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# The quenched limiting distributions of a charged-polymer model in one and two dimensions

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**Summary.** The limit distributions of the charged-polymer Hamiltonian of Kantor and Kardar [Bernoulli case] and Derrida, Griffiths and Higgs [Gaussian case] are considered. Two sources of randomness enter in the definition: a random field  $q = (q_i)_{i \geq 1}$  of i.i.d. random variables, which is called the random charges, and a random walk  $S = (S_n)_{n \in \mathbb{N}}$  evolving in  $\mathbb{Z}^d$ , independent of the charges. The energy or Hamiltonian  $K = (K_n)_{n \geq 2}$  is then defined as

$$K_n := \sum_{1 \leq i < j \leq n} q_i q_j \mathbf{1}_{S_i = S_j}.$$

The law of  $K$  under the joint law of  $q$  and  $S$  is called “annealed”, and the conditional law given  $q$  is called “quenched”. Recently, strong approximations under the annealed law were proved for  $K$ . In this paper we consider the limit distributions of  $K$  under the quenched law for  $d = 1$  and  $2$ .

**Keywords:** Random walk, polymer model, self-intersection local time, limit theorems, law of the iterated logarithm, martingale.

**2011 Mathematics Subject Classification:** 60G50, 60K35, 60F05.

## 1. INTRODUCTION

Let  $d \geq 1$  and  $q = (q_i)_{i \geq 1}$  be a collection of i.i.d. real random variables, hereafter referred to as charges, and  $S = (S_n)_{n \geq 0}$  be a random walk in  $\mathbb{Z}^d$  starting at 0, i.e.,  $S_0 = 0$  and  $(S_n - S_{n-1})_{n \geq 1}$  is a sequence of i.i.d.  $\mathbb{Z}^d$ -valued random variables, independent of  $q$ . We are interested in the limit distributions of the sequence  $K := (K_n)_{n \geq 1}$  defined by setting  $K_1 := 0$  and, for  $n \geq 2$ ,

$$K_n := \sum_{1 \leq i < j \leq n} q_i q_j \mathbf{1}_{\{S_i = S_j\}}. \quad (1)$$

In the physics literature this sum is known as the Hamiltonian of the so-called *charged polymer model*; see Kantor and Kardar [19] in the case of Bernoulli random charges and Derrida, Griffiths and Higgs [11] in the Gaussian case. This model has been largely studied by physicists since it is believed that a protein molecule looks like a random walk with random charges attached at the vertices of the walk; these charges are interacting through local interactions mimicking chemical reactions [23].

Results were first established under the *annealed* measure, that is when one averages at the same time over the charges and the random walk. Chen [7], Chen and Khoshnevisan [8] proved that the one-dimensional limiting distributions are closely related to the model of *Random walk*

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*in random scenery*. Hu and Khoshnevisan [18] then established that in dimension one the limit process of the (correctly renormalized) Hamiltonian  $K_n$  is strongly approximated by a Brownian motion, time-changed by the self-intersection local time process of an independent Brownian motion. Especially, it differs from the so-called Kesten and Spitzer's process [20] obtained as the continuous limit process of the one-dimensional random walk in random scenery.

To our knowledge distributional limit theorems for *quenched* charges (that is, conditionally given the charges) are not known. Let us note that in the physicists' usual setting the charges are usually quenched: a typical realization of the charges is fixed, and the average is over the walk. In the case of dimension one, we determine the quenched weak limits of  $K_n$  by applying Strassen [25]'s functional law of the iterated logarithm. As a consequence, conditioned on the random charges, the Hamiltonian  $K_n$  does not converge in law. In contrast with the one-dimensional setting, in the case of a planar random walk with a finite non-singular covariance matrix and charges with a moment of order strictly greater than two, we are able to prove a quenched central limit theorem with the unusual  $\sqrt{n \log n}$ -scaling and Gaussian law in the limit. Especially, our assumptions on the charges' moments are slightly better than the ones in [8, 18]; an alternative proof of the annealed functional central limit theorem is then given in the appendix.

## 2. CASE OF THE DIMENSION ONE

**2.1. Results.** In this section we study the case of the dimension one,  $S = (S_k, k \geq 0)$  is the simple one-dimensional random walk. Moreover we assume that

$$\mathbb{E}(q_0) = 0, \quad \mathbb{E}(q_0^2) = 1 \quad \text{and} \quad \mathbb{E}(|q_0|^6) < \infty.$$

We prove that under these assumptions, there is no quenched distributional limit theorem for  $K$ . In the sequel, for  $0 < b \leq \infty$ , we will denote by  $\mathcal{AC}([0, b] \rightarrow \mathbb{R})$  the set of absolutely continuous functions defined on the interval  $[0, b]$  with values in  $\mathbb{R}$ . Recall that if  $f \in \mathcal{AC}([0, b] \rightarrow \mathbb{R})$ , then the derivative of  $f$  (denoted by  $\dot{f}$ ) exists almost everywhere and is Lebesgue integrable on  $[0, b]$ . Define

$$\mathcal{K}^* := \left\{ f \in \mathcal{AC}(\mathbb{R}_+ \rightarrow \mathbb{R}) : f(0) = 0, \int_0^\infty (\dot{f}(x))^2 dx \leq 1 \right\}. \quad (2)$$

**Theorem 1.** *For  $\mathbb{P}$ -a.e.  $q$ , under the quenched probability  $\mathbb{P}(\cdot | q)$ , the process*

$$\tilde{K}_n := \frac{K_n}{(n^{3/2} \log \log n)^{1/2}}, \quad n > e^e,$$

*does not converge in law. More precisely, for  $\mathbb{P}$ -a.e.  $q$ , under the quenched probability  $\mathbb{P}(\cdot | q)$ , the limit points of the law of  $\tilde{K}_n$ , as  $n \rightarrow \infty$ , under the topology of weak convergence of measures, are equal to the set of the laws of random variables in  $\Theta_B$ , with*

$$\Theta_B := \left\{ f(V_1) : f \in \mathcal{K}^* \right\}, \quad (3)$$

*where  $V_1$  denotes the self-intersection local time at time 1 of a one-dimensional Brownian motion  $B$  starting from 0.*

*The set  $\Theta_B$  is closed for the topology of weak convergence of measures, and is a compact subset of  $L^2((B_t)_{t \in [0,1]})$ .*

Instead of Theorem 1, we shall prove that there is no quenched limit theorem for the continuous analogue of  $K$  introduced by Hu and Khoshnevisan [18] and deduce Theorem 1 by using a strong approximation. Let us define this continuous analogue: Assume that  $B := (B(t))_{t \geq 0}$ ,  $W := (W(t))_{t \geq 0}$  are two real Brownian motions starting from 0, defined on the same probability space and independent of each other. We denote by  $\mathbb{P}_B, \mathbb{P}_W$  the law of these processes. We will

also denote by  $(L_t(x))_{t \geq 0, x \in \mathbb{R}}$  a continuous version with compact support of the local time of the process  $B$ , and  $(V_t)_{t \geq 0}$  its self-intersection local time up to time  $t$ , that is

$$V_t := \int_{\mathbb{R}} L_t(x)^2 dx.$$

We define the continuous version of the sequence  $K_n$  as

$$Z_t := W(V_t), t \geq 0.$$

In dimension one, under the annealed measure, Hu and Khoshnevisan [18] proved that the process  $(n^{-3/4}K([nt]))_{t \geq 0}$  weakly converges in the space of continuous functions to the continuous process  $Z = (2^{-1/2}Z_t)_{t \geq 0}$ . They gave a stronger version of this result more precisely, they proved that there is a coupling of  $q$ ,  $S$ ,  $B$  and  $W$  such that  $(q, W)$  is independent of  $(S, B)$  and for any  $\varepsilon \in (0, 1/24)$ , almost surely,

$$K_n = 2^{-1/2}Z_n + o(n^{\frac{3}{4}-\varepsilon}), \quad n \rightarrow +\infty. \quad (4)$$

Theorem 1 will follow from this strong approximation and the following result.

**Theorem 2.**  $\mathbb{P}_W$ -almost surely, under the quenched probability  $\mathbb{P}(\cdot|W)$ , the limit points of the law of

$$\tilde{Z}_t := \frac{Z_t}{(2t^{3/2} \log \log t)^{1/2}}, \quad t \rightarrow \infty,$$

under the topology of weak convergence of measures, are equal to the set of the laws of random variables in  $\Theta_B$  defined in Theorem 1. Consequently, under  $\mathbb{P}(\cdot|W)$ , as  $t \rightarrow \infty$ ,  $\tilde{Z}_t$  does not converge in law.

To prove Theorem 2, we shall apply Strassen [25]'s functional law of the iterated logarithm applied to the Brownian motion  $W$ .

**2.2. Proofs.** For a one-dimensional Brownian motion  $(W(t), t \geq 0)$  starting from 0, let us define for any  $\lambda > e^e$ ,

$$W_\lambda(t) := \frac{W(\lambda t)}{(2\lambda \log \log \lambda)^{1/2}}, \quad t \geq 0.$$

**Lemma 3.** (i) Almost surely, for any  $r > 0$  rational numbers,  $(W_\lambda(t), 0 \leq t \leq r)$  is relatively compact in the uniform topology and the set of its limit points is  $\mathcal{K}_{0,r}$ , with

$$\mathcal{K}_{0,r} := \left\{ f \in \mathcal{AC}([0, r] \rightarrow \mathbb{R}) : f(0) = 0, \int_0^r (\dot{f}(x))^2 dx \leq 1 \right\}.$$

(ii) There exists some finite random variable  $\mathcal{A}_W$  only depending on  $(W(x), x \geq 0)$  such that for all  $\lambda \geq e^{36}$ ,

$$\sup_{t > 0} \frac{|W_\lambda(t)|}{\sqrt{|t| \log \log (|t| + \frac{1}{|t|} + 36)}} =: \mathcal{A}_W < \infty.$$

The proof of this lemma can be found in [16].

Let us define for all  $\lambda > e^e$  and  $n \geq 1$ ,

$$H_\lambda := W_\lambda(V_1), \quad H_\lambda^{(n)} := W_\lambda(V_1) \mathbf{1}_{\{V_1 \leq n\}},$$

**Lemma 4.** *There exist some positive constants  $c_1, c_2$  such that for any  $\lambda > e^{36}$  and  $n \geq 1$ , we have*

$$\mathbb{E}_B \left| H_\lambda - H_\lambda^{(n)} \right| \leq c_1 e^{-c_2 n^2} \mathcal{A}_W, \quad (5)$$

$$\mathbb{E}_B \left| f(V_1) \mathbf{1}_{\{V_1 > n\}} \right| \leq c_1 e^{-c_2 n^2}, \quad (6)$$

for any function  $f \in \mathcal{K}^*$ .

**Proof:** By Lemma 3 (ii),  $\mathbb{E}_B [(W_\lambda(V_1))^2] \leq \mathcal{A}_W^2 \mathbb{E}_B [V_1 |\log \log(|V_1| + \frac{1}{|V_1|} + 36)] \leq c_1 \mathcal{A}_W^2$ , since  $V_1$  has finite moments of any order. Then by Cauchy-Schwarz' inequality, we have that

$$\begin{aligned} \mathbb{E}_B \left| H_\lambda - H_\lambda^{(n)} \right| &= \mathbb{E}_B \left[ W_\lambda(V_1) \mathbf{1}_{\{V_1 > n\}} \right] \\ &\leq \sqrt{\mathbb{E}_B [W_\lambda(V_1)^2]} \sqrt{\mathbb{P}_B(V_1 > n)} \\ &\leq c_1 \mathcal{A}_W e^{-c_2 n^2}, \end{aligned}$$

by the fact that:  $\mathbb{P}_B(V_1 > x) \leq c_1 e^{-c_2 x^2}$  for any  $x > 0$  (see Corollary 5.6 in [21]). Then we get (5).

For the other part of the lemma, let  $f \in \mathcal{K}^*$ , observe that  $|f(x)| \leq \sqrt{|x \int_0^x (\dot{f}(y))^2 dy|} \leq \sqrt{|x|}$  for all  $x \in \mathbb{R}_+$ . Then by Cauchy-Schwarz' inequality, we have that

$$\begin{aligned} \mathbb{E}_B \left[ |f(V_1)| \mathbf{1}_{\{V_1 > n\}} \right] &\leq \sqrt{\mathbb{E}_B [f(V_1)^2]} \sqrt{\mathbb{P}_B(V_1 > n)} \\ &\leq \sqrt{\mathbb{E}_B [V_1]} \sqrt{\mathbb{P}_B(V_1 > n)} \\ &\leq c_1 e^{-c_2 n^2}. \end{aligned}$$

Then (6) follows.

**Lemma 5.**  *$\mathbb{P}_W$ -almost surely,*

$$d_{L^1(B)}(H_\lambda, \Theta_B) \rightarrow 0, \quad \text{as } \lambda \rightarrow \infty,$$

where  $\Theta_B$  is defined in (3). Moreover,  $\mathbb{P}_W$ -almost surely, for any  $\zeta \in \Theta_B$ ,

$$\liminf_{\lambda \rightarrow \infty} d_{L^1(B)}(H_\lambda, \zeta) = 0.$$

**Proof:** Let  $\varepsilon > 0$ . Choose a large  $n = n(\varepsilon)$  such that  $c_1 e^{-c_2 n^2} \leq \varepsilon$ , where  $c_1, c_2$  are the constants defined in Lemma 4. By Lemma 3 (i), for all large  $\lambda \geq \lambda_0(W, \varepsilon, n)$ , there exists some function  $g = g_{\lambda, W, \varepsilon, n} \in \mathcal{K}_{0, n}$  such that  $\sup_{x \in [0, n]} |W_\lambda(x) - g(x)| \leq \varepsilon$ . We get that

$$\mathbb{E}_B \left| H_\lambda^{(n)} - g(V_1) \mathbf{1}_{\{V_1 \leq n\}} \right| \leq \varepsilon.$$

We extend  $g$  to  $\mathbb{R}_+$  by letting  $g(x) = g(n)$  if  $x \geq n$ , then  $g \in \mathcal{K}^*$ . By the triangular inequality, (5) and (6),

$$\mathbb{E}_B \left| H_\lambda - g(V_1) \right| \leq (2 + \mathcal{A}_W) \varepsilon.$$

It follows that  $d_{L^1(B)}(H_\lambda, \Theta_B) \leq (2 + \mathcal{A}_W) \varepsilon$ . Hence  $\mathbb{P}_W$ -a.s.,  $\limsup_{\lambda \rightarrow \infty} d_{L^1(B)}(H_\lambda, \Theta_B) \leq (2 + \mathcal{A}_W) \varepsilon$ , showing the first part in the lemma.

For the other part of the Lemma, let  $h \in \mathcal{K}^*$  such that  $\zeta = h(V_1)$ . For any  $\varepsilon > 0$ , we may use (6) and choose an integer  $n = n(\varepsilon)$  such that  $c_1 e^{-c_2 n^2} \leq \varepsilon$  and

$$d_{L^1(B)}(\zeta, \zeta_n) \leq \varepsilon,$$

where  $\zeta_n := h(V_1) \mathbf{1}_{\{V_1 \leq n\}}$ . Applying Lemma 3 (i) to the restriction of  $h$  on  $[0, n]$ , we may find a sequence  $\lambda_j = \lambda_j(\varepsilon, W, n) \rightarrow \infty$  such that  $\sup_{|x| \leq n} |W_{\lambda_j}(x) - h(x)| \leq \varepsilon$ , then

$$d_{L^1(B)}(H_{\lambda_j}^{(n)}, \zeta_n) \leq \varepsilon.$$

By (5) and the choice of  $n$ ,  $d_{L^1(B)}(H_{\lambda_j}^{(n)}, H_{\lambda_j}) \leq \varepsilon \mathcal{A}_W$  for all large  $\lambda_j$ , it follows from the triangular inequality that

$$d_{L^1(B)}(\zeta, H_{\lambda_j}) \leq (2 + \mathcal{A}_W)\varepsilon,$$

implying that  $\mathbb{P}_W$ -a.s.,  $\liminf_{\lambda \rightarrow \infty} d_{L^1(B)}(H_\lambda, \zeta) \leq (2 + \mathcal{A}_W)\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .  $\square$

We now are ready to give the proof of Theorems 2 and 1.

**Proof of Theorem 2.** Remark that  $\mathbb{P}_W$ -a.s.,

$$W(V_t) \stackrel{(d)}{=} W(V_1 t^{3/2}) \tag{7}$$

from the scaling property of the self-intersection local time of the Brownian motion  $B$ . The first part of Theorem 2 directly follows from Lemma 5.

**Proof of Theorem 1.** We use the strong approximation of [18] : there exists on a suitably enlarged probability space, a coupling of  $q, S, B$  and  $W$  such that  $(q, W)$  is independent of  $(S, B)$  and for any  $\varepsilon \in (0, 1/24)$ , almost surely,

$$K_n = 2^{-1/2} Z_n + o(n^{\frac{3}{4}-\varepsilon}), \quad n \rightarrow +\infty.$$

From the independence of  $(q, W)$  and  $(S, B)$ , we deduce that for  $\mathbb{P}$ -a.e.  $(q, W)$ , under the quenched probability  $\mathbb{P}(\cdot | q, W)$ , the limit points of the laws of  $\tilde{K}_n$  and  $\tilde{Z}_n$  are the same ones. Now, by adapting the proof of Theorem 2, we have that for  $\mathbb{P}$ -a.e.  $(q, W)$ , under the quenched probability  $\mathbb{P}(\cdot | q, W)$ , the limit points of the laws of  $\tilde{Z}_n$ , as  $n \rightarrow \infty$ , under the topology of weak convergence of measures, are equal to the set of the laws of random variables in  $\Theta_B$ . It gives that for  $\mathbb{P}$ -a.e.  $(q, W)$ , under the quenched probability  $\mathbb{P}(\cdot | q, W)$ , the limit points of the laws of  $\tilde{K}_n$ , as  $n \rightarrow \infty$ , under the topology of weak convergence of measures, are equal to the set of the laws of random variables in  $\Theta_B$  and Theorem 1 follows.

Let  $(\zeta_n)_n$  be a sequence of random variables in  $\Theta_B$ , each  $\zeta_n$  being associated to a function  $f_n \in \mathcal{K}^*$ . The sequence of the (almost everywhere) derivatives of  $f_n$  is then a bounded sequence in the Hilbert space  $L^2(\mathbb{R}_+)$ , so we can extract a subsequence which weakly converges to a limit whose integral is in  $\mathcal{K}^*$ . Using the definition of the weak convergence and the fact that  $\zeta_n = \int_{\mathbb{R}_+} \mathbf{1}_{[0, V_1]}(y) \dot{f}_n(y) dy$ ,  $(\zeta_n)_n$  converges almost surely. Since the sequence  $(\zeta_n)_n$  is bounded in  $L^p(B)$  for any  $p \geq 1$ , the convergence also holds in  $L^2(B)$ , and compactness follows.  $\square$

### 3. CASE OF DIMENSION TWO

**3.1. Assumptions and results.** We will make the following two assumptions on the random walk and on the random scenery:

(A1). The random walk increment  $S_1$  takes its values in  $\mathbb{Z}^2$  and has a centered law with a finite and non-singular covariance matrix  $\Sigma$ . We further suppose that the random walk is aperiodic in the sense of Spitzer [24], which amounts to requiring that  $\varphi(u) = 1$  if and only if  $u \in 2\pi\mathbb{Z}^2$ , where  $\varphi$  is the characteristic function of  $S_1$ .

(A2).  $\mathbb{E}[q_1] = 0$ ,  $\mathbb{E}[q_1^2] = 1$  and  $\mathbb{E}[|q_1|^\gamma] < \infty$  for some  $\gamma > 2$ .

Our aim is to prove the following quenched central limit theorem.

**Theorem 3.1.** *Under assumptions (A1) and (A2), for  $\mathbb{P}$ -a.e.  $q$ , for all  $x \in \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( K_n \geq x \sqrt{n \log n} \mid q \right) = \frac{1}{\sqrt{2\pi\sigma}} \int_x^\infty e^{-u^2/2\sigma^2} du, \quad (8)$$

where  $\sigma^2 = (2\pi\sqrt{\det \Sigma})^{-1}$ .

**Remark:** The conclusion of this theorem still holds if, alternatively, the (A1) assumption is replaced by the following one:

(A1'). The sequence  $S = (S_n)_{n \geq 0}$  is an aperiodic random walk in  $\mathbb{Z}$  starting from 0 such that the sequence  $\left(\frac{S_n}{n}\right)_n$  converges in distribution to a random variable with characteristic function given by  $t \mapsto \exp(-a|t|)$  with  $a > 0$ , in that case  $\sigma^2$  is given by  $(2\pi a)^{-1}$ .

Indeed, the proof of Theorem 3.1 depends on  $S$  through properties of the self-intersection local time and of the intersection local time of the random walk  $S$  which are known to be the same under assumptions (A1) or (A1').

An ingredient in the proof of Theorem 3.1 is the following functional central limit theorem under  $\mathbb{P}(\cdot|S)$ , which is of independent interest. Indeed, it implies the same result under the annealed law, improving the previously known assumptions for such a theorem to hold (see [18]).

Let

$$s_n^2 := \begin{cases} n \log n & \text{if } d = 2, \\ n & \text{if } d \geq 3, \end{cases} \quad (9)$$

and

$$\sigma^2 := \begin{cases} (2\pi\sqrt{\det \Sigma})^{-1} & \text{if } d = 2, \\ \frac{1}{2} \left( \sum_{j=1}^\infty j^2 \chi^2 (1 - \chi)^{j-1} - 1 \right) & \text{if } d \geq 3, \end{cases} \quad (10)$$

where  $\chi := \mathbb{P}(S_n \neq 0 \forall n \geq 1)$ .

**Theorem 3.2.** *Under conditions (A1)–(A2) or (A1')–(A2), for a.e. realization of  $S$ , the process*

$$B_t^{(n)} := s_n^{-1} K_{[nt]}, \quad 0 \leq t \leq 1, \quad (11)$$

*converges weakly under  $\mathbb{P}(\cdot|S)$  in the Skorohod topology as  $n \rightarrow \infty$  to a Brownian motion with variance  $\sigma^2$ .*

The proof of Theorem 3.2 is an application of the martingale CLT, and is given in Appendix A.

For the proof of Theorem 3.1, we proceed as follows. Define the subsequence

$$\tau_n := \lceil \exp n^\alpha \rceil, \quad \frac{1}{2} \vee \frac{2}{\gamma} < \alpha < 1. \quad (12)$$

The following two propositions directly imply Theorem 3.1. Both assume (A1)–(A2).

**Proposition 6.** *For  $\mathbb{P}$ -a.e.  $q$ , for all  $x \in \mathbb{R}$ ,*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( K_{\tau_n} \geq x \sqrt{\tau_n \log \tau_n} \mid q \right) = \frac{1}{\sqrt{2\pi\sigma}} \int_x^\infty e^{-u^2/2\sigma^2} du. \quad (13)$$

**Proposition 7.** *Define  $i(n) \in \mathbb{N}$  by  $\tau_{i(n)} \leq n < \tau_{i(n)+1}$ . Then, for  $\mathbb{P}$ -a.e.  $q$ , the difference*

$$\frac{K_n}{\sqrt{n \log n}} - \frac{K_{\tau_{i(n)}}}{\sqrt{\tau_{i(n)} \log \tau_{i(n)}}} \quad (14)$$

*converges in probability to 0 as  $n \rightarrow \infty$  under  $\mathbb{P}(\cdot|q)$ .*

Propositions 6 and 7 are proved in Sections 3.3 and 3.4, respectively.

**3.2. Two-dimensional random walks.** We gather here some useful facts concerning the local times of two-dimensional random walks. In the following we always assume (A1). Analogous results hold under the alternative assumption (A1').

**3.2.1. Maximum local times.** Let  $N_n(x) := \sum_{i=1}^n \mathbf{1}_{\{S_i=x\}}$  be the local times of the random walk  $S$  up to time  $n$  and

$$N_n^* := \sup_{x \in \mathbb{Z}^2} N_n(x) \quad (15)$$

be the maximum among them.

**Lemma 8.** (i) For all  $k \in \mathbb{N}$ , there exists a  $K := K(k) > 0$  such that

$$\mathbb{E} \left[ (N_n^*)^k \right] \leq K (\log n)^{2k} \quad \forall n \geq 2. \quad (16)$$

(ii) There exists a  $K > 0$  such that

$$\mathbb{P} \left( N_n^* > K (\log n)^2 \right) \leq n^{-2} \quad \forall n \geq 1. \quad (17)$$

*Proof.* The two statements follow from Lemma 18, (b) in [13].  $\square$

**3.2.2. Self-intersection local times.** For  $p \geq 1$ , the  $p$ -fold self-intersection local time  $I_n^{[p]}$  of  $S$  up to time  $n$  is defined by

$$I_n^{[p]} := \sum_{x \in \mathbb{Z}^2} N_n^p(x) = \sum_{1 \leq i_1, \dots, i_p \leq n} \mathbf{1}_{\{S_{i_1} = \dots = S_{i_p}\}}. \quad (18)$$

When  $p = 2$  we will omit the superscript and write  $I_n$ .

**Lemma 9.** For all  $p \geq 2$  and  $k \in \mathbb{N}$ , there exists a  $K > 0$  such that

$$\mathbb{E} \left[ (I_n^{[p]})^k \right] \leq K n^k (\log n)^{k(p-1)} \quad \forall n \geq 2. \quad (19)$$

*Proof.* The statement can be found in [14] (Proposition 2.3).  $\square$

We will also need the following lemma about the self-intersection local times of higher-dimensional random walks.

**Lemma 10.** Let  $\tilde{S}$  be a random walk with a finite, non-singular covariance matrix in dimension  $d \geq 3$ , and let  $\tilde{I}_n^{[p]}$  denote its  $p$ -fold self-intersection local time up to time  $n$ . Then, for all  $p \geq 2$  and  $k \in \mathbb{N}$ , there exists a  $K > 0$  such that

$$\mathbb{E} \left[ (\tilde{I}_n^{[p]})^k \right] \leq K n^k \quad \forall n \geq 2. \quad (20)$$

*Proof.* We can follow the proof of item (i) of Proposition 2.3 in [14], using the fact that, for all  $k \in \mathbb{N}$ ,  $\sup_n \mathbb{E} \left[ \tilde{N}_n(0)^k \right] = \mathbb{E} \left[ \tilde{N}_\infty(0)^k \right] < \infty$  since  $\tilde{N}_\infty(0) := \sum_{n=1}^{\infty} \mathbf{1}_{\{\tilde{S}_n=0\}}$  follows a geometric law with parameter  $\mathbb{P}(\tilde{S}_n \neq 0 \forall n \geq 1) > 0$ .  $\square$

### 3.3. Proof of Proposition 6.



3.3.1. *Truncation.* Fix  $\beta \in (0, 1/4)$ . For  $n \geq 1$ , set  $b_n := n^\beta$ , define  $q^{(n)} \in \mathbb{R}^{\mathbb{N}^*}$  by

$$q_i^{(n)} := q_i \mathbf{1}_{\{|q_i| \leq b_n\}} - \mathbb{E} [q_i \mathbf{1}_{\{|q_i| \leq b_n\}}], \quad i \geq 1, \quad (21)$$

and  $K^{(n)}$  by

$$K_k^{(n)} := \sum_{1 \leq i < j \leq k} q_i^{(n)} q_j^{(n)} \mathbf{1}_{\{S_i = S_j\}}, \quad k \geq 2. \quad (22)$$

The following proposition shows that, in order to prove Proposition 6 for  $K_n$ , it is enough to prove the same statement for  $K_n^{(n)}$ .

**Proposition 11.** (Comparison between  $K$  and  $K^{(n)}$ )

$$\lim_{n \rightarrow \infty} \sup_{2 \leq k \leq \tau_n} \frac{|K_k - K_k^{(\tau_n)}|}{\sqrt{\tau_n} \log \tau_n} = 0 \quad \mathbb{P}\text{-a.s.} \quad (23)$$

*Proof.* Let

$$q_i^{(n)>} := q_i \mathbf{1}_{\{|q_i| > b_n\}} - \mathbb{E} [q_i \mathbf{1}_{\{|q_i| > b_n\}}] \quad (24)$$

and note that, since  $q_1$  is centered,

$$q_i q_j - q_i^{(n)} q_j^{(n)} = -q_i^{(n)>} q_j^{(n)>} + q_i q_j^{(n)>} + q_i^{(n)>} q_j. \quad (25)$$

Write

$$K_k - K_k^{(n)} = -\mathcal{E}_k^{(n,1)} + \mathcal{E}_k^{(n,2)} + \mathcal{E}_k^{(n,3)} \quad (26)$$

where

$$\mathcal{E}_k^{(n,1)} := \sum_{1 \leq i < j \leq k} q_i^{(n)>} q_j^{(n)>} \mathbf{1}_{\{S_i = S_j\}} \quad (27)$$

and  $\mathcal{E}_k^{(n,2)}$ ,  $\mathcal{E}_k^{(n,3)}$  are defined analogously from the corresponding terms in (25). Let us focus for the moment on  $\mathcal{E}_k^{(n,1)}$ . Note that it is a martingale under  $\mathbb{P}$ . Therefore, by Doob's maximal inequality,

$$\mathbb{E} \left[ \sup_{2 \leq k \leq n} |\mathcal{E}_k^{(n,1)}|^2 \right] \leq \mathbb{E} [|\mathcal{E}_n^{(n,1)}|^2] \leq \mathbb{E} [ |q_1^{(n)>}|^2 ]^2 \mathbb{E} [I_n]. \quad (28)$$

Since

$$\mathbb{E} [ |q_1^{(n)>}|^2 ] \leq 2\mathbb{E} [ |q_1|^2 \mathbf{1}_{\{|q_1| > b_n\}} ] \leq \frac{C}{b_n^{\gamma-2}}, \quad (29)$$

by (27)–(29) and Lemma 9(i) we get

$$\mathbb{E} \left[ \sup_{2 \leq k \leq n} \frac{|\mathcal{E}_k^{(n,1)}|^2}{n \log n} \right] \leq \frac{C}{n^{2\beta(\gamma-2)}} \quad (30)$$

which is summable along  $\tau_n$  since  $\gamma > 2$ . Analogously, we can show a similar inequality for  $\mathcal{E}^{(n,2)}$  and  $\mathcal{E}^{(n,3)}$  with the bound  $Cn^{-\beta(\gamma-2)}$  instead, which is also summable along  $\tau_n$ . The proof is concluded by applying the Borel-Cantelli lemma.  $\square$

3.3.2. *Decomposition of quenched moments.* From now on, we will work with the truncated and recentered version  $K^{(n)}$  of the energy. Since the  $q_i^{(n)}$  are bounded, the quenched moments

$$m_n^{(p)} := \sum_{1 \leq i_1 < j_1 \leq n} \cdots \sum_{1 \leq i_p < j_p \leq n} \prod_{\ell=1}^p q_{i_\ell}^{(n)} q_{j_\ell}^{(n)} \mathbb{P} \left( \bigcap_{\ell=1}^p \{S_{i_\ell} = S_{j_\ell}\} \right) \quad (31)$$

are all well defined and satisfy

$$m_n^{(p)} = \mathbb{E} \left[ (K_n^{(n)})^p \mid q \right] \quad \mathbb{P}\text{-a.s.} \quad (32)$$

We aim to prove that the  $m_n^{(p)}$  when properly normalized converge a.s. along  $\tau_n$  to the corresponding moments of a Gaussian random variable. In order to do that, we will first show how they can be decomposed into sums of terms that are easier to control.

In the following we will use the notation  $\mathbf{i} = (i_1, \dots, i_p)$ ,  $\mathbf{j} = (j_1, \dots, j_p)$ , and we will write  $\mathbf{i} < \mathbf{j}$  to mean that  $i_\ell < j_\ell$  for all  $\ell = 1, \dots, p$ .

For fixed  $J \in \{2, \dots, 2p\}$ ,  $L = (L_1, \dots, L_J) \in (\mathbb{N}^*)^J$  such that  $L_1 + \dots + L_J = 2p$ , and  $a = (a_1, \dots, a_J) \in \mathbb{N}^J$  such that  $a_1 < \dots < a_J$ , we will say that a pair  $\mathbf{i} < \mathbf{j}$  is *compatible with  $J, L$  and  $a$*  if the following hold:

$$\begin{aligned} (i) \quad & \{i_1, j_1, \dots, i_p, j_p\} = \{a_1, \dots, a_J\}; \\ (ii) \quad & \#\{\ell: i_\ell = a_k\} + \#\{\ell: j_\ell = a_k\} = L_k, \quad k = 1, \dots, J. \end{aligned} \quad (33)$$

With this definition, we can see that  $m_n^{(p)}$  may be written as follows:

$$m_n^{(p)} = \sum_{J=2}^{2p} \sum_{\substack{L \in (\mathbb{N}^*)^J \\ L_1 + \dots + L_J = 2p}} \sum_{1 \leq a_1 < \dots < a_J \leq n} \prod_{\ell=1}^J (q_{a_\ell}^{(n)})^{L_\ell} \sum_{\substack{\mathbf{i} < \mathbf{j} \\ \text{compat. with } J, L, a}} \mathbb{P} \left( \bigcap_{\ell=1}^p \{S_{i_\ell} = S_{j_\ell}\} \right). \quad (34)$$

Next, given  $\mathbf{i} < \mathbf{j}$  compatible with  $J, L$  and  $a$ , we define on the set  $\{1, \dots, J\}$  a graph structure depending on  $\mathbf{i}$  and  $\mathbf{j}$  as follows. For  $u \neq v \in \{1, \dots, J\}$ , declare the edge  $\{u, v\}$  present if there exist  $\ell, k \in \{1, \dots, p\}$  (possibly equal) such that:

$$\begin{aligned} (i) \quad & \{a_u, a_v\} \subset \{i_\ell, j_\ell\} \cup \{i_k, j_k\}; \\ (ii) \quad & \{i_\ell, j_\ell\} \cap \{i_k, j_k\} \neq \emptyset. \end{aligned} \quad (35)$$

Let  $\mathcal{P}_{\mathbf{i}\mathbf{j}}$  be the partition of  $\{1, \dots, J\}$  into the connected components of the graph thus constructed. Note that, if  $P \in \mathcal{P}_{\mathbf{i}\mathbf{j}}$ , then  $|P| \geq 2$ . For an arbitrary partition  $\mathcal{P}$  of  $\{1, \dots, J\}$ , we say that  $\mathbf{i} < \mathbf{j}$  is *compatible with  $\mathcal{P}$*  if  $\mathcal{P}_{\mathbf{i}\mathbf{j}} = \mathcal{P}$ .

We will now show that a consequence of the previous definitions is that

$$\begin{aligned} \bigcap_{\ell=1}^p \{S_{i_\ell} = S_{j_\ell}\} &= \bigcap_{P \in \mathcal{P}_{\mathbf{i}\mathbf{j}}} \bigcup_{x \in \mathbb{Z}^2} \bigcap_{u \in P} \{S_{a_u} = x\} \\ &= \bigcap_{P \in \mathcal{P}_{\mathbf{i}\mathbf{j}}} \bigcap_{u, v \in P} \{S_{a_u} = S_{a_v}\}. \end{aligned} \quad (36)$$

To see this, first note, setting  $a(P) := \{a_u: u \in P\}$ , that

$$\bigcap_{\ell=1}^p \{S_{i_\ell} = S_{j_\ell}\} = \bigcap_{P \in \mathcal{P}_{\mathbf{i}\mathbf{j}}} \bigcap_{\ell: i_\ell \in a(P)} \{S_{i_\ell} = S_{j_\ell}\}. \quad (37)$$

Now, if  $i_k \neq i_{k'} \in a(P)$  and  $S_{i_k} = x$ , we claim that, on the event  $E_P := \bigcap_{\ell: i_\ell \in a(P)} \{S_{i_\ell} = S_{j_\ell}\}$ ,  $S_{i_{k'}} = x$  as well. Indeed, letting  $v, v'$  such that  $a_v = i_k$ ,  $a_{v'} = i_{k'}$ , we have that  $v, v' \in P$  and so

there exist  $v = v_0, v_1, \dots, v_q = v'$  such that  $v_\ell \in P$  and  $\{v_{\ell-1}, v_\ell\}$  is an edge for each  $\ell = 1, \dots, q$ . Using (35) and the definition of the graph structure, we can see that, on  $E_P \cap \{S_{a_{v_0}} = x\}$ ,  $S_{v_1} = x$  as well. Proceeding analogously by induction, we conclude the claim. The equality (36) then follows by noting that  $\cup_{\ell: i_\ell \in a(P)} \{i_\ell, j_\ell\} = a(P)$ .

In addition, the numbers

$$F(J, L, \mathcal{P}, a) := \#\{\mathbf{i} < \mathbf{j} \text{ compatible with } J, L, a \text{ and } \mathcal{P}\} \quad (38)$$

do not depend on  $a$ . This can be seen by considering, for  $a \neq a'$ , the bijection between  $\{\mathbf{i} < \mathbf{j} \text{ compat. w/ } J, L, \mathcal{P}, a\}$  and  $\{\mathbf{i}' < \mathbf{j}' \text{ compat. w/ } J, L, \mathcal{P}, a'\}$  given by  $i'_\ell = a'_u$  if and only if  $i_\ell = a_u$ , and analogously for  $\mathbf{j}'$ .

Thus we see that we may decompose  $m_n^{(p)}$  in the following manner:

$$m_n^{(p)} = \sum_{J=2}^{2p} \sum_{\substack{\mathcal{P} \text{ partition of } \{1, \dots, J\} \\ \text{s.t. } |P| \geq 2 \forall P \in \mathcal{P}}} \sum_{\substack{L \in (\mathbb{N}^*)^J \\ L_1 + \dots + L_J = 2p}} F(J, L, \mathcal{P}) m_n^{(p)}(J, L, \mathcal{P}) \quad (39)$$

where  $F(J, L, \mathcal{P}) := F(J, L, \mathcal{P}, a)$  as in (38) with any  $a$ , and

$$m_n^{(p)}(J, L, \mathcal{P}) := \sum_{1 \leq a_1 < \dots < a_J \leq n} \prod_{\ell=1}^J (q_{a_\ell}^{(n)})^{L_\ell} \mathbb{P} \left( \bigcap_{P \in \mathcal{P}} \bigcap_{u, v \in P} \{S_{a_u} = S_{a_v}\} \right). \quad (40)$$

Next, using the identity

$$\prod_{\ell \in \mathcal{I}} (c_\ell + d_\ell) = \sum_{A \subset \mathcal{I}} \prod_{\ell \in A} c_\ell \prod_{\ell \notin A} d_\ell, \quad (41)$$

we see that we may further decompose  $m_n^{(p)}(J, L, \mathcal{P})$  as

$$m_n^{(p)}(J, L, \mathcal{P}) = \sum_{A \subset \{1, \dots, J\}} m_n^{(p)}(J, L, \mathcal{P}, A), \quad (42)$$

where  $m_n^{(p)}(J, L, \mathcal{P}, A) :=$

$$\sum_{1 \leq a_1 < \dots < a_J \leq n} \prod_{\ell \in A} \mathbb{E} \left[ (q_{a_\ell}^{(n)})^{L_\ell} \right] \prod_{\ell \notin A} \left\{ (q_{a_\ell}^{(n)})^{L_\ell} - \mathbb{E} \left[ (q_{a_\ell}^{(n)})^{L_\ell} \right] \right\} \mathbb{P} \left( \bigcap_{P \in \mathcal{P}} \bigcap_{u, v \in P} \{S_{a_u} = S_{a_v}\} \right). \quad (43)$$

For fixed  $J$  and  $L$ , let

$$\mathcal{A} := \{\ell \in \{1, \dots, J\} : L_\ell > 1\}. \quad (44)$$

Since  $m_n^{(p)}(J, L, \mathcal{P}, A) = 0$  if  $A \cap \mathcal{A}^c \neq \emptyset$  and  $m_n^{(p)}(J, L, \mathcal{P}, \{1, \dots, J\}) = \mathbb{E} \left[ m_n^{(p)}(J, L, \mathcal{P}) \right]$ ,

$$m_n^{(p)}(J, L, \mathcal{P}) - \mathbb{E} \left[ m_n^{(p)}(J, L, \mathcal{P}) \right] = \sum_{A \subset \mathcal{A}: A^c \neq \emptyset} m_n^{(p)}(J, L, \mathcal{P}, A). \quad (45)$$

Moreover, when  $A^c \neq \emptyset$  we may write

$$m_n^{(p)}(J, L, \mathcal{P}, A) = \sum_{a_\ell: \ell \notin A} \prod_{\ell \notin A} \left\{ (q_{a_\ell}^{(n)})^{L_\ell} - \mathbb{E} \left[ (q_{a_\ell}^{(n)})^{L_\ell} \right] \right\} \mathcal{W}_n((a_\ell)_{\ell \notin A}, L, \mathcal{P}, A), \quad (46)$$

where  $\mathcal{W}_n((a_\ell)_{\ell \notin A}, L, \mathcal{P}, A) :=$

$$\sum_{a_\ell: \ell \in A} \prod_{\ell \in A} \mathbb{E} \left[ (q_{a_\ell}^{(n)})^{L_\ell} \right] \mathbb{P} \left( \bigcap_{P \in \mathcal{P}} \bigcap_{u, v \in P} \{S_{a_u} = S_{a_v}\} \right) \mathbf{1}_{\{1 \leq a_1 < \dots < a_J \leq n\}}. \quad (47)$$

Finally, using the i.i.d. structure of  $q$  and the fact that we are taking ordered sums, we may write

$$\left\| m_n^{(p)}(J, L, \mathcal{P}, A) \right\|_2^2 = \prod_{\ell \notin A} \left\| (q_1^{(n)})^{L_\ell} - \mathbb{E} \left[ (q_1^{(n)})^{L_\ell} \right] \right\|_2^2 \sum_{a_\ell: \ell \notin A} \mathcal{W}_n^2((a_\ell)_{\ell \notin A}, L, \mathcal{P}, A). \quad (48)$$

3.3.3. *Analysis of the terms.* We begin with the terms in which  $A^c = \emptyset$ , i.e., the ones corresponding to  $\mathbb{E}[m_n^{(p)}(J, L, \mathcal{P})]$ .

**Proposition 12.** *For all  $p \in \mathbb{N}^*$ , there exists a constant  $K \in (0, \infty)$  such that*

$$\mathbb{E} \left[ m_n^{(p)}(J, L, \mathcal{P}) \right] \leq K(n \log n)^{p/2} \quad \forall n \geq 2. \quad (49)$$

*Proof.* Integrating (40) we get

$$\mathbb{E} \left[ m_n^{(p)}(J, L, \mathcal{P}) \right] = \prod_{\ell=1}^J \mathbb{E} \left[ \left( q_{a_\ell}^{(n)} \right)^{L_\ell} \right] \sum_{1 \leq a_1 < \dots < a_J \leq n} \mathbb{P} \left( \bigcap_{P \in \mathcal{P}} \bigcap_{u, v \in P} \{S_{a_u} = S_{a_v}\} \right). \quad (50)$$

We may suppose that  $L_\ell \geq 2$  for all  $1 \leq \ell \leq J$  since otherwise  $\mathbb{E} \left[ m_n^{(p)}(J, L, \mathcal{P}) \right] = 0$ . In particular,  $J \leq p$ . Estimating

$$\mathbb{E} \left[ |q_1^{(n)}|^{L_\ell} \right] \leq 2^{L_\ell} \mathbb{E} \left[ |q_1 \mathbf{1}_{\{|q_1| \leq b_n\}}|^{L_\ell} \right] \leq 2^{L_\ell} \mathbb{E} \left[ |q_1 \mathbf{1}_{\{|q_1| \leq b_n\}}|^{L_\ell-2} q_1^2 \right] \leq 2^{L_\ell} b_n^{L_\ell-2}, \quad (51)$$

we see that the first term with the product in (50) is at most  $C b_n^{2(p-J)}$ . On the other hand, the second term is smaller than

$$\begin{aligned} \sum_{1 \leq a_1, \dots, a_J \leq n} \mathbb{E} \left[ \prod_{P \in \mathcal{P}} \sum_{x \in \mathbb{Z}^2} \prod_{u \in P} \mathbf{1}_{\{S_{a_u} = x\}} \right] &= \mathbb{E} \left[ \prod_{P \in \mathcal{P}} \sum_{x \in \mathbb{Z}^2} N_n^{|P|}(x) \right] \\ &= \mathbb{E} \left[ \prod_{P \in \mathcal{P}} I_n^{[|P|]} \right] \leq C n^{|\mathcal{P}|} (\log n)^{J-|\mathcal{P}|} \end{aligned} \quad (52)$$

where we used Hölder's inequality and Lemma 9(i).

Combining (50)–(52) we obtain

$$\mathbb{E} \left[ m_n^{(p)}(J, L, \mathcal{P}) \right] \leq C b_n^{2(p-J)} n^{|\mathcal{P}|} (\log n)^{J-|\mathcal{P}|}. \quad (53)$$

We now split into different cases. Note that  $|\mathcal{P}| \leq \lfloor p/2 \rfloor$ . If  $J = p$  and  $|\mathcal{P}| = p/2$ , then (49) holds by (53). If  $J = p$  and  $|\mathcal{P}| < p/2$ , then (53) divided by  $(n \log n)^{\frac{p}{2}}$  goes to zero as  $n \rightarrow \infty$ . Lastly, if  $J < p$ , then

$$\frac{\mathbb{E} \left[ m_n^{(p)}(J, L, \mathcal{P}) \right]}{(n \log n)^{\frac{p}{2}}} \leq C n^{-\frac{p-J}{2}(1-4\beta)} (\log n)^{\frac{p}{2}} \quad (54)$$

which goes to zero as  $n \rightarrow \infty$  since  $\beta < 1/4$ .  $\square$

The rest of the analysis consists in showing that all other terms with  $A^c \neq \emptyset$  converge to zero a.s. along  $\tau_n$  when normalized.

**Proposition 13.** *For any fixed choice of  $p, J, L, \mathcal{P}$ , if  $A^c \neq \emptyset$  then*

$$\lim_{n \rightarrow \infty} \frac{m_{\tau_n}^{(p)}(J, L, \mathcal{P}, A)}{(\tau_n \log \tau_n)^{p/2}} = 0 \quad \mathbb{P}\text{-a.s.} \quad (55)$$

*Proof.* We may suppose that  $A \subset \mathcal{A}$ . Recall that the variance of  $m_n^{(p)}(J, L, \mathcal{P}, A)$  is given by (48). Ignoring the  $\langle \cdot \rangle$ 's in the indicator function in (47), we may estimate

$$\left\| m_n^{(p)}(J, L, \mathcal{P}, A) \right\|_2^2 \leq \prod_{\ell \notin A} \left\| (q_1^{(n)})^{L_\ell} - \mathbb{E} \left[ (q_1^{(n)})^{L_\ell} \right] \right\|_2^2 \prod_{\ell \in A} \mathbb{E} \left[ (q_1^{(n)})^{L_\ell} \right]^2 B_n(J, L, \mathcal{P}, A) \quad (56)$$

where

$$\begin{aligned} B_n(J, L, \mathcal{P}, A) &:= \sum_{1 \leq a_\ell \leq n: \ell \notin A} \left\{ \sum_{1 \leq a_\ell \leq n: \ell \in A} \mathbb{E} \left[ \prod_{P \in \mathcal{P}} \sum_{x \in \mathbb{Z}^2} \prod_{u \in P} \mathbf{1}_{\{S_{a_u} = x\}} \right] \right\}^2 \\ &= \sum_{1 \leq a_\ell \leq n: \ell \notin A} \mathbb{E} \left[ \prod_{P \in \mathcal{P}} \sum_{x \in \mathbb{Z}^2} N_n^{|P \cap A|}(x) \prod_{u \in P \cap A^c} \mathbf{1}_{\{S_{a_u} = x\}} \right]^2. \end{aligned} \quad (57)$$

We proceed to bound  $B_n(J, L, \mathcal{P}, A)$ . Denoting by  $\widehat{N}_n(x)$  the local times of an independent copy  $\widehat{S}$  of  $S$ , and by  $\widetilde{N}_n(x, y)$  the local times of the 4-dimensional random walk  $\widetilde{S}_n = (S_n, \widehat{S}_n)$ , we can rewrite (57) as

$$\begin{aligned} &\sum_{1 \leq a_\ell \leq n: \ell \notin A} \mathbb{E}^{\otimes 2} \left[ \prod_{P \in \mathcal{P}} \sum_{x, y \in \mathbb{Z}^2} \left( N_n(x) \widehat{N}_n(y) \right)^{|P \cap A|} \prod_{u \in P \cap A^c} \mathbf{1}_{\{S_{a_u} = x, \widehat{S}_{a_u} = y\}} \right] \\ &= E^{\otimes 2} \left[ \prod_{P \in \mathcal{P}} \sum_{x, y \in \mathbb{Z}^2} \left( N_n(x) \widehat{N}_n(y) \right)^{|P \cap A|} \widetilde{N}_n^{|P \cap A^c|}(x, y) \right] \\ &= E^{\otimes 2} \left[ \prod_{P \subset A} I_n^{[|P|]} \widehat{I}_n^{[|P|]} \prod_{P \subset A^c} \widetilde{I}_n^{[|P|]} \prod_{P: \emptyset \neq P \cap A \neq P} \sum_{x, y \in \mathbb{Z}^2} \left( N_n(x) \widehat{N}_n(y) \right)^{|P \cap A|} \widetilde{N}_n^{|P \cap A^c|}(x, y) \right] \\ &\leq E^{\otimes 2} \left[ \prod_{P \subset A} I_n^{[|P|]} \widehat{I}_n^{[|P|]} \prod_{P \subset A^c} \widetilde{I}_n^{[|P|]} \prod_{P: \emptyset \neq P \cap A \neq P} (N_n^* \widehat{N}_n^*)^{|P \cap A|} \widetilde{I}_n^{[|P \cap A^c|]} \right]. \end{aligned} \quad (58)$$

where  $\widehat{I}_n^{[k]}, \widetilde{I}_n^{[k]}$  are the analogues of  $I_n^{[k]}$  for the corresponding random walks, and  $\widehat{N}_n^* = \sup_x \widehat{N}_n(x)$ . Using Hölder's inequality, Lemmas 8(i), 9(i) and 10, and the fact that  $\mathcal{P}$  is a partition, we see that (58) is at most

$$Cn^{|\mathcal{P}| + |\{P: P \subset A\}|} (\log n)^{2(|A| + \sum_{P: \emptyset \neq P \cap A \neq P} |P \cap A| - |\{P: P \subset A\}|)}. \quad (59)$$

Now we note that  $|\mathcal{P}| \leq \lfloor J/2 \rfloor$ ,  $|\{P: P \cap A \neq \emptyset\}| \leq \lfloor |A|/2 \rfloor$  and  $2|A| + |A^c| \leq 2p$ . Therefore,

$$t := p - |\mathcal{P}| - |\{P: P \subset A\}| \geq 0 \quad (60)$$

and there is equality if and only if

- (1)  $|A|$  and  $|A^c|$  are even;
- (2)  $L_\ell = 2 \forall \ell \in A$  and  $L_\ell = 1 \forall \ell \in A^c$ ;
- (3)  $|P| = 2 \forall P \in \mathcal{P}$ ;
- (4) for any  $P \in \mathcal{P}$ , either  $P \subset A$  or  $P \subset A^c$ .

Thus by (57)–(59)

$$\frac{B_n(J, L, \mathcal{P}, A)}{(n \log n)^p} \leq \begin{cases} C(\log n)^{-\frac{|A^c|}{2}} & \text{if } t = 0, \\ Cn^{-t}(\log n)^{3p} & \text{if } t > 0. \end{cases} \quad (61)$$

We will consider three cases separately:

- Case 1:  $t \geq 1$ ;
- Case 2:  $t = 0$  and  $|A^c| \geq 4$ ;
- Case 3:  $t = 0$  and  $|A^c| = 2$ .

For each of these cases we will show that, for every  $\epsilon > 0$ ,

$$\sum_{n=1}^{\infty} \mathbb{P} \left( |m_{\tau_n}^{(p)}(J, L, \mathcal{P}, A)| > \epsilon \sqrt{\tau_n \log \tau_n} \right) < \infty. \quad (62)$$

This will be done via Markov's inequality together with (56) and (61). In the third case, the variance estimate (56) is not good enough, but we will get a better bound estimating a higher moment.

**Case 1:** ( $t \geq 1$ ) For  $\ell \in A$ , we have

$$\mathbb{E} \left[ (q_1^{(n)})^{L_\ell} \right]^2 \leq 2^{2L_\ell} b_n^{2(L_\ell-2)} \quad (63)$$

as in (51) and, for  $\ell \notin A$ , we can estimate in a similar fashion

$$\begin{aligned} \left\| (q_1^{(n)})^{L_\ell} - \mathbb{E} \left[ (q_1^{(n)})^{L_\ell} \right] \right\|_2^2 &\leq 2^{2L_\ell} \mathbb{E} \left[ (q_1 \mathbf{1}_{\{|q_1| \leq b_n\}})^{2L_\ell} \right] \\ &\leq 2^{2L_\ell} \mathbb{E} \left[ (q_1 \mathbf{1}_{\{|q_1| \leq b_n\}})^{2(L_\ell-1)} q_1^2 \right] \\ &\leq 2^{2L_\ell} b_n^{2(L_\ell-1)}. \end{aligned} \quad (64)$$

Using (56), (61) and  $|P| + |\{P: P \subset A\}| \leq |A| + |A^c|/2$ , we get

$$\begin{aligned} \frac{\left\| m_n^{(p)}(J, L, \mathcal{P}, A) \right\|_2^2}{(n \log n)^p} &\leq C b_n^{4(p-|A|-|A^c|/2)} n^{-t} (\log n)^{3p} \leq C \left( \frac{b_n^4}{n} \right)^t (\log n)^{3p} \\ &= C n^{-t(1-4\beta)} (\log n)^{3p} \end{aligned} \quad (65)$$

which is summable along  $\tau_n$  since  $\beta < 1/4$ .

**Case 2:** ( $t = 0, |A^c| \geq 4$ ) As mentioned above, in this case  $L_\ell = 2$  for  $\ell \in A$  and  $L_\ell = 1$  for  $\ell \notin A$ . Using  $\mathbb{E}[|q_1^{(n)}|^2] \leq \mathbb{E}[|q_1|^2] = 1$ , we get from (56) and (61) that

$$\frac{\left\| m_n^{(p)}(j, l, A) \right\|_2^2}{(n \log n)^p} \leq C (\log n)^{-2} \quad (66)$$

which is summable along  $\tau_n$  since  $\alpha > 1/2$ .

**Case 3:** ( $t = 0, |A^c| = 2$ ) In this case, (56) is not enough to prove (62). However, since  $|A^c| = 2$  and  $t = 0$ , these terms are of the form

$$m_n^{(p)}(J, L, \mathcal{P}, A) = \sum_{a_{k_1}, a_{k_2}} q_{a_{k_1}}^{(n)} q_{a_{k_2}}^{(n)} \mathcal{W}_n(a_{k_1}, a_{k_2}) \quad (67)$$

where

$$\mathcal{W}_n(a_{k_1}, a_{k_2}) = \mathbb{E} \left[ (q_1^{(n)})^2 \right]^{(J-2)} \sum_{a_\ell: \ell \neq k_1, k_2} \mathbb{E} \left[ \prod_{P \in \mathcal{P}} \sum_{x \in \mathbb{Z}^2} \prod_{u \in P} \mathbf{1}_{\{S_{a_u} = x\}} \right] \mathbf{1}_{\{a_1 < \dots < a_J\}} \quad (68)$$

and  $\{k_1, k_2\} \in \mathcal{P}$ . Rewrite

$$m_n^{(p)}(J, L, \mathcal{P}, A) = \sum_{1 \leq i < j \leq n} q_i^{(n)} q_j^{(n)} \mathcal{W}_n(i, j) = \sum_{j=2}^n q_j^{(n)} \sum_{i=1}^{j-1} q_i^{(n)} \mathcal{W}_n(i, j). \quad (69)$$

Since

$$k \mapsto \sum_{j=2}^k q_j^{(n)} \sum_{i=1}^{j-1} q_i^{(n)} \mathcal{W}_n(i, j) \quad \text{and} \quad k \mapsto \sum_{i=1}^k q_i^{(n)} \mathcal{W}_n(i, j) \quad (70)$$

are centered martingales, by Burkholder's inequality applied twice we have

$$\begin{aligned} \left\| m_n^{(p)}(J, L, \mathcal{P}, A) \right\|_{\gamma}^2 &\leq C \sum_{j=2}^n \left\| q_j^{(n)} \sum_{i=1}^{j-1} q_i^{(n)} \mathcal{W}_n(i, j) \right\|_{\gamma}^2 \\ &= C \mathbb{E} \left[ |q_1^{(n)}|^{\gamma} \right]^{\frac{2}{\gamma}} \sum_{j=2}^n \left\| \sum_{i=1}^{j-1} q_i^{(n)} \mathcal{W}_n(i, j) \right\|_{\gamma}^2 \\ &\leq C \sum_{1 \leq i < j \leq n} \mathcal{W}_n^2(i, j). \end{aligned} \quad (71)$$

Since, by (61),

$$\sum_{1 \leq i < j \leq n} \mathcal{W}_n^2(i, j) \leq B_n(J, L, \mathcal{P}, A) \leq C n^p (\log n)^{p-1}, \quad (72)$$

combining (71)–(72) we get

$$\frac{\left\| m_n^{(p)}(J, L, \mathcal{P}, A) \right\|_{\gamma}^{\gamma}}{(n \log n)^p} \leq C (\log n)^{-\frac{\gamma}{2}} \quad (73)$$

which is summable along  $\tau_n$  since  $\alpha > 2/\gamma$ .  $\square$

**3.3.4. Conclusion.** From the results of Section 3.3.3 we obtain the following two propositions. Together with Proposition 11, they will allow us to finish the proof of Proposition 6.

**Proposition 14.** (Convergence of annealed moments)

For every  $p \in \mathbb{N}^*$ ,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \left[ m_n^{(p)} \right]}{(n \log n)^{p/2}} = \frac{1}{\sqrt{2\pi\sigma}} \int x^p e^{-x^2/2\sigma^2} dx. \quad (74)$$

*Proof.* First we note that, because of the annealed CLT for  $K_n$  (see Appendix A) and Proposition 11,  $K_n^{(n)}$  satisfies a CLT with variance  $\sigma^2$  under  $\mathbb{P}$ . Integrating (39) and applying Proposition 12, we see that, for all  $p \geq 1$ ,

$$\sup_{n \geq 2} \frac{\mathbb{E} \left[ m_n^{(p)} \right]}{(n \log n)^{p/2}} < \infty, \quad (75)$$

hence  $(K_n^{(n)})^p / (n \log n)^{p/2}$  is uniformly integrable for all  $p \geq 1$ .  $\square$

**Proposition 15.** (Convergence of quenched moments)

For every  $p \in \mathbb{N}^*$ ,

$$\lim_{n \rightarrow \infty} \frac{m_{\tau_n}^{(p)} - \mathbb{E} \left[ m_{\tau_n}^{(p)} \right]}{(\tau_n \log \tau_n)^{p/2}} = 0 \quad \mathbb{P}\text{-a.s.} \quad (76)$$

*Proof.* Combining (39) and (45), we see that  $m_n^{(p)} - \mathbb{E} \left[ m_n^{(p)} \right]$  is a sum of terms  $m_n^{(p)}(J, L, \mathcal{P}, A)$  with  $A \subset \mathcal{A}$ ,  $A^c \neq \emptyset$ , so the result follows from Proposition 13.  $\square$

*Proof of Proposition 6.* The conclusion is now straightforward: Propositions 14–15 give us (13) with  $K_n^{(n)}$  in place of  $K_n$  by the method of moments, and this is passed to  $K_n$  by Proposition 11.  $\square$

**3.4. Proof of Proposition 7.** Before we start, we note some properties of the subsequence  $\tau_n$  that will be used in the sequel: there exist positive constants  $K_1, K_2$  such that

$$\begin{aligned} \text{(p1)} \quad & \lim_{n \rightarrow \infty} \tau_{n+1}/\tau_n = 1; \\ \text{(p2)} \quad & K_1 \exp(n^\alpha/2) \leq \tau_{n+1} - \tau_n \leq K_2 \tau_n / n^{1-\alpha} \quad \forall n \in \mathbb{N}^*; \\ \text{(p3)} \quad & \tau_n \leq K_2 \exp(n^\alpha) \quad \forall n \in \mathbb{N}^*. \end{aligned} \tag{77}$$

*Proof.* For integers  $b \geq a \geq 2$ , let

$$K_{a,b} := K_b - K_a. \tag{78}$$

Once we show that

$$\lim_{n \rightarrow \infty} \sup_{\tau_n \leq k \leq \tau_{n+1}} \frac{|K_{\tau_n, k}|}{\sqrt{\tau_n \log \tau_n}} = 0 \quad \mathbb{P}\text{-a.s.}, \tag{79}$$

Proposition 7 will follow by noting that

$$\begin{aligned} \left| \frac{K_n}{\sqrt{n \log n}} - \frac{K_{\tau_{i(n)}}}{\sqrt{\tau_{i(n)} \log \tau_{i(n)}}} \right| &\leq \frac{|K_n - K_{\tau_{i(n)}}|}{\sqrt{n \log n}} \\ &+ \frac{|K_{\tau_{i(n)}}|}{\sqrt{\tau_{i(n)} \log \tau_{i(n)}}} \left( 1 - \sqrt{\frac{\tau_{i(n)} \log \tau_{i(n)}}{n \log n}} \right). \end{aligned} \tag{80}$$

Then, by (79) and since  $\lim_{n \rightarrow \infty} n^{-1} \tau_{i(n)} = 1$ , the first term in the r.h.s. of (80) converges a.s. to 0. Moreover, the second term converges for  $\mathbb{P}$ -a.e.  $q$  in  $\mathbb{P}(\cdot|q)$ -probability to 0 since, by Proposition 6,  $K_{\tau_n}/\sqrt{\tau_n \log \tau_n}$  is a.s. tight under  $\mathbb{P}(\cdot|q)$ . Therefore, we only need to show (79). By Proposition 11, it is enough to prove (79) for the sequence  $q_i^{(\tau_{n+1})}$ ,  $i \geq 1$ , i.e.

$$\lim_{n \rightarrow \infty} \sup_{\tau_n \leq k \leq \tau_{n+1}} \frac{|K_{\tau_n, k}^{(\tau_{n+1})}|}{\sqrt{\tau_n \log \tau_n}} = 0 \quad \mathbb{P}\text{-a.s.} \tag{81}$$

To this end, we will make use of a maximal inequality for demimartingales due to Newman and Wright [22], as well as Bernstein's inequality.

The sequence  $(K_{a,k}^{(n)})_{k \geq a}$  is a zero-mean martingale under both  $\mathbb{P}$  and  $\mathbb{P}(\cdot|S)$  with respect to the filtration  $\sigma((q_i)_{i \leq k}, S)$ . Indeed,

$$K_{a,k+1}^{(n)} - K_{a,k}^{(n)} = K_{k+1}^{(n)} - K_k^{(n)} = q_{k+1}^{(n)} \sum_{i=1}^k q_i^{(n)} 1_{\{S_i = S_{k+1}\}}$$



and the r.v.'s  $q_i^{(n)}$ ,  $i \geq 1$  are independent and centered. Therefore,

$$\begin{aligned} \mathbb{E} \left[ \left( K_{a,b}^{(n)} \right)^2 \middle| S \right] &= \sum_{k=a+1}^b \mathbb{E} \left[ \left( q_k^{(n)} \sum_{i=1}^{k-1} q_i^{(n)} \mathbf{1}_{\{S_i=S_k\}} \right)^2 \middle| S \right] \\ &\leq C \sum_{k=a+1}^b \sum_{i=1}^{k-1} \mathbf{1}_{\{S_i=S_k\}} = C \sum_{k=a+1}^b N_{k-1}(S_k) \\ &\leq C \sum_{x \in \mathbb{Z}^2} N_b(x) N_{a,b}(x) \leq C \sqrt{I_b} \sqrt{I_{a,b}} \end{aligned} \quad (82)$$

where  $N_{a,b}(x) := \sum_{k=a+1}^b \mathbf{1}_{\{S_k=x\}}$ ,  $I_{a,b} := \sum_{x \in \mathbb{Z}^2} N_{a,b}^2(x)$  and for the last step we used the Cauchy-Schwarz inequality. Integrating (82) and using Hölder's inequality we get

$$\begin{aligned} \mathbb{E} \left[ \left( K_{a,b}^{(n)} \right)^2 \right] &\leq C \mathbb{E} \left[ \sqrt{I_b} \sqrt{I_{a,b}} \right] \leq C \sqrt{\mathbb{E} [I_b] \mathbb{E} [I_{a,b}]} \\ &\leq C \sqrt{b \log b} \sqrt{(b-a) \log(b-a)} \leq C b \log b, \end{aligned} \quad (83)$$

where for the third inequality we used Lemma 9(i) and that  $I_{a,b}$  has the same law as  $I_{b-a}$ .

Since  $K_{a,k}^{(n)}$  is in particular a demimartingale under  $\mathbb{P}$ , by Corollary 6 in [22] we get

$$\begin{aligned} \mathbb{P} \left( \sup_{\tau_n \leq k \leq \tau_{n+1}} |K_{\tau_n, k}^{(\tau_{n+1})}| \geq 2\varepsilon \sqrt{\tau_n \log \tau_n} \right) \\ \leq \sqrt{\frac{2\mathbb{E} \left[ \left( K_{\tau_n, \tau_{n+1}}^{(\tau_{n+1})} \right)^2 \right]}{\varepsilon^2 \tau_n \log \tau_n}} \sqrt{\mathbb{P} \left( |K_{\tau_n, \tau_{n+1}}^{(\tau_{n+1})}| \geq \sqrt{\tau_n \log \tau_n} \right)} \\ \leq C \sqrt{\mathbb{P} \left( |K_{\tau_n, \tau_{n+1}}^{(\tau_{n+1})}| \geq \sqrt{\tau_n \log \tau_n} \right)} \end{aligned} \quad (84)$$

by (83) and the properties of  $\tau_n$ .

Now note that  $K_{a,b}^{(n)}$  can be rewritten as

$$\sum_{x \in \mathbb{Z}^2} \left( \Lambda_b(x) - \Lambda_a(x) \right)$$

where

$$\Lambda_k(x) = \sum_{i < j \in \mathcal{L}_k(x)} q_i^{(n)} q_j^{(n)}, \quad \mathcal{L}_k(x) := \{1 \leq i \leq k : S_i = x\}.$$

Given the random walk  $S$ , the random variables  $\Lambda_b(x) - \Lambda_a(x)$ ,  $x \in \mathbb{Z}^2$  are independent, centered and uniformly bounded by  $(b_n N_b^*)^2$ . Furthermore, by (82),

$$\sum_{x \in \mathbb{Z}^2} \mathbb{E}[(\Lambda_b(x) - \Lambda_a(x))^2 | S] \leq C(\sqrt{I_{a,b}} \sqrt{I_b}). \quad (85)$$

Thus we may use Bernstein's inequality under  $\mathbb{P}(\cdot | S)$  to estimate the probability in the right hand side of (84), obtaining that, for all  $u > 0$ ,

$$\mathbb{P} \left( |K_{a,b}^{(n)}| \geq u \middle| S \right) \leq \exp \left( -\frac{1}{2} \frac{u^2}{\sqrt{I_{a,b} I_b} + u(b_n N_b^*)^2} \right).$$

Integrating with respect to the random walk, we get, for every  $\varepsilon > 0$ ,

$$\mathbb{P}\left(|K_{\tau_n, \tau_{n+1}}^{(\tau_{n+1})}| \geq \varepsilon \sqrt{\tau_n \log \tau_n}\right) \leq \mathbb{E}\left[\exp\left(-C \frac{\tau_n \log \tau_n}{\sqrt{I_{\tau_n, \tau_{n+1}} I_{\tau_n}} + \sqrt{\tau_n \log \tau_n} (b_{\tau_{n+1}} N_{\tau_{n+1}}^*)^2}\right)\right]. \quad (86)$$

Recall that, by Lemma 8(ii), there exists  $C > 0$  such that

$$P(N_k^* > C(\log k)^2) \leq k^{-2} \quad \forall k \geq 1. \quad (87)$$

Now fix  $0 < \delta < \frac{1}{2}(\alpha^{-1} - 1)$  and an integer  $\theta > 2/(\alpha\delta)$ . By Markov's inequality and Lemma 9(i), we have

$$P(I_k > k(\log k)^{1+\delta}) \leq \frac{E[I_k^\theta]}{k^\theta (\log k)^{(1+\delta)\theta}} \leq \frac{C}{(\log k)^{\theta\delta}} \quad \forall k \geq 2. \quad (88)$$

By (86)–(88), the subadditivity of  $\sqrt{\cdot}$  and the fact that  $e^{-2x/(y+z)} \leq e^{-x/y} + e^{-x/z}$  for any  $x, y, z > 0$ , we see that (84) is at most

$$C_1(\tau_{n+1})^{-1} + C_2(\log(\tau_{n+1} - \tau_n))^{-\frac{\theta\delta}{2}} + C_3(\log \tau_n)^{-\frac{\theta\delta}{2}} + e^{-C_4 d_n/e_n} + e^{-C_4 d_n/f_n}, \quad (89)$$

where  $C_1$ – $C_4$  are positive constants and

$$\begin{aligned} d_n &:= \tau_n \log \tau_n, \\ e_n &:= \sqrt{\tau_n(\tau_{n+1} - \tau_n)} [\log(\tau_{n+1} - \tau_n) \log(\tau_n)]^{(1+\delta)/2}, \\ f_n &:= \sqrt{\tau_n \log \tau_n} (\tau_{n+1})^{2\beta} (\log \tau_{n+1})^4. \end{aligned} \quad (90)$$

Using the properties of  $\tau_n$ , we see that the first term of (89) is summable; by our choice of  $\theta$ , so are the second and the third. Furthermore,

$$d_n/e_n \geq C n^{\frac{1-\alpha}{2}} / (\log \tau_n)^\delta \geq C n^{\frac{1-\alpha(1+2\delta)}{2}}, \quad (91)$$

and so the fourth term is summable by our choice of  $\delta$ . As for the last term, note that

$$d_n/f_n \geq C(\tau_n)^{\frac{1}{2}(1-4\beta)} (\log \tau_n)^{-7/2} \quad (92)$$

so the fifth term is summable since  $\beta < 4$ . Thus, by the Borel-Cantelli lemma, (79) holds.  $\square$

## APPENDIX A. FUNCTIONAL CLT UNDER THE CONDITIONAL LAW GIVEN $S$

In this appendix we prove Theorem 3.2.

*Proof.* We will apply the martingale functional CLT in the Lindeberg-Feller formulation as in e.g. [12], Theorem 7.3 on page 411. We will tacitly use the laws of large numbers for  $I_n^{[p]}$  proven in [6] for  $d = 2$  and [2] for  $d \geq 3$ . Note that  $I_n^{[p]}$  can be extended to  $p \in (0, \infty)$ .

Let us define  $K_1 := 0$  and

$$\Delta_{n,k} := s_n^{-1}(K_k - K_{k-1}) = s_n^{-1} q_k \sum_{i=1}^{k-1} q_i \mathbf{1}_{\{S_i = S_k\}}, \quad k \geq 2. \quad (93)$$

Then  $\Delta_{n,m}$  is a martingale difference array under  $\mathbb{P}(\cdot | S)$  w.r.t. the filtration  $\mathcal{F}_m := \sigma(q_i, i \leq m)$ . The corresponding quadratic variations are given by

$$Q_{n,m} := \sum_{k=1}^m \mathbb{E}[\Delta_{n,k}^2 | S, \mathcal{F}_{k-1}]. \quad (94)$$

According to [12], the proof will be finished once we show that, for all  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \sum_{k=2}^n \mathbb{E} \left[ \Delta_{n,k}^2 \mathbf{1}_{\{|\Delta_{n,k}| > \epsilon\}} \mid S \right] = 0, \quad (95)$$

and that, for all  $t \geq 0$ ,

$$\lim_{n \rightarrow \infty} Q_{n, [nt]} = \sigma^2 t \quad \text{in probability under } \mathbb{P}(\cdot | S). \quad (96)$$

To verify (95), write

$$\begin{aligned} \sum_{k=2}^n \mathbb{E} \left[ \Delta_{n,k}^2 \mathbf{1}_{\{|\Delta_{n,k}| > \epsilon\}} \mid S \right] &\leq C \sum_{k=2}^n \mathbb{E} [|\Delta_{n,k}|^\gamma \mid S] \\ &= C s_n^{-\gamma} \sum_{k=2}^n \mathbb{E} \left[ \left| \sum_{i=1}^{k-1} q_i \mathbf{1}_{\{S_i = S_k\}} \right|^\gamma \mid S \right] \\ &\leq C s_n^{-\gamma} \sum_{k=2}^n \left( \sum_{i=1}^{k-1} \mathbf{1}_{\{S_i = S_k\}} \right)^{\gamma/2} \\ &= C s_n^{-\gamma} \sum_{x \in \mathbb{Z}^d} \sum_{k=2}^n \mathbf{1}_{\{S_k = x\}} N_{k-1}^{\gamma/2}(x) \\ &\leq C s_n^{-\gamma} I_n^{[1+\gamma/2]}, \end{aligned} \quad (97)$$

where for the third step we used Burkholder's inequality. Since  $I_n^{[p]}/s_n^\gamma$  goes to 0 a.s. as  $n \rightarrow \infty$ , (95) follows.

Let us now verify (96). Write

$$Q_{n,m} = s_n^{-2} \left( \frac{I_m - m}{2} + R_m^{(1)} + R_m^{(2)} \right), \quad (98)$$

where

$$\begin{aligned} R_m^{(1)} &:= \sum_{k=2}^m \sum_{i=1}^{k-1} (q_i^2 - 1) \mathbf{1}_{\{S_i = S_k\}} \\ &= \sum_{i=1}^{m-1} (q_i^2 - 1) N_{i,m}(S_i) \end{aligned} \quad (99)$$

with  $N_{a,b}(x) := \sum_{k=a+1}^b \mathbf{1}_{\{S_k = x\}}$ , and

$$\begin{aligned} R_m^{(2)} &:= 2 \sum_{k=2}^m \sum_{1 \leq i < j \leq k-1} q_i q_j \mathbf{1}_{\{S_i = S_j = S_k\}} \\ &= 2 \sum_{x \in \mathbb{Z}^d} \sum_{1 \leq i < j \leq m-1} q_i q_j \mathbf{1}_{\{S_i = S_j = x\}} N_{j,m}(x). \end{aligned} \quad (100)$$

Since  $(I_{[nt]} - [nt])/2s_n^2 \rightarrow \sigma^2 t$  a.s., we only need to show that the remaining terms in (98) converge to 0.

Let us first deal with  $R_m^{(2)}$ . Note that, under  $\mathbb{P}(\cdot|S)$ , the summands in the r.h.s. of (100) are independent and centered for different  $x$  to write

$$\begin{aligned} \mathbb{E} \left[ (R_m^{(2)})^2 \mid S \right] &= 4 \sum_{x \in \mathbb{Z}^d} \sum_{1 \leq i < j \leq m-1} \mathbf{1}_{\{S_i = S_j = x\}} N_{j,m}^2(x) \\ &\leq C \sum_{x \in \mathbb{Z}^d} N_m^4(x) = CI_m^{[4]} \end{aligned} \quad (101)$$

and conclude that  $R_{\lfloor nt \rfloor}^{(2)}/s_n^2$  goes to 0 in probability under  $\mathbb{P}(\cdot|S)$ . To control  $R_m^{(1)}$ , we split into two cases. If  $\gamma \geq 4$ , then reasoning as before we get

$$\mathbb{E} \left[ (R_m^{(1)})^2 \mid S \right] \leq CI_m^{[3]}$$

and we conclude as for  $R_m^{(2)}$ . If  $\gamma < 4$ , we use Theorem 1(c) in [17]. Note that

$$\begin{aligned} \sum_{i=1}^{m-1} N_{i,m}^{\gamma/2}(S_i) &= \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{m-1} \mathbf{1}_{\{S_i = x\}} N_{i,m}^{\gamma/2}(x) \\ &\leq \sum_{x \in \mathbb{Z}^d} N_m^{1+\gamma/2}(x) = I_m^{[1+\gamma/2]}, \end{aligned} \quad (102)$$

and also

$$\mathbb{P}(|q^2 - 1| \geq u) \leq Cu^{-\gamma/2} \quad \forall u > 0. \quad (103)$$

Letting

$$a_{n,i} := \begin{cases} N_{i,\lfloor nt \rfloor}(S_i)/s_n^2 & \text{if } i \leq \lfloor nt \rfloor, \\ 0 & \text{otherwise,} \end{cases} \quad (104)$$

we obtain from the aforementioned theorem that, for some constant  $C > 0$ ,

$$\mathbb{P} \left( |R_{\lfloor nt \rfloor}^{(1)}| > \epsilon s_n^2 \mid S \right) \leq CI_{\lfloor nt \rfloor}^{[1+\gamma/2]}/s_n^\gamma, \quad (105)$$

which goes to 0 as  $n \rightarrow \infty$ .  $\square$

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