore the system (8)–(9) is equivalent to the system

$$\frac{\partial(h_1+h_2)^2}{\partial\xi_1} - \frac{\partial(h_1^2+h_2^2)}{\partial\xi_1} = \Big(\big(h_1+h_2\big)^2 - \big(h_1^2+h_2^2\big)\Big)\frac{\partial(h_1+h_2)}{\partial\xi_0}, \tag{10}$$

$$\frac{\partial(h_1^2+h_2^2)}{\partial\xi_0} = 2\frac{\partial(h_1+h_2)}{\partial\xi_1}. \tag{11}$$

We substitute into this system $h_1^2+h_2^2$ and h_1+h_2 from equation (4), using notations $P_1(\xi_0,\xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$, $P_2(\xi_0,\xi_1) = C_{21}(\xi_1) + C_{22}(\xi_1)\xi_0 + C_{23}(\xi_1)\xi_0^2$. Equation (11) restricted on $\Omega_l^{(0)}$ takes the form

$$\frac{\partial G_2}{\partial\xi_0}(\xi_0,0) - c_{22} - 2c_{23}\xi_0 = 2\Big(\frac{\partial G_1}{\partial\xi_1}(\xi_0,0) - \dot C_{11}(0) - \dot C_{12}(0)\xi_0\Big).$$

Divide this equation into ξ_0 and tend $\xi_0\to\infty$. We obtain the equality $\dot C_{12}(0) = c_{23}$. In this equation, using last equality, in the passage to the limit for $\xi_0\to\infty$ we obtain the equality $\dot C_{11}(0) = \frac{1}{2}c_{22}$.

Now substitute into (10) expression for $h_1^2+h_2^2$ and h_1+h_2 and restrict obtained formula to $\Omega_l^{(0)}$. We obtain the equality

$$2\Big(G_1 - c_{11} - c_{12}\xi_0\Big)\Big(\frac{\partial G_1}{\partial\xi_1} - \dot C_{11}(0) - \dot C_{12}(0)\xi_0\Big) - \frac{\partial G_2}{\partial\xi_1} + \dot C_{21}(0) + \dot C_{22}(0)\xi_0 + \dot C_{23}(0)\xi_0^2 =$$
$$= \Big(\big(G_1 - c_{11} - c_{12}\xi_0\big)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2\Big)\Big(\frac{\partial G_1}{\partial\xi_0} + c_{12}\Big). \tag{12}$$

Divide this equation into ξ_0^2 and in passage to the limit for $\xi_0\to\infty$ we obtain the equality

$$2c_{12}\dot C_{12}(0) + \dot C_{23}(0) = \big(c_{12}^2 + c_{23}\big)c_{12}.$$

Using last equality, divide (12) into ξ_0 and in passage to the limit for $\xi_0\to\infty$ we obtain the equality

$$2c_{11}\dot C_{12}(0) + 2c_{12}\dot C_{11}(0) + \dot C_{22}(0) = \big(-2c_{11} + c_{22}\big)c_{12}.$$

Taking into account obtained equalities one can rewrite (12) in the form

$$2\big(G_1-c_{11}-c_{12}\xi_0\big)\frac{\partial G_1}{\partial\xi_1}-2G_1\big(\dot C_{11}(0)+\dot C_{12}(0)\xi_0\big)+2c_{11}\dot C_{11}(0)-\frac{\partial G_2}{\partial\xi_1}+\dot C_{21}(0)=$$
$$=\bigg(\big(G_1-c_{11}-c_{12}\xi_0\big)^2-G_2+c_{21}+c_{22}\xi_0+c_{23}\xi_0^2\bigg)\frac{\partial G_1}{\partial\xi_0}+$$
$$+\bigg(\big(G_1-c_{11}\big)^2-2G_1c_{12}\xi_0-G_2+c_{21}\bigg)c_{12}. \quad (13)$$

Take $\xi_0\to\infty$ in this equality and note that the following relations are valid

$$\lim_{\xi_0\to\infty}\xi_0\frac{\partial G_1}{\partial\xi_1}=\lim_{\xi_0\to\infty}\xi_0\frac{1}{2\pi i}\int\limits_\gamma\frac{z_1\,dz_1}{\xi_0+z_2}=\frac{1}{2\pi i}\int\limits_\gamma z_1\,dz_1=\langle b\gamma,\frac{1}{4\pi i}z_1^2\rangle=0,$$
$$\lim_{\xi_0\to\infty}\xi_0G_1=\lim_{\xi_0\to\infty}\xi_0\frac{1}{2\pi i}\int\limits_\gamma\frac{z_1\,dz_2}{\xi_0+z_2}=\frac{1}{2\pi i}\int\limits_\gamma z_1\,dz_2,$$
$$\lim_{\xi_0\to\infty}\xi_0^2\frac{\partial G_1}{\partial\xi_0}=-\lim_{\xi_0\to\infty}\xi_0^2\frac{1}{2\pi i}\int\limits_\gamma\frac{z_1\,dz_2}{(\xi_0+z_2)^2}=-\frac{1}{2\pi i}\int\limits_\gamma z_1\,dz_2.$$

We obtain

$$-2æ\dot C_{12}(0)+2c_{11}\dot C_{11}(0)+\dot C_{21}(0)=-\big(c_{12}^2+c_{23}\big)æ+\big(c_{11}^2-2c_{12}æ+c_{21}\big)c_{12}.$$

In obtained equations we express constants $\dot C_{ij}(0)$ through c_{ij}:

$$\dot C_{11}(0)=\frac{1}{2}c_{22},$$
$$\dot C_{12}(0)=c_{23},$$
$$\dot C_{23}(0)=c_{12}^3-c_{12}c_{23},$$
$$\dot C_{22}(0)=-2c_{11}\big(c_{12}+c_{23}\big),$$
$$\dot C_{21}(0)=\big(c_{11}^2-3c_{12}æ+c_{21}\big)c_{12}+æc_{23}-c_{11}c_{22}.$$

Substituting these constants to (13), we obtain the third statement of theorem 3.1. □

Let us describe algorithm of reconstruction of complex curve X in $\mathbb{C}P^2$ without compact components (satisfying minimality condition $(*)$). As above, $\gamma=bX$ is compact real curve.

Algorithm for reconstruction of curve X permits, instead of original curves, to find curve coinciding with original curve in the finite number of points and obtained by interpolation in other points. Let $\{\xi^k\}_{k=1}^N$, $\xi^k\in\mathbb{C}$ be arbitrary frame on $\mathbb{C}$, $\xi^i\neq\xi^j$, $i\neq j$, and $\xi^k\notin\pi_2\gamma$, $k=1,\ldots,N$. Complex curve X intersects complex line $\{z_2=\xi^k\}$ in $N_+(\xi_k,0)$ points. Algorithm allows to find these points.

For all algorithms in input we have points $\{\xi^k\}_{k=1}^N$ and curve γ (for example, as finite number of points belonging to γ). In output we have a set of points $(h_s(\xi^k,0),\xi^k)$, $k=1,\ \ldots,\ N$; $s=1,\ \ldots,\ N_+(\xi^k,0)$, belonging to complex curve X.

The case of $\mu_0=0$.

1. *Calculation of* μ_l. By formula (3) for every domain Ω_l, $l=1,\ \ldots,\ L$, number μ_l is equal to index of curve $\pi_2\gamma$ with respect to $\xi_0\in\Omega_l$:

$$\mu_l\equiv N_+(\xi_0,0)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\mathrm{d}z_2}{z_2-\xi_0}\equiv\frac{1}{2\pi i}\int\limits_{\pi_2\gamma}\frac{\mathrm{d}z}{z-\xi_0},\quad \xi_0\in\Omega_l.$$

2. *Calculation of power sums.* For the case $\mu_0=0$ for every point $\xi^k\in\Omega_l$, $l=1,\ \ldots,\ L$, by theorem 3.1 we have equalities $P_m\equiv 0$. By formula (4) we have the following formulas for power sums:

$$s_m(\xi^k)\equiv h_1^m(\xi^k,0)+\cdots+h_{\mu_l}^m(\xi^k,0)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{z_1^m\,dz_2}{z_2-\xi^k},\quad m=\overline{1,\mu_l}.$$

   By theorem 2.1 the points $(h_s(\xi^k,0),\xi^k)$, $s=1,\ \ldots,\ N_+(\xi^k,0)$; $k=1,\ \ldots,\ N$, are required points of X.

3. *Calculation of symmetric functions.* For every point $\xi^k\in\Omega_l$, $l=1,\ \ldots,\ L$, the Newton formulas

$$k\sigma_k(\xi^k)=\sum_{i=1}^{k}(-1)^{i-1}\sigma_{k-i}(\xi^k)s_i(\xi^k),\quad k=1,\ldots,N_+(\xi^k).$$

   allow to reconstruct elementary symmetric functions:

$$\begin{aligned}\sigma_1(\xi^k)&=h_1(\xi^k,0)+\cdots+h_{\mu_l}(\xi^k,0),\\ &\cdots=\cdots\\ \sigma_{\mu_l}(\xi^k)&=h_1(\xi^k,0)\times\cdots\times h_{\mu_l}(\xi^k,0).\end{aligned}$$

4. *Desymmetrisation.* For any point $\xi^k\in\Omega_l$ using Vieta formulas one can find numbers $h_1(\xi^k,0),\ \ldots,\ h_{\mu_l}(\xi^k,0)$. Required points $(h_s(\xi^k,0),\xi^k)$, $s=1,\ \ldots,\ N_+(\xi^k,0)$; $k=1,\ \ldots,\ N$, of complex curve X are obtained.

The cases for $\mu_0=1,\ 2$. These cases are reduced to the case $\mu_0=0$ by the following way. Since $\pi_2\gamma\subset\mathbb{C}$ is compact real curve, there exists such $R>0$, that the set $B_R^c(0)=\{z\in\mathbb{C}\mid |z|\geqslant R\}$ belongs to Ω_0. Without restriction of generality, one can suppose that $|\xi^k|<R$ for all $k=1,\ \ldots,\ N$. Otherwise one can increase R.

Let us define auxiliary complex curve $X_R=\{(z_1,z_2)\in X\mid |z_2|\leqslant R\}$. Its boundary γ_R consists in two disjoint parts (possibly, multiconnected): γ and

real curve , obtained by lifting of the circle $S_R = \{z \in \mathbb{C} \mid |z| = R\}$ on surface X, by inversion of projection $\pi_2 \colon X \to \mathbb{C}$. Complex curve X_R does not intersect infinity and, as a consequence $\mu_0(X_R) = 0$. Moreover, any point $(z_1(\xi^k), \xi^k)$, $k = 1, \ldots, N$, belongs to X iff it belongs to X_R. Therefore, for reconstruction of complex curve X_R it is sufficient to reconstruct, at first, real curve obtained by lifting of S_R on X and after to solve reconstruction problem, as in the case $\mu_0 = 0$. Finally, we come to the following algorithm.

1. *New boundary.* We choose constant R so large, that the exterior of the disk of radius R with center in zero belong to Ω_0 and all ξ^k belong to this disk. Denote boundary of this disk by S_R. In the case $\mu_0 = 1$ for points $\xi_0 \in S_R$ by formula (4) we have equality

$$h_1(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_\gamma \frac{z_1\, dz_2}{z_2 - \xi_0} + P_1(\xi_0, 0).$$

   In the case $\mu_0 = 2$ we have two equalities:

$$h_1(\xi_0, 0) + h_2(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_\gamma \frac{z_1\, dz_2}{z_2 - \xi_0} + P_1(\xi_0, 0),$$

$$h_1^2(\xi_0, 0) + h_2^2(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_\gamma \frac{z_1^2\, dz_2}{z_2 - \xi_0} + P_2(\xi_0, 0),$$

   where polynomials can be found from theorem 3.1. In the case $\mu_0 = 1$ by lifting the curve S_R on X we obtain at once the real curve of the form $\{(h_1(\xi_0, 0), \xi_0) \mid \xi_0 \in S_R\}$. In the case $\mu_0 = 2$ we have to apply, at first, Newton formula and after Vieta formula in order from functions $h_1 + h_2$ and $h_1^2 + h_2^2$ to obtain h_1 and h_2.

2. *Reduction.* For finding of complex curve X_R with boundary $bX_R = \gamma_R$ we apply algorithm of reconstruction for the case $\mu_0 = 0$. The points obtained in such a way will be those, which we need due to explanation before algorithm.

4 Visualization.

Let us write a little bit about algorithm of visualization of Riemann surface by selection of appropriate finite number of points. For illustration of this algorithm we consider visualization of Riemann surface of the function

$$f(z) = \sqrt{\exp\left(\frac{z}{4}\right) + \sqrt{z^2 + 1}}, \quad |z| \leqslant 2.$$

Note, firstly, that Riemann surface of this function has six algebraic points of ramification of index 2 in the domain $\{|z| \leqslant 2\}$, two of which correspond to

points $z = i$ and $z = -i$ of nulling of argument of interior radical, and four others correspond to points of nulling of argument of exterior radical.

We assume that we have grid $\{z_1^{ij}\}$ in variable z_1 and a set of points $\{(z_1^{ij}, z_2^{ij})\}$ on Riemann surface $X = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = f(z_1),\ |z_1| \leqslant 2\}$ of function $f(z)$, where $1 \leqslant i \leqslant N_1$, $1 \leqslant j \leqslant N_2$, and to neighbor values of i, j correspond neighbor points of grid. Suppose, that for each point z_1^{ij} the number of corresponding points (z_1^{ij}, z_2^{ij}) is equal to number of leafs of Riemann surface X. Moreover, one can accept that some point z_1^{ij} may be a projection of some point of ramification on X, permitting also multiple points (z_1^{ij}, z_2^{ij}). Suppose also, that points of ramification can be only interior points of the grid $\{z_1^{ij}\}$. We will solve further the problem of visualization in `Matlab`.

1. **Finding of ramification points.** The first step of visualization algorithm consists in finding of ramification points. Choose an arbitrary point $(z_1^{i_0 j_0}, z_2^{i_0 j_0})$ such, that point $z_1^{i_0 j_0}$ is interior point of the grid. Let us go around the point $z_1^{i_0 j_0}$ along discrete circle of sufficiently small radius with center in $z_1^{i_0 j_0}$ (such that inside of the circle would not be more than one projection of ramification point of X). Choose an arbitrary point from the grid $\{z_1^{i_1 j_1}, z_2^{i_1 j_1}\}$, corresponding to the first point $z_1^{i_1 j_1}$ of this circuit and let us in the process of circuit associate with current point $z_1^{i_k j_k}$ the point $z_2^{i_k j_k}$ such, that the distance between points $(z_1^{i_{k-1} j_{k-1}}, z_2^{i_{k-1} j_{k-1}})$ and $(z_1^{i_k j_k}, z_2^{i_k j_k})$ be minimal.We associate the initial point of circuit $(z_1^{i_1 j_1}, z_2^{i_1 j_1})$ with the final point $(z_1^{i_1 j_1}, z_2^{i_N, j_N})$ of circuit. If in this case $z_2^{i_1 j_1} \neq z_2^{i_N j_N}$, we mark the disk with center in $z_1^{i_0 j_0}$ with the same radius as for circuit of circle by the marked disk containing projection of some ramification point on X. This method of obtaining from discret circle on the grid $\{z_1^{ij}\}$ of the set of points $\{(z_1^{ij}, z_2^{ij})\}$ will be called by lifting of this circle .

2. **Realization of slits.** We joint the disks marked in the first item with boundary of the grid by some discrete curves on the grid $\{z_1^{ij}\}$, in such a way that different discrete curves don't intersect each other and call these curves by slits. It is impossible now to go around any point z_1^{ij} on discrete grid by closed discrete curve without intersections with these slits and the marked disks such that for describing lifting of this discrete curve into the set of points $\{(z_1^{ij}, z_2^{ij}\}$ the initial and final points would be lifted into different points of this set of points.

3. **Representation of leafs.** Due to previous item the discrete grid $\{z_1^{ij}\}$ without marked disks and without slits could be uniquely lifted into the set $\{z_1^{ij}, z_2^{ij}\}$ by fixing of lifting of some arbitrary point for each connected component on the set $\{z_1^{ij}\}$. These lifted sets of points we call leafs. These leafs one can visualize as usual surfaces.

4. **Lifting of slits.** By definition of slits $\{z_1^{ij}\}$ every slit admits unique lifting into the set $\{(z_1^{ij}, z_2^{ij})\}$ if for any slit to fix lifting of one of its points.

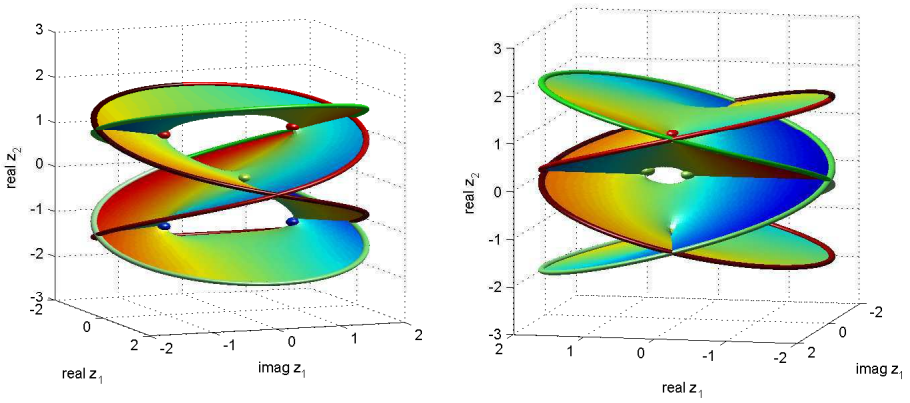

Figure 1: Riemann surface for function $f(z) = \sqrt{\exp\left(\frac{z}{4}\right) + \sqrt{z^2+1}}$, $|z| \leqslant 2$, obtained by visualization algorithm. Red and green curves— two connected components of surface boundary, colored small balls — points of ramification.

This item is similar to previous one.

5 Examples.

5.1 The case $\mu_0 = 1$.

Consider example of reconstruction of Riemann surface for the case $\mu_0 = 1$. The case $\mu_0 = 2$ will be considered further. The case $\mu_0 = 0$ is directly included in considered cases because of specific character of reconstruction algorithm. Let us reconstruct surface

$$X_1 = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid (z_1 - 1)z_2 = \exp(z_1^2), \quad |z_1| \leqslant 2\right\}.$$

The boundary $\gamma_1 = bX_1$ is given in the form of discrete number of points (see further fig. 3).

Note, that if point $(z_1, z_2) \in X_1$ is such that z_1 approaches to 1, then z_2 increases indefinitely. Choose R large enough, for example $R = 60$, and reconstruct real curve $\Gamma = \{(z_1, z_2) \in X_1 \mid |z_2| = R\}$. Compute, first, for two

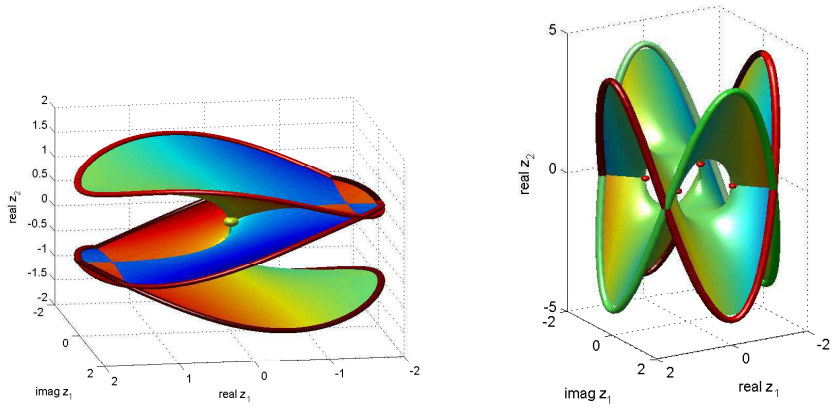

Figure 2: From the left Riemann surface of function $f(z) = \sqrt{\sin(z)}$, $|z| \leqslant 2$, from the right — functions $f(z) = \sqrt{z^4 + 1}$, $|z| \leqslant 2$, obtained by visualization algorithm.

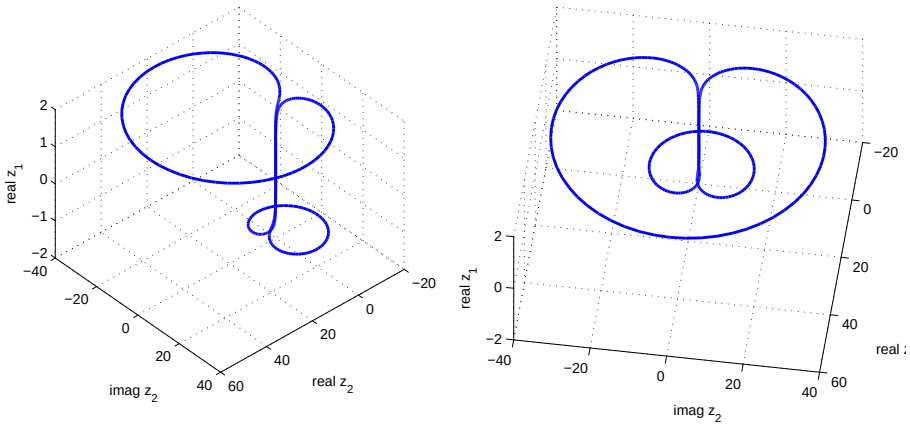

Figure 3: Boundary $\gamma_1 = bX_1$ of the surface X_1 by which the surfaceX_1 is to reconstruct.

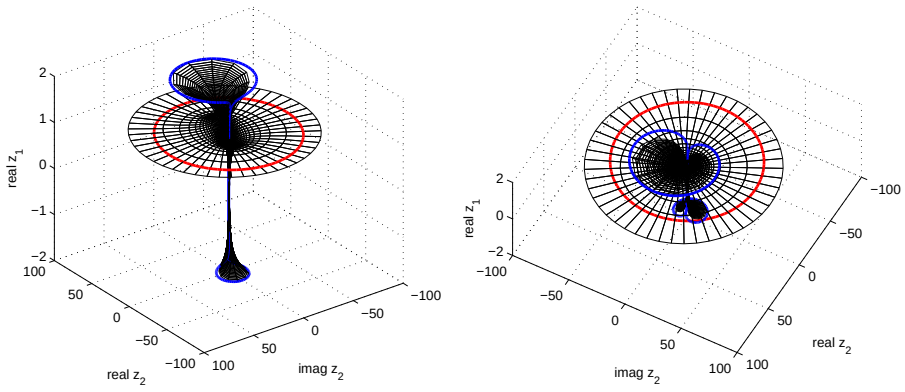

Figure 4: By black is represented contour of surface X_1, by blue — γ_1, by red — reconstructed curve Γ, belonging to X_1.

different ξ_0^1, ξ_0^2, $|\xi_0^1| = |\xi_0^2| = R$, the values of functions

$$G_1(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma_2} \frac{z_1 dz_2}{z_2 + \xi_0},$$

$$\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) = -\frac{1}{2\pi i} \int\limits_{\gamma_2} \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2},$$

$$\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma_2} \left(\frac{z_1 dz_1}{z_2 + \xi_0} - \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2} \right)$$

and find from linear system for c_{11} and c_{12}

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) + c_{12} \left(\xi_0^1 \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) + G_1(\xi_0^1, 0) \right) = G_1(\xi_0^1, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0^1, 0),$$

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) + c_{12} \left(\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) + G_1(\xi_0^2, 0) \right) = G_1(\xi_0^2, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0^2, 0)$$

values $c_{11} = 1$, $c_{12} = 0$. Now calculate the values of functions $G_1(\xi_0, 0)$ on the circle $|\xi_0| = R$ and find function $h_1(\xi_0) = G_1(\xi_0, 0) + c_{11} + c_{12}\xi_0$, $|\xi_0| = R$. So we reconstruct a real curve $\Gamma = \{(h_1(\xi_0), -\xi_0) \mid |\xi_0| = R\} \subseteq X_1$ (see fig. 4).

We can apply further the reconstruction algorithm of Riemann surface for the case $\mu_0 = 0$ to the surface $X_1^R = \{z \in X_1 \mid |z_2| \leqslant 60\}$ with boundary $bX_1^R = \gamma_1 + \Gamma$. We can calculate the values of function

$$\sigma_0(\xi_0) = \frac{1}{2\pi i} \int\limits_{\gamma_1 + \Gamma} \frac{dz_2}{z_2 + \xi_0}.$$

The value of function $\sigma_0(\xi_0)$ in point ξ_0 is equal to number $N_+(\xi_0)$ of points of surface X_1^R, projected into the point $-\xi_0$ under projection $(z_1, z_2) \mapsto z_2$. Further for every point with $N_+(\xi_0) > 0$ we calculate functions

$$s_k(\xi_0) = \frac{1}{2\pi i} \int\limits_{\gamma_1+\Gamma} \frac{z_1^k dz_2}{z_2 + \xi_0}, \quad k = 1, \ldots, N_+(\xi_0).$$

From functions $s_k(\xi_0)$ we can find functions $\sigma_k(\xi_0)$, using Newton formulas:

$$k\sigma_k(\xi_0) = \sum_{i=1}^{k} (-1)^{i-1} \sigma_{k-i}(\xi_0) s_i(\xi_0), \quad k = 1, \ldots, N_+(\xi_0).$$

After we find roots $h_1(\xi_0)$, $\ldots$, $h_{\sigma_0(\xi_0)}(\xi_0)$ of polynomial

$$t^{N_+(\xi_0)} - \sigma_1(\xi_0) t^{N_+(\xi_0)-1} + \cdots + (-1)^{N_+(\xi_0)} \sigma_{N_+(\xi_0)}(\xi_0) = 0.$$

The points $\{(h_k(\xi_0), -\xi_0) \mid k = 1, \ldots, N_+(\xi_0)\}$ represent the set of all points of X_1^R, projected into $-\xi_0$ by projection $(z_1, z_2) \to z_2$. Visualization, obtained for all ξ_0, of set of points can be realized by visualization algorithm described in previous section. Reconstructed surface is represented in fig. 5.

5.2 The case $\mu_0 = 2$.

Consider example of reconstruction of Riemann surface for the case $\mu_0 = 2$. As surface for reconstruction we choose the reconstruction

$$X_2 = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid z_2(z_1^2 - 1) = z_1 \exp(z_1^2), \quad |z_1| \leqslant 2 \right\}.$$

From definition of X_2 one can see that if $(z_1, z_2) \in X_2$ and z_1 approaches to 1 or to -1, then z_2 increases indefinitely. From given boundary $\gamma_2 = bX_2$ (see fig. 6), we need to reconstruct surface X_2 as a finite number of points.

Choose R large enough, for example, $R = 60$,and consider circle C_R of radius R in the plane of variable z_2 with center in zero. Calculate for $-\xi_0 \in C_R$ the values of functions

$$G_1(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma_2} \frac{z_1 dz_2}{z_2 + \xi_0},$$

$$G_2(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma_2} \frac{z_1^2 dz_2}{z_2 + \xi_0},$$

$$\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) = -\frac{1}{2\pi i} \int\limits_{\gamma_2} \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2},$$

$$\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma_2} \left(\frac{z_1 dz_1}{z_2 + \xi_0} - \frac{z_1^2 dz_2}{(z_2 + \xi_0)^2} \right),$$

$$\frac{\partial G_2}{\partial \xi_1}(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma_2} \left(\frac{z_1^2 dz_1}{z_2 + \xi_0} - \frac{z_1^3 dz_2}{(z_2 + \xi_0)^2} \right)$$

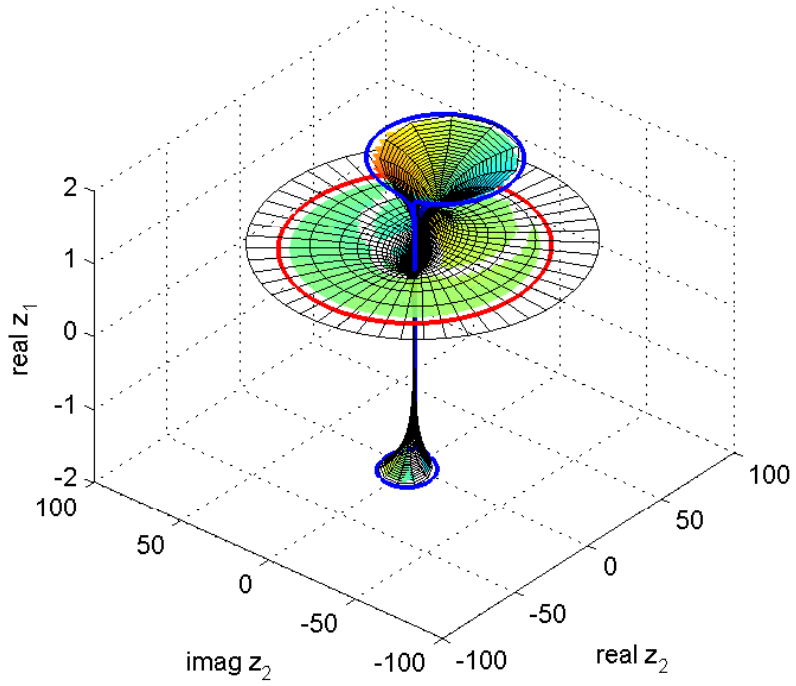

Figure 5: By black is represented contour of surface X_1, by blue — γ_1, by red — reconstructed curve Γ, belonging to X_1, colored domains — reconstructed leafs of surface X_1.

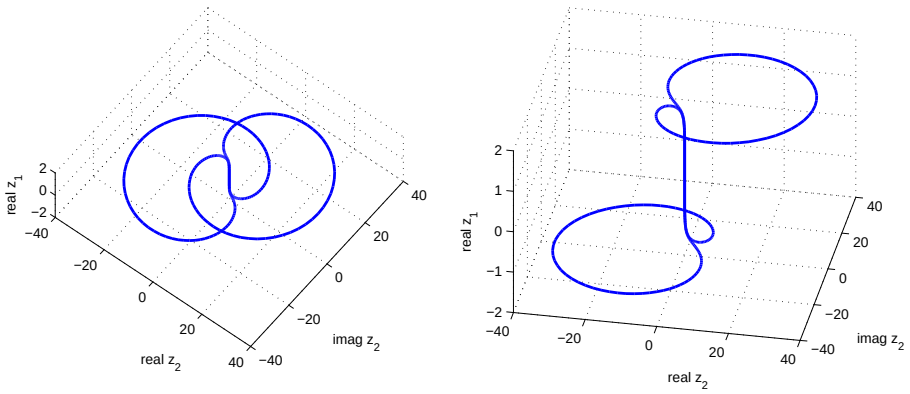

Figure 6: Boundary bX_2 of surface X_2 by which surfaceX_2 is to reconstruct.

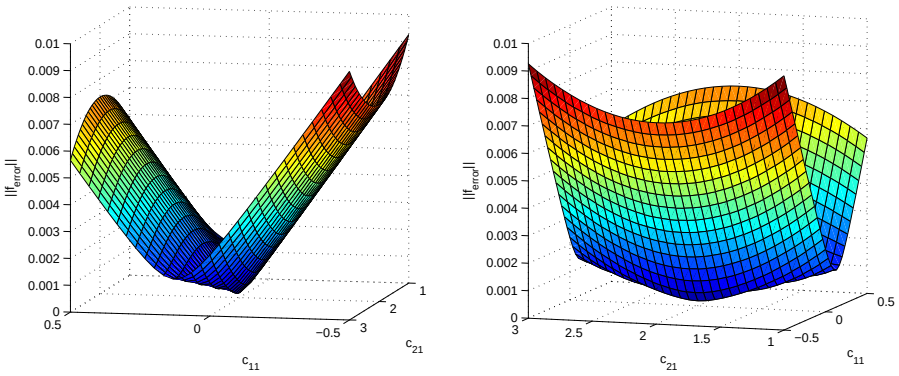

Figure 7: Graph of function $\|f_{error}\|(c_{11}, c_{21})$ in the neighborhood of point of global minimum.

and the value of constant $æ = \frac{1}{2\pi i}\int_{\gamma_2} z_1 dz_2$, for example, using the method of rectangles. We will solve numerically the problem of minimization of $L_2(C_R)$-norm of function

$$f_{error}(\xi_0) = 2\big(G_1 - c_{11} - c_{12}\xi_0\big)\frac{\partial G_1}{\partial \xi_1} - G_1\big(c_{22} + 2c_{23}\xi_0\big) - \frac{\partial G_2}{\partial \xi_1} - 3c_{12}^2 æ + æ c_{23} + \\ + (-1)\Big(\big(G_1 - c_{11} - c_{12}\xi_0\big)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2\Big)\frac{\partial G_1}{\partial \xi_0} + \\ + (-1)\Big(G_1^2 - 2G_1c_{11} - 2G_1c_{12}\xi_0 - G_2\Big)c_{12},$$

in variables c_{11}, c_{12}, c_{21}, c_{22}, c_{23}. As a result of solving of minimization problem we find $c_{11} = 0$, $c_{12} = 0$, $c_{21} = 2$, $c_{22} = 0$, $c_{23} = 0$ (see fig 7).

Let us calculate symmetric functions $\sigma_1 = G_1 + c_{11} + c_{12}\xi_0$, $\sigma_2 = G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2$ and σ_0 on the circle C_R and we do its desymmetrization, using Newton formulas. Value of $\sigma_0(\xi_0)$ is a sum of $\mu_0 = 2$ and index of point $-\xi_0$ with respect to curve γ_2, which by definition is equal to $G_0(\xi_0, 0) = \frac{1}{2\pi i}\int_{\gamma_2}\frac{dz_2}{z_2+\xi_0}$. As a result of desymmetrization we obtain functions $h_1(\xi_0)$ and $h_2(\xi_0)$ on the circle C_R, which realize lifting of circle C_R on the surface X_2. Denote by $\Gamma_{1,2} = \{(h_{1,2}(z_2), -z_2) \mid z_2 \in C_R\}$ curves, obtained by corresponding lifting C_R on X_2 (see fig. 8).

Now we consider the curve $\gamma_2 + \Gamma_1 + \Gamma_2$ as a new initial curve and reconstruct the surface $X_2^R = \{z \in X_2 \mid |z_2| \leqslant R\}$. Further, our considerations are similar to those for the case $\mu_0 = 1$. We calculate, first, functions s_k, then symmetric functions σ_k, then we solve algebraic equation (numerically) and find functions $h_k(\xi_0)$, $k = 1, \ldots, N_+(\xi_0)$. Reconstructed surface is given by fig. 9.

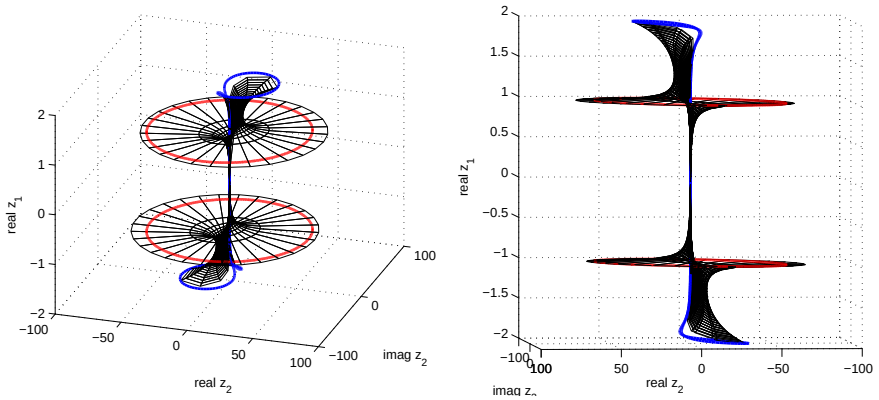

Figure 8: By black is represented contour of surface X_2, by blue — γ_2, by red — reconstructed curves $\Gamma_{1,2}$, belonging to X_2.

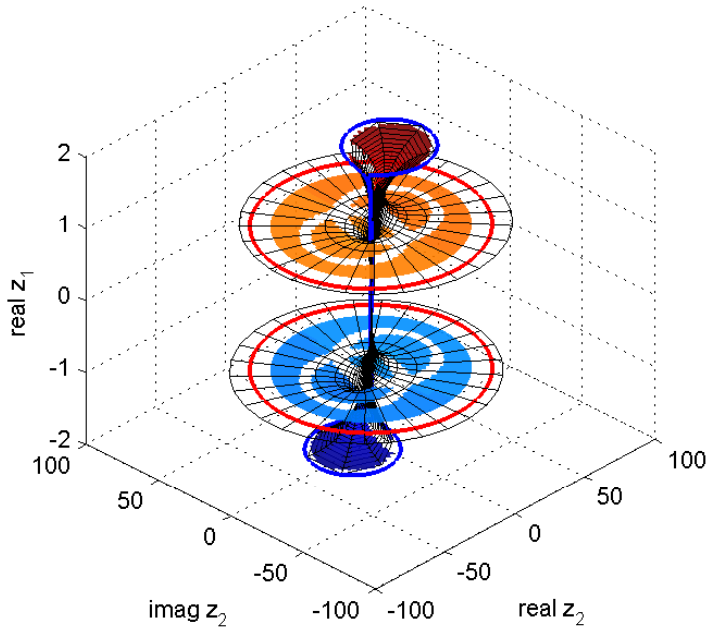

Figure 9: By black is represented contour of surface X_2, by blue — γ_2, by red — reconstructed curves $\Gamma_{1,2}$, belonging to X_2, colored by dark-blue, orange, red and blue domains — reconstructed leafs of surface.

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