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Explicit reconstruction of Riemann surface with given boundary in complex projective space[1]

A. D. Agaltsov[2], *G. M. Henkin*[3]

In this paper we propose a numerically realizable method for reconstruction of a complex curve with known boundary and without compact components in complex projective space.

Keywords: Riemann surface, reconstruction algorithm, Burgers equation, Cauchy-type formulas

1 Introduction

Let $\mathbb{C}P^2$ be the complex projective plane with homogeneous coordinates $(w_0 : w_1 : w_2)$. Let $X \subset \mathbb{C}P^2$ be a complex curve with rectifiable boundary $\gamma = \partial X$. Without loss of generality, we suppose that the following conditions of general position hold:

$$(0 : 1 : 0) \notin X, \quad w_0|_\gamma \neq 0.$$

Put $\mathbb{C}^2 = \{w \in \mathbb{C}P^2 \colon w_0 \neq 0\}$ with coordinates $z_1 = \frac{w_1}{w_0}$, $z_2 = \frac{w_2}{w_0}$. For almost all $\xi = (\xi_0, \xi_1) \in \mathbb{C}^2$ the points of intersection of X with complex line $\mathbb{C}^1_\xi = \{z \in \mathbb{C}^2 \colon \xi_0 + \xi_1 z_1 + z_2 = 0\}$ form a finite set of points

$$\big(z_1^{(j)}(\xi), z_2^{(j)}(\xi)\big) = \big(h_j(\xi), -\xi_0 - \xi_1 h_j(\xi)\big), \quad \xi = (\xi_0, \xi_1),\ j = 1, \dots, N_+(\xi).$$

By Darboux lemma [2, 4] functions $\{h_j\}$ satisfy the following equations

$$\frac{\partial h_j(\xi)}{\partial \xi_1} = h_j(\xi) \frac{\partial h_j(\xi)}{\partial \xi_0}, \quad \xi = (\xi_0, \xi_1),\ j = 1, \dots, N_+(\xi), \tag{1}$$

which are often called the shock-wave equations, the inviscid Burgers equations or the Riemann-Burgers equations. In this interpretation ξ_1 is the time variable and ξ_0 is the space variable.

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The following Cauchy-type formula from [4] plays the essential role in the reconstruction of X from γ:

$$G_m(\xi) \stackrel{\text{def}}{=\!=} \frac{1}{2\pi i}\int\limits_{\gamma} \frac{z_1^m d(\xi_0+\xi_1 z_1+z_2)}{\xi_0+\xi_1 z_1+z_2} = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi) + P_m(\xi), \quad m \geq 1, \tag{2}$$

where $N_+(\xi) = |X \cap \mathbb{C}^1_\xi|$ and $P_m(\xi_0, \xi_1)$ is a polynomial of degree at most m with respect to ξ_0 at fixed ξ_1. In addition, $P_0(\xi_0, \xi_1) = -N_-$, where $N_- = |X \cap \mathbb{C}P^1_\infty|$, $\mathbb{C}P^1_\infty = \{w \in \mathbb{C}P^2 \colon w_0 = 0\}$, and

$$P_1(\xi_0, \xi_1) = -\sum_{k=1}^{N_-} \frac{a_k\xi_0 + b_k}{1 - a_k\xi_1}, \quad \text{if } h_k(\xi_0, 0) \sim a_k\xi_0 + b_k + O(\xi_0^{-1}) \text{ at } \infty. \tag{3}$$

In particular, it follows from (2) that:

$$G_0(\xi) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{d(\xi_0+\xi_1 z_1+z_2)}{\xi_0+\xi_1 z_1+z_2} = N_+(\xi) - N_-. \tag{4}$$

Let $\pi_2 \colon \mathbb{C}^2 \to \mathbb{C}$ be the projection onto the second coordinate: $\pi_2(z_1, z_2) = -z_2$. We have that $\mathbb{C} \setminus \pi_2\gamma = \cup_{l=0}^{L} \Omega_l$, where $\Omega_{l\geq 0}$ are connected and Ω_0 is unbounded. From the definition of $N_\pm$ it follows that

$$N_+(\xi_0, 0) = N_-, \quad \xi_0 \in \Omega_0. \tag{5}$$

Assume that complex curve X does not contain compact components without boundary, or equivalently, satisfies the following condition of minimality:

$$\begin{gathered}\text{for any complex curve } \widetilde{X} \subset \mathbb{C}P^2 \text{ such that } \partial\widetilde{X} = \partial X = \gamma \\ \text{and for almost all } \xi \in \mathbb{C}^2 \text{ we have } |\widetilde{X} \cap \mathbb{C}^1_\xi| \geq |X \cap \mathbb{C}^1_\xi|.\end{gathered} \tag{$*$}$$

Condition of minimality $(*)$ is a condition of general position and is fulfilled for X if, for example, every irreducible component of X is a transcendental complex curve. Note that from theorems of Chow [1] and Harvey-Shiffman [7] it follows that an arbitrary complex curve $\widetilde{X} \subset \mathbb{C}P^2$ satisfying $\partial\widetilde{X} = \partial X$ admits the unique representation $\widetilde{X} = X \cup V$, where X is a curve satisfying $(*)$, and V is a compact algebraic curve, possibly, with multiple components.

The main result of [3] gives a solution to the important problem of J. King [9], when a real curve $\gamma \subset \mathbb{C}P^2$ is the boundary of a complex curve $X \subset \mathbb{C}P^2$. Let $\gamma \subset \mathbb{C}^2$. Then $\gamma = \partial X$ for some open connected complex curve X in $\mathbb{C}P^2$ if and only if in a

neighborhood W_{ξ^*} of some point $\xi^* \in \mathbb{C}^2$ one can find mutually distinct holomorphic functions $h_1, \ldots, h_{N(\xi^*)}$ satisfying shock-wave equations (1) and also the equation

$$\frac{\partial^2}{\partial \xi_0^2}\big(G_1(\xi_0,\xi_1) - \sum_{j=1}^{p} h_j(\xi_0,\xi_1)\big) = 0, \quad \xi = (\xi_0,\xi_1) \in W_{\xi^*}.$$

In this work in development of [3, 4] we propose a numerically realizable algorithm for reconstruction of a complex curve $X \subset \mathbb{C}P^2$ from the known boundary and satisfying the condition of minimality. This algorithm permits, in particular, to make applicable the result of [8] about principal possibility to reconstruct the topology and the conformal structure of a two-dimensional bordered surface X in $\mathbb{R}^3$ with constant scalar conductivity from measurements on ∂X of electric current densities, induced by three potentials in general position.

Our algorithm depends on the number of points at infinity N_- of the complex curve X. It was tested on many examples and admits a simple and complete justification for $N_- = 0, 1, 2$. Despite a cumbersome description for $N_- \geq 3$, we show that, in principle, there are no obstacles for the justification and numerical realization for any $N_- \geq 0$. Moreover, in Theorem 3.2 we propose a method for finding the parameter N_- in terms of γ. This makes the algorithm much more applicable.

2 Cauchy-type formulas and Riemann-Burgers equations

We begin by giving a new proof of the Cauchy type formulas (2) from [4], which allows to obtain explicit expressions for functions P_m.

Theorem 2.1. *Let $X \subset \mathbb{C}P^2 \setminus (0:1:0)$ be a complex curve with rectifiable boundary $\gamma \subset \mathbb{C}^2$ and satisfying $(*)$. Suppose that for almost all $\xi \in \mathbb{C}^2$ all the points of intersection of X with $\mathbb{C}^1_\xi$ have multiplicity at most one. Then the following formulas hold for almost all $\xi = (\xi_0,\xi_1) \in \mathbb{C}^2$:*

$$G_m(\xi) = \sum\nolimits_{j=1}^{N_+(\xi)} h_j^m(\xi) + P_m(\xi), \quad m \geq 1, \tag{6}$$

where $P_m(\xi)$ is holomorphic in a neighborhood of almost all $(\xi_0, 0)$, $\xi_0 \in \mathbb{C} \setminus \pi_2(\gamma)$, and is polynomial in ξ_0 of degree at most m for any fixed ξ_1. Furthermore, the following

explicit formulas hold:

$$P_m(\xi_0,\xi_1) = \sum_{s=1}^{N_-}\sum_{k=0}^{m-1}\sum_{i_1+\dots+i_m=k}\frac{\frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s)\cdots\frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)!}\frac{d^{m-k}}{dw_0^{m-k}}\ln(\xi_0w_0+\xi_1w_1+w_2)|_{q_s} - \sum_{s=1}^{\mu_0}\sum_{i_1+\dots+i_m=m}\frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s)\cdots\frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s),$$

where $X\cap\mathbb{C}P^1_\infty=\{q_1,\dots,q_{N_-}\}$. *In particular, if* $N_-=0$ *then* $P_m=0$, $m\geq 1$.

Proof. Put $\widetilde{g}=\xi_0w_0+\xi_1w_1+w_2$ and $g=\frac{\widetilde{g}}{w_0}=\xi_0+\xi_1z_1+z_2$. Consider differential forms

$$\omega_m \stackrel{\text{def}}{=\!=} z_1^m\frac{dg}{g}=\frac{w_1^m}{w_0^m}\frac{w_0}{\widetilde{g}}d\left(\frac{\widetilde{g}}{w_0}\right)=\frac{w_1^m}{w_0^m}\frac{d\widetilde{g}}{\widetilde{g}}-\frac{w_1^m}{w_0^{m+1}}dw_0,\quad m=0,1,\dots$$

Then $G_m(\xi)=\frac{1}{2\pi i}\int_\gamma\omega_m$. Let us compute this integral explicitly. Denote by p_j, $j=1$, $\dots$, $N_+(\xi)$ the points of intersection of X with $\mathbb{C}P^1_\xi$, and by q_s, $s=1$, $\dots$, N_- the points of intersection of X with infinity $\mathbb{C}P^1_\infty$. Denote by B_j^ε the intersection of X with the ball of radius ε in $\mathbb{C}P^2$ centered at p_j and by D_s^ε the intersection of X with the ball of radius ε centered at q_s. The restriction of form ω_m on X is meromorphic with poles at points p_j and q_s. Thus the following equality is valid:

$$G_m(\xi)=\frac{1}{2\pi i}\int\limits_\gamma\omega_m=\sum_{j=1}^{N_+(\xi)}\frac{1}{2\pi i}\int\limits_{bB_j^\varepsilon}\omega_m+\sum_{s=1}^{N_-}\frac{1}{2\pi i}\int\limits_{bD_s^\varepsilon}\omega_m.$$

If $N_-=0$, then the second group of terms is absent. The integral $\int_{bB_j^\varepsilon}\omega_m$ can be calculated as a residue at the first order pole:

$$\int\limits_{bB_j^\varepsilon}\omega_m=\int\limits_{bB_j^\varepsilon}z_1^m\frac{d\widetilde{g}}{\widetilde{g}}-\int\limits_{bB_j^\varepsilon}\frac{w_1^m}{w_0^{m+1}}dw_0=\int\limits_{bB_j^\varepsilon}z_1^m\frac{d\widetilde{g}}{\widetilde{g}}=2\pi i\,h_j^m(\xi).$$

Let $N_->0$. Computation of integral $\int_{bD_s^\varepsilon}\omega_1$ will be done in two steps. Let us calculate first $\int_{bD_s^\varepsilon}\frac{w_1^m}{w_0^{m+1}}dw_0$. Consider the expansion of $w_1(w_0)$ into power series in w_0 in the neighborhood of point q_s:

$$w_1(w_0)=w_1(q_s)+\frac{dw_1}{dw_0}(q_s)w_0+\frac{d^2w_1}{dw_0^2}(q_s)w_0^2+\cdots.$$

Note further that

$$w_1^m(w_0) = \sum_{k=0}^{\infty} \sum_{i_1+\cdots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s)\cdots\frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)w_0^k.$$

The coefficient near w_0^m can be presented in the form

$$\int\limits_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}}dw_0 = 2\pi i \sum_{i_1+\cdots+i_m=m} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s)\cdots\frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s).$$

Now we can calculate the integral $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m}\frac{d\widetilde{g}}{\widetilde{g}}$. Using relation $d\widetilde{g} = \frac{d\widetilde{g}}{dw_0}dw_0$ and expansion of $w_1(w_0)$ into power series in w_0 we obtain:

$$\int\limits_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m}\frac{d\widetilde{g}}{\widetilde{g}} = \int_{bD_s^\varepsilon} \frac{1}{w_0^m}\left(w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2w_1}{dw_0^2}(q_s)w_0^2 + \cdots\right)^m \frac{d\widetilde{g}}{dw_0}\frac{1}{\widetilde{g}}dw_0 =$$

$$\sum_{k=0}^{\infty}\int_{bD_s^\varepsilon} \sum_{i_1+\cdots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s)\cdots\frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)w_0^{k-m}\frac{d\widetilde{g}}{dw_0}\frac{1}{\widetilde{g}}dw_0 =$$

$$\sum_{k=0}^{m-1}\int_{bD_s^\varepsilon} \sum_{i_1+\cdots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s)\cdots\frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)w_0^{k-m}\frac{d\widetilde{g}}{dw_0}\frac{1}{\widetilde{g}}dw_0 =$$

$$\sum_{k=0}^{m-1}\frac{2\pi i}{(m-k-1)!}\sum_{i_1+\cdots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s)\cdots\frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)\lim_{w_0\to 0}\frac{d^{m-k-1}}{dw_0^{m-k-1}}\left(\frac{d\widetilde{g}}{dw_0}\frac{1}{\widetilde{g}}\right).$$

From here, taking into account the relation $\frac{d\widetilde{g}}{dw_0}\frac{1}{\widetilde{g}} = \frac{d\ln\widetilde{g}}{dw_0}$, we obtain, finally

$$\int\limits_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m}\frac{d\widetilde{g}}{\widetilde{g}} = 2\pi i\sum_{k=0}^{m-1}\sum_{i_1+\cdots+i_m=k} \frac{\frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s)\cdots\frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)!}\frac{d^{m-k}}{dw_0^{m-k}}\ln(\xi_0w_0+\xi_1w_1+w_2)|_{q_s}.$$

It is a polynomial of degree at most m with respect to ξ_0. □

Remark 2.1. We excluded the case $(0:1:0)\in X$ because the point $(0:1:0)$ can lead to a non-polynomial contribution in ξ_0 in functions P_m.

Consider, for example, the projective curve $\widetilde{X}$: $w_0^2 = w_1w_2$ and the meromorphic 1-form Ω_{ξ_0} on it:

$$\Omega_{\xi_0} = \frac{z_1dz_2}{z_2+\xi_0}, \quad \xi_0\in\mathbb{C}\setminus 0,$$

where $z_1 = \frac{w_1}{w_0}$, $z_2 = \frac{w_2}{w_0}$. The form Ω_{ξ_0} is holomorphic on X except the points $q_1 = (0:0:1)$, $q_2 = (0:1:0)$ and $p_{\xi_0} = (-\xi_0 : 1 : \xi_0^2)$.

One can see that the residues of Ω_{ξ_0} at q_1, q_2 are equal to 0 and $1/\xi_0$, respectively. It follows that the residue at p_{ξ_0} is equal to $-1/\xi_0$.

Now, consider the part X of $\widetilde{X}$ obtained by cutting off some small neighborhood of the point q_1. Let γ denote the boundary of X. Choose any $\xi = (\xi_0, 0)$ such that $X \cap \mathbb{C}^1_\xi \neq \varnothing$. Then $X \cap \mathbb{C}^1_\xi = \{(-1/\xi_0, -\xi_0)\}$ and, as far as the residue of Ω_{ξ_0} at q_1 is zero, we have that

$$0 = G_1(\xi_0, 0) = \frac{1}{2\pi i}\int_\gamma \Omega_{\xi_0} = -\frac{1}{\xi_0} + P_1(\xi_0, 0),$$

so that $P_1(\xi_0, 0) = 1/\xi_0$.

We will also use the following result of [8], which gives an effective characterization of functions $h_{j\geq 1}$ of Theorem 2.1.

Theorem 2.2 (Remark 4 to Theorem 3a of [8])**.** *Let $X \subset \mathbb{C}P^2 \setminus (0:1:0)$ be a complex curve with rectifiable boundary $\gamma \subset \mathbb{C}^2$ and satisfying $(*)$. Fix any $\xi_0^* \in \Omega_0$ and let W_{ξ^*} be a neighborhood of $\xi^* = (\xi_0^*, 0)$ in $\mathbb{C}^2$. Let $G_{k\geq 1}$, $h_{j\geq 1}$ be defined as in* (2) *and* (6)*. Suppose that there exist functions $\widetilde{h}_1, \ldots, \widetilde{h}_p$, holomorphic in W_{ξ^*} and satisfying*

$$\tfrac{\partial^2}{\partial \xi_0^2}(G_1 - \widetilde{h}_1 - \cdots - \widetilde{h}_p) = 0 \quad \text{in } W_{\xi^*}, \tag{7}$$

$$\tfrac{\partial \widetilde{h}_k}{\partial \xi_1} = \widetilde{h}_k \tfrac{\partial \widetilde{h}_k}{\partial \xi_0} \quad \text{in } W_{\xi^*},\ k = 1, \ldots, p. \tag{8}$$

Then $p \geq N_-$. Furthermore, if $p = N_- := |X \cap \mathbb{C}P^1_\infty|$, then $h_1, \ldots, h_{N_-}$ coincide with $\widetilde{h}_1, \ldots, \widetilde{h}_{N_-}$ in W_{ξ^*} (up to order).

3 Reconstruction algorithm

We now pass to the reconstruction algorithm for a complex curve $X \subseteq \mathbb{C}P^2$ with given boundary ∂X and satisfying the condition of minimality $(*)$. Let us consider the cases $N_- = 0, 1, 2$.

The reconstruction algorithm is based on formulas (6). The next theorem permits to find the functions P_m of (6). We will use the notation

$$æ_{kl} = \frac{1}{2\pi i}\int\limits_\gamma z_1^k z_2^l \, dz_2, \quad k, l \geq 0. \tag{9}$$

Theorem 3.1. *Let $X \subset \mathbb{C}P^2$, $(0:1:0) \notin X$, be a complex curve with rectifiable boundary $\gamma \subset \mathbb{C}^2$ and satisfying $(*)$. Let $h_{j\geq 1}$ and $P_{m\geq 1}$ be the functions defined in Theorem 2.1. Then the following statements are valid:*

1. *If $N_- = 0$, then $P_m = 0$ for all $m \geq 1$. Besides, $G_1 = 0$ in a neighborhood of any $\xi^* = (\xi_0^*, 0)$ with $\xi_0^* \in \Omega_0$.*

2. *If $N_- = 1$, then $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$, where constants c_{11} and c_{12} satisfy the following identity in $\xi = (\xi_0, 0) \in \Omega_0 \times 0$:*

$$c_{11}\tfrac{\partial G_1}{\partial \xi_0}(\xi) + c_{12}\big(\xi_0 \tfrac{\partial G_1}{\partial \xi_0}(\xi) + G_1(\xi)\big) = G_1(\xi)\tfrac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \tfrac{\partial G_1}{\partial \xi_1}(\xi). \tag{10}$$

3. *If $N_- = 2$, then $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$, $P_2(\xi_0, 0) = c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2$, where constants c_{11}, c_{12}, c_{21}, c_{22}, c_{23} satisfy the following identity in $\xi = (\xi_0, 0) \in \Omega_0 \times 0$:*

$$\begin{aligned} æ_{10}(c_{12}^2 + c_{23}) = \tfrac{\partial G_2}{\partial \xi_1} - 2\tfrac{\partial G_1}{\partial \xi_1}(G_1 - c_{11} - c_{12}\xi_0) + G_1(c_{22} + 2c_{23}\xi_0) \\ + \frac{\partial G_1}{\partial \xi_0} \cdot \big((G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2\big) \\ + \big(G_1^2 - 2c_{11}G_1 - 2c_{12}G_1\xi_0 - G_2\big) \cdot (-c_{12}), \end{aligned} \tag{11}$$

where all the functions are evaluated at point $\xi = (\xi_0, 0)$.

Proof. 1. By Theorem 2.1, if $N_- = 0$ then $P_m = 0$ for all $m \geq 1$. It also follows from formula (6) that $G_1 = 0$ in a neighborhood of any $\xi^* = (\xi_0^*, 0)$ with $\xi_0^* \in \Omega_0$.

2. It follows from Theorem 2.1 that $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$ in a neighborhood of almost all $\xi^* = (\xi_0^*, 0) \in \Omega_l \times 0$, where C_{11}, C_{12} are holomorphic in a neighborhood of zero. We need to find constants $c_{11} = C_{11}(0)$ and $c_{12} = C_{12}(0)$.

We differentiate the equation (6) with respect to ξ_0 and restrict this equation and its differentiated version to $\Omega_l \times 0$:

$$\begin{aligned} h_1(\xi_0, 0) &= G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0, \\ \frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0, \\ \frac{\partial h_1}{\partial \xi_0}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12}, \end{aligned} \tag{12}$$

where $\xi_0 \in \Omega_0$. By (1) there is the equality $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) = h_1(\xi_0, 0)\frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$. We substitute $h_1(\xi_0, 0)$, $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0)$, $\frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$ in this equation by their expressions (12), and we obtain the equation

$$\begin{aligned} \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 = \\ = \Big(G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0\Big)\Big(\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12}\Big). \end{aligned} \tag{13}$$

This equation is valid, in particular, for $\xi_0 \in \Omega_0$. We divide it by ξ_0 and tend $\xi_0 \to \infty$. As a result, we obtain the equality $\dot{C}_{12}(0) = -c_{12}^2$. Taking into account this equality, we can rewrite equation (13) in the form

$$\begin{aligned} &\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) = \\ &= \Big(G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0\Big)\frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \Big(G_1(\xi_0, 0) - c_{11}\Big)c_{12}. \end{aligned} \tag{14}$$

Taking into account that $\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) \to 0$ as $\xi_0 \to \infty$ and passing $\xi_0 \to \infty$ in (14), we obtain the equality $\dot{C}_{11}(0) = -c_{11}c_{12}$. Substituting ths explicit expression for $\dot{C}_{11}(0)$ into (14), we get (10).

3. By (1) functions $h_1(\xi)$ and $h_2(\xi)$ satisfy the Riemann-Burgers equation in a neighborhood of any $\xi^* = (\xi_0^*, 0) \in \Omega_0 \times 0$, so that the following equalities are valid:

$$\frac{\partial(h_1 h_2)}{\partial \xi_1} = h_1 \frac{\partial h_2}{\partial \xi_1} + \frac{\partial h_1}{\partial \xi_1} h_2 = h_1 h_2 \frac{\partial(h_1 + h_2)}{\partial \xi_0}, \tag{15}$$

$$\frac{\partial(h_1^2 + h_2^2)}{\partial \xi_0} = 2h_1 \frac{\partial h_1}{\partial \xi_0} + 2h_2 \frac{\partial h_2}{\partial \xi_0} = 2\frac{\partial(h_1 + h_2)}{\partial \xi_1}. \tag{16}$$

Note that $h_1 h_2 = \frac{1}{2}\big(h_1 + h_2\big)^2 - \frac{1}{2}\big(h_1^2 + h_2^2\big)$. Therefore the system (15)–(16) is equivalent to the system

$$\frac{\partial(h_1 + h_2)^2}{\partial \xi_1} - \frac{\partial(h_1^2 + h_2^2)}{\partial \xi_1} = \Big(\big(h_1 + h_2\big)^2 - \big(h_1^2 + h_2^2\big)\Big)\frac{\partial(h_1 + h_2)}{\partial \xi_0}, \tag{17}$$

$$\frac{\partial(h_1^2 + h_2^2)}{\partial \xi_0} = 2\frac{\partial(h_1 + h_2)}{\partial \xi_1}. \tag{18}$$

We substitute the expressions of (6) for $h_1^2 + h_2^2$ and $h_1 + h_2$ into (17), (18), using the notations $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$, $P_2(\xi_0, \xi_1) = C_{21}(\xi_1) + C_{22}(\xi_1)\xi_0 + C_{23}(\xi_1)\xi_0^2$. Then, equation (18) restricted to $\Omega_0 \times 0$ takes the form

$$\frac{\partial G_2}{\partial \xi_0}(\xi_0, 0) - c_{22} - 2c_{23}\xi_0 = 2\Big(\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0\Big). \tag{19}$$

We divide this equation by ξ_0 and tend $\xi_0 \to \infty$. It leads to the equality $\dot{C}_{12}(0) = c_{23}$. Taking this equality into account and passing $\xi_0 \to \infty$ in (19), we obtain the equality $\dot{C}_{11}(0) = \frac{1}{2}c_{22}$.

Next, we substitute the expressions of (6) for $h_1^2 + h_2^2$ and $h_1 + h_2$ into (17) and restrict the obtained formula to $\Omega_0 \times 0$. It leads to the equality

$$\begin{aligned} &2\big(G_1 - c_{11} - c_{12}\xi_0\big)\Big(\frac{\partial G_1}{\partial \xi_1} - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0\Big) - \frac{\partial G_2}{\partial \xi_1} + \dot{C}_{21}(0) + \dot{C}_{22}(0)\xi_0 + \dot{C}_{23}(0)\xi_0^2 = \\ &= \Big(\big(G_1 - c_{11} - c_{12}\xi_0\big)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2\Big)\Big(\frac{\partial G_1}{\partial \xi_0} - c_{12}\Big). \end{aligned} \tag{20}$$

We divide this equation by ξ_0^2 and tend $\xi_0 \to \infty$. This leads to the equality

$$2c_{12}\dot{C}_{12}(0) + \dot{C}_{23}(0) = -\big(c_{12}^2 + c_{23}\big)c_{12}.$$

Taking this equality into account, dividing (20) by ξ_0 and passing $\xi_0 \to \infty$, we obtain the equality

$$2c_{11}\dot{C}_{12}(0) + 2c_{12}\dot{C}_{11}(0) + \dot{C}_{22}(0) = -\big(2c_{11}c_{12} + c_{22}\big)c_{12}.$$

Using the obtained equalities, one can rewrite (20) in the form

$$\begin{aligned}
2\big(G_1 - c_{11} - c_{12}\xi_0\big)\frac{\partial G_1}{\partial \xi_1} - 2G_1\big(\dot{C}_{11}(0) + \dot{C}_{12}(0)\xi_0\big) + 2c_{11}\dot{C}_{11}(0) - \frac{\partial G_2}{\partial \xi_1} + \dot{C}_{21}(0) = \\
= \Big(\big(G_1 - c_{11} - c_{12}\xi_0\big)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2\Big)\frac{\partial G_1}{\partial \xi_0} + \\
+ \Big(\big(G_1 - c_{11}\big)^2 - 2G_1c_{12}\xi_0 - G_2 + c_{21}\Big)(-c_{12}). \quad (21)
\end{aligned}$$

We pass $\xi_0 \to \infty$ and note that the following relations are valid

$$\begin{aligned}
\lim_{\xi_0\to\infty} \xi_0 \frac{\partial G_1}{\partial \xi_1} &= \lim_{\xi_0\to\infty} \xi_0 \frac{1}{2\pi i}\int_\gamma \frac{z_1\,dz_1}{\xi_0 + z_2} = \frac{1}{2\pi i}\int_\gamma z_1\,dz_1 = 0, \\
\lim_{\xi_0\to\infty} \xi_0 G_1 &= \lim_{\xi_0\to\infty} \xi_0 \frac{1}{2\pi i}\int_\gamma \frac{z_1\,dz_2}{\xi_0 + z_2} = \frac{1}{2\pi i}\int_\gamma z_1\,dz_2 = æ_{10}, \\
\lim_{\xi_0\to\infty} \xi_0^2 \frac{\partial G_1}{\partial \xi_0} &= -\lim_{\xi_0\to\infty} \xi_0^2 \frac{1}{2\pi i}\int_\gamma \frac{z_1\,dz_2}{(\xi_0 + z_2)^2} = -\frac{1}{2\pi i}\int_\gamma z_1\,dz_2 = -æ_{10}.
\end{aligned}$$

As a result, we obtain

$$-2æ_{10}\dot{C}_{12}(0) + 2c_{11}\dot{C}_{11}(0) + \dot{C}_{21}(0) = -\big(c_{12}^2 + c_{23}\big)æ_{10} - \big(c_{11}^2 - 2c_{12}æ_{10} + c_{21}\big)c_{12}.$$

Due to the obtained relations, we can express constants $\dot{C}_{ij}(0)$ as functions of c_{ij}:

$$\begin{aligned}
\dot{C}_{11}(0) &= \frac{1}{2}c_{22}, \\
\dot{C}_{12}(0) &= c_{23}, \\
\dot{C}_{23}(0) &= -c_{12}^3 - 3c_{12}c_{23}, \\
\dot{C}_{22}(0) &= -2\big(c_{11}c_{12}^2 + c_{12}c_{22} + c_{11}c_{23}\big), \\
\dot{C}_{21}(0) &= æ_{10}(c_{12}^2 + c_{23}) - c_{12}(c_{11}^2 + c_{21}) - c_{11}c_{22}.
\end{aligned} \quad (22)$$

Substituting these constants to (21), we obtain the third statement of Theorem 3.1. □

Complement 3.1. The statement of Theorem 3.1 admits a development for the case $N_- \geq 3$. In this case

$$P_k(\xi_0, \xi_1) = C_{k1}(\xi_1) + C_{k2}(\xi_1)\xi_0 + \cdots + C_{k,k+1}(\xi_1)\xi_0^k, \quad k = 1, \ldots, N_-.$$

Denote $\dot{C}_{ij}(0) = \frac{\partial C_{ij}}{\partial \xi_1}(0)$ and $c_{ij} = C_{ij}(0)$ for $i = 1, \ldots, N_-$ and $j = 1, \ldots, i+1$.

Let us indicate the following general procedure for finding of constants c_{ij}. Due to the Riemann–Burgers equations (1) the following identities in $\xi_0 \in \Omega_0$ hold for $k = 1, \ldots, N_- - 1$:

$$\begin{aligned} &-\frac{\partial G_k}{\partial \xi_1}(\xi_0, 0) + \dot{C}_{k1}(0) + \dot{C}_{k2}(0)\xi_0 + \cdots + \dot{C}_{k,k+1}(0)\xi_0^k \\ &= \frac{k}{k+1}\left(-\frac{\partial G_{k+1}}{\partial \xi_0}(\xi_0, 0) + c_{k+1,2} + 2c_{k+1,3}\xi_0 + \cdots + (k+1)c_{k+1,k+2}\xi_0^k\right). \end{aligned}$$

Taking into account that $\frac{\partial G_k}{\partial \xi_1}(\xi_0, 0) \to 0$ and $\frac{\partial G_{k+1}}{\partial \xi_0}(\xi_0, 0) \to 0$ as $\xi_0 \to +\infty$ we obtain the equalities

$$\dot{C}_{k,m}(0) = \frac{km}{k+1}c_{k+1,m+1}, \quad k = 1, \ldots, \mu_0 - 1, \quad m = 1, \ldots, k+1.$$

Due to the Riemann–Burgers equations (1) the following identity in $\xi_0 \in \Omega_0$ holds:

$$\frac{\partial e_{\mu_0}}{\partial \xi_1}(\xi_0, 0) = e_{\mu_0}(\xi_0, 0)\frac{\partial p_1}{\partial \xi_0}(\xi_0, 0), \tag{23}$$

where functions e_k are given by the following formulas:

$$\begin{aligned} ke_k(\xi_0, \xi_1) &= \sum_{i=1}^{k-1}(-1)^{i+1}e_{k-i}(\xi_0, \xi_1)p_i(\xi_0, \xi_1) + (-1)^{k+1}p_k(\xi_0, \xi_1), \\ p_k(\xi_0, \xi_1) &= G_k(\xi_0, \xi_1) - C_{k1}(\xi_1) - C_{k2}(\xi_1)\xi_0 - \cdots - C_{k,k+1}(\xi_1)\xi_0^k, \end{aligned} \tag{24}$$

where $k = 1, \ldots, \mu_0$.

Equality (23) allows to represent constants $\{\dot{C}_{\mu_0,j}(0)\}$ as functions of constants $\{c_{ij}\}$. Finally, substituting the obtained expressions for constants $\{\dot{C}_{ij}(0)\}$ via constants $\{c_{ij}\}$ into equation (23) we obtain the identity in $\xi_0 \in \Omega_0$ for computation of constants $\{c_{ij}\}$.

For example, in the case $N_- = 3$ the identity (23) in $\xi_0 \in \Omega_0$ for finding of constants c_{ij} takes the form

$$\begin{aligned} &\dot{C}_{31}(0) + \dot{C}_{32}(0)\xi_0 + \dot{C}_{33}(0)\xi_0^2 + \dot{C}_{34}(0)\xi_0^3 = \frac{\partial G_3}{\partial \xi_1} + \\ &+\frac{3}{4}(p_1^2 - p_2)\frac{\partial p_2}{\partial \xi_0} - p_1\frac{\partial p_3}{\partial \xi_0} - \frac{1}{2}\big(p_1^3 - 3p_1p_2 + 2p_3\big)\frac{\partial p_1}{\partial \xi_0}, \end{aligned}$$

where all functions are evaluated at point $(\xi_0, 0)$, the functions p_k are defined in formula (24) and the constants $\dot{C}_{31}(0)$, $\dot{C}_{32}(0)$, $\dot{C}_{33}(0)$, $\dot{C}_{34}(0)$ are given by formulas

$$\begin{aligned}
\dot{C}_{31}(0) &= \frac{1}{2}æ_{10}\big(3c_{11}c_{12}^2 + 3c_{12}c_{22} + 3c_{11}c_{23} + 2c_{33}\big) \\
&\quad - \frac{1}{2}æ_{11}\big(c_{12}^3 + 3c_{12}c_{23} + 2c_{34}\big) - \frac{3}{2}\frac{1}{2\pi i}\int_\gamma z_1 z_2\, dz_1\big(c_{12}^2 + c_{23}\big) \\
&\quad - \frac{1}{2}c_{11}^3 c_{12} - \frac{3}{2}c_{11}c_{12}c_{21} - \frac{3}{4}c_{11}^2 c_{22} - \frac{3}{4}c_{21}c_{22} - c_{12}c_{31} - c_{11}c_{32}, \\
\dot{C}_{32}(0) &= æ_{10}\big(c_{12}^3 + 3c_{12}c_{23} + 2c_{34}\big) - \frac{3}{2}c_{11}^2 c_{12}^2 - \frac{3}{2}c_{12}^2 c_{21} - 3c_{11}c_{12}c_{22} \\
&\quad - \frac{3}{4}c_{22}^2 - \frac{3}{2}c_{11}^2 c_{23} - \frac{3}{2}c_{21}c_{23} - 2c_{12}c_{32} - 2c_{11}c_{33}, \\
\dot{C}_{33}(0) &= -\frac{3}{2}c_{11}c_{12}^3 - \frac{9}{4}c_{12}^2 c_{22} - \frac{9}{2}c_{11}c_{12}c_{23} - \frac{9}{4}c_{22}c_{23} - 3c_{12}c_{33} - 3c_{11}c_{34} \\
\dot{C}_{34}(0) &= -\frac{1}{2}c_{12}^4 - 3c_{12}^2 c_{23} - \frac{3}{2}c_{23}^2 - 4c_{12}c_{34},
\end{aligned}$$

where $æ_{10}$ and $æ_{11}$ are defined in formula (9).

The next theorem permits to find $N_- = |X \cap \mathbb{C}P^1_\infty|$ from $\gamma = \partial X$.

Theorem 3.2. *Let $X \subset \mathbb{C}P^2 \setminus (0:1:0)$ be a complex curve with rectifiable boundary $\gamma \subset \mathbb{C}^2$ and satisfying $(*)$. Let $G_{m\geq 1}$ be the functions defined in (2) and let $N_- = |X \cap \mathbb{C}P^1_\infty|$. Fix any $\xi_0^* \in \Omega_0$ and let W_{ξ^*} be a neighborhood of $\xi^* = (\xi_0^*, 0)$ in $\mathbb{C}^2$. Then the following statements are valid:*

1. *If $G_1 = 0$ in W_{ξ^*}, then either $N_- = 0$, or γ' bounds a complex curve in $\mathbb{C}^2$, where γ' denotes γ with the opposite orientation.*

2. *If there exist complex constants c_{11}, c_{12} such that*

$$\tfrac{\partial}{\partial \xi_1}\big(G_1 - P_1\big) = (G_1 - P_1)\tfrac{\partial}{\partial \xi_0}(G_1 - P_1) \quad \text{in } W_{\xi^*}, \tag{25}$$

   where $P_1(\xi_0, \xi_1) = \frac{c_{11}\xi_0 + c_{12}}{1 + c_{11}\xi_1}$, then $N_- \leq 1$. Furthermore, c_{11}, c_{12} are the same constants as in Theorem 3.1.

3. *If there exist complex constants a_1, a_2, b_1, b_2, c_1, c_2 such that $c_1 + c_2 = æ_{10}$,*

$$\begin{aligned}
&\tfrac{\partial}{\partial \xi_1}((G_1 - P_1)^2 - G_2 - P_2) \\
&\qquad = ((G_1 - P_1)^2 - G_2 - P_2)\tfrac{\partial}{\partial \xi_0}(G_1 - P_1) \quad \text{in } W_{\xi^*}, \\
&\qquad P_1(\xi_0, \xi_1) = -\tfrac{a_1\xi_0 + b_1}{1 - a_1\xi_1} - \tfrac{a_2\xi_0 + b_2}{1 - a_2\xi_0}, \\
&\qquad P_2(\xi_0, \xi_1) = -\sum_{j=1}^{2}\big[\big(\tfrac{a_j\xi_0 + b_j}{1 - a_j\xi_1}\big)^2 + \tfrac{2a_jc_j}{1 - a_j\xi_1}\big],
\end{aligned} \tag{26}$$

then $N_- \leq 2$. Furthermore, these constants are related to the constants of Theorem 3.1 by the equations:

$$\begin{gathered} a_1 + a_2 = -c_{12}, \quad b_1 + b_2 = -c_{11}, \\ a_1^2 + a_2^2 = -c_{23}, \quad a_1 b_1 + a_2 b_2 = -\tfrac{1}{2} c_{22}, \\ b_1^2 + b_2^2 + 2a_1 c_1 + 2a_2 c_2 = -c_{21}, \quad c_1 + c_2 = æ_{10}. \end{gathered} \tag{27}$$

Proof. 1. The equality $G_1 = 0$ in W_{ξ^*} implies that $G_1 = 0$ for all $\xi \in \Omega_0 \times \mathbb{C}$. In turn, it implies, according to [4], the moment condition

$$\int_\gamma z_1^{k_1} z_2^{k_2}\, dz_2 = 0, \quad k_1, k_2 \geq 0.$$

Then, according to [13] and [6], for an appropriate choice of orientation, γ is the boundary of a complex curve in $\mathbb{C}^2$.

2. Set $h = G_1 - P_1$ in a neighborhood W_{ξ^*} of ξ^*. Then h satisfies

$$\begin{gathered} \tfrac{\partial^2}{\partial \xi_0^2}(G_1 - h) = 0 \quad \text{in } W_{\xi^*}, \\ \tfrac{\partial h}{\partial \xi_1} = h \tfrac{\partial h}{\partial \xi_0} \quad \text{in } W_{\xi^*}. \end{gathered}$$

It follows from Theorem 2.2 that $N_- \leq 1$. Note also that equation (10) is the restriction of (25) to $W_{\xi^*} \cap (\Omega_0 \times 0)$.

3. Consider the following quadratic equation in variable t:

$$t^2 - (G_1 - P_1)t + \tfrac{1}{2}((G_1 - P_1)^2 - G_2 - P_2) = 0. \tag{28}$$

Suppose that the discriminant is non-zero at ξ^*. Then, without loss of generality, it is non-zero in W_{ξ^*} (we can always choose a smaller neighborhood). We denote two different roots of this equation as $\widetilde{h}_1 = \widetilde{h}_1(\xi)$, $\widetilde{h}_2 = \widetilde{h}_2(\xi)$. Clearly, $\widetilde{h}_1$ and $\widetilde{h}_2$ are holomorphic in W_{ξ^*}. Furthermore, by the Vi?te formulas we have

$$\begin{gathered} \widetilde{h}_1 + \widetilde{h}_2 = \widetilde{e}_1 := G_1 - P_1 \quad \text{in } W_{\xi^*}, \\ \widetilde{h}_1 \widetilde{h}_2 = \widetilde{e}_2 := \tfrac{1}{2}((G_1 - P_1)^2 - G_2 - P_2) \quad \text{in } W_{\xi^*}. \end{gathered}$$

Note that by definition $\frac{\partial P_1}{\partial \xi_1} = \frac{1}{2}\frac{\partial P_2}{\partial \xi_0}$. Note also that by Lemma 3.3.1 of [4] we have $\frac{\partial G_1}{\partial \xi_1} = \frac{1}{2}\frac{\partial G_2}{\partial \xi_0}$. It leads to the equation

$$\frac{\partial \widetilde{e}_1}{\partial \xi_1} = \frac{1}{2}\frac{\partial}{\partial \xi_0}(\widetilde{e}_1^2 - 2\widetilde{e}_2) \quad \text{in } W_{\xi^*}. \tag{29}$$

Furthermore, equation (26) can be rewritten in the form

$$\frac{\partial \widetilde{e}_2}{\partial \xi_1} = \widetilde{e}_2 \frac{\partial \widetilde{e}_1}{\partial \xi_0} \quad \text{in } W_{\xi^*}. \tag{30}$$

Now denote by $\widehat{h}_1$, $\widehat{h}_2$ the shock-wave extensions of $\widetilde{h}_1(\cdot,0)$ and $\widetilde{h}_1(\cdot,0)$ to W_{ξ^*} which exist and are unique by the Cauchy-Kowalevski theorem. Set $\widehat{e}_1 = \widehat{h}_1 + \widehat{h}_2$, $\widehat{e}_2 = \widehat{h}_1\widehat{h}_2$. Due to the shock-wave equations for $\widehat{h}_1$ and $\widehat{h}_2$, the functions $\widehat{e}_1$ and $\widehat{e}_2$ satisfy

$$\frac{\partial \widehat{e}_1}{\partial \xi_1} = \frac{1}{2}\frac{\partial}{\partial \xi_0}(\widehat{e}_1^2 - 2\widehat{e}_2) \quad \text{in } W_{\xi^*},$$
$$\frac{\partial \widehat{e}_2}{\partial \xi_1} = \widehat{e}_1\frac{\partial \widehat{e}_2}{\partial \xi_0} \quad \text{in } W_{\xi^*}.$$

Thus, $\widetilde{e}_1$, $\widetilde{e}_2$ and $\widehat{e}_1$, $\widehat{e}_2$ are holomorphic solutions to the same system with the same restrictions at $\xi_1 = 0$. By the Cauchy-Kowalevski theorem, $\widetilde{e}_1 = \widehat{e}_1$ and $\widetilde{e}_2 = \widehat{e}_2$. It follows from the Vi?te formulas that $\widetilde{h}_1$, $\widetilde{h}_2$ coincide with $\widehat{h}_1$, $\widehat{h}_2$ (up to order). Hence, $\widetilde{h}_1$, $\widetilde{h}_2$ satisfy the shock-wave equations.

Applying Theorem 2.2, we obtain that $N_- \leq 2$.

It remains to consider the case when the determinant of equation (28) vanishes in W_{ξ^*}. Otherwise, it vanishes on (at most) a dimension one analytic set and in any neighborhood of ξ^* there are balls where it does not vanish.

The zero discriminant condition reads

$$(G_1 - P_1)^2 = 2(G_2 - P_1) \quad \text{in } W_{\xi^*}.$$

We set $\widetilde{h} = \frac{1}{2}(G_1 - P_1)$. Then by definition of P_1 and from the discriminant condition we get

$$\tfrac{\partial}{\partial \xi_0^2}(G_1 - 2\widetilde{h}) = 0 \quad \text{in } W_{\xi^*},$$
$$\tfrac{\partial \widetilde{h}}{\partial \xi_1} = \widetilde{h}\tfrac{\partial \widetilde{h}}{\partial \xi_0} \quad \text{in } W_{\xi^*}.$$

By Theorem 2.2 it implies $N_- \leq 2$. Note also that if $N_- = 2$, then all intersections of X with $\mathbb{C}^1_\xi$, $\xi \in W_{\xi^*}$, are double.

Finally, note that equation (11) is the restriction of (26) to the set $W_{\xi^*} \cap (\Omega_0 \times 0)$. □

Complement 3.2. The statement of Theorem 3.2 can be generalized to the case $N_- \geq 3$ in the spirit of cases $N_- \leq 2$. Such a generalization will be developed in a separate paper together with a statement of Theorem 3.1 for $N_- \geq 3$, indicated in Complement 3.1.

We pass to the description of the algorithm of reconstruction of a complex curve $X \subset \mathbb{C}P^2$ satisfying the minimality condition $(*)$ from the known boundary $\gamma = \partial X \subset \mathbb{C}^2$.

Let $\{\xi_0^k\}_{k=1}^N$, $\xi_0^k \in \mathbb{C}$ be an arbitrary grid in $\mathbb{C}$, $\xi_0^i \neq \xi_0^j$, $i \neq j$, and $\xi_0^k \notin \pi_2\gamma$, $k = 1, \ldots, N$. The complex curve X intersects complex line $\{z_2 = -\xi_0^k\}$ at points $(h_s(\xi_0^k, 0), -\xi_0^k)$, $1 \leq s \leq N_+(\xi_0^k, 0)$. We are going to present the formulas for finding these points.

The algorithm takes as input the points $\{\xi_0^k\}_{k=1}^N$ and the curve γ (for example, represented as a finite number of points belonging to γ). On the output of the algorithm we obtain the set of points $(h_s(\xi_0^k, 0), -\xi_0^k)$, $1 \le k \le N$; $1 \le s \le N_+(\xi_0^k, 0)$.

1. No points at infinity

1. *Computation of N_+.* According to formula (4), for every domain $\Omega_{l\ge 0}$, the number $\mu_l = N_+(\xi_0, 0)$, $\xi_0 \in \Omega_l$, is equal to the winding number of the curve $\pi_2\gamma$ with respect to a point $\xi_0 \in \Omega_l$:

$$\mu_l \equiv N_+(\xi_0, 0) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{\mathrm{d}z_2}{z_2 + \xi_0} \equiv \frac{1}{2\pi i}\int\limits_{\pi_2\gamma} \frac{\mathrm{d}z}{z - \xi_0}, \quad \xi_0 \in \Omega_l.$$

2. *Computation of power sums.* If $N_- = 0$ then, according to Theorem 3.1, for every point $\xi_0^k \in \Omega_{l\ge 0}$ we have that $P_m(\xi_0^k, 0) = 0$. Using formula (6) we obtain the following formulas for the power sums of the functions to be determined:

$$s_m(\xi_0^k) \equiv h_1^m(\xi_0^k, 0) + \cdots + h_{\mu_l}^m(\xi_0^k, 0) = \frac{1}{2\pi i}\int\limits_{\gamma} \frac{z_1^m\, dz_2}{z_2 + \xi_0^k}, \quad 1 \le m \le \mu_l.$$

3. *Computation of symmetric functions.* For every point $\xi_0^k \in \Omega_{\ge 1}$, the Newton identities

$$k\sigma_k(\xi_0^k) = \sum_{i=1}^{k} (-1)^{i-1}\sigma_{k-i}(\xi_0^k)s_i(\xi_0^k), \quad 1 \le k \le N_+(\xi_0^k, 0).$$

allow to reconstruct the elementary symmetric functions:

$$\begin{aligned} \sigma_1(\xi_0^k) &= h_1(\xi_0^k, 0) + \cdots + h_{\mu_l}(\xi_0^k, 0), \\ \cdots &= \cdots \\ \sigma_{\mu_l}(\xi_0^k) &= h_1(\xi_0^k, 0) \times \cdots \times h_{\mu_l}(\xi_0^k, 0). \end{aligned}$$

4. *Desymmetrisation.* For every point $\xi_0^k \in \Omega_l$, using Vi?te formulas, one can find the complex numbers $h_1(\xi_0^k, 0)$, $\ldots$, $h_{\mu_l}(\xi_0^k, 0)$ (up to order). The points $\big(h_s(\xi_0^k, 0), -\xi_0^k\big)$, $1 \le s \le N_+(\xi_0^k, 0)$, $1 \le k \le N$, are the required points of the complex curve X.

2. One or two points at infinity

These cases can be reduced to the case $N_- = 0$ in the following way. Since $\pi_2\gamma \subset \mathbb{C}$ is a compact real curve, there exists such $R > 0$, that the set $B_R^c(0) = \{z \in \mathbb{C} \mid |z| \geqslant R\}$

in contained in Ω_0. Without loss of generality, one can suppose that $|\xi_0^k| < R$ for all $k = 1, \ldots, N$.

Consider an auxiliary complex curve $X_R = \{(z_1, z_2) \in X \mid |z_2| \leqslant R\}$. Its boundary γ_R consists of two disjoint parts (possibly, multiconnected): the first part is γ and the second part γ'_R is obtained by lifting the circle $S_R = \{z \in \mathbb{C} \mid |z| = R\}$ to X via the projection $\pi_2 \colon X \to \mathbb{C}$.

The complex curve X_R does not intersect infinity. Moreover, points of the form $\left(a, -\xi_0^k\right)$, $k = 1, \ldots, N$ belong to X if and only if they belong to X_R.

Therefore, in order to reconstruct X it is sufficient to reconstruct γ'_R and then to reconstruct X_R, using the algorithm for the case when $N_- = 0$. The algorithm can be formulated as follows:

1. *New boundary.* Choose a sufficiently large $R > 0$, so that $B_R^c \subset \Omega_0$ and all ξ_0^k belong to B_R. In the case of $N_- = 1$, by virtue of formulas (6), we have that

$$h_1(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{z_1\, dz_2}{z_2 + \xi_0} - P_1(\xi_0, 0), \quad |\xi_0| = R,$$

   where P_1 can be found using Theorem 3.1. This formula allows to recover γ'_R and, as a corollary, $\gamma_R = \gamma \sqcup \gamma'_R = \partial X_R$.

   In the case of $N_- = 2$ we have two equalities:

$$h_1(\xi_0, 0) + h_2(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{z_1\, dz_2}{z_2 + \xi_0} - P_1(\xi_0, 0), \quad |\xi_0| = R,$$

$$h_1^2(\xi_0, 0) + h_2^2(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{z_1^2\, dz_2}{z_2 + \xi_0} - P_2(\xi_0, 0), \quad |\xi_0| = R,$$

   where P_1 and P_2 can be found using Theorem 3.1. Applying Newton identities and Vi?te formulas, we find $h_1(\xi_0, 0)$ and $h_2(\xi_0, 0)$. Thus, we have recovered γ'_R and, as a corollary, $\gamma_R = \gamma \sqcup \gamma'_R = \partial X_R$.

2. *Reduction.* In order to find the complex curve X_R with boundary $\partial X_R = \gamma_R$ we apply the algorithm of reconstruction for the case of $N_- = 0$.

4 Visualization

To our knowledge, there are, at least, two known algorithms for automatic visualisation of complex curves. The first one was proposed by Trott [12] and requires the knowledge of an analytic expression for the curve. The second algorithm was proposed by Nieser-Poelke-Polthier [11] and requires the knowledge of the branching points and their

indices for some fixed projection to $\mathbb{C}$. Our algorithm requires the knowledge of the unordered set of points of intersections of the curve with complex lines $\mathbb{C}^1_\xi$.

Let us describe in few words the algorithm of visualisation of complex curves that we use in our examples. Denote by $\pi_1\colon \mathbb{C}^2 \to \mathbb{C}$ the projection into the first factor: $\pi_1(z_1, z_2) = z_1$. Suppose that X is a complex curve in $\mathbb{C}^2$ such that the covering $\pi_1\colon X \setminus \{\text{ramification points}\} \to \mathbb{C}$ has multiplicity L. Consider, for simplicity, a rectangular grid Λ in $\mathbb{C}$:

$$\Lambda = \big\{z_1^{ij}\colon \operatorname{Re} z_1^{ij} = \frac{i}{N},\ \operatorname{Im} z_1^{ij} = \frac{j}{N},\ i, j = 0, \dots, N\big\},$$

where N is a natural number. Suppose now that we are given the set $X_\Lambda = \pi_1^{-1}(\Lambda) \cap X$ and we need to visualize the part of X lying above the rectangle $0 \le \operatorname{Re} z_1 \le 1$, $0 \le \operatorname{Im} z_1 \le 1$.

Let us introduce some terminology. We define a path in Λ as a map $\gamma\colon \{1, \dots, M\} \to \Lambda$ such that $|\gamma(k+1) - \gamma(k)| = \frac{1}{N}$ for all admissable k, where M is some natural number.

Let $\gamma\colon \{1, \dots, M\} \to \Lambda$ be a path in Λ and let $i\colon \{1, \dots, M\} \to [1, M]$ be the inclusion map. Define the function $i_*\gamma\colon [1, M] \to \mathbb{C}$ such that $i_*\gamma(k) = \gamma(k)$ for integer k and $i_*\gamma|_{[k,k+1]}$ is linear for all admissable k. It is clear that $i_*\gamma$ is a continuous function and hence it can be lifted to X by the map π_1.

We define a path in X_Λ as a map $\Gamma\colon \{1, \dots, M\} \to X_\Lambda$ such that $\gamma = \pi_1 \circ \Gamma$ is a path in Λ and $\Gamma = i^* L(i_*\gamma)$, where i^* is the pullback map with respect to i and $L(i_*\gamma)$ is some lift of $i_*\gamma$ to X by π_1, i. e. $L(i_*\gamma)$ is a continuous map from $[1, M]$ to X such that $\pi_1 \circ L(i_*\gamma) = i_*\gamma$. We also say that Γ is obtained by lifting of γ.

We will call subsets of Λ and X_Λ path-connected if every two points of these sets can be connected by a path in Λ and X_Λ, respectively.

Let us describe the practical way to lift paths in Λ to paths in X_Λ. Suppose that N is sufficiently large. Let $\gamma\colon \{1, \dots, M\} \to \Lambda$ be a path in Λ and let $\Gamma(1) \in \pi^{-1}(\gamma(1)) \cap X$ be an arbitrary point. We select $\Gamma(k) \in \pi^{-1}(\gamma(k)) \cap X$ in such a way that

$$|\Gamma(k) - \Gamma(k-1)| = \min\big\{|z - \Gamma(k-1)|\colon z \in \pi^{-1}(\gamma(k)) \cap X\big\}, \quad k = 2, \dots, M.$$

Then Γ is a path in X_Λ obtained by lifting of γ. All possible lifts of γ may be obtained by varying $\Gamma(1)$. Note that if γ is closed, i. e. $\gamma(1) = \gamma(M)$, Γ need not to be closed.

Finding of ramification points and making branch cuts. The first step in visualization procedure consists in finding of ramification points of X with respect to projection π_1. Since we have only a finite number of points on X we can find ramification points only approximately. More precisely, we will localize them in small circles.

Without restriction of generality we suppose that all ramification points are projected by π_1 into interior points of Λ. Take any interior point $z_1 \in \Lambda$ and select a small closed path $\gamma\colon \{1, \dots, M\} \to \Lambda$ around z_1 so that there is at most one ramification point inside the polygon $\overline{\gamma(1)\dots\gamma(M)}$. For example, one can take as γ the following path:

$$z_1 + \frac{1}{N} \to z_1 + \frac{1+i}{N} \to z_1 + \frac{i}{N} \to \dots \to z_1 + \frac{1-i}{N} \to z_1 + \frac{1}{N},$$

where i is the imaginary unit.

Now consider different lifts of γ to X_Λ. If at least one lift is not closed, mark z_1 as a possible ramification point (meaning that it is situated near the projection of some ramification point of X). Now vary z_1 and mark all possible ramification points. The resulting set will consist of several path-connected components each of which localizes the position of one ramification point of X with respect to π_1.

Next, we need to cut the grid Λ with branching points removed into simply connected domains. The natural choice is the shortest cut graph, which can be computed by the algorithm proposed in [10], which, in turn, generalizes the algorithm for closed surfaces of [5].

Denote the cut graph by Λ_c. An important observation is that every closed path in $\Lambda \setminus \Lambda_c$ always lifts to a closed path in X_Λ since it doesn't contain π_1-projections of ramification points inside.

Visualization. Now denote $\Lambda \setminus \Lambda_c = \cup_{s=1}^{S} \Lambda_s$, where Λ_s are different path-connected components. Take any $z_1^s \in \Lambda_s$ and $z_2^s \in \pi_1^{-1}(z_1^s) \cap X$. Now take other $z_1 \in \Lambda_s$ and connect z_1^s with z_1 by some path γ. Then γ lifts to a path Γ with $\Gamma(1) = (z_1^s, z_2^s)$ and $\Gamma(2) = (z_1, z_2)$ for some $z_2 \in \pi_1^{-1}(z_1) \cap X$ and z_2 doesn't depend on γ. Varying z_1 we thus obtain the map $\Sigma(z_1^s, z_2^s)\colon \Lambda_s \to X_\Lambda$ which allows us to visualize the part of X.

Varying $z_2^s \in \pi_1^{-1}(z_1^s) \cap X$ (the latter is the finite set, namely, it consists of L elements) we obtain the other maps $\Sigma(z_1^s, z_2^s)$ which allow us to visualize other parts of X. Clearly, the set of obtained maps doesn't depend on the choice of $z_1^s \in \Lambda_s$. Hence we can denote the obtained maps by Σ_s^l, $l = 1, \dots, L$. It is clear that $\cup_{l=1}^{L} \Sigma_s^l(\Lambda_s) = \pi^{-1}(\Lambda_s) \cap X$. Now vary s to visualize

$$\cup_{s=1}^{S} \cup_{l=1}^{L} \Sigma_s^l(\Lambda_s) = \pi^{-1}(\cup_{s=1}^{S} \Lambda_s) \cap X = X_\Lambda \setminus \pi^{-1}(\Lambda_c).$$

The part $\pi^{-1}(\Lambda_c) \cap X_\Lambda$ consists of cuts and preimages of possible ramification points. The cuts can be visualized as the already visualized part of the surface. The only problem is the visualization of π_1-preimages of possible ramification points. But the latter take a little part of the surface when N is large and one can just forget about their visualization. On the other hand, in our examples they were visualized using low-level graphics approach.

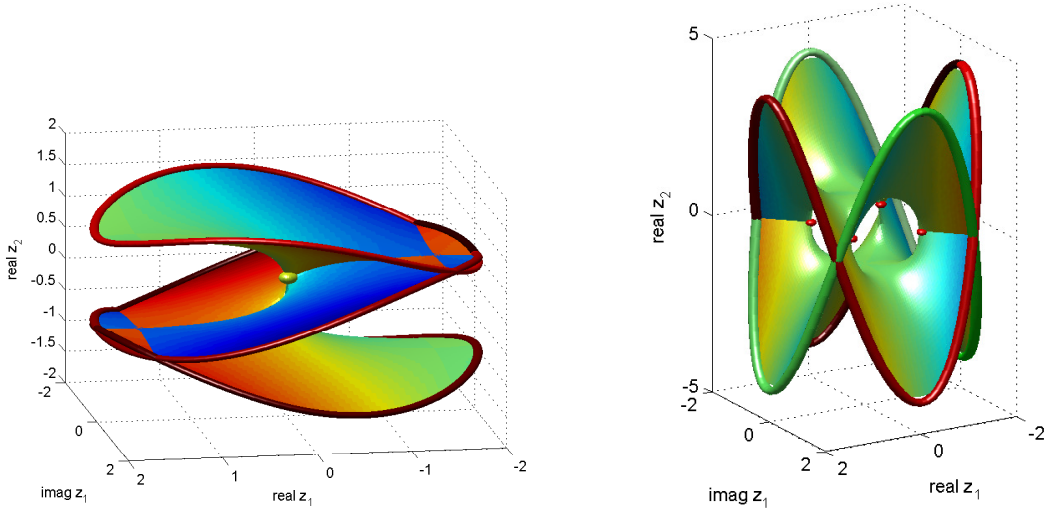

Figure 1: Rieman surfaces of functions $f(z) = \sqrt{\sin(z)}$, $|z| \leqslant 2$ (left) and $f(z) = \sqrt{z^4+1}$, $|z| \leqslant 2$ (right) obtained by the visualization algorithm.

Examples of application of this algorithm are given at Fig. 1 and in Section 5. The visualization algorithm can be easily generalized to the case of general grids. For instance, in our examples we have used a modification with periodic grid.

5 Examples

1. No points at infinity. We consider the complex curve given by the equation

$$X_1 = \big\{(z_1, z_2) \in \mathbb{C}^2 \colon z_2 + 1 = \big(z_1^2 - \exp(\tfrac{z_2}{4})\big)^2,\ |z_2| \leq 1.9\big\}. \tag{31}$$

The boundary $\gamma_1 = \partial X_1$ of this curve is the disjoint union of two real curves γ_1^r and γ_1^g, see Fig. 2. We are going to recover X_1 from γ_1.

First, note that the real curve $|z_2| = 1.9$ in $\mathbb{C}$ divides $\mathbb{C}$ into two parts $\Omega_0 = \{|z_2| > 1.9\}$ and $\Omega_1 = \{|z_2| < 1.9\}$. We check numerically that $G_1(\xi)$ is zero in a neighborhood W_{ξ^*} of some $\xi^* = (\xi_0^*, 0)$, $\xi_0 \in \Omega_0$, and thus $N_- = 0$ by Theorem 3.2.

Next, we compute the value

$$\mu_1 = \sigma_0(\xi_0) = \frac{1}{2\pi i} \int\limits_{\gamma_1} \frac{dz_2}{z_2 + \xi_0},$$

for some $\xi_0 \in \Omega_1$ (it will be the same for any $\xi_0 \in \Omega_1$). It turns out that $\mu_1 = 4$ and thus $\pi_2 \colon X_1 \to \Omega_1$ is a (ramified) covering with 4 sheets. Denote $X_1 \cap \mathbb{C}^1_\xi = \{(h_j(\xi), -\xi_0)\}_{j=1}^4$, $\xi = (\xi_0, 0)$, $|\xi_0| < 1.9$.

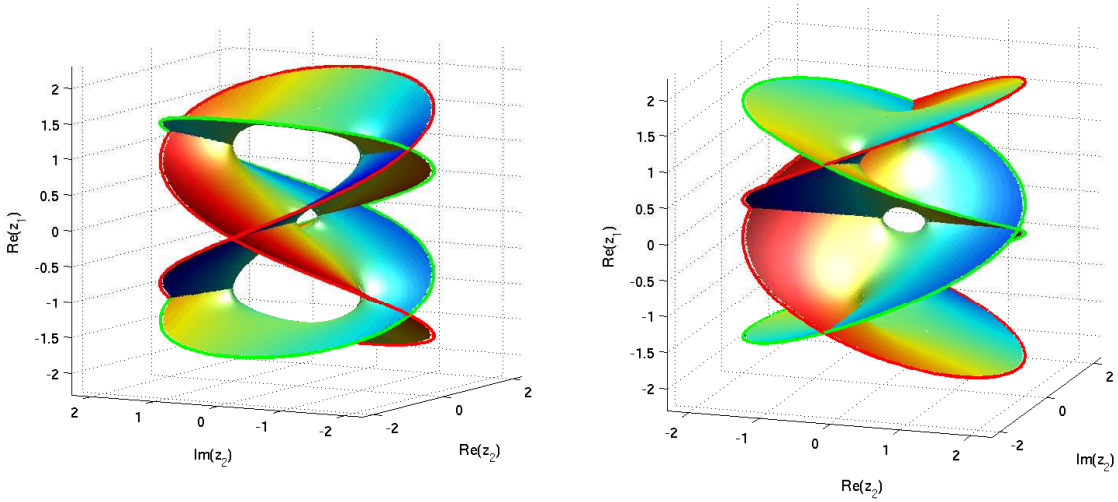

Figure 2: The given boundary $\gamma_1 = \gamma_1^r \sqcup \gamma_1^g$ (red and green) and the reconstructed complex curve $X_1 \subset \mathbb{C}^2$. The color indicates $\mathrm{Im}(z_1)$.

Next, we compute the values of the power sum functions s_k:

$$s_k(\xi_0) := \sum_{j=1}^{4} h_j^k(\xi_0, 0) = \frac{1}{2\pi i} \int\limits_{\gamma_1} \frac{z_1^k dz_2}{z_2 + \xi_0}, \quad |\xi_0| < 1.9, \ 1 \le k \le 4.$$

Using Newton identities, we determine the values of symmetric functions σ_k:

$$\sigma_1 = s_1, \quad \sigma_2 = \tfrac{1}{2}(s_1^2 - s_2),$$
$$\sigma_3 = \tfrac{1}{6}(s_1^3 - 3s_1 s_2 + 2s_3),$$
$$\sigma_4 = \tfrac{1}{24}(s_1^4 - 6s_1^2 s_2 + 3s_2^2 + 8s_1 s_3 - 6s_4).$$

Finally, taking into account Vi?te formulas, we find $h_1(\xi_0, 0), \ldots, h_4(\xi_0, 0)$ as the roots of the following polynomial in t:

$$t^{N_+(\xi_0)} - \sigma_1(\xi_0) t^{N_+(\xi_0)-1} + \cdots + (-1)^{N_+(\xi_0)} \sigma_{N_+(\xi_0,0)}(\xi_0) = 0.$$

Reconstructed surface is represented at Fig. 2.

2. One point at infinity. Consider the the complex curve X_2 which is the part of the curve

$$\widetilde{X}_2 = \left\{(w_0 : w_1 : w_2) \in \mathbb{C}P^2 \colon w_1^2 = w_2^2 + w_0^2\right\},$$

bounded by the real curve

$$\gamma_2 = \{(z_1(t), z_2(t)) \in \mathbb{C}^2 \colon t \in [0, 2\pi]\},$$
$$z_2(t) = 1.9e^{it}, \quad z_1(t) = \sqrt{z_2^2(t) + 1}, \quad z_1(0) > 0,$$

intersecting infinity at $(0:1:1)$. We suppose that the number of points at infinity is apriori unknown. We are going to reconstruct $X_{2,R} = X_2 \cap \{z_2 < R\}$, where $R = 3$, from γ_2.

Note that the curve $\pi_2\gamma_2 = \{|z_2| = 1.9\}$ divides $\mathbb{C}$ into two connected components $\Omega_0 = \{z_2 > 1.9\}$ and $\Omega_1 = \{z_2 < 1.9\}$.

According to the algorithm, the first step is to determine the number of points at infinity. We choose some $\xi^* = (\xi_0^*, 0)$ with $\xi_0^* \in \Omega_0$ and check that in some small neighborhood W_{ξ^*} of ξ^* we have $G_1 \not\equiv 0$ and thus $N_- \geq 1$ by Theorem 3.2.

Next, we fix two generic ξ_0^1, $\xi_0^2 \in \Omega_0$ and determine c_{11}, c_{12} from the linear system

$$\begin{aligned} c_{11}\tfrac{\partial G_1}{\partial \xi_0}(\xi_0^1,0) + c_{12}\big(\xi_0^1\tfrac{\partial G_1}{\partial \xi_0}(\xi_0^1,0) + G_1(\xi_0^1,0)\big) &= G_1(\xi_0^1,0)\tfrac{\partial G_1}{\partial \xi_0}(\xi_0^1,0) - \tfrac{\partial G_1}{\partial \xi_1}(\xi_0^2,0), \\ c_{11}\tfrac{\partial G_1}{\partial \xi_0}(\xi_0^2,0) + c_{12}\big(\xi_0^2\tfrac{\partial G_1}{\partial \xi_0}(\xi_0^2,0) + G_1(\xi_0^2,0)\big) &= G_1(\xi_0^2,0)\tfrac{\partial G_1}{\partial \xi_0}(\xi_0^2,0) - \tfrac{\partial G_1}{\partial \xi_1}(\xi_0^2,0). \end{aligned} \tag{32}$$

This system is uniquely solvable and gives $c_{11} \approx 1$, $c_{12} \approx 0$.

Next, we check numerically that in the neighborhood W_{ξ^*} of ξ^* the following identity holds:

$$\begin{gathered} \tfrac{\partial}{\partial \xi_1}\big(G_1 - P_1\big) = (G_1 - P_1)\tfrac{\partial}{\partial \xi_0}(G_1 - P_1), \\ \text{where } P_1(\xi_0,\xi_1) = \tfrac{c_{11}\xi_0 + c_{12}}{1 + c_{11}\xi_1}. \end{gathered} \tag{33}$$

Theorem 3.2 together with the estimate $N_- \geq 1$ imply that $N_- = 1$.

Next, according to the algorithm, in order to reduce the problem to the case of no points at infinity, we need to recover the auxilary curve $\gamma'_{2,R} = X_2 \cap \{z_2 = R\}$. Note that $\gamma'_{2,R} = \{(h_1(\xi_0,0), -\xi_0)\colon |\xi_0| = R\}$, where $h_1(\xi_0,0)$ can be found from the explicit formula

$$h_1(\xi_0,0) = G_1(\xi_0,0) - c_{11}\xi_0 - c_{12}, \quad |\xi_0| = R.$$

Next, we set $\gamma_{2,R} = \gamma_2 \sqcup \gamma'_{2,R}$ so that $\partial X_{2,R} = \gamma_{2,R}$, $X_{2,R} \cap \{|z_2| < R\} = X_2 \cap \{|z_2| < R\}$ and $X_{2,R}$ does not intersect $\mathbb{C}P^1_\infty$. Thus, it remains to recover $X_{2,R}$ from the known boundary $\gamma_{2,R}$ using the algorithm for the case of no points at infinity, see the previous example. The reconstructed curve is depicted at Fig. 3.

3. Two points at infinity. Consider the complex curve X_3, which is the part of the algebraic curve

$$\widetilde{X}_3 = \big\{(w_0\colon w_1\colon w_3) \in \mathbb{C}P^2\colon w_1^3 = (w_2 - iw_0)(w_2 + iw_0)(w_2 - w_0)\big\},$$

bounded by the real curve

$$\begin{gathered} \gamma_3 = \big\{(z_1(t), z_2(t))\colon t \in [0, 2\pi]\big\}, \\ z_2(t) = 1.9e^{it},\ z_1(t) = \sqrt[3]{(z_2(t) - i)(z_2(t) + i)(z_2(t) - 1)}, \quad z_1(0) > 0. \end{gathered}$$

The curve X_3 has two points at infinity, namely, $q_1 = (0:1:e^{\frac{2\pi i}{3}})$, $q_2 = (0:1:e^{-\frac{2\pi i}{3}})$. We suppose that the number of points at infinity is a priori unknown. We are going to reconstruct X_3 from γ_3.

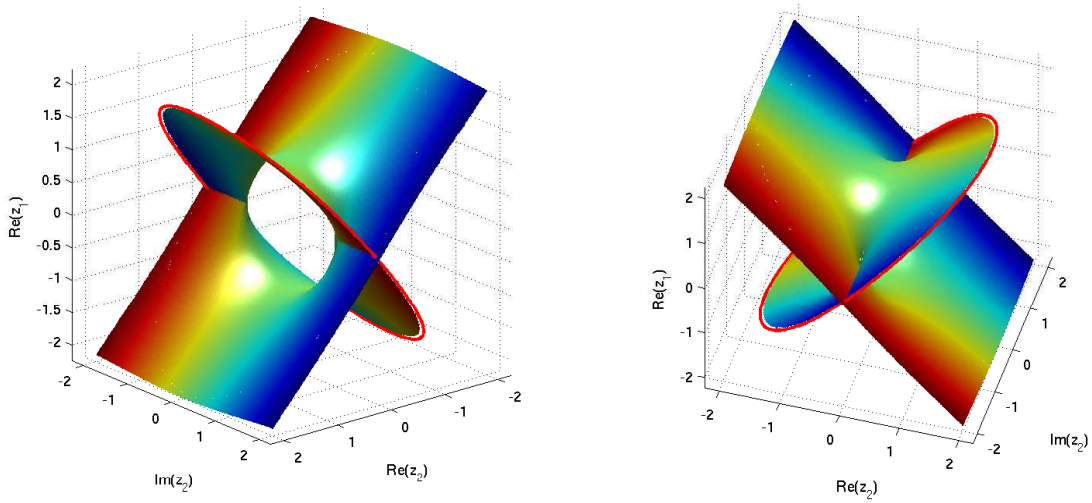

Figure 3: The real curve γ_2 (red) and the reconstructed complex curve X_2 with one point at infinity. The color indicates $\mathrm{Im}(z_1)$.

As in the previous examples, the real curve $\pi_2\gamma_2$ divides $\mathbb{C}$ into two parts $\Omega_0 = \{|z_2| > 1.3\}$ and $\Omega_1 = \{|z_2| < 1.9\}$.

The first step is to determine the number of points at infinity. Choose a generic $\xi^* = (\xi_0^*, 0)$, $\xi_0^* \in \Omega_0$, and some neighborhood W_{ξ^*} of ξ^*.

1. We check that $G_1 \not\equiv 0$ in W_{ξ^*} so that $N_- \geq 1$ by Theorem 3.2.

2. We choose two generic ξ_0^1, $\xi_0^2 \in \Omega_0$ and solve (32) for c_{11}, c_{12}. It turns out that system (32) is uniquely solvable. However, identity (33) does not hold in W_{ξ^*}. Using Theorem 3.2, we obtain that $N_- \geq 2$.

3. We consider (11) as an identity in a neighborhood of ξ_0^*. We find numerically that
$$c_{11} \approx 0.9974, \quad c_{12} \approx -0.3335,$$
$$c_{21} \approx 1.0133, \quad c_{22} \approx -0.6699, \quad c_{23} \approx 0.5567.$$
We check that (26) holds in a neighborhood of W_{ξ^*} with constants a_1, a_2, b_1, b_2, c_1, c_2 given by equations (27). If follows from Theorem 3.2 and from the inequality $N_- \geq 2$ that $N_- = 2$.

Suppose that we want to recover $X_{3,R} = X_3 \cap \{|z_2| < R\}$, $R = 3$. According to the algorithm, we need to find the auxilary boundary $\gamma'_{3,R} = X_3 \cap \{|z_2| = R\}$. Denote $\gamma'_{3,R} = \{(h_j(\xi_0, 0), -\xi_0) \colon |\xi_0| = R, j = 1, 2\}$. We compute the values of symmetric functions $\sigma_1(\xi_0) = h_1(\xi_0, 0) + h_2(\xi_0, 0)$, $\sigma_2(\xi_0) = h_1(\xi_0, 0)h_2(\xi_0, 0)$ using the explicit

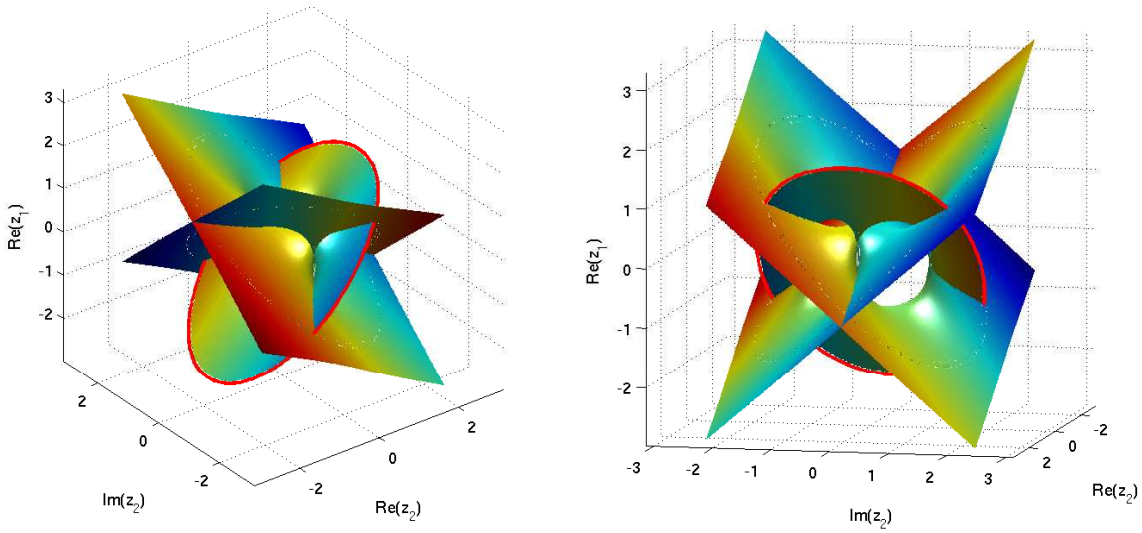

Figure 4: The real curve γ_3 (red) and the reconstructed complex curve X_3 with two points at infinity. The color indicates $\mathrm{Im}(z_1)$.

formulas

$$\sigma_1 = s_1, \quad \sigma_2 = \tfrac{1}{2}(s_1^2 - s_2),$$
$$s_1(\xi_0) := h_1(\xi_0, 0) + h_2(\xi_0, 0) = \frac{1}{2\pi i}\int_\gamma \frac{z_1 dz_2}{z_2 + \xi_0} - c_{11} - c_{12}\xi_0, \quad |\xi_0| = R,$$
$$s_2(\xi_0) := h_1^2(\xi_0, 0) + h_2^2(\xi_0, 0) = \frac{1}{2\pi i}\int_\gamma \frac{z_1^2 dz_2}{z_2 + \xi_0} - c_{21} - c_{22}\xi_0 - c_{23}\xi_0^2, \quad |\xi_0| = R.$$

Next, taking into account the Vi?te formulas, we find the values $h_1(\xi_0, 0)$, $h_2(\xi_0, 0)$, $|\xi_0| = R$, as the roots of the following polynomial in t:

$$t^2 - \sigma_1(\xi_0)t + \sigma_2(\xi_0) = 0.$$

Then we put $\gamma_{3,R} = \gamma_3 \sqcup \gamma'_{3,R}$ and recover $X_{3,R}$ from $\gamma_{3,R}$ using the algorithm for the case of no points at infinity. The result of reconstruction is depicted at Fig. 4.

References

[1] W. L. Chow, "On compact complex analytic varieties," *Amer. J. of Math.*, vol. 71, no. 4, pp. 893–914, 1949.

[2] L. Darboux, *Théorie des surfaces.* Paris: Gauthier-Villars, 1914.

[3] P. Dolbeault and G. Henkin, "Surfaces de Riemann de bord donné dans $\mathbb{C}P^n$," in *Contributions to complex analysis and analytic geometry*, ser. Aspects Math. Braunschweig: Friedr. Vieweg, 1994, vol. E26, pp. 163–187.

[4] ——, "Chaînes holomorphes de bord donné dans $\mathbb{C}P^n$," *Bull. Sci. Math.*, vol. 125, pp. 383–445, 1997.

[5] J. Erickson and K. Whittlesey, "Greedy optimal homotopy and homology generators," in *SODA*, 2005, pp. 1038–1046.

[6] F. R. Harwey and H. Lawson, "Boundaries of complex analytic varieties," *Ann. of Math.*, vol. 102, no. 2, pp. 223–290, 1975.

[7] F. R. Harwey and B. Shiffman, "A characterization of holomorphic chaines," *Ann. of Math.*, vol. 99, no. 3, pp. 553–587, 1974.

[8] G. Henkin and V. Michel, "On the explicit reconstruction of a Riemann surface from its Dirichlet-to-Neumann operator," *Geom. Funct. Anal.*, vol. 17, no. 1, pp. 116–155, 2007.

[9] J. King, "Open problems in geomtric function theory," in *Conference on geometric function theory, Problem D-1*, Katata, September, 1-6, 1978, p. 4.

[10] F. Kälberer, M. Nieser, and K. Polthier, "Stripe parametrization of tubular surfaces," in *Topology-Based Methods in Visualization III. Mathematics and Visualization.* Springer-Verlag, 2010.

[11] M. Nieser, K. Poelke, and K. Polthier, "Automatic generation of Riemann surface meshes," in *Advances in geometric modeling and processing*, ser. Lecture Notes in Computer Science. Berlin/Heidelberg: Springer, 2010, vol. 6130, pp. 161–178.

[12] M. Trott, "Visualization of Riemann surfaces of algebraic functions," *Mathematica in education and research*, vol. 6, pp. 15–36, 1997.

[13] J. Wermer, "The hull of a curve in $\mathbb{C}^n$," *Ann. of Math.*, vol. 58, no. 3, pp. 550–561, 1958.