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► **To cite this version:**

Alexey Agaltsov, Guennadi Khenkine. Explicit reconstruction of Riemann surface with given boundary in complex projective space. *Journal of Geometric Analysis*, 2015, 25 (4), pp.2450-2473. 10.1007/s12220-014-9522-1 . hal-00912925v3

**HAL Id: hal-00912925**

**<https://hal.science/hal-00912925v3>**

Submitted on 24 Mar 2017

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# Explicit reconstruction of Riemann surface with given boundary in complex projective space<sup>1</sup>

A. D. Agaltsov<sup>2</sup>, G. M. Henkin<sup>3</sup>

In this paper we propose a numerically realizable method for reconstruction of a complex curve with known boundary and without compact components in complex projective space.

**Keywords:** Riemann surface, reconstruction algorithm, Burgers equation, Cauchy-type formulas

## 1 Introduction

Let  $\mathbb{C}P^2$  be the complex projective plane with homogeneous coordinates  $(w_0 : w_1 : w_2)$ . Let  $X \subset \mathbb{C}P^2$  be a complex curve with rectifiable boundary  $\gamma = \partial X$ . Without loss of generality, we suppose that the following conditions of general position hold:

$$(0 : 1 : 0) \notin X, \quad w_0|_\gamma \neq 0.$$

Put  $\mathbb{C}^2 = \{w \in \mathbb{C}P^2 : w_0 \neq 0\}$  with coordinates  $z_1 = \frac{w_1}{w_0}$ ,  $z_2 = \frac{w_2}{w_0}$ . For almost all  $\xi = (\xi_0, \xi_1) \in \mathbb{C}^2$  the points of intersection of  $X$  with complex line  $\mathbb{C}_\xi^1 = \{z \in \mathbb{C}^2 : \xi_0 + \xi_1 z_1 + z_2 = 0\}$  form a finite set of points

$$(z_1^{(j)}(\xi), z_2^{(j)}(\xi)) = (h_j(\xi), -\xi_0 - \xi_1 h_j(\xi)), \quad \xi = (\xi_0, \xi_1), \quad j = 1, \dots, N_+(\xi).$$

By Darboux lemma [2, 4] functions  $\{h_j\}$  satisfy the following equations

$$\frac{\partial h_j(\xi)}{\partial \xi_1} = h_j(\xi) \frac{\partial h_j(\xi)}{\partial \xi_0}, \quad \xi = (\xi_0, \xi_1), \quad j = 1, \dots, N_+(\xi), \quad (1)$$

which are often called the shock-wave equations, the inviscid Burgers equations or the Riemann-Burgers equations. In this interpretation  $\xi_1$  is the time variable and  $\xi_0$  is the space variable.

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<sup>1</sup>This is an improved version of the article *A. D. Agaltsov, G. M. Henkin, Explicit reconstruction of Riemann surface with given boundary in complex projective space, Journal of Geometric Analysis 55 (10), 2015, 2450-2473*

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The following Cauchy-type formula from [4] plays the essential role in the reconstruction of  $X$  from  $\gamma$ :

$$G_m(\xi) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^m d(\xi_0 + \xi_1 z_1 + z_2)}{\xi_0 + \xi_1 z_1 + z_2} = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi) + P_m(\xi), \quad m \geq 1, \quad (2)$$

where  $N_+(\xi) = |X \cap \mathbb{C}_{\xi}^1|$  and  $P_m(\xi_0, \xi_1)$  is a polynomial of degree at most  $m$  with respect to  $\xi_0$  at fixed  $\xi_1$ . In addition,  $P_0(\xi_0, \xi_1) = -N_-$ , where  $N_- = |X \cap \mathbb{C}P_{\infty}^1|$ ,  $\mathbb{C}P_{\infty}^1 = \{w \in \mathbb{C}P^2: w_0 = 0\}$ , and

$$P_1(\xi_0, \xi_1) = - \sum_{k=1}^{N_-} \frac{a_k \xi_0 + b_k}{1 - a_k \xi_1}, \quad \text{if } h_k(\xi_0, 0) \sim a_k \xi_0 + b_k + O(\xi_0^{-1}) \text{ at } \infty. \quad (3)$$

In particular, it follows from (2) that:

$$G_0(\xi) = \frac{1}{2\pi i} \int_{\gamma} \frac{d(\xi_0 + \xi_1 z_1 + z_2)}{\xi_0 + \xi_1 z_1 + z_2} = N_+(\xi) - N_-. \quad (4)$$

Let  $\pi_2: \mathbb{C}^2 \rightarrow \mathbb{C}$  be the projection onto the second coordinate:  $\pi_2(z_1, z_2) = -z_2$ . We have that  $\mathbb{C} \setminus \pi_2 \gamma = \cup_{l=0}^L \Omega_l$ , where  $\Omega_{l \geq 0}$  are connected and  $\Omega_0$  is unbounded. From the definition of  $N_{\pm}$  it follows that

$$N_+(\xi_0, 0) = N_-, \quad \xi_0 \in \Omega_0. \quad (5)$$

Assume that complex curve  $X$  does not contain compact components without boundary, or equivalently, satisfies the following condition of minimality:

$$\begin{aligned} & \text{for any complex curve } \tilde{X} \subset \mathbb{C}P^2 \text{ such that } \partial \tilde{X} = \partial X = \gamma \\ & \text{and for almost all } \xi \in \mathbb{C}^2 \text{ we have } |\tilde{X} \cap \mathbb{C}_{\xi}^1| \geq |X \cap \mathbb{C}_{\xi}^1|. \end{aligned} \quad (*)$$

Condition of minimality (\*) is a condition of general position and is fulfilled for  $X$  if, for example, every irreducible component of  $X$  is a transcendental complex curve. Note that from theorems of Chow [1] and Harvey-Shiffman [7] it follows that an arbitrary complex curve  $\tilde{X} \subset \mathbb{C}P^2$  satisfying  $\partial \tilde{X} = \partial X$  admits the unique representation  $\tilde{X} = X \cup V$ , where  $X$  is a curve satisfying (\*), and  $V$  is a compact algebraic curve, possibly, with multiple components.

The main result of [3] gives a solution to the important problem of J. King [9], when a real curve  $\gamma \subset \mathbb{C}P^2$  is the boundary of a complex curve  $X \subset \mathbb{C}P^2$ . Let  $\gamma \subset \mathbb{C}^2$ . Then  $\gamma = \partial X$  for some open connected complex curve  $X$  in  $\mathbb{C}P^2$  if and only if in a

neighborhood  $W_{\xi^*}$  of some point  $\xi^* \in \mathbb{C}^2$  one can find mutually distinct holomorphic functions  $h_1, \dots, h_{N(\xi^*)}$  satisfying shock-wave equations (1) and also the equation

$$\frac{\partial^2}{\partial \xi_0^2} (G_1(\xi_0, \xi_1) - \sum_{j=1}^p h_j(\xi_0, \xi_1)) = 0, \quad \xi = (\xi_0, \xi_1) \in W_{\xi^*}.$$

In this work in development of [3, 4] we propose a numerically realizable algorithm for reconstruction of a complex curve  $X \subset \mathbb{C}P^2$  from the known boundary and satisfying the condition of minimality. This algorithm permits, in particular, to make applicable the result of [8] about principal possibility to reconstruct the topology and the conformal structure of a two-dimensional bordered surface  $X$  in  $\mathbb{R}^3$  with constant scalar conductivity from measurements on  $\partial X$  of electric current densities, induced by three potentials in general position.

Our algorithm depends on the number of points at infinity  $N_-$  of the complex curve  $X$ . It was tested on many examples and admits a simple and complete justification for  $N_- = 0, 1, 2$ . Despite a cumbersome description for  $N_- \geq 3$ , we show that, in principle, there are no obstacles for the justification and numerical realization for any  $N_- \geq 0$ . Moreover, in Theorem 3.2 we propose a method for finding the parameter  $N_-$  in terms of  $\gamma$ . This makes the algorithm much more applicable.

## 2 Cauchy-type formulas and Riemann-Burgers equations

We begin by giving a new proof of the Cauchy type formulas (2) from [4], which allows to obtain explicit expressions for functions  $P_m$ .

**Theorem 2.1.** *Let  $X \subset \mathbb{C}P^2 \setminus (0 : 1 : 0)$  be a complex curve with rectifiable boundary  $\gamma \subset \mathbb{C}^2$  and satisfying (\*). Suppose that for almost all  $\xi \in \mathbb{C}^2$  all the points of intersection of  $X$  with  $\mathbb{C}_\xi^1$  have multiplicity at most one. Then the following formulas hold for almost all  $\xi = (\xi_0, \xi_1) \in \mathbb{C}^2$ :*

$$G_m(\xi) = \sum_{j=1}^{N_+(\xi)} h_j^m(\xi) + P_m(\xi), \quad m \geq 1, \quad (6)$$

where  $P_m(\xi)$  is holomorphic in a neighborhood of almost all  $(\xi_0, 0)$ ,  $\xi_0 \in \mathbb{C} \setminus \pi_2(\gamma)$ , and is polynomial in  $\xi_0$  of degree at most  $m$  for any fixed  $\xi_1$ . Furthermore, the following

explicit formulas hold:

$$\begin{aligned}
P_m(\xi_0, \xi_1) = & \\
& \sum_{s=1}^{N_-} \sum_{k=0}^{m-1} \sum_{i_1+\dots+i_m=k} \frac{\frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)!} \frac{d^{m-k}}{dw_0^{m-k}} \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s} \\
& - \sum_{s=1}^{\mu_0} \sum_{i_1+\dots+i_m=m} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s),
\end{aligned}$$

where  $X \cap \mathbb{C}P_\infty^1 = \{q_1, \dots, q_{N_-}\}$ . In particular, if  $N_- = 0$  then  $P_m = 0$ ,  $m \geq 1$ .

*Proof.* Put  $\tilde{g} = \xi_0 w_0 + \xi_1 w_1 + w_2$  and  $g = \frac{\tilde{g}}{w_0} = \xi_0 + \xi_1 z_1 + z_2$ . Consider differential forms

$$\omega_m \stackrel{\text{def}}{=} z_1^m \frac{dg}{g} = \frac{w_1^m w_0}{w_0^m \tilde{g}} d\left(\frac{\tilde{g}}{w_0}\right) = \frac{w_1^m d\tilde{g}}{w_0^m \tilde{g}} - \frac{w_1^m}{w_0^{m+1}} dw_0, \quad m = 0, 1, \dots$$

Then  $G_m(\xi) = \frac{1}{2\pi i} \int_\gamma \omega_m$ . Let us compute this integral explicitly. Denote by  $p_j$ ,  $j = 1, \dots, N_+(\xi)$  the points of intersection of  $X$  with  $\mathbb{C}P_\xi^1$ , and by  $q_s$ ,  $s = 1, \dots, N_-$  the points of intersection of  $X$  with infinity  $\mathbb{C}P_\infty^1$ . Denote by  $B_j^\varepsilon$  the intersection of  $X$  with the ball of radius  $\varepsilon$  in  $\mathbb{C}P^2$  centered at  $p_j$  and by  $D_s^\varepsilon$  the intersection of  $X$  with the ball of radius  $\varepsilon$  centered at  $q_s$ . The restriction of form  $\omega_m$  on  $X$  is meromorphic with poles at points  $p_j$  and  $q_s$ . Thus the following equality is valid:

$$G_m(\xi) = \frac{1}{2\pi i} \int_\gamma \omega_m = \sum_{j=1}^{N_+(\xi)} \frac{1}{2\pi i} \int_{bB_j^\varepsilon} \omega_m + \sum_{s=1}^{N_-} \frac{1}{2\pi i} \int_{bD_s^\varepsilon} \omega_m.$$

If  $N_- = 0$ , then the second group of terms is absent. The integral  $\int_{bB_j^\varepsilon} \omega_m$  can be calculated as a residue at the first order pole:

$$\int_{bB_j^\varepsilon} \omega_m = \int_{bB_j^\varepsilon} z_1^m \frac{d\tilde{g}}{\tilde{g}} - \int_{bB_j^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = \int_{bB_j^\varepsilon} z_1^m \frac{d\tilde{g}}{\tilde{g}} = 2\pi i h_j^m(\xi).$$

Let  $N_- > 0$ . Computation of integral  $\int_{bD_s^\varepsilon} \omega_m$  will be done in two steps. Let us calculate first  $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0$ . Consider the expansion of  $w_1(w_0)$  into power series in  $w_0$  in the neighborhood of point  $q_s$ :

$$w_1(w_0) = w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2w_1}{dw_0^2}(q_s)w_0^2 + \dots$$

Note further that

$$w_1^m(w_0) = \sum_{k=0}^{\infty} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^k.$$

The coefficient near  $w_0^m$  can be presented in the form

$$\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^{m+1}} dw_0 = 2\pi i \sum_{i_1+\dots+i_m=m} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s).$$

Now we can calculate the integral  $\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{\tilde{g}}$ . Using relation  $d\tilde{g} = \frac{d\tilde{g}}{dw_0} dw_0$  and expansion of  $w_1(w_0)$  into power series in  $w_0$  we obtain:

$$\begin{aligned} \int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{\tilde{g}} &= \int_{bD_s^\varepsilon} \frac{1}{w_0^m} \left( w_1(q_s) + \frac{dw_1}{dw_0}(q_s)w_0 + \frac{d^2w_1}{dw_0^2}(q_s)w_0^2 + \cdots \right)^m \frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} dw_0 = \\ &= \sum_{k=0}^{\infty} \int_{bD_s^\varepsilon} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} dw_0 = \\ &= \sum_{k=0}^{m-1} \int_{bD_s^\varepsilon} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) w_0^{k-m} \frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} dw_0 = \\ &= \sum_{k=0}^{m-1} \frac{2\pi i}{(m-k-1)!} \sum_{i_1+\dots+i_m=k} \frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s) \lim_{w_0 \rightarrow 0} \frac{d^{m-k-1}}{dw_0^{m-k-1}} \left( \frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} \right). \end{aligned}$$

From here, taking into account the relation  $\frac{d\tilde{g}}{dw_0} \frac{1}{\tilde{g}} = \frac{d \ln \tilde{g}}{dw_0}$ , we obtain, finally

$$\int_{bD_s^\varepsilon} \frac{w_1^m}{w_0^m} \frac{d\tilde{g}}{\tilde{g}} = 2\pi i \sum_{k=0}^{m-1} \sum_{i_1+\dots+i_m=k} \frac{\frac{d^{i_1}w_1}{dw_0^{i_1}}(q_s) \cdots \frac{d^{i_m}w_1}{dw_0^{i_m}}(q_s)}{(m-k-1)!} \frac{d^{m-k}}{dw_0^{m-k}} \ln(\xi_0 w_0 + \xi_1 w_1 + w_2)|_{q_s}.$$

It is a polynomial of degree at most  $m$  with respect to  $\xi_0$ . □

**Remark 2.1.** We excluded the case  $(0 : 1 : 0) \in X$  because the point  $(0 : 1 : 0)$  can lead to a non-polynomial contribution in  $\xi_0$  in functions  $P_m$ .

Consider, for example, the projective curve  $\tilde{X}: w_0^2 = w_1 w_2$  and the meromorphic 1-form  $\Omega_{\xi_0}$  on it:

$$\Omega_{\xi_0} = \frac{z_1 dz_2}{z_2 + \xi_0}, \quad \xi_0 \in \mathbb{C} \setminus 0,$$

where  $z_1 = \frac{w_1}{w_0}$ ,  $z_2 = \frac{w_2}{w_0}$ . The form  $\Omega_{\xi_0}$  is holomorphic on  $X$  except the points  $q_1 = (0 : 0 : 1)$ ,  $q_2 = (0 : 1 : 0)$  and  $p_{\xi_0} = (-\xi_0 : 1 : \xi_0^2)$ .

One can see that the residues of  $\overline{\Omega_{\xi_0}}$  at  $q_1, q_2$  are equal to 0 and  $1/\xi_0$ , respectively. It follows that the residue at  $p_{\xi_0}$  is equal to  $-1/\xi_0$ .

Now, consider the part  $X$  of  $\tilde{X}$  obtained by cutting off some small neighborhood of the point  $q_1$ . Let  $\gamma$  denote the boundary of  $X$ . Choose any  $\xi = (\xi_0, 0)$  such that  $X \cap \mathbb{C}_\xi^1 \neq \emptyset$ . Then  $X \cap \mathbb{C}_\xi^1 = \{(-1/\xi_0, -\xi_0)\}$  and, as far as the residue of  $\Omega_{\xi_0}$  at  $q_1$  is zero, we have that

$$0 = G_1(\xi_0, 0) = \frac{1}{2\pi i} \int_\gamma \Omega_{\xi_0} = -\frac{1}{\xi_0} + P_1(\xi_0, 0),$$

so that  $P_1(\xi_0, 0) = 1/\xi_0$ .

We will also use the following result of [8], which gives an effective characterization of functions  $h_{j \geq 1}$  of Theorem 2.1.

**Theorem 2.2** (Remark 4 to Theorem 3a of [8]). *Let  $X \subset \mathbb{C}P^2 \setminus (0 : 1 : 0)$  be a complex curve with rectifiable boundary  $\gamma \subset \mathbb{C}^2$  and satisfying (\*). Fix any  $\xi_0^* \in \Omega_0$  and let  $W_{\xi^*}$  be a neighborhood of  $\xi^* = (\xi_0^*, 0)$  in  $\mathbb{C}^2$ . Let  $G_{k \geq 1}, h_{j \geq 1}$  be defined as in (2) and (6). Suppose that there exist functions  $\tilde{h}_1, \dots, \tilde{h}_p$ , holomorphic in  $W_{\xi^*}$  and satisfying*

$$\frac{\partial^2}{\partial \xi_0^2} (G_1 - \tilde{h}_1 - \dots - \tilde{h}_p) = 0 \quad \text{in } W_{\xi^*}, \quad (7)$$

$$\frac{\partial \tilde{h}_k}{\partial \xi_1} = \tilde{h}_k \frac{\partial \tilde{h}_k}{\partial \xi_0} \quad \text{in } W_{\xi^*}, \quad k = 1, \dots, p. \quad (8)$$

*Then  $p \geq N_-$ . Furthermore, if  $p = N_- := |X \cap \mathbb{C}P_\infty^1|$ , then  $h_1, \dots, h_{N_-}$  coincide with  $\tilde{h}_1, \dots, \tilde{h}_{N_-}$  in  $W_{\xi^*}$  (up to order).*

### 3 Reconstruction algorithm

We now pass to the reconstruction algorithm for a complex curve  $X \subseteq \mathbb{C}P^2$  with given boundary  $\partial X$  and satisfying the condition of minimality (\*). Let us consider the cases  $N_- = 0, 1, 2$ .

The reconstruction algorithm is based on formulas (6). The next theorem permits to find the functions  $P_m$  of (6). We will use the notation

$$\mathfrak{a}_{kl} = \frac{1}{2\pi i} \int_\gamma z_1^k z_2^l dz_2, \quad k, l \geq 0. \quad (9)$$

**Theorem 3.1.** *Let  $X \subset \mathbb{C}P^2$ ,  $(0 : 1 : 0) \notin X$ , be a complex curve with rectifiable boundary  $\gamma \subset \mathbb{C}^2$  and satisfying (\*). Let  $h_{j \geq 1}$  and  $P_{m \geq 1}$  be the functions defined in Theorem 2.1. Then the following statements are valid:*

1. If  $N_- = 0$ , then  $P_m = 0$  for all  $m \geq 1$ . Besides,  $G_1 = 0$  in a neighborhood of any  $\xi^* = (\xi_0^*, 0)$  with  $\xi_0^* \in \Omega_0$ .
2. If  $N_- = 1$ , then  $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$ , where constants  $c_{11}$  and  $c_{12}$  satisfy the following identity in  $\xi = (\xi_0, 0) \in \Omega_0 \times 0$ :

$$c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi) + c_{12}(\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi) + G_1(\xi)) = G_1(\xi) \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi). \quad (10)$$

3. If  $N_- = 2$ , then  $P_1(\xi_0, 0) = c_{11} + c_{12}\xi_0$ ,  $P_2(\xi_0, 0) = c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2$ , where constants  $c_{11}$ ,  $c_{12}$ ,  $c_{21}$ ,  $c_{22}$ ,  $c_{23}$  satisfy the following identity in  $\xi = (\xi_0, 0) \in \Omega_0 \times 0$ :

$$\begin{aligned} \mathfrak{ae}_{10}(c_{12}^2 + c_{23}) &= \frac{\partial G_2}{\partial \xi_1} - 2 \frac{\partial G_1}{\partial \xi_1} (G_1 - c_{11} - c_{12}\xi_0) + G_1(c_{22} + 2c_{23}\xi_0) \\ &+ \frac{\partial G_1}{\partial \xi_0} \cdot ((G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2) \\ &+ (G_1^2 - 2c_{11}G_1 - 2c_{12}G_1\xi_0 - G_2) \cdot (-c_{12}), \end{aligned} \quad (11)$$

where all the functions are evaluated at point  $\xi = (\xi_0, 0)$ .

*Proof.* 1. By Theorem 2.1, if  $N_- = 0$  then  $P_m = 0$  for all  $m \geq 1$ . It also follows from formula (6) that  $G_1 = 0$  in a neighborhood of any  $\xi^* = (\xi_0^*, 0)$  with  $\xi_0^* \in \Omega_0$ .

2. It follows from Theorem 2.1 that  $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$  in a neighborhood of almost all  $\xi^* = (\xi_0^*, 0) \in \Omega_l \times 0$ , where  $C_{11}$ ,  $C_{12}$  are holomorphic in a neighborhood of zero. We need to find constants  $c_{11} = C_{11}(0)$  and  $c_{12} = C_{12}(0)$ .

We differentiate the equation (6) with respect to  $\xi_0$  and restrict this equation and its differentiated version to  $\Omega_l \times 0$ :

$$\begin{aligned} h_1(\xi_0, 0) &= G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0, \\ \frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0, \\ \frac{\partial h_1}{\partial \xi_0}(\xi_0, 0) &= \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12}, \end{aligned} \quad (12)$$

where  $\xi_0 \in \Omega_0$ . By (1) there is the equality  $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0) = h_1(\xi_0, 0) \frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$ . We substitute  $h_1(\xi_0, 0)$ ,  $\frac{\partial h_1}{\partial \xi_1}(\xi_0, 0)$ ,  $\frac{\partial h_1}{\partial \xi_0}(\xi_0, 0)$  in this equation by their expressions (12), and we obtain the equation

$$\begin{aligned} &\frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 = \\ &= \left( G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0 \right) \left( \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - c_{12} \right). \end{aligned} \quad (13)$$



This equation is valid, in particular, for  $\xi_0 \in \Omega_0$ . We divide it by  $\xi_0$  and tend  $\xi_0 \rightarrow \infty$ . As a result, we obtain the equality  $\dot{C}_{12}(0) = -c_{12}^2$ . Taking into account this equality, we can rewrite equation (13) in the form

$$\begin{aligned} & \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) = \\ & = \left( G_1(\xi_0, 0) - c_{11} - c_{12}\xi_0 \right) \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) - \left( G_1(\xi_0, 0) - c_{11} \right) c_{12}. \end{aligned} \quad (14)$$

Taking into account that  $\xi_0 \frac{\partial G_1}{\partial \xi_0}(\xi_0, 0) \rightarrow 0$  as  $\xi_0 \rightarrow \infty$  and passing  $\xi_0 \rightarrow \infty$  in (14), we obtain the equality  $\dot{C}_{11}(0) = -c_{11}c_{12}$ . Substituting this explicit expression for  $\dot{C}_{11}(0)$  into (14), we get (10).

3. By (1) functions  $h_1(\xi)$  and  $h_2(\xi)$  satisfy the Riemann-Burgers equation in a neighborhood of any  $\xi^* = (\xi_0^*, 0) \in \Omega_0 \times 0$ , so that the following equalities are valid:

$$\frac{\partial(h_1 h_2)}{\partial \xi_1} = h_1 \frac{\partial h_2}{\partial \xi_1} + \frac{\partial h_1}{\partial \xi_1} h_2 = h_1 h_2 \frac{\partial(h_1 + h_2)}{\partial \xi_0}, \quad (15)$$

$$\frac{\partial(h_1^2 + h_2^2)}{\partial \xi_0} = 2h_1 \frac{\partial h_1}{\partial \xi_0} + 2h_2 \frac{\partial h_2}{\partial \xi_0} = 2 \frac{\partial(h_1 + h_2)}{\partial \xi_1}. \quad (16)$$

Note that  $h_1 h_2 = \frac{1}{2}(h_1 + h_2)^2 - \frac{1}{2}(h_1^2 + h_2^2)$ . Therefore the system (15)–(16) is equivalent to the system

$$\frac{\partial(h_1 + h_2)^2}{\partial \xi_1} - \frac{\partial(h_1^2 + h_2^2)}{\partial \xi_1} = \left( (h_1 + h_2)^2 - (h_1^2 + h_2^2) \right) \frac{\partial(h_1 + h_2)}{\partial \xi_0}, \quad (17)$$

$$\frac{\partial(h_1^2 + h_2^2)}{\partial \xi_0} = 2 \frac{\partial(h_1 + h_2)}{\partial \xi_1}. \quad (18)$$

We substitute the expressions of (6) for  $h_1^2 + h_2^2$  and  $h_1 + h_2$  into (17), (18), using the notations  $P_1(\xi_0, \xi_1) = C_{11}(\xi_1) + C_{12}(\xi_1)\xi_0$ ,  $P_2(\xi_0, \xi_1) = C_{21}(\xi_1) + C_{22}(\xi_1)\xi_0 + C_{23}(\xi_1)\xi_0^2$ . Then, equation (18) restricted to  $\Omega_0 \times 0$  takes the form

$$\frac{\partial G_2}{\partial \xi_0}(\xi_0, 0) - c_{22} - 2c_{23}\xi_0 = 2 \left( \frac{\partial G_1}{\partial \xi_1}(\xi_0, 0) - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 \right). \quad (19)$$

We divide this equation by  $\xi_0$  and tend  $\xi_0 \rightarrow \infty$ . It leads to the equality  $\dot{C}_{12}(0) = c_{23}$ . Taking this equality into account and passing  $\xi_0 \rightarrow \infty$  in (19), we obtain the equality  $\dot{C}_{11}(0) = \frac{1}{2}c_{22}$ .

Next, we substitute the expressions of (6) for  $h_1^2 + h_2^2$  and  $h_1 + h_2$  into (17) and restrict the obtained formula to  $\Omega_0 \times 0$ . It leads to the equality

$$\begin{aligned} & 2(G_1 - c_{11} - c_{12}\xi_0) \left( \frac{\partial G_1}{\partial \xi_1} - \dot{C}_{11}(0) - \dot{C}_{12}(0)\xi_0 \right) - \frac{\partial G_2}{\partial \xi_1} + \dot{C}_{21}(0) + \dot{C}_{22}(0)\xi_0 + \dot{C}_{23}(0)\xi_0^2 = \\ & = \left( (G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \left( \frac{\partial G_1}{\partial \xi_0} - c_{12} \right). \end{aligned} \quad (20)$$

We divide this equation by  $\xi_0^2$  and tend  $\xi_0 \rightarrow \infty$ . This leads to the equality

$$2c_{12}\dot{C}_{12}(0) + \dot{C}_{23}(0) = -(c_{12}^2 + c_{23})c_{12}.$$

Taking this equality into account, dividing (20) by  $\xi_0$  and passing  $\xi_0 \rightarrow \infty$ , we obtain the equality

$$2c_{11}\dot{C}_{12}(0) + 2c_{12}\dot{C}_{11}(0) + \dot{C}_{22}(0) = -(2c_{11}c_{12} + c_{22})c_{12}.$$

Using the obtained equalities, one can rewrite (20) in the form

$$\begin{aligned} 2(G_1 - c_{11} - c_{12}\xi_0)\frac{\partial G_1}{\partial \xi_1} - 2G_1(\dot{C}_{11}(0) + \dot{C}_{12}(0)\xi_0) + 2c_{11}\dot{C}_{11}(0) - \frac{\partial G_2}{\partial \xi_1} + \dot{C}_{21}(0) = \\ = \left( (G_1 - c_{11} - c_{12}\xi_0)^2 - G_2 + c_{21} + c_{22}\xi_0 + c_{23}\xi_0^2 \right) \frac{\partial G_1}{\partial \xi_0} + \\ + \left( (G_1 - c_{11})^2 - 2G_1c_{12}\xi_0 - G_2 + c_{21} \right) (-c_{12}). \end{aligned} \quad (21)$$

We pass  $\xi_0 \rightarrow \infty$  and note that the following relations are valid

$$\begin{aligned} \lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{\partial G_1}{\partial \xi_1} &= \lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_1}{\xi_0 + z_2} = \frac{1}{2\pi i} \int_{\gamma} z_1 dz_1 = 0, \\ \lim_{\xi_0 \rightarrow \infty} \xi_0 G_1 &= \lim_{\xi_0 \rightarrow \infty} \xi_0 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{\xi_0 + z_2} = \frac{1}{2\pi i} \int_{\gamma} z_1 dz_2 = \mathfrak{a}_{10}, \\ \lim_{\xi_0 \rightarrow \infty} \xi_0^2 \frac{\partial G_1}{\partial \xi_0} &= - \lim_{\xi_0 \rightarrow \infty} \xi_0^2 \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{(\xi_0 + z_2)^2} = - \frac{1}{2\pi i} \int_{\gamma} z_1 dz_2 = -\mathfrak{a}_{10}. \end{aligned}$$

As a result, we obtain

$$-2\mathfrak{a}_{10}\dot{C}_{12}(0) + 2c_{11}\dot{C}_{11}(0) + \dot{C}_{21}(0) = -(c_{12}^2 + c_{23})\mathfrak{a}_{10} - (c_{11}^2 - 2c_{12}\mathfrak{a}_{10} + c_{21})c_{12}.$$

Due to the obtained relations, we can express constants  $\dot{C}_{ij}(0)$  as functions of  $c_{ij}$ :

$$\begin{aligned} \dot{C}_{11}(0) &= \frac{1}{2}c_{22}, \\ \dot{C}_{12}(0) &= c_{23}, \\ \dot{C}_{23}(0) &= -c_{12}^3 - 3c_{12}c_{23}, \\ \dot{C}_{22}(0) &= -2(c_{11}c_{12}^2 + c_{12}c_{22} + c_{11}c_{23}), \\ \dot{C}_{21}(0) &= \mathfrak{a}_{10}(c_{12}^2 + c_{23}) - c_{12}(c_{11}^2 + c_{21}) - c_{11}c_{22}. \end{aligned} \quad (22)$$

Substituting these constants to (21), we obtain the third statement of Theorem 3.1.  $\square$

**Complement 3.1.** The statement of Theorem 3.1 admits a development for the case  $N_- \geq 3$ . In this case

$$P_k(\xi_0, \xi_1) = C_{k1}(\xi_1) + C_{k2}(\xi_1)\xi_0 + \cdots + C_{k,k+1}(\xi_1)\xi_0^k, \quad k = 1, \dots, N_-.$$

Denote  $\dot{C}_{ij}(0) = \frac{\partial C_{ij}}{\partial \xi_1}(0)$  and  $c_{ij} = C_{ij}(0)$  for  $i = 1, \dots, N_-$  and  $j = 1, \dots, i + 1$ .

Let us indicate the following general procedure for finding of constants  $c_{ij}$ . Due to the Riemann–Burgers equations (1) the following identities in  $\xi_0 \in \Omega_0$  hold for  $k = 1, \dots, N_- - 1$ :

$$\begin{aligned} & -\frac{\partial G_k}{\partial \xi_1}(\xi_0, 0) + \dot{C}_{k1}(0) + \dot{C}_{k2}(0)\xi_0 + \cdots + \dot{C}_{k,k+1}(0)\xi_0^k \\ &= \frac{k}{k+1} \left( -\frac{\partial G_{k+1}}{\partial \xi_0}(\xi_0, 0) + c_{k+1,2} + 2c_{k+1,3}\xi_0 + \cdots + (k+1)c_{k+1,k+2}\xi_0^k \right). \end{aligned}$$

Taking into account that  $\frac{\partial G_k}{\partial \xi_1}(\xi_0, 0) \rightarrow 0$  and  $\frac{\partial G_{k+1}}{\partial \xi_0}(\xi_0, 0) \rightarrow 0$  as  $\xi_0 \rightarrow +\infty$  we obtain the equalities

$$\dot{C}_{k,m}(0) = \frac{km}{k+1}c_{k+1,m+1}, \quad k = 1, \dots, \mu_0 - 1, \quad m = 1, \dots, k + 1.$$

Due to the Riemann–Burgers equations (1) the following identity in  $\xi_0 \in \Omega_0$  holds:

$$\frac{\partial e_{\mu_0}}{\partial \xi_1}(\xi_0, 0) = e_{\mu_0}(\xi_0, 0) \frac{\partial p_1}{\partial \xi_0}(\xi_0, 0), \quad (23)$$

where functions  $e_k$  are given by the following formulas:

$$\begin{aligned} k e_k(\xi_0, \xi_1) &= \sum_{i=1}^{k-1} (-1)^{i+1} e_{k-i}(\xi_0, \xi_1) p_i(\xi_0, \xi_1) + (-1)^{k+1} p_k(\xi_0, \xi_1), \\ p_k(\xi_0, \xi_1) &= G_k(\xi_0, \xi_1) - C_{k1}(\xi_1) - C_{k2}(\xi_1)\xi_0 - \cdots - C_{k,k+1}(\xi_1)\xi_0^k, \end{aligned} \quad (24)$$

where  $k = 1, \dots, \mu_0$ .

Equality (23) allows to represent constants  $\{\dot{C}_{\mu_0,j}(0)\}$  as functions of constants  $\{c_{ij}\}$ . Finally, substituting the obtained expressions for constants  $\{\dot{C}_{ij}(0)\}$  via constants  $\{c_{ij}\}$  into equation (23) we obtain the identity in  $\xi_0 \in \Omega_0$  for computation of constants  $\{c_{ij}\}$ .

For example, in the case  $N_- = 3$  the identity (23) in  $\xi_0 \in \Omega_0$  for finding of constants  $c_{ij}$  takes the form

$$\begin{aligned} & \dot{C}_{31}(0) + \dot{C}_{32}(0)\xi_0 + \dot{C}_{33}(0)\xi_0^2 + \dot{C}_{34}(0)\xi_0^3 = \frac{\partial G_3}{\partial \xi_1} + \\ & + \frac{3}{4}(p_1^2 - p_2) \frac{\partial p_2}{\partial \xi_0} - p_1 \frac{\partial p_3}{\partial \xi_0} - \frac{1}{2}(p_1^3 - 3p_1 p_2 + 2p_3) \frac{\partial p_1}{\partial \xi_0}, \end{aligned}$$

where all functions are evaluated at point  $(\xi_0, 0)$ , the functions  $p_k$  are defined in formula (24) and the constants  $\dot{C}_{31}(0)$ ,  $\dot{C}_{32}(0)$ ,  $\dot{C}_{33}(0)$ ,  $\dot{C}_{34}(0)$  are given by formulas

$$\begin{aligned}\dot{C}_{31}(0) &= \frac{1}{2}\mathfrak{a}_{10}(3c_{11}c_{12}^2 + 3c_{12}c_{22} + 3c_{11}c_{23} + 2c_{33}) \\ &\quad - \frac{1}{2}\mathfrak{a}_{11}(c_{12}^3 + 3c_{12}c_{23} + 2c_{34}) - \frac{3}{2} \frac{1}{2\pi i} \int_{\gamma} z_1 z_2 dz_1 (c_{12}^2 + c_{23}) \\ &\quad - \frac{1}{2}c_{11}^3 c_{12} - \frac{3}{2}c_{11}c_{12}c_{21} - \frac{3}{4}c_{11}^2 c_{22} - \frac{3}{4}c_{21}c_{22} - c_{12}c_{31} - c_{11}c_{32}, \\ \dot{C}_{32}(0) &= \mathfrak{a}_{10}(c_{12}^3 + 3c_{12}c_{23} + 2c_{34}) - \frac{3}{2}c_{11}^2 c_{12}^2 - \frac{3}{2}c_{12}^2 c_{21} - 3c_{11}c_{12}c_{22} \\ &\quad - \frac{3}{4}c_{22}^2 - \frac{3}{2}c_{11}^2 c_{23} - \frac{3}{2}c_{21}c_{23} - 2c_{12}c_{32} - 2c_{11}c_{33}, \\ \dot{C}_{33}(0) &= -\frac{3}{2}c_{11}c_{12}^3 - \frac{9}{4}c_{12}^2 c_{22} - \frac{9}{2}c_{11}c_{12}c_{23} - \frac{9}{4}c_{22}c_{23} - 3c_{12}c_{33} - 3c_{11}c_{34} \\ \dot{C}_{34}(0) &= -\frac{1}{2}c_{12}^4 - 3c_{12}^2 c_{23} - \frac{3}{2}c_{23}^2 - 4c_{12}c_{34},\end{aligned}$$

where  $\mathfrak{a}_{10}$  and  $\mathfrak{a}_{11}$  are defined in formula (9).

The next theorem permits to find  $N_- = |X \cap \mathbb{C}P_{\infty}^1|$  from  $\gamma = \partial X$ .

**Theorem 3.2.** *Let  $X \subset \mathbb{C}P^2 \setminus (0 : 1 : 0)$  be a complex curve with rectifiable boundary  $\gamma \subset \mathbb{C}^2$  and satisfying (\*). Let  $G_{m \geq 1}$  be the functions defined in (2) and let  $N_- = |X \cap \mathbb{C}P_{\infty}^1|$ . Fix any  $\xi_0^* \in \Omega_0$  and let  $W_{\xi^*}$  be a neighborhood of  $\xi^* = (\xi_0^*, 0)$  in  $\mathbb{C}^2$ . Then the following statements are valid:*

1. *If  $G_1 = 0$  in  $W_{\xi^*}$ , then either  $N_- = 0$ , or  $\gamma'$  bounds a complex curve in  $\mathbb{C}^2$ , where  $\gamma'$  denotes  $\gamma$  with the opposite orientation.*
2. *If there exist complex constants  $c_{11}, c_{12}$  such that*

$$\frac{\partial}{\partial \xi_1}(G_1 - P_1) = (G_1 - P_1) \frac{\partial}{\partial \xi_0}(G_1 - P_1) \quad \text{in } W_{\xi^*}, \quad (25)$$

*where  $P_1(\xi_0, \xi_1) = \frac{c_{11}\xi_0 + c_{12}}{1 + c_{11}\xi_1}$ , then  $N_- \leq 1$ . Furthermore,  $c_{11}, c_{12}$  are the same constants as in Theorem 3.1.*

3. *If there exist complex constants  $a_1, a_2, b_1, b_2, c_1, c_2$  such that  $c_1 + c_2 = \mathfrak{a}_{10}$ ,*

$$\begin{aligned}\frac{\partial}{\partial \xi_1}((G_1 - P_1)^2 - G_2 - P_2) &= ((G_1 - P_1)^2 - G_2 - P_2) \frac{\partial}{\partial \xi_0}(G_1 - P_1) \quad \text{in } W_{\xi^*}, \\ P_1(\xi_0, \xi_1) &= -\frac{a_1\xi_0 + b_1}{1 - a_1\xi_1} - \frac{a_2\xi_0 + b_2}{1 - a_2\xi_0}, \\ P_2(\xi_0, \xi_1) &= -\sum_{j=1}^2 \left[ \left( \frac{a_j\xi_0 + b_j}{1 - a_j\xi_1} \right)^2 + \frac{2a_j c_j}{1 - a_j\xi_1} \right],\end{aligned} \quad (26)$$

then  $N_- \leq 2$ . Furthermore, these constants are related to the constants of Theorem 3.1 by the equations:

$$\begin{aligned} a_1 + a_2 &= -c_{12}, & b_1 + b_2 &= -c_{11}, \\ a_1^2 + a_2^2 &= -c_{23}, & a_1 b_1 + a_2 b_2 &= -\frac{1}{2}c_{22}, \\ b_1^2 + b_2^2 + 2a_1 c_1 + 2a_2 c_2 &= -c_{21}, & c_1 + c_2 &= \mathfrak{a}_{10}. \end{aligned} \quad (27)$$

*Proof.* 1. The equality  $G_1 = 0$  in  $W_{\xi^*}$  implies that  $G_1 = 0$  for all  $\xi \in \Omega_0 \times \mathbb{C}$ . In turn, it implies, according to [4], the moment condition

$$\int_{\gamma} z_1^{k_1} z_2^{k_2} dz_2 = 0, \quad k_1, k_2 \geq 0.$$

Then, according to [13] and [6], for an appropriate choice of orientation,  $\gamma$  is the boundary of a complex curve in  $\mathbb{C}^2$ .

2. Set  $h = G_1 - P_1$  in a neighborhood  $W_{\xi^*}$  of  $\xi^*$ . Then  $h$  satisfies

$$\begin{aligned} \frac{\partial^2}{\partial \xi_0^2}(G_1 - h) &= 0 \quad \text{in } W_{\xi^*}, \\ \frac{\partial h}{\partial \xi_1} &= h \frac{\partial h}{\partial \xi_0} \quad \text{in } W_{\xi^*}. \end{aligned}$$

It follows from Theorem 2.2 that  $N_- \leq 1$ . Note also that equation (10) is the restriction of (25) to  $W_{\xi^*} \cap (\Omega_0 \times 0)$ .

3. Consider the following quadratic equation in variable  $t$ :

$$t^2 - (G_1 - P_1)t + \frac{1}{2}((G_1 - P_1)^2 - G_2 - P_2) = 0. \quad (28)$$

Suppose that the discriminant is non-zero at  $\xi^*$ . Then, without loss of generality, it is non-zero in  $W_{\xi^*}$  (we can always choose a smaller neighborhood). We denote two different roots of this equation as  $\tilde{h}_1 = \tilde{h}_1(\xi)$ ,  $\tilde{h}_2 = \tilde{h}_2(\xi)$ . Clearly,  $\tilde{h}_1$  and  $\tilde{h}_2$  are holomorphic in  $W_{\xi^*}$ . Furthermore, by the Vi?te formulas we have

$$\begin{aligned} \tilde{h}_1 + \tilde{h}_2 &= \tilde{e}_1 := G_1 - P_1 \quad \text{in } W_{\xi^*}, \\ \tilde{h}_1 \tilde{h}_2 &= \tilde{e}_2 := \frac{1}{2}((G_1 - P_1)^2 - G_2 - P_2) \quad \text{in } W_{\xi^*}. \end{aligned}$$

Note that by definition  $\frac{\partial P_1}{\partial \xi_1} = \frac{1}{2} \frac{\partial P_2}{\partial \xi_0}$ . Note also that by Lemma 3.3.1 of [4] we have  $\frac{\partial G_1}{\partial \xi_1} = \frac{1}{2} \frac{\partial G_2}{\partial \xi_0}$ . It leads to the equation

$$\frac{\partial \tilde{e}_1}{\partial \xi_1} = \frac{1}{2} \frac{\partial}{\partial \xi_0} (\tilde{e}_1^2 - 2\tilde{e}_2) \quad \text{in } W_{\xi^*}. \quad (29)$$

Furthermore, equation (26) can be rewritten in the form

$$\frac{\partial \tilde{e}_2}{\partial \xi_1} = \tilde{e}_2 \frac{\partial \tilde{e}_1}{\partial \xi_0} \quad \text{in } W_{\xi^*}. \quad (30)$$

Now denote by  $\widehat{h}_1, \widehat{h}_2$  the shock-wave extensions of  $\widetilde{h}_1(\cdot, 0)$  and  $\widetilde{h}_2(\cdot, 0)$  to  $W_{\xi^*}$  which exist and are unique by the Cauchy-Kowalevski theorem. Set  $\widehat{e}_1 = \widehat{h}_1 + \widehat{h}_2$ ,  $\widehat{e}_2 = \widehat{h}_1 \widehat{h}_2$ . Due to the shock-wave equations for  $\widehat{h}_1$  and  $\widehat{h}_2$ , the functions  $\widehat{e}_1$  and  $\widehat{e}_2$  satisfy

$$\begin{aligned}\frac{\partial \widehat{e}_1}{\partial \xi_1} &= \frac{1}{2} \frac{\partial}{\partial \xi_0} (\widehat{e}_1^2 - 2\widehat{e}_2) \quad \text{in } W_{\xi^*}, \\ \frac{\partial \widehat{e}_2}{\partial \xi_1} &= \widehat{e}_1 \frac{\partial \widehat{e}_2}{\partial \xi_0} \quad \text{in } W_{\xi^*}.\end{aligned}$$

Thus,  $\widetilde{e}_1, \widetilde{e}_2$  and  $\widehat{e}_1, \widehat{e}_2$  are holomorphic solutions to the same system with the same restrictions at  $\xi_1 = 0$ . By the Cauchy-Kowalevski theorem,  $\widetilde{e}_1 = \widehat{e}_1$  and  $\widetilde{e}_2 = \widehat{e}_2$ . It follows from the Vi?te formulas that  $\widetilde{h}_1, \widetilde{h}_2$  coincide with  $\widehat{h}_1, \widehat{h}_2$  (up to order). Hence,  $\widetilde{h}_1, \widetilde{h}_2$  satisfy the shock-wave equations.

Applying Theorem 2.2, we obtain that  $N_- \leq 2$ .

It remains to consider the case when the determinant of equation (28) vanishes in  $W_{\xi^*}$ . Otherwise, it vanishes on (at most) a dimension one analytic set and in any neighborhood of  $\xi^*$  there are balls where it does not vanish.

The zero discriminant condition reads

$$(G_1 - P_1)^2 = 2(G_2 - P_1) \quad \text{in } W_{\xi^*}.$$

We set  $\widetilde{h} = \frac{1}{2}(G_1 - P_1)$ . Then by definition of  $P_1$  and from the discriminant condition we get

$$\begin{aligned}\frac{\partial}{\partial \xi_0^2} (G_1 - 2\widetilde{h}) &= 0 \quad \text{in } W_{\xi^*}, \\ \frac{\partial \widetilde{h}}{\partial \xi_1} &= \widetilde{h} \frac{\partial \widetilde{h}}{\partial \xi_0} \quad \text{in } W_{\xi^*}.\end{aligned}$$

By Theorem 2.2 it implies  $N_- \leq 2$ . Note also that if  $N_- = 2$ , then all intersections of  $X$  with  $\mathbb{C}_\xi^1$ ,  $\xi \in W_{\xi^*}$ , are double.

Finally, note that equation (11) is the restriction of (26) to the set  $W_{\xi^*} \cap (\Omega_0 \times 0)$ .  $\square$

**Complement 3.2.** The statement of Theorem 3.2 can be generalized to the case  $N_- \geq 3$  in the spirit of cases  $N_- \leq 2$ . Such a generalization will be developed in a separate paper together with a statement of Theorem 3.1 for  $N_- \geq 3$ , indicated in Complement 3.1.

We pass to the description of the algorithm of reconstruction of a complex curve  $X \subset \mathbb{C}P^2$  satisfying the minimality condition (\*) from the known boundary  $\gamma = \partial X \subset \mathbb{C}^2$ .

Let  $\{\xi_0^k\}_{k=1}^N$ ,  $\xi_0^k \in \mathbb{C}$  be an arbitrary grid in  $\mathbb{C}$ ,  $\xi_0^i \neq \xi_0^j$ ,  $i \neq j$ , and  $\xi_0^k \notin \pi_2 \gamma$ ,  $k = 1, \dots, N$ . The complex curve  $X$  intersects complex line  $\{z_2 = -\xi_0^k\}$  at points  $(h_s(\xi_0^k, 0), -\xi_0^k)$ ,  $1 \leq s \leq N_+(\xi_0^k, 0)$ . We are going to present the formulas for finding these points.

The algorithm takes as input the points  $\{\xi_0^k\}_{k=1}^N$  and the curve  $\gamma$  (for example, represented as a finite number of points belonging to  $\gamma$ ). On the output of the algorithm we obtain the set of points  $(h_s(\xi_0^k, 0), -\xi_0^k)$ ,  $1 \leq k \leq N$ ;  $1 \leq s \leq N_+(\xi_0^k, 0)$ .

## 1. No points at infinity

1. *Computation of  $N_+$ .* According to formula (4), for every domain  $\Omega_{l \geq 0}$ , the number  $\mu_l = N_+(\xi_0, 0)$ ,  $\xi_0 \in \Omega_l$ , is equal to the winding number of the curve  $\pi_2\gamma$  with respect to a point  $\xi_0 \in \Omega_l$ :

$$\mu_l \equiv N_+(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz_2}{z_2 + \xi_0} \equiv \frac{1}{2\pi i} \int_{\pi_2\gamma} \frac{dz}{z - \xi_0}, \quad \xi_0 \in \Omega_l.$$

2. *Computation of power sums.* If  $N_- = 0$  then, according to Theorem 3.1, for every point  $\xi_0^k \in \Omega_{l \geq 0}$  we have that  $P_m(\xi_0^k, 0) = 0$ . Using formula (6) we obtain the following formulas for the power sums of the functions to be determined:

$$s_m(\xi_0^k) \equiv h_1^m(\xi_0^k, 0) + \dots + h_{\mu_l}^m(\xi_0^k, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^m dz_2}{z_2 + \xi_0^k}, \quad 1 \leq m \leq \mu_l.$$

3. *Computation of symmetric functions.* For every point  $\xi_0^k \in \Omega_{\geq 1}$ , the Newton identities

$$k\sigma_k(\xi_0^k) = \sum_{i=1}^k (-1)^{i-1} \sigma_{k-i}(\xi_0^k) s_i(\xi_0^k), \quad 1 \leq k \leq N_+(\xi_0^k, 0).$$

allow to reconstruct the elementary symmetric functions:

$$\begin{aligned} \sigma_1(\xi_0^k) &= h_1(\xi_0^k, 0) + \dots + h_{\mu_l}(\xi_0^k, 0), \\ &\dots = \dots \\ \sigma_{\mu_l}(\xi_0^k) &= h_1(\xi_0^k, 0) \times \dots \times h_{\mu_l}(\xi_0^k, 0). \end{aligned}$$

4. *Desymmetrisation.* For every point  $\xi_0^k \in \Omega_l$ , using Vi?te formulas, one can find the complex numbers  $h_1(\xi_0^k, 0), \dots, h_{\mu_l}(\xi_0^k, 0)$  (up to order). The points  $(h_s(\xi_0^k, 0), -\xi_0^k)$ ,  $1 \leq s \leq N_+(\xi_0^k, 0)$ ,  $1 \leq k \leq N$ , are the required points of the complex curve  $X$ .

## 2. One or two points at infinity

These cases can be reduced to the case  $N_- = 0$  in the following way. Since  $\pi_2\gamma \subset \mathbb{C}$  is a compact real curve, there exists such  $R > 0$ , that the set  $B_R^c(0) = \{z \in \mathbb{C} \mid |z| \geq R\}$

in contained in  $\Omega_0$ . Without loss of generality, one can suppose that  $|\xi_0^k| < R$  for all  $k = 1, \dots, N$ .

Consider an auxiliary complex curve  $X_R = \{(z_1, z_2) \in X \mid |z_2| \leq R\}$ . Its boundary  $\gamma_R$  consists of two disjoint parts (possibly, multiconnected): the first part is  $\gamma$  and the second part  $\gamma'_R$  is obtained by lifting the circle  $S_R = \{z \in \mathbb{C} \mid |z| = R\}$  to  $X$  via the projection  $\pi_2: X \rightarrow \mathbb{C}$ .

The complex curve  $X_R$  does not intersect infinity. Moreover, points of the form  $(a, -\xi_0^k)$ ,  $k = 1, \dots, N$  belong to  $X$  if and only if they belong to  $X_R$ .

Therefore, in order to reconstruct  $X$  it is sufficient to reconstruct  $\gamma'_R$  and then to reconstruct  $X_R$ , using the algorithm for the case when  $N_- = 0$ . The algorithm can be formulated as follows:

1. *New boundary.* Choose a sufficiently large  $R > 0$ , so that  $B_R^c \subset \Omega_0$  and all  $\xi_0^k$  belong to  $B_R$ . In the case of  $N_- = 1$ , by virtue of formulas (6), we have that

$$h_1(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{z_2 + \xi_0} - P_1(\xi_0, 0), \quad |\xi_0| = R,$$

where  $P_1$  can be found using Theorem 3.1. This formula allows to recover  $\gamma'_R$  and, as a corollary,  $\gamma_R = \gamma \sqcup \gamma'_R = \partial X_R$ .

In the case of  $N_- = 2$  we have two equalities:

$$h_1(\xi_0, 0) + h_2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{z_2 + \xi_0} - P_1(\xi_0, 0), \quad |\xi_0| = R,$$

$$h_1^2(\xi_0, 0) + h_2^2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^2 dz_2}{z_2 + \xi_0} - P_2(\xi_0, 0), \quad |\xi_0| = R,$$

where  $P_1$  and  $P_2$  can be found using Theorem 3.1. Applying Newton identities and Vi?te formulas, we find  $h_1(\xi_0, 0)$  and  $h_2(\xi_0, 0)$ . Thus, we have recovered  $\gamma'_R$  and, as a corollary,  $\gamma_R = \gamma \sqcup \gamma'_R = \partial X_R$ .

2. *Reduction.* In order to find the complex curve  $X_R$  with boundary  $\partial X_R = \gamma_R$  we apply the algorithm of reconstruction for the case of  $N_- = 0$ .

## 4 Visualization

To our knowledge, there are, at least, two known algorithms for automatic visualisation of complex curves. The first one was proposed by Trott [12] and requires the knowledge of an analytic expression for the curve. The second algorithm was proposed by Nieser-Poelke-Polthier [11] and requires the knowledge of the branching points and their



indices for some fixed projection to  $\mathbb{C}$ . Our algorithm requires the knowledge of the unordered set of points of intersections of the curve with complex lines  $\mathbb{C}_\xi^1$ .

Let us describe in few words the algorithm of visualisation of complex curves that we use in our examples. Denote by  $\pi_1: \mathbb{C}^2 \rightarrow \mathbb{C}$  the projection into the first factor:  $\pi_1(z_1, z_2) = z_1$ . Suppose that  $X$  is a complex curve in  $\mathbb{C}^2$  such that the covering  $\pi_1: X \setminus \{\text{ramification points}\} \rightarrow \mathbb{C}$  has multiplicity  $L$ . Consider, for simplicity, a rectangular grid  $\Lambda$  in  $\mathbb{C}$ :

$$\Lambda = \{z_1^{ij} : \operatorname{Re} z_1^{ij} = \frac{i}{N}, \operatorname{Im} z_1^{ij} = \frac{j}{N}, i, j = 0, \dots, N\},$$

where  $N$  is a natural number. Suppose now that we are given the set  $X_\Lambda = \pi_1^{-1}(\Lambda) \cap X$  and we need to visualize the part of  $X$  lying above the rectangle  $0 \leq \operatorname{Re} z_1 \leq 1, 0 \leq \operatorname{Im} z_1 \leq 1$ .

Let us introduce some terminology. We define a path in  $\Lambda$  as a map  $\gamma: \{1, \dots, M\} \rightarrow \Lambda$  such that  $|\gamma(k+1) - \gamma(k)| = \frac{1}{N}$  for all admissible  $k$ , where  $M$  is some natural number.

Let  $\gamma: \{1, \dots, M\} \rightarrow \Lambda$  be a path in  $\Lambda$  and let  $i: \{1, \dots, M\} \rightarrow [1, M]$  be the inclusion map. Define the function  $i_*\gamma: [1, M] \rightarrow \mathbb{C}$  such that  $i_*\gamma(k) = \gamma(k)$  for integer  $k$  and  $i_*\gamma|_{[k, k+1]}$  is linear for all admissible  $k$ . It is clear that  $i_*\gamma$  is a continuous function and hence it can be lifted to  $X$  by the map  $\pi_1$ .

We define a path in  $X_\Lambda$  as a map  $\Gamma: \{1, \dots, M\} \rightarrow X_\Lambda$  such that  $\gamma = \pi_1 \circ \Gamma$  is a path in  $\Lambda$  and  $\Gamma = i^*L(i_*\gamma)$ , where  $i^*$  is the pullback map with respect to  $i$  and  $L(i_*\gamma)$  is some lift of  $i_*\gamma$  to  $X$  by  $\pi_1$ , i. e.  $L(i_*\gamma)$  is a continuous map from  $[1, M]$  to  $X$  such that  $\pi_1 \circ L(i_*\gamma) = i_*\gamma$ . We also say that  $\Gamma$  is obtained by lifting of  $\gamma$ .

We will call subsets of  $\Lambda$  and  $X_\Lambda$  path-connected if every two points of these sets can be connected by a path in  $\Lambda$  and  $X_\Lambda$ , respectively.

Let us describe the practical way to lift paths in  $\Lambda$  to paths in  $X_\Lambda$ . Suppose that  $N$  is sufficiently large. Let  $\gamma: \{1, \dots, M\} \rightarrow \Lambda$  be a path in  $\Lambda$  and let  $\Gamma(1) \in \pi_1^{-1}(\gamma(1)) \cap X$  be an arbitrary point. We select  $\Gamma(k) \in \pi_1^{-1}(\gamma(k)) \cap X$  in such a way that

$$|\Gamma(k) - \Gamma(k-1)| = \min\{|z - \Gamma(k-1)| : z \in \pi_1^{-1}(\gamma(k)) \cap X\}, \quad k = 2, \dots, M.$$

Then  $\Gamma$  is a path in  $X_\Lambda$  obtained by lifting of  $\gamma$ . All possible lifts of  $\gamma$  may be obtained by varying  $\Gamma(1)$ . Note that if  $\gamma$  is closed, i. e.  $\gamma(1) = \gamma(M)$ ,  $\Gamma$  need not to be closed.

**Finding of ramification points and making branch cuts.** The first step in visualization procedure consists in finding of ramification points of  $X$  with respect to projection  $\pi_1$ . Since we have only a finite number of points on  $X$  we can find ramification points only approximately. More precisely, we will localize them in small circles.

Without restriction of generality we suppose that all ramification points are projected by  $\pi_1$  into interior points of  $\Lambda$ . Take any interior point  $z_1 \in \Lambda$  and select a small closed path  $\gamma: \{1, \dots, M\} \rightarrow \Lambda$  around  $z_1$  so that there is at most one ramification point inside the polygon  $\gamma(1) \dots \gamma(M)$ . For example, one can take as  $\gamma$  the following path:

$$z_1 + \frac{1}{N} \rightarrow z_1 + \frac{1+i}{N} \rightarrow z_1 + \frac{i}{N} \rightarrow \dots \rightarrow z_1 + \frac{1-i}{N} \rightarrow z_1 + \frac{1}{N},$$

where  $i$  is the imaginary unit.

Now consider different lifts of  $\gamma$  to  $X_\Lambda$ . If at least one lift is not closed, mark  $z_1$  as a possible ramification point (meaning that it is situated near the projection of some ramification point of  $X$ ). Now vary  $z_1$  and mark all possible ramification points. The resulting set will consist of several path-connected components each of which localizes the position of one ramification point of  $X$  with respect to  $\pi_1$ .

Next, we need to cut the grid  $\Lambda$  with branching points removed into simply connected domains. The natural choice is the shortest cut graph, which can be computed by the algorithm proposed in [10], which, in turn, generalizes the algorithm for closed surfaces of [5].

Denote the cut graph by  $\Lambda_c$ . An important observation is that every closed path in  $\Lambda \setminus \Lambda_c$  always lifts to a closed path in  $X_\Lambda$  since it doesn't contain  $\pi_1$ -projections of ramification points inside.

**Visualization.** Now denote  $\Lambda \setminus \Lambda_c = \cup_{s=1}^S \Lambda_s$ , where  $\Lambda_s$  are different path-connected components. Take any  $z_1^s \in \Lambda_s$  and  $z_2^s \in \pi_1^{-1}(z_1^s) \cap X$ . Now take other  $z_1 \in \Lambda_s$  and connect  $z_1^s$  with  $z_1$  by some path  $\gamma$ . Then  $\gamma$  lifts to a path  $\Gamma$  with  $\Gamma(1) = (z_1^s, z_2^s)$  and  $\Gamma(2) = (z_1, z_2)$  for some  $z_2 \in \pi_1^{-1}(z_1) \cap X$  and  $z_2$  doesn't depend on  $\gamma$ . Varying  $z_1$  we thus obtain the map  $\Sigma(z_1^s, z_2^s): \Lambda_s \rightarrow X_\Lambda$  which allows us to visualize the part of  $X$ .

Varying  $z_2^s \in \pi_1^{-1}(z_1^s) \cap X$  (the latter is the finite set, namely, it consists of  $L$  elements) we obtain the other maps  $\Sigma(z_1^s, z_2^s)$  which allow us to visualize other parts of  $X$ . Clearly, the set of obtained maps doesn't depend on the choice of  $z_1^s \in \Lambda_s$ . Hence we can denote the obtained maps by  $\Sigma_s^l$ ,  $l = 1, \dots, L$ . It is clear that  $\cup_{l=1}^L \Sigma_s^l(\Lambda_s) = \pi^{-1}(\Lambda_s) \cap X$ . Now vary  $s$  to visualize

$$\cup_{s=1}^S \cup_{l=1}^L \Sigma_s^l(\Lambda_s) = \pi^{-1}(\cup_{s=1}^S \Lambda_s) \cap X = X_\Lambda \setminus \pi^{-1}(\Lambda_c).$$

The part  $\pi^{-1}(\Lambda_c) \cap X_\Lambda$  consists of cuts and preimages of possible ramification points. The cuts can be visualized as the already visualized part of the surface. The only problem is the visualization of  $\pi_1$ -preimages of possible ramification points. But the latter take a little part of the surface when  $N$  is large and one can just forget about their visualization. On the other hand, in our examples they were visualized using low-level graphics approach.

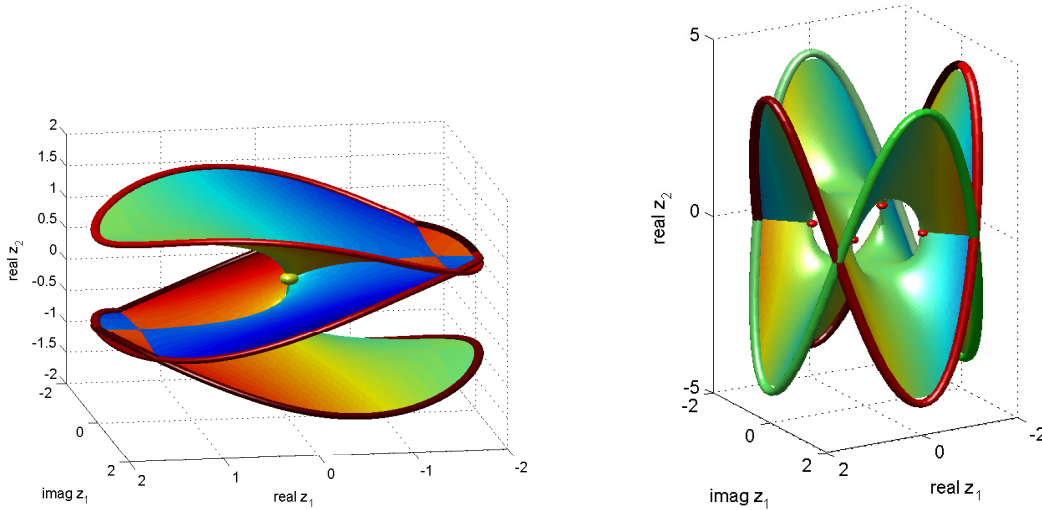


Figure 1: Riemann surfaces of functions  $f(z) = \sqrt{\sin(z)}$ ,  $|z| \leq 2$  (left) and  $f(z) = \sqrt{z^4 + 1}$ ,  $|z| \leq 2$  (right) obtained by the visualization algorithm.

Examples of application of this algorithm are given at Fig. 1 and in Section 5. The visualization algorithm can be easily generalized to the case of general grids. For instance, in our examples we have used a modification with periodic grid.

## 5 Examples

**1. No points at infinity.** We consider the complex curve given by the equation

$$X_1 = \{(z_1, z_2) \in \mathbb{C}^2 : z_2 + 1 = (z_1^2 - \exp(\frac{z_2}{4}))^2, |z_2| \leq 1.9\}. \quad (31)$$

The boundary  $\gamma_1 = \partial X_1$  of this curve is the disjoint union of two real curves  $\gamma_1^r$  and  $\gamma_1^g$ , see Fig. 2. We are going to recover  $X_1$  from  $\gamma_1$ .

First, note that the real curve  $|z_2| = 1.9$  in  $\mathbb{C}$  divides  $\mathbb{C}$  into two parts  $\Omega_0 = \{|z_2| > 1.9\}$  and  $\Omega_1 = \{|z_2| < 1.9\}$ . We check numerically that  $G_1(\xi)$  is zero in a neighborhood  $W_{\xi^*}$  of some  $\xi^* = (\xi_0^*, 0)$ ,  $\xi_0 \in \Omega_0$ , and thus  $N_- = 0$  by Theorem 3.2.

Next, we compute the value

$$\mu_1 = \sigma_0(\xi_0) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{dz_2}{z_2 + \xi_0},$$

for some  $\xi_0 \in \Omega_1$  (it will be the same for any  $\xi_0 \in \Omega_1$ ). It turns out that  $\mu_1 = 4$  and thus  $\pi_2: X_1 \rightarrow \Omega_1$  is a (ramified) covering with 4 sheets. Denote  $X_1 \cap \mathbb{C}_\xi^1 = \{(h_j(\xi), -\xi_0)\}_{j=1}^4$ ,  $\xi = (\xi_0, 0)$ ,  $|\xi_0| < 1.9$ .

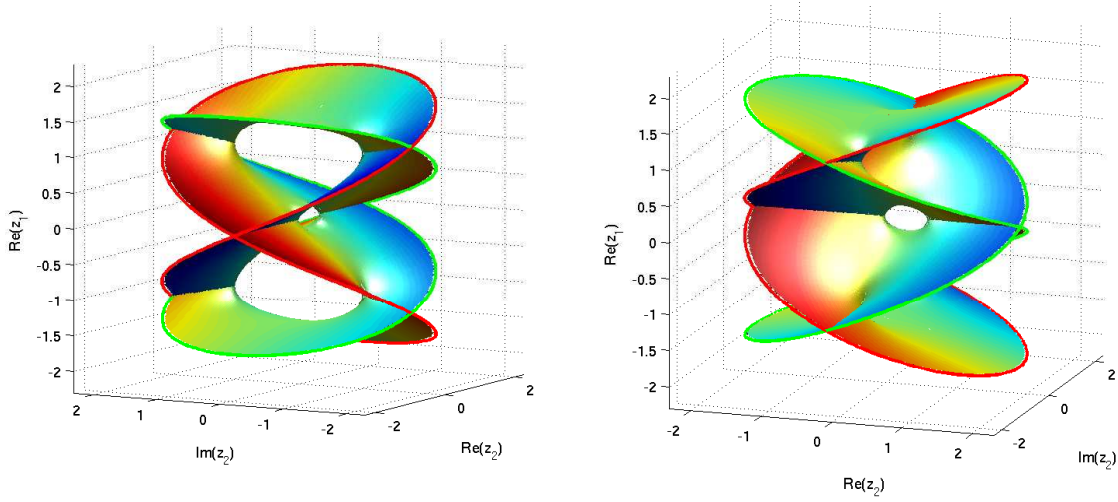


Figure 2: The given boundary  $\gamma_1 = \gamma_1^r \sqcup \gamma_1^g$  (red and green) and the reconstructed complex curve  $X_1 \subset \mathbb{C}^2$ . The color indicates  $\text{Im}(z_1)$ .

Next, we compute the values of the power sum functions  $s_k$ :

$$s_k(\xi_0) := \sum_{j=1}^4 h_j^k(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{z_1^k dz_2}{z_2 + \xi_0}, \quad |\xi_0| < 1.9, \quad 1 \leq k \leq 4.$$

Using Newton identities, we determine the values of symmetric functions  $\sigma_k$ :

$$\begin{aligned} \sigma_1 &= s_1, & \sigma_2 &= \frac{1}{2}(s_1^2 - s_2), \\ \sigma_3 &= \frac{1}{6}(s_1^3 - 3s_1s_2 + 2s_3), \\ \sigma_4 &= \frac{1}{24}(s_1^4 - 6s_1^2s_2 + 3s_2^2 + 8s_1s_3 - 6s_4). \end{aligned}$$

Finally, taking into account Vi?te formulas, we find  $h_1(\xi_0, 0), \dots, h_4(\xi_0, 0)$  as the roots of the following polynomial in  $t$ :

$$t^{N+(\xi_0)} - \sigma_1(\xi_0)t^{N+(\xi_0)-1} + \dots + (-1)^{N+(\xi_0)}\sigma_{N+(\xi_0,0)}(\xi_0) = 0.$$

Reconstructed surface is represented at Fig. 2.

**2. One point at infinity.** Consider the the complex curve  $X_2$  which is the part of the curve

$$\tilde{X}_2 = \{(w_0 : w_1 : w_2) \in \mathbb{C}P^2 : w_1^2 = w_2^2 + w_0^2\},$$

bounded by the real curve

$$\begin{aligned} \gamma_2 &= \{(z_1(t), z_2(t)) \in \mathbb{C}^2 : t \in [0, 2\pi]\}, \\ z_2(t) &= 1.9e^{it}, \quad z_1(t) = \sqrt{z_2^2(t) + 1}, \quad z_1(0) > 0, \end{aligned}$$

intersecting infinity at  $(0 : 1 : 1)$ . We suppose that the number of points at infinity is a priori unknown. We are going to reconstruct  $X_{2,R} = X_2 \cap \{z_2 < R\}$ , where  $R = 3$ , from  $\gamma_2$ .

Note that the curve  $\pi_2\gamma_2 = \{|z_2| = 1.9\}$  divides  $\mathbb{C}$  into two connected components  $\Omega_0 = \{z_2 > 1.9\}$  and  $\Omega_1 = \{z_2 < 1.9\}$ .

According to the algorithm, the first step is to determine the number of points at infinity. We choose some  $\xi^* = (\xi_0^*, 0)$  with  $\xi_0^* \in \Omega_0$  and check that in some small neighborhood  $W_{\xi^*}$  of  $\xi^*$  we have  $G_1 \neq 0$  and thus  $N_- \geq 1$  by Theorem 3.2.

Next, we fix two generic  $\xi_0^1, \xi_0^2 \in \Omega_0$  and determine  $c_{11}, c_{12}$  from the linear system

$$\begin{aligned} c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) + c_{12} \left( \xi_0^1 \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) + G_1(\xi_0^1, 0) \right) &= G_1(\xi_0^1, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0^1, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0^1, 0), \\ c_{11} \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) + c_{12} \left( \xi_0^2 \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) + G_1(\xi_0^2, 0) \right) &= G_1(\xi_0^2, 0) \frac{\partial G_1}{\partial \xi_0}(\xi_0^2, 0) - \frac{\partial G_1}{\partial \xi_1}(\xi_0^2, 0). \end{aligned} \quad (32)$$

This system is uniquely solvable and gives  $c_{11} \approx 1, c_{12} \approx 0$ .

Next, we check numerically that in the neighborhood  $W_{\xi^*}$  of  $\xi^*$  the following identity holds:

$$\begin{aligned} \frac{\partial}{\partial \xi_1}(G_1 - P_1) &= (G_1 - P_1) \frac{\partial}{\partial \xi_0}(G_1 - P_1), \\ \text{where } P_1(\xi_0, \xi_1) &= \frac{c_{11}\xi_0 + c_{12}}{1 + c_{11}\xi_1}. \end{aligned} \quad (33)$$

Theorem 3.2 together with the estimate  $N_- \geq 1$  imply that  $N_- = 1$ .

Next, according to the algorithm, in order to reduce the problem to the case of no points at infinity, we need to recover the auxiliary curve  $\gamma'_{2,R} = X_2 \cap \{z_2 = R\}$ . Note that  $\gamma'_{2,R} = \{(h_1(\xi_0, 0), -\xi_0) : |\xi_0| = R\}$ , where  $h_1(\xi_0, 0)$  can be found from the explicit formula

$$h_1(\xi_0, 0) = G_1(\xi_0, 0) - c_{11}\xi_0 - c_{12}, \quad |\xi_0| = R.$$

Next, we set  $\gamma_{2,R} = \gamma_2 \sqcup \gamma'_{2,R}$  so that  $\partial X_{2,R} = \gamma_{2,R}$ ,  $X_{2,R} \cap \{|z_2| < R\} = X_2 \cap \{|z_2| < R\}$  and  $X_{2,R}$  does not intersect  $\mathbb{C}P^1_\infty$ . Thus, it remains to recover  $X_{2,R}$  from the known boundary  $\gamma_{2,R}$  using the algorithm for the case of no points at infinity, see the previous example. The reconstructed curve is depicted at Fig. 3.

**3. Two points at infinity.** Consider the complex curve  $X_3$ , which is the part of the algebraic curve

$$\tilde{X}_3 = \{(w_0 : w_1 : w_3) \in \mathbb{C}P^2 : w_1^3 = (w_2 - iw_0)(w_2 + iw_0)(w_2 - w_0)\},$$

bounded by the real curve

$$\begin{aligned} \gamma_3 &= \{(z_1(t), z_2(t)) : t \in [0, 2\pi]\}, \\ z_2(t) &= 1.9e^{it}, \quad z_1(t) = \sqrt[3]{(z_2(t) - i)(z_2(t) + i)(z_2(t) - 1)}, \quad z_1(0) > 0. \end{aligned}$$

The curve  $X_3$  has two points at infinity, namely,  $q_1 = (0 : 1 : e^{\frac{2\pi i}{3}})$ ,  $q_2 = (0 : 1 : e^{-\frac{2\pi i}{3}})$ . We suppose that the number of points at infinity is a priori unknown. We are going to reconstruct  $X_3$  from  $\gamma_3$ .

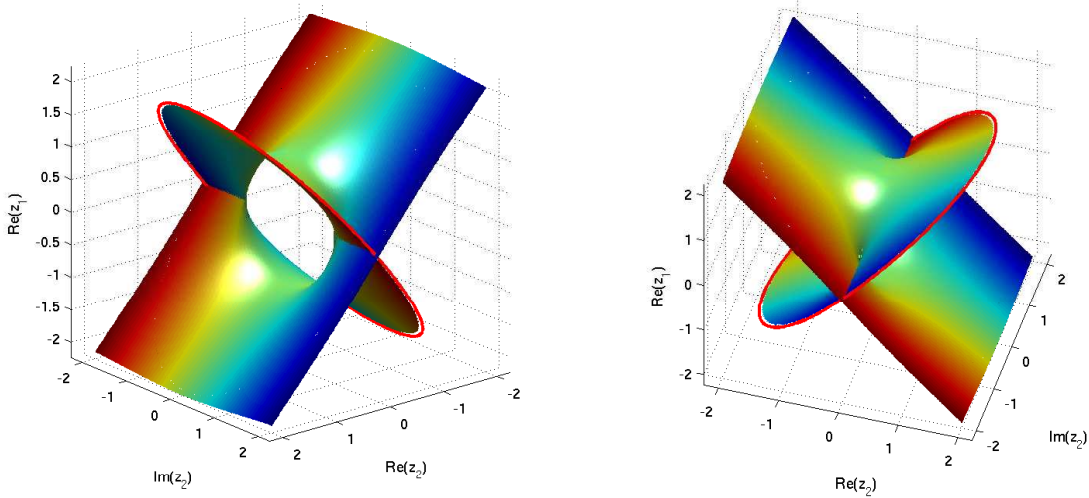


Figure 3: The real curve  $\gamma_2$  (red) and the reconstructed complex curve  $X_2$  with one point at infinity. The color indicates  $\text{Im}(z_1)$ .

As in the previous examples, the real curve  $\pi_2\gamma_2$  divides  $\mathbb{C}$  into two parts  $\Omega_0 = \{|z_2| > 1.3\}$  and  $\Omega_1 = \{|z_2| < 1.9\}$ .

The first step is to determine the number of points at infinity. Choose a generic  $\xi^* = (\xi_0^*, 0)$ ,  $\xi_0^* \in \Omega_0$ , and some neighborhood  $W_{\xi^*}$  of  $\xi^*$ .

1. We check that  $G_1 \not\equiv 0$  in  $W_{\xi^*}$  so that  $N_- \geq 1$  by Theorem 3.2.
2. We choose two generic  $\xi_0^1, \xi_0^2 \in \Omega_0$  and solve (32) for  $c_{11}, c_{12}$ . It turns out that system (32) is uniquely solvable. However, identity (33) does not hold in  $W_{\xi^*}$ . Using Theorem 3.2, we obtain that  $N_- \geq 2$ .
3. We consider (11) as an identity in a neighborhood of  $\xi_0^*$ . We find numerically that

$$\begin{aligned} c_{11} &\approx 0.9974, & c_{12} &\approx -0.3335, \\ c_{21} &\approx 1.0133, & c_{22} &\approx -0.6699, & c_{23} &\approx 0.5567. \end{aligned}$$

We check that (26) holds in a neighborhood of  $W_{\xi^*}$  with constants  $a_1, a_2, b_1, b_2, c_1, c_2$  given by equations (27). It follows from Theorem 3.2 and from the inequality  $N_- \geq 2$  that  $N_- = 2$ .

Suppose that we want to recover  $X_{3,R} = X_3 \cap \{|z_2| < R\}$ ,  $R = 3$ . According to the algorithm, we need to find the auxiliary boundary  $\gamma'_{3,R} = X_3 \cap \{|z_2| = R\}$ . Denote  $\gamma'_{3,R} = \{(h_j(\xi_0, 0), -\xi_0) : |\xi_0| = R, j = 1, 2\}$ . We compute the values of symmetric functions  $\sigma_1(\xi_0) = h_1(\xi_0, 0) + h_2(\xi_0, 0)$ ,  $\sigma_2(\xi_0) = h_1(\xi_0, 0)h_2(\xi_0, 0)$  using the explicit

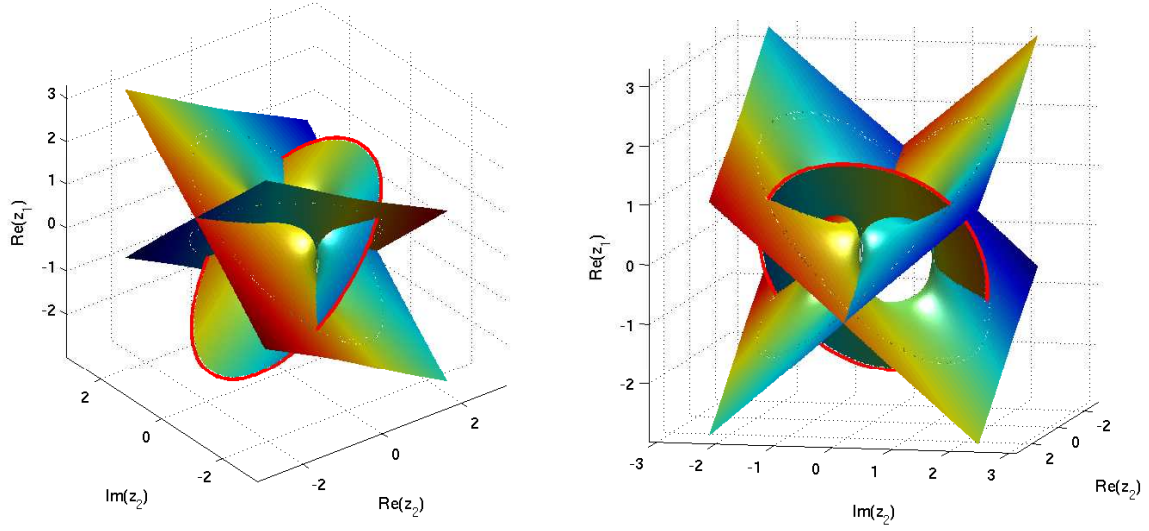


Figure 4: The real curve  $\gamma_3$  (red) and the reconstructed complex curve  $X_3$  with two points at infinity. The color indicates  $\text{Im}(z_1)$ .

formulas

$$\begin{aligned} \sigma_1 &= s_1, & \sigma_2 &= \frac{1}{2}(s_1^2 - s_2), \\ s_1(\xi_0) &:= h_1(\xi_0, 0) + h_2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1 dz_2}{z_2 + \xi_0} - c_{11} - c_{12}\xi_0, & |\xi_0| &= R, \\ s_2(\xi_0) &:= h_1^2(\xi_0, 0) + h_2^2(\xi_0, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{z_1^2 dz_2}{z_2 + \xi_0} - c_{21} - c_{22}\xi_0 - c_{23}\xi_0^2, & |\xi_0| &= R. \end{aligned}$$

Next, taking into account the Vi?te formulas, we find the values  $h_1(\xi_0, 0)$ ,  $h_2(\xi_0, 0)$ ,  $|\xi_0| = R$ , as the roots of the following polynomial in  $t$ :

$$t^2 - \sigma_1(\xi_0)t + \sigma_2(\xi_0) = 0.$$

Then we put  $\gamma_{3,R} = \gamma_3 \sqcup \gamma'_{3,R}$  and recover  $X_{3,R}$  from  $\gamma_{3,R}$  using the algorithm for the case of no points at infinity. The result of reconstruction is depicted at Fig. 4.

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