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# FITTING SETS IN $\mathfrak{L}$ -GROUPS

OLIVIER FRÉCON

ABSTRACT. We consider different natural definitions for Fitting sets, and we prove the existence and conjugacy of the associated injectors in the largest possible classes of locally finite groups.

A *Fitting set*  $\mathfrak{X}$  of a finite group  $G$  is a non-empty set of subgroups which is closed under taking normal products, taking subnormal subgroups and conjugation in  $G$ . This concept of Anderson generalized the work of Fischer, Gaschütz and Hartley about *Fitting classes* [2, p.535–537], where a *Fitting class* is a class of finite groups closed under taking normal products and taking subnormal subgroups.

The theory of Fitting sets concern the study of *injectors*: for a collection  $\mathfrak{X}$  of finite groups, a subgroup  $V$  of a finite group  $G$  is an  $\mathfrak{X}$ -*injector* of  $G$  if  $V \cap A$  is a maximal  $\mathfrak{X}$ -subgroup of  $A$  for every subnormal subgroup  $A$ . The basic theorem provides the existence and conjugacy of  $\mathfrak{X}$ -injectors in any finite soluble group  $G$  for each Fitting set  $\mathfrak{X}$  [2, Theorem 2.9 p.539].

In this paper, we consider the natural generalizations of Fitting sets and injectors to locally finite groups. For each definition of a Fitting set, we prove the existence and conjugacy of associated injectors in a class of locally finite groups. It is shown in [3] that our classes of locally finite groups are the largest in which the results about injectors in finite soluble groups hold.

Note that, thanks to our new approach, we generalize in the last section the theorem of Hartley and Tomkinson about the existence and conjugacy of injectors in  $\mathfrak{L}$ -groups [6] from *Fitting classes* to Fitting sets.

## 1. NORMAL FITTING SETS

In this section, we analyze the injectors associated to *normal Fitting sets*.

*Notation* 1.1. Let  $\mathfrak{X}$  be a collection of groups. For each group  $G$ , we denote by  $G_{\mathfrak{X}n}$  the join of its normal  $\mathfrak{X}$ -subgroups.

**Definition 1.2.** Let  $G$  be a locally finite group. A nonempty set  $\mathfrak{X}$  of subgroups of  $G$  is a *normal Fitting set* of  $G$  if:

- (NF1) every normal subgroup of an  $\mathfrak{X}$ -group belongs to  $\mathfrak{X}$ ;
- (NF2) when  $H$  is a subgroup of  $G$  then  $H_{\mathfrak{X}n}$  is an  $\mathfrak{X}$ -subgroup of  $H$ ;
- (NF3) every conjugate of an  $\mathfrak{X}$ -group is an  $\mathfrak{X}$ -group.

**Definition 1.3.** Let  $G$  be a locally finite group. If  $\mathfrak{X}$  is a set of subgroups of  $G$ , a subgroup  $V$  of  $G$  is an  $\mathfrak{X}$ -*n-injector* of  $G$  if, for every normal subgroup  $A$  of  $G$ ,  $V \cap A$  is a maximal  $\mathfrak{X}$ -subgroup of  $A$ .

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The main result of this section is the following.

**Theorem 1.4.** *Let  $G$  be a subsoluble  $\mathfrak{U}$ -group, and  $\mathfrak{X}$  a normal Fitting set of  $G$ . Then  $G$  has exactly one conjugacy class of  $\mathfrak{X}$ - $n$ -injectors.*

*It is likely that the class of subsoluble  $\mathfrak{U}$ -groups is the largest class of groups satisfying this result. A partial answer is obtained in [3].*

We recall that a group is called a *Baer group* (resp. a *Gruenberg group*) if every cyclic subgroup is subnormal (resp. ascendant). A group  $G$  is said to be *subsoluble* (resp. an *SN\*-group*) if it possesses an ascending series with each term subnormal (resp. ascendant) in  $G$  and each factor abelian.

In any group  $G$ , there is a *Baer radical*  $\beta(G)$  which is a Baer subgroup containing all the subnormal Baer subgroups of  $G$ , and a *Gruenberg radical*  $\gamma(G)$  which is a Gruenberg subgroup containing all the ascendant Gruenberg subgroups of  $G$ . We refer to [7, §2.3] for more details.

As in [1, p.15], we denote by  $\rho(G)$  the *Hirsch-Plotkin radical* of any group  $G$ , and  $\rho_0(G) = 1$  and  $\rho_{n+1}(G)/\rho_n(G) = \rho(G/\rho_n(G))$  for each integer  $n$ .

We recall that the class  $\mathfrak{U}$  was introduced as below in [4], and that by an Hartley's Theorem [1, Theorem 4.4.7 p.163], the first condition is redundant.

**Definition 1.5.** The class  $\mathfrak{U}$  is the largest subgroup-closed class of locally finite groups satisfying the conditions:

(U1) if  $G \in \mathfrak{U}$  then  $G = \rho_n(G)$  for an integer  $n$ ;

(U2) if  $G \in \mathfrak{U}$  and  $\pi$  is any set of primes, then the Sylow  $\pi$ -subgroups of  $G$  are conjugate in  $G$ .

*Fact 1.6.* [5, Theorem E] If  $G$  is a  $\mathfrak{U}$ -group, then  $G/\rho(G)$  is (2-step soluble)-by-finite. In particular  $G/\rho(G)$  is soluble. Moreover,  $G/\rho(G)$  is hyperfinite.

We need to introduce *normal\* Fitting sets* as the sets  $\mathfrak{X}$  of subgroups of a locally finite group  $G$  satisfying the conditions (NF1), (NF3) and

(NF2\*) *if  $H$  is a subgroup of  $G$  such that  $H/H_{\mathfrak{X}n}$  is locally nilpotent, then  $H_{\mathfrak{X}n}$  is a maximal  $\mathfrak{X}$ -subgroup of  $H$ .*

Any normal\* Fitting set of a locally finite group  $G$  is a normal Fitting set too. By the result below, the converse is true in subsoluble  $\mathfrak{U}$ -groups.

**Proposition 1.7.** *If  $\mathfrak{X}$  is a normal Fitting set of a subsoluble  $\mathfrak{U}$ -group  $G$ , then  $\mathfrak{X}$  is a normal\* Fitting set.*

*Proof.* Let  $H$  be a subgroup of  $G$  such that  $H/H_{\mathfrak{X}n}$  is locally nilpotent. We define  $\delta_0(H) = H_{\mathfrak{X}n}$ ,  $\delta_{i+1}(H)/\delta_i(H) = \beta(H/\delta_i(H))$  for every ordinal  $i$ , and  $\delta_\mu(H) = \cup_{i < \mu} \delta_i(H)$  for every limit ordinal  $\mu$ . Since  $G$  is subsoluble, then  $H$  is subsoluble too, and there is an ordinal  $\alpha$  such that  $\delta_\alpha(H) = H$ . Suppose toward a contradiction that  $H_{\mathfrak{X}n}$  is not a maximal  $\mathfrak{X}$ -subgroup of  $H$ . Since it is an  $\mathfrak{X}$ -subgroup by (NF2), there is a smallest ordinal  $\gamma$  such that  $H_{\mathfrak{X}n}$  is strictly contained in an  $\mathfrak{X}$ -subgroup  $K$  of  $\delta_\gamma(H)$ . By (NF1), we have  $K \cap \delta_i(H) = H_{\mathfrak{X}n}$  for all  $i < \gamma$ , so there is an ordinal  $\nu$  such that  $\gamma = \nu + 1$ . Now  $K/H_{\mathfrak{X}n} \simeq K\delta_\nu(H)/\delta_\nu(H)$  is a Baer group. Hence, if  $g$  is an element of  $K$ , then  $\langle g \rangle H_{\mathfrak{X}n}$  is subnormal in  $K$ , and  $\langle g \rangle H_{\mathfrak{X}n}$  is an  $\mathfrak{X}$ -group by (NF1). By the minimality of  $\gamma$ , it is a maximal  $\mathfrak{X}$ -subgroup of  $\langle g \rangle \delta_i(H)$  for all  $i < \gamma$ .

For any element  $b$  of  $\delta_\nu(H)$ , the local nilpotence of  $H/H_{\mathfrak{X}n}$  implies the nilpotence of  $\langle g, b \rangle H_{\mathfrak{X}n}/H_{\mathfrak{X}n}$ . Therefore  $\langle g \rangle H_{\mathfrak{X}n}$  is a subnormal  $\mathfrak{X}$ -subgroup of  $\langle g, b \rangle H_{\mathfrak{X}n}$ . By

the maximality of  $\langle g \rangle H_{\mathfrak{X}_n}$  in  $\langle g \rangle \delta_\nu(H)$ , we have  $\langle g \rangle H_{\mathfrak{X}_n} = (\langle g, b \rangle H_{\mathfrak{X}_n})_{\mathfrak{X}_n}$  and  $b$  normalizes  $\langle g \rangle H_{\mathfrak{X}_n}$ , so  $\langle g \rangle H_{\mathfrak{X}_n} = (\langle g \rangle \delta_\nu(H))_{\mathfrak{X}_n}$ .

Since  $\delta_\gamma(H)/\delta_\nu(H)$  is a Baer group, then  $\langle g \rangle \delta_\nu(H)$  is subnormal in  $\delta_\gamma(H)$ . Hence  $\langle g \rangle H_{\mathfrak{X}_n}$  is a subnormal  $\mathfrak{X}$ -subgroup of  $\delta_\gamma(H)$  and we have  $\langle g \rangle H_{\mathfrak{X}_n} \leq (\delta_\gamma(H))_{\mathfrak{X}_n}$ . As  $\delta_\gamma(H)$  is a normal subgroup of  $H$ , we obtain  $\langle g \rangle H_{\mathfrak{X}_n} \leq H_{\mathfrak{X}_n}$ , contradicting the choice of  $g$ . This proves the proposition.  $\square$

For this section, we fix a normal\* Fitting set  $\mathfrak{F}$  of a fixed  $\mathfrak{U}$ -group  $G$ .

**Lemma 1.8.** *Let  $A$  a subgroup of  $G$  containing  $G'$ . If  $A$  possesses a maximal  $\mathfrak{F}$ -subgroup  $W$ , then every  $\mathfrak{F}$ -subgroup of  $G$  containing  $W$  is contained in a maximal  $\mathfrak{F}$ -subgroup of  $G$ .*

*Proof.* Let  $(V_i)_{i < \alpha}$  be an increasing sequence of  $\mathfrak{F}$ -subgroups of  $G$  containing  $W$ , for an ordinal  $\alpha$ . For each  $i < \alpha$ , since  $A$  is normal in  $G$ ,  $V_i \cap A$  is an  $\mathfrak{F}$ -subgroup of  $A$  containing  $W$  and we have  $V_i \cap A = W$  by the maximality of  $W$ . As  $A$  contains  $G'$ , then  $W$  contains  $V'$  where  $V = \cup_{i < \alpha} V_i$ . So  $V_i$  is normal in  $V$  for all  $i < \alpha$ . We obtain  $V = V_{\mathfrak{F}_n}$  and  $V$  is an  $\mathfrak{F}$ -group by  $(NF2^*)$ .  $\square$

A Carter subgroup of a group  $H$  is a locally nilpotent and self-serializing subgroup of  $H$ . The main result concerning these subgroups is the following:

*Fact 1.9.* [4, Theorem 5.4] Let  $H$  be a  $\mathfrak{U}$ -group, and  $A$  a normal subgroup of  $H$ . Then  $H$  has a unique conjugacy class of Carter subgroups. Moreover, if  $C$  is a Carter subgroup of  $H$ , then  $CA/A$  is a Carter subgroup of  $H/A$  and each Carter subgroup of  $H/A$  has this form.

**Lemma 1.10.** *Let  $A$  be a subgroup of  $G$  containing  $G'$ . If  $A$  has a maximal  $\mathfrak{F}$ -subgroup  $W$ , then the maximal  $\mathfrak{F}$ -subgroups of  $G$  containing  $W$  are conjugate.*

*Proof.* By Lemma 1.8,  $G$  has a maximal  $\mathfrak{F}$ -subgroup  $V$  containing  $W$ , and  $W = V \cap A$  is normal in  $V$ . Let  $U = N_G(W)$  and  $N/W = N_{U/W}(V/W)$ . We show that there exists a Carter subgroup  $C/W$  of  $U/W$  such that  $V = C_{\mathfrak{F}_n}$ . As  $[V, N]$  is contained in  $V \cap G' \leq W$ ,  $N/W$  centralizes  $V/W$ . By Fact 1.9,  $N/W$  has a Carter subgroup  $C/W$ . Then  $C/W$  contains  $V/W$ , and  $V$  is a normal and maximal  $\mathfrak{F}$ -subgroup of  $C$ . In particular, we obtain  $V = C_{\mathfrak{F}_n}$  and  $V/W$  is normal in  $N_{U/W}(C/W)$ . Thus we have  $N_{U/W}(C/W) = N_{N/W}(C/W) = C/W$  and  $C/W$  is a Carter subgroup of  $U/W$ .

Let  $V_1$  and  $V_2$  be two maximal  $\mathfrak{F}$ -subgroups of  $G$  containing  $W$ . Then there are two Carter subgroups  $C_1/W$  and  $C_2/W$  of  $U/W$  such that  $V_1 = (C_1)_{\mathfrak{F}_n}$  and  $V_2 = (C_2)_{\mathfrak{F}_n}$  respectively. The conjugacy of  $C_1$  and  $C_2$  is given by Fact 1.9, and the lemma is proved.  $\square$

For every group  $H$ , we denote by  $H^{\text{Ln}}$  the intersection of all the normal subgroups  $K$  of  $H$  such that  $H/K$  is locally nilpotent.

We recall that a subgroup  $H$  of a group  $G$  is *pronormal* in  $G$  if  $H$  and  $H^g$  are conjugate in  $\langle H, H^g \rangle$  for every  $g \in G$ .

**Lemma 1.11.** *Let  $V$  be a pronormal subgroup of  $G$  such that  $V \cap G^{\text{Ln}}$  is normal in  $G$ . Then  $V/(V \cap G^{\text{Ln}})$  is a normal subgroup of a Carter subgroup of  $G/(V \cap G^{\text{Ln}})$ .*

*Proof.* We may assume  $V \cap G^{\text{Ln}} = 1$ . Let  $G_0 = G$  and, for all  $i \in \mathbb{N}$ ,  $G_{i+1} = G_i^{\text{Ln}}$ . We show that, for all  $i \in \mathbb{N}$ ,  $V G_i / G_i$  is a normal subgroup of a Carter subgroup of  $G/G_i$ . It is true if  $i = 0$ . Assume that the result is true for  $i \in \mathbb{N}$ .

Then  $VG_i/G_i$  is a normal subgroup of a Carter subgroup  $C/G_i$  of  $G/G_i$ . By the pronormality of  $V$  in  $G$ , for every  $c \in C$ , there is  $x \in G_i$  such that  $V^x = V^c$ , and we obtain  $C = N_C(V)G_i$ . By Fact 1.9,  $N_C(V)$  has a Carter subgroup  $D_0$ , and as  $C/G_i$  is locally nilpotent, we have  $C = D_0G_i$ . Let  $D/G_{i+1}$  be a maximal locally nilpotent subgroup of  $C/G_{i+1}$  containing  $D_0G_{i+1}/G_{i+1}$ . Now  $D/G_{i+1}$  is a Carter subgroup of  $C/G_{i+1}$  [4, Theorem 5.12], and  $D/G_{i+1}$  is a Carter subgroup of  $G/G_{i+1}$  by Fact 1.9. But we have  $(D_0V)^{L^{\mathfrak{N}}} \leq V \cap G^{L^{\mathfrak{N}}} = 1$ , hence  $D_0V$  is locally nilpotent, and  $D_0$  contains  $V$  by the definition of a Carter subgroup. Therefore,  $VG_{i+1}/G_{i+1}$  is contained in a Carter subgroup  $D/G_{i+1}$  of  $G/G_{i+1}$ . Since  $D/G_{i+1}$  is locally nilpotent,  $VG_{i+1}/G_{i+1}$  is a serial subgroup of  $D/G_{i+1}$ . As  $VG_{i+1}/G_{i+1}$  is a pronormal subgroup of  $D/G_{i+1}$ , it is a normal subgroup of  $D/G_{i+1}$ . This proves that  $VG_{i+1}/G_{i+1}$  is a normal subgroup of a Carter subgroup of  $G/G_{i+1}$  for each  $i \in \mathbb{N}$ . In particular,  $V$  is a normal subgroup of a Carter subgroup of  $G$ .  $\square$

**Lemma 1.12.** *Let  $A$  be a subgroup of  $G$  containing  $G'$ ,  $L$  a normal subgroup of  $G$  containing  $G^{L^{\mathfrak{N}}}$  and  $V$  a maximal  $\mathfrak{F}$ -subgroup of  $G$ . Suppose that, if  $B$  is a normal subgroup of  $A$  and if  $B_1$  is a subgroup of  $B$  containing  $V \cap B$ , then  $V \cap B$  is an  $\mathfrak{F}$ - $n$ -injector of  $B_1$ . Then  $V \cap L$  is a maximal  $\mathfrak{F}$ -subgroup of  $L$ .*

*Proof.* We show that  $V$  is pronormal in  $G$ . For every  $g \in G$ ,  $V$  and  $V^g$  are two maximal  $\mathfrak{F}$ -subgroups of  $\langle V, V^g \rangle$ . By our hypothesis, for all  $k \in \mathbb{N}$ ,  $V \cap A^{(k)}$  is an  $\mathfrak{F}$ - $n$ -injector of  $\langle V, V^g \rangle \cap A^{(k)}$  and of  $\langle V^{g^{-1}}, V \rangle \cap A^{(k)}$ . So  $V^g \cap A^{(k)}$  is an  $\mathfrak{F}$ - $n$ -injector of  $\langle V, V^g \rangle \cap A^{(k)}$ . Since there exists  $r \in \mathbb{N}$  such that  $A^{(r)}$  is locally nilpotent (Fact 1.6), the unique  $\mathfrak{F}$ - $n$ -injector of  $\langle V, V^g \rangle \cap A^{(r)}$  is  $(\langle V, V^g \rangle \cap A^{(r)})_{\mathfrak{F}n}$  by  $(NF2^*)$ , and we obtain  $V \cap A^{(r)} = V^g \cap A^{(r)}$ . Suppose that for an element  $k \in \{1, \dots, r\}$  there exists  $\alpha \in A^{(k)}$  such that  $V \cap A^{(k)} = V^{g\alpha} \cap A^{(k)}$ . By hypothesis,  $V \cap A^{(k)}$  is an  $\mathfrak{F}$ - $n$ -injector of  $\langle V, V^{g\alpha} \rangle \cap A^{(k)}$ . Then  $V \cap A^{(k-1)}$  and  $V^{g\alpha} \cap A^{(k-1)}$  are conjugate in  $A^{(k-1)}$  by Lemma 1.10. Thus  $V \cap A$  and  $V^g \cap A$  are conjugate in  $A$ . Now a last application of Lemma 1.10 provides the conjugacy of  $V$  and  $V^g$  in  $\langle V, V^g \rangle$ , as desired.

Let  $N = N_G(V \cap L \cap A)$ . Since  $V \cap L \cap A$  is an  $\mathfrak{F}$ - $n$ -injector of  $L \cap A$ , Lemma 1.8 shows that  $V \cap L$  is contained in a maximal  $\mathfrak{F}$ -subgroup  $V_1$  of  $L$ . We show that  $V_1$  is pronormal in  $N$ . By hypothesis,  $V_1 \cap A = V \cap L \cap A$  is an  $\mathfrak{F}$ - $n$ -injector of  $\langle V_1, V_1^g \rangle \cap A$  for all  $g \in G$ . So, for every  $g \in N$ ,  $V_1$  and  $V_1^g$  are two maximal  $\mathfrak{F}$ -subgroups of  $\langle V_1, V_1^g \rangle \leq L$  containing the  $\mathfrak{F}$ - $n$ -injector  $V_1 \cap A = V_1^g \cap A$  of  $\langle V_1, V_1^g \rangle \cap A$ . By Lemma 1.10, we obtain conjugacy of  $V_1$  and  $V_1^g$  in  $\langle V_1, V_1^g \rangle$ . Hence  $V_1$  is pronormal in  $N$ .

Since  $G^{L^{\mathfrak{N}}}$  is contained in  $A$  and  $L$ ,  $V \cap N^{L^{\mathfrak{N}}}$  and  $V_1 \cap N^{L^{\mathfrak{N}}}$  are contained in  $V \cap L \cap A = V_1 \cap A$ . So  $V \cap N^{L^{\mathfrak{N}}} = V_1 \cap N^{L^{\mathfrak{N}}}$  is a normal subgroup of  $N$ . By Lemma 1.11,  $V/(V \cap N^{L^{\mathfrak{N}}})$  and  $V_1/(V \cap N^{L^{\mathfrak{N}}})$  are normal subgroups of Carter subgroups  $C/(V \cap N^{L^{\mathfrak{N}}})$  and  $C_1/(V \cap N^{L^{\mathfrak{N}}})$  of  $N/(V \cap N^{L^{\mathfrak{N}}})$  respectively, and Fact 1.9 gives  $x \in N$  such that  $C_1^x = C$ . As  $V$  is a maximal  $\mathfrak{F}$ -subgroup of  $G$  and as  $C_{\mathfrak{F}n} \in \mathfrak{F}$  contains  $V$  and  $V_1^x$ , we find  $V_1^x \leq V$ . But  $V_1^x$  is a maximal  $\mathfrak{F}$ -subgroup of  $L$ , so  $V \cap L = V_1^x$  is a maximal  $\mathfrak{F}$ -subgroup of  $L$ .  $\square$

**Lemma 1.13.** *Let  $A$  be a subgroup of  $G$  containing  $G'$  and let  $V$  be a maximal  $\mathfrak{F}$ -subgroup of  $G$ . We suppose that, if  $B$  is a normal subgroup of  $G$  contained in  $A$ , then  $V \cap B$  is an  $\mathfrak{F}$ - $n$ -injector of  $B$ . Then  $V \cap L$  is a maximal  $\mathfrak{F}$ -subgroup of  $L$  for every normal subgroup  $L$  of  $G$  such that  $G/L$  is soluble.*

*Proof.* We proceed by induction on the solubility class  $k$  of  $G/L$ . We may assume  $k \geq 1$ . Let  $M = G^{(k-1)}L$ . By induction,  $V \cap M$  is a maximal  $\mathfrak{F}$ -subgroup of  $M$ . Since  $V \cap M'$  is an  $\mathfrak{F}$ - $n$ -injector of  $M'$  by our hypothesis, Lemma 1.8 says that  $V \cap M'$  is contained in a maximal  $\mathfrak{F}$ -subgroup  $V_1$  of  $L$ . Since  $V_1$  is contained in a maximal  $\mathfrak{F}$ -subgroup  $V_2$  of  $M$  (Lemma 1.8), we have two maximal  $\mathfrak{F}$ -subgroups  $V \cap M$  and  $V_2$  of  $M$  containing  $V \cap M'$ . Lemma 1.10 gives  $g \in M$  such that  $(V \cap M)^g = V_2$ . As  $L$  is normal in  $G$ , we obtain  $(V \cap L)^g = V_2 \cap L = V_1$ , and  $V \cap L$  is a maximal  $\mathfrak{F}$ -subgroup of  $L$ .  $\square$

The following proposition is an analogue of [2, Proposition 2.12].

**Proposition 1.14.** *Let  $(A_i)_{i=0, \dots, n}$  ( $n \in \mathbb{N}$ ) be a finite increasing series of subgroups of  $G$  such that  $A_0$  is locally nilpotent and normal,  $G/A_0$  is soluble,  $A_n = G$  and, for every  $i \in \{0, \dots, n-1\}$ ,  $A_i$  is normal in  $A_{i+1}$  and  $A_{i+1}/A_i$  is abelian. Suppose that there is a subgroup  $V$  of  $G$  such that  $V \cap A_0 = (A_0)_{\mathfrak{F}n}$  and such that  $V \cap A_i$  is a maximal  $\mathfrak{F}$ -subgroup of  $A_i$  for every  $i \in \{1, \dots, n\}$ . If  $B$  is a normal subgroup of  $G$  and if  $B_1$  is a subgroup of  $B$  containing  $V \cap B$ , then  $V \cap B$  is an  $\mathfrak{F}$ - $n$ -injector of  $B_1$ .*

*Proof.* We proceed by induction on  $n$ . We may assume that  $G$  is not locally nilpotent and  $n \geq 1$ . First, we show that  $V$  is an  $\mathfrak{F}$ - $n$ -injector of  $G$ . Let  $L$  be a normal subgroup of  $G$ . By induction, for  $i \in \{0, \dots, n-1\}$ , if  $N$  is a normal subgroup of  $A_i$ , and if  $N_1$  is a subgroup of  $N$  containing  $V \cap N$ , then  $V \cap N$  is an  $\mathfrak{F}$ - $n$ -injector of  $N_1$ . In particular,  $V \cap \rho(G)L \cap A_i$  is a maximal  $\mathfrak{F}$ -subgroup of  $\rho(G)L \cap A_i$ , and we have  $V \cap \rho(G)L \cap A_0 = \rho(G)L \cap (A_0)_{\mathfrak{F}n} = (\rho(G)L \cap A_0)_{\mathfrak{F}n}$ . Moreover, by Lemma 1.13 and Fact 1.6,  $V \cap \rho(G)L$  is a maximal  $\mathfrak{F}$ -subgroup of  $\rho(G)L$ . By induction, if  $E$  is a normal subgroup of  $\rho(G)L \cap A_{n-1}$  and if  $E_1$  is a subgroup of  $E$  containing  $V \cap E$ , then  $V \cap E$  is an  $\mathfrak{F}$ - $n$ -injector of  $E_1$ . By an application of Lemma 1.12 to  $\rho(G)L$ ,  $V \cap L$  is a maximal  $\mathfrak{F}$ -subgroup of  $L$ . Hence  $V$  is an  $\mathfrak{F}$ - $n$ -injector of  $G$ .

Let  $B$  be a normal subgroup of  $G$  and let  $B_1$  be a subgroup of  $B$  containing  $V \cap B$ . We show that  $V \cap B$  is an  $\mathfrak{F}$ - $n$ -injector of  $B_1$ . As  $V$  is an  $\mathfrak{F}$ - $n$ -injector of  $G$ ,  $V \cap B \cap A_i = V \cap B_1 \cap A_i$  is a maximal  $\mathfrak{F}$ -subgroup of  $B \cap A_i$  and of  $B_1 \cap A_i$  for all  $i \in \{0, \dots, n\}$ , and we have

$$V \cap B_1 \cap A_0 = V \cap B \cap A_0 = B \cap (A_0)_{\mathfrak{F}n} = (B_1 \cap A_0)_{\mathfrak{F}n}.$$

By induction, if  $D$  is a normal subgroup of  $B_1 \cap A_{n-1}$  and if  $D_1$  is a subgroup of  $D$  containing  $V \cap D$ , then  $V \cap D$  is an  $\mathfrak{F}$ - $n$ -injector of  $D_1$ . In particular, if  $D$  is a normal subgroup of  $B_1$  contained in  $B_1 \cap A_{n-1}$ , then  $V \cap D$  is an  $\mathfrak{F}$ - $n$ -injector of  $D$ . Let  $L$  be a normal subgroup of  $B_1$ . Then, by Lemma 1.13 and Fact 1.6,  $V \cap \rho(B_1)L \cap A_i$  is a maximal  $\mathfrak{F}$ -subgroup of  $\rho(B_1)L \cap A_i$  for all  $i \in \{0, \dots, n\}$  and we have

$$V \cap \rho(B_1)L \cap A_0 = \rho(B_1)L \cap (A_0)_{\mathfrak{F}n} = (\rho(B_1)L \cap A_0)_{\mathfrak{F}n}.$$

By induction, if  $F$  is a normal subgroup of  $\rho(B_1)L \cap A_{n-1}$  and if  $F_1$  is a subgroup of  $F$  containing  $V \cap F$ , then  $V \cap F$  is an  $\mathfrak{F}$ - $n$ -injector of  $F_1$ . By an application of Lemma 1.12 to  $\rho(B_1)L$ ,  $V \cap L$  is a maximal  $\mathfrak{F}$ -subgroup of  $L$ . Hence  $V \cap B$  is an  $\mathfrak{F}$ - $n$ -injector of  $B_1$ .  $\square$

**Corollary 1.15.** *If  $G$  possesses an  $\mathfrak{F}$ - $n$ -injector  $V$ , then  $V \cap A$  is a maximal  $\mathfrak{F}$ -subgroup of  $A$  for every subnormal subgroup  $A$  of  $G$ .*

*Proof.* We may assume  $A$  normal in  $G$ . Then  $V \cap A^{(k)}$  is a maximal  $\mathfrak{F}$ -subgroup of  $A^{(k)}$  for every  $k \in \mathbb{N}$ . Let  $n \in \mathbb{N}$  be such that  $A^{(n)}$  is locally nilpotent ( $n$  exists by Fact 1.6). Then we have  $V \cap A^{(n)} = (A^{(n)})_{\mathfrak{F}n}$  by  $(NF2^*)$ , and Proposition 1.14 says that  $V \cap A$  is an  $\mathfrak{F}$ - $n$ -injector of  $A$ .  $\square$

**Theorem 1.16.**  *$G$  possesses a unique conjugacy class of  $\mathfrak{F}$ - $n$ -injectors.*

*Proof.* We proceed by induction on the solubility class  $k$  of  $G/\rho(G)$ . By  $(NF2^*)$  we may assume that  $G$  is not locally nilpotent, that is  $k \geq 1$ . By induction,  $G'$  has an  $\mathfrak{F}$ - $n$ -injector  $W$ , and  $W$  is contained in a maximal  $\mathfrak{F}$ -subgroup  $V$  of  $G$  by Lemma 1.8. In particular,  $V \cap G^{(i)} = W \cap G^{(i)}$  is a maximal  $\mathfrak{F}$ -subgroup of  $G^{(i)}$  for every  $i \in \{0, \dots, k\}$ , and we have  $V \cap G^{(k)} = (G^{(k)})_{\mathfrak{F}n}$ . Hence  $V$  is an  $\mathfrak{F}$ - $n$ -injector of  $G$  by Proposition 1.14.

Let  $V_1$  and  $V_2$  be two  $\mathfrak{F}$ - $n$ -injectors of  $G$ . Then  $V_1 \cap G'$  and  $V_2 \cap G'$  are  $\mathfrak{F}$ - $n$ -injectors of  $G'$  (Corollary 1.15), and we can suppose  $V_1 \cap G' = V_2 \cap G'$  by induction. Lemma 1.10 gives the result.  $\square$

Now we obtain Theorem 1.4 by Proposition 1.7 and Theorem 1.16.

*Remark 1.17.* If the word *normal* is replaced by *subnormal* in Definition 1.2 then clearly we do not obtain a new notion of Fitting set. Moreover, Corollary 1.15 shows that if we do the same thing in Definition 1.3, then *in the context of subsoluble  $\mathfrak{U}$ -groups*, we do not obtain a new notion of injector.

## 2. ASCENDANT FITTING SETS

In this section we replace the word *normal* in §1 by *ascendant*.

*Notation 2.1.* Let  $\mathfrak{X}$  be a collection of groups. For each group  $G$ , we denote by  $G_{\mathfrak{X}a}$  the join of its ascendant  $\mathfrak{X}$ -subgroups.

**Definition 2.2.** Let  $G$  be a locally finite group. A nonempty set  $\mathfrak{X}$  of subgroups of  $G$  is an *ascendant Fitting set* of  $G$  if it satisfies  $(NF3)$  and:

- (AF1) every ascendant subgroup of an  $\mathfrak{X}$ -group belongs to  $\mathfrak{X}$ ;
- (AF2) when  $H$  is a subgroup of  $G$  then  $H_{\mathfrak{X}a}$  is an  $\mathfrak{X}$ -subgroup of  $H$ .

**Definition 2.3.** Let  $G$  be a locally finite group. If  $\mathfrak{X}$  is a set of subgroups of  $G$ , a subgroup  $V$  of  $G$  is an  *$\mathfrak{X}$ - $a$ -injector* of  $G$  if, for every ascendant subgroup  $A$  of  $G$ ,  $V \cap A$  is a maximal  $\mathfrak{X}$ -subgroup of  $A$ .

We will prove the following result.

**Theorem 2.4.** *Let  $G$  be both an  $SN^*$ -group and a  $\mathfrak{U}$ -group, and  $\mathfrak{X}$  an ascendant Fitting set. Then  $G$  has exactly one conjugacy class of  $\mathfrak{X}$ - $a$ -injectors.*

*It is likely that the class  $SN^* \cap \mathfrak{U}$  is the largest class of groups satisfying this result. A partial answer is obtained in [3].*

Similarly to §1, we define an *ascendant\* Fitting set* as a set  $\mathfrak{X}$  of subgroups of a locally finite group  $G$  satisfying  $(AF1)$ ,  $(NF3)$  and  $(AF2^*)$  if  $H$  is a subgroup of  $G$  such that  $H/H_{\mathfrak{X}a}$  is locally nilpotent, then  $H_{\mathfrak{X}a}$  is a maximal  $\mathfrak{X}$ -subgroup of  $H$ .

We note that in any locally finite group  $G$ , an ascendant Fitting set is a normal Fitting set, and an ascendant\* Fitting set is both a normal\* Fitting set and an ascendant Fitting set.

Moreover, by a similar proof of Proposition 1.7, we show that if  $G$  is both an  $SN^*$ -group and a  $\mathfrak{U}$ -group, then its an ascendant\* Fitting sets are precisely its ascendant Fitting sets.

**Theorem 2.5.** *Let  $G$  be a  $\mathfrak{U}$ -group. For any ascendant\* Fitting set  $\mathfrak{F}$  of  $G$ , the  $\mathfrak{F}$ -n-injectors of  $G$  are precisely its  $\mathfrak{F}$ -a-injectors.*

*Proof.* Let  $A$  be an ascendant subgroup of  $G$ . Let  $(A_i)_{i \leq \alpha}$  ( $\alpha$  ordinal) be an ascending series from  $A$  to  $G$ . Let  $W$  be an  $\mathfrak{F}$ -n-injector of  $A$  (Theorem 1.16). We prove that there is an ascending series  $(W_i)_{i \leq \alpha}$  such that  $W_0 = W$  and  $W_i$  is an  $\mathfrak{F}$ -n-injector of  $A_i$  for every  $i \leq \alpha$ . Assume  $W_i$  has been constructed. Let  $U_{i+1}$  be an  $\mathfrak{F}$ -n-injector of  $A_{i+1}$ . Then  $U_{i+1} \cap A_i$  is an  $\mathfrak{F}$ -n-injector of  $A_i$  by Corollary 1.15. By Theorem 1.16, there is  $a \in A_i$  such that  $(U_{i+1} \cap A_i)^a = W_i$ . So  $U_{i+1}^a$  is an  $\mathfrak{F}$ -n-injector of  $A_{i+1}$  containing  $W_i$ , and we can take  $W_{i+1} = U_{i+1}^a$ . We must show that, if  $j$  is a limit ordinal, then  $W_j = \cup_{i < j} W_i$  is an  $\mathfrak{F}$ -n-injector of  $A_j$ . We note that  $W_j = (W_j)_{\mathfrak{F}a}$  is an  $\mathfrak{F}$ -subgroup of  $A_j$  by (AF2\*). Let  $L$  be a normal subgroup of  $A_j$  and let  $K$  be an  $\mathfrak{F}$ -subgroup of  $L$  containing  $W_j \cap L$ . Since  $W_i$  is an  $\mathfrak{F}$ -n-injector of  $A_i$  for every  $i < j$ ,  $W_i \cap L$  is a maximal  $\mathfrak{F}$ -subgroup of  $A_i \cap L$  for all  $i < j$ . Therefore, since  $K \cap A_i$  is an  $\mathfrak{F}$ -subgroup of  $A_i \cap L$  containing  $W_i \cap L$  for all  $i < j$  by (AF1), we have  $K \cap A_i = W_i \cap L$  for all  $i < j$ . Hence we obtain  $K = W_j \cap L$  and  $W_j \cap L$  is a maximal  $\mathfrak{F}$ -subgroup of  $L$ . So  $W_j$  is an  $\mathfrak{F}$ -n-injector of  $A_j$ .

Let  $V$  be an  $\mathfrak{F}$ -n-injectors of  $G$ . We show that  $V \cap A$  is an  $\mathfrak{F}$ -n-injector of  $A$ . Otherwise there is a smallest ordinal  $\beta$  and an  $\mathfrak{F}$ -n-injector  $U$  of  $A_\beta$  such that  $U \cap A$  is not an  $\mathfrak{F}$ -n-injector of  $A$ . If  $\beta = \delta + 1$  for an ordinal  $\delta$ , then  $U \cap A_\delta$  is an  $\mathfrak{F}$ -n-injector of  $A_\delta$  by Corollary 1.15, and  $U \cap A = (U \cap A_\delta) \cap A$  is an  $\mathfrak{F}$ -n-injector of  $A$  by the minimality of  $\beta$ , contradicting the choice of  $\beta$ . So  $\beta$  is a limit ordinal. By Theorem 1.16, there exists  $b \in A_\beta$  such that  $U^b = W_\beta$ . So there is  $i < \beta$  such that  $b \in A_i$  and we have  $(U \cap A_i)^b = W_i$ . Hence  $U \cap A_i$  is an  $\mathfrak{F}$ -n-injector of  $A_i$  and  $U \cap A = (U \cap A_i) \cap A$  is an  $\mathfrak{F}$ -n-injector of  $A$  by the minimality of  $\beta$ . Our result follows from this contradiction.  $\square$

Now Theorem 2.4 is obtained from Theorems 1.4 and 2.5.

### 3. SERIAL FITTING SETS

In this section we replace the word *normal* in §1 by *ascendant*.

*Notation 3.1.* Let  $\mathfrak{X}$  be a collection of groups. For each group  $G$ , we denote by  $G_{\mathfrak{X}}$  the join of its serial  $\mathfrak{X}$ -subgroups.

**Definition 3.2.** Let  $G$  be a locally finite group. A nonempty set  $\mathfrak{X}$  of subgroups of  $G$  is a *serial Fitting set* of  $G$  if it satisfies (NF3) and:

- (SF1) every serial subgroup of an  $\mathfrak{X}$ -group belongs to  $\mathfrak{X}$ ;
- (SF2) when  $H$  is a subgroup of  $G$  then  $H_{\mathfrak{X}s}$  is an  $\mathfrak{X}$ -subgroup of  $H$ .

We note that in any locally finite group  $G$ , a serial Fitting set is an ascendant\* Fitting set too.

**Definition 3.3.** Let  $G$  be a locally finite group. If  $\mathfrak{X}$  is a set of subgroups of  $G$ , a subgroup  $V$  of  $G$  is an  $\mathfrak{X}$ -injector of  $G$  if, for every serial subgroup  $A$  of  $G$ ,  $V \cap A$  is a maximal  $\mathfrak{X}$ -subgroup of  $A$ .

We will prove the following result.



**Theorem 3.4.** *Let  $G$  be a  $\mathfrak{U}$ -group, and  $\mathfrak{X}$  a serial Fitting set of  $G$ . Then  $G$  has exactly one conjugacy class of  $\mathfrak{X}$ -injectors.*

*It is proven in [3] that the class of  $\mathfrak{U}$ -groups is the largest class of groups satisfying this result. The difficulty of the proof in [3] is that without the conjugacy of Sylow subgroups, a Sylow subgroup might not be an injector.*

From now on,  $G$  is a fixed  $\mathfrak{U}$ -group and  $\mathfrak{F}$  is a serial Fitting set of  $G$ .

**Lemma 3.5.** *Let  $N$  be a normal subgroup of  $G$  such that  $G/N$  is a  $p$ -group for a prime  $p$  and  $N_{\mathfrak{F}}$  is a maximal  $\mathfrak{F}$ -subgroup of  $N$ . If  $N$  contains a subgroup  $A$  of  $G$  and if  $V$  is an  $\mathfrak{F}$ -a-injector of  $G$ , then  $V \cap A$  is an  $\mathfrak{F}$ -a-injector of  $A$ .*

*Proof.* By Theorems 1.16 and 2.5,  $A$  has an  $\mathfrak{F}$ -a-injector  $W$ . Then we have  $W \cap N = N_{\mathfrak{F}}$ , so  $W/N_{\mathfrak{F}}$  is a  $p$ -group and there is a maximal  $p$ -subgroup  $R/N_{\mathfrak{F}}$  of  $G/N_{\mathfrak{F}}$  containing  $W/N_{\mathfrak{F}}$ . Likewise we have  $V \cap N = N_{\mathfrak{F}}$  and there is a maximal  $p$ -subgroup  $S/N_{\mathfrak{F}}$  of  $G/N_{\mathfrak{F}}$  containing  $V/N_{\mathfrak{F}}$ . As  $G/N$  is a  $p$ -group, we have  $G = RN = SN$  [4, Lemma 2.1 (ii)], and there is  $g \in N$  such that  $R^g = S$ . But  $S/N_{\mathfrak{F}}$  is locally nilpotent, so  $S_{\mathfrak{F}}$  contains all the  $\mathfrak{F}$ -subgroups of  $S$  containing  $N_{\mathfrak{F}}$ . Hence  $S_{\mathfrak{F}}$  contains  $V$  and  $W^g$ . By the maximality of  $V$ , we obtain  $V = S_{\mathfrak{F}}$  and  $W^g \leq V$ . But  $W^g$  is an  $\mathfrak{F}$ -a-injector of  $A$  since  $g \in N \leq A$ . As  $A$  contains  $N$  and as  $G/N$  is locally nilpotent,  $A$  is a serial subgroup of  $G$ . Thus  $V \cap A$  is an  $\mathfrak{F}$ -subgroup of  $A$ . Since  $V \cap A$  contains an  $\mathfrak{F}$ -a-injector  $W^g$  of  $A$ , we obtain  $V \cap A = W^g$  and the proof is complete.  $\square$

**Lemma 3.6.** *Assume that  $G$  has a normal  $p$ -subgroup  $M$  for a prime  $p$  and a serial subgroup  $A$  such that  $G = MA$ . If  $V$  is an  $\mathfrak{F}$ -a-injector of  $G$ , then  $V \cap A$  is an  $\mathfrak{F}$ -a-injector of  $A$ .*

*Proof.* Let  $P = \langle S : S \text{ is a } p'\text{-subgroup of } G \rangle$ . We show that  $P$  is a subgroup of  $A$ . By the seriality of  $A$ , for each  $x \in G \setminus A$ , there exist  $U$  and  $V$  two subgroups of  $G$  containing  $A$  such that  $V \trianglelefteq U$  and  $x \in U \setminus V$ . But  $U = (M \cap U)V$ , so  $U/V$  is a  $p$ -group and  $x$  is not a  $p'$ -element. This proves that each  $p'$ -subgroup of  $G$  is contained in  $A$ , hence  $P$  is a subgroup of  $A$ .

We show that  $V \cap N_A(V \cap P)$  is an  $\mathfrak{F}$ -a-injector of  $N_A(V \cap P)$ . Let  $G_1 = N_G(V \cap P)$  and  $N = N_P(V \cap P)$ . As  $P$  is normal in  $G$ ,  $V \cap P$  is an  $\mathfrak{F}$ -a-injector of  $P$ , so we have  $N_{\mathfrak{F}} = V \cap P$ , and  $N_{\mathfrak{F}}$  is a maximal  $\mathfrak{F}$ -subgroup of  $P$  and of  $N$ . Since  $N = G_1 \cap P$  contains all the  $p'$ -elements of  $G_1$ ,  $G_1/N$  is a  $p$ -group. But, as  $A$  contains  $P$ ,  $N_A(V \cap P)$  is a subgroup of  $G_1$  containing  $N$  and, since  $V$  is an  $\mathfrak{F}$ -a-injector of  $G_1$  (Corollary 1.15 and Theorem 2.5), Lemma 3.5 says that  $V \cap N_A(V \cap P)$  is an  $\mathfrak{F}$ -a-injector of  $N_A(V \cap P)$ .

By Theorems 1.16 and 2.5,  $A$  has an  $\mathfrak{F}$ -a-injector  $W$ . Then  $W \cap P$  and  $V \cap P$  are two  $\mathfrak{F}$ -a-injectors of  $P$  and there is  $a \in P$  such that  $(W \cap P)^a = V \cap P$  (Theorem 2.5). Now  $W^a$  is an  $\mathfrak{F}$ -a-injector of  $N_A(V \cap P)$  by Corollary 1.15 and Theorem 2.5. Since  $V \cap N_A(V \cap P)$  is an  $\mathfrak{F}$ -a-injector of  $N_A(V \cap P)$ , Theorem 2.5 says that  $W^a$  and  $V \cap N_A(V \cap P)$  are conjugate in  $N_A(V \cap P)$ . Thus  $V \cap N_A(V \cap P)$  is an  $\mathfrak{F}$ -a-injector of  $A$  contained in the  $\mathfrak{F}$ -subgroup  $V \cap A$ . Hence  $V \cap N_A(V \cap P) = V \cap A$  and  $V \cap A$  is as desired.  $\square$

**Theorem 3.7.** *The  $\mathfrak{F}$ -a-injectors of  $G$  are precisely its  $\mathfrak{F}$ -injectors.*

*Proof.* It is sufficient to prove that any  $\mathfrak{F}$ -a-injector  $V$  of  $G$  is an  $\mathfrak{F}$ -injector. We prove that, if  $A$  is a serial subgroup of  $G$ , then  $V \cap A$  is an  $\mathfrak{F}$ -a-injector of  $A$ . Let

$W$  be an  $\mathfrak{F}$ -a-injector of  $A$ . We show that  $W$  is contained in an  $\mathfrak{F}$ -a-injector  $U$  of  $A\rho(G)$ . Let  $N_0 = 1$  and, for every  $j \geq 1$ ,  $N_j = \mathcal{O}_{\pi_j}(\rho(G))$  where  $\pi_j = \{q : q \text{ is a the } i^{\text{th}} \text{ prime for } i \leq j\}$ . We construct an increasing sequence  $(W_i)_{i \in \mathbb{N}}$  of  $\mathfrak{F}$ -subgroups of  $G$  such that  $W_0 = W$  and, for every  $i \in \mathbb{N}$ ,  $W_i$  is an  $\mathfrak{F}$ -a-injector of  $N_i A$ . Suppose  $W_i$  constructed for  $i \in \mathbb{N}$ . Let  $W_{i+1,0}$  be an  $\mathfrak{F}$ -a-injector of  $N_{i+1} A$ . Let  $p$  be the  $(i+1)^{\text{th}}$  prime. Then we have  $N_{i+1} = N_i \times \mathcal{O}_p(\rho(G))$ , and since  $N_i A$  is a serial subgroup of  $N_{i+1} A$ , then  $W_{i+1,0} \cap N_i A$  is an  $\mathfrak{F}$ -a-injector of  $N_i A$  by Lemma 3.6. So there is  $u \in N_i A$  such that  $W_i = (W_{i+1,0} \cap N_i A)^u$  (Theorem 2.5). Now we may choose  $W_{i+1} = W_{i+1,0}^u$ , and  $W_i = W_{i+1} \cap N_i A$  is serial in  $W_{i+1}$ . Let  $U = \cup_{i \in \mathbb{N}} W_i$ . Then  $W_i$  is a serial  $\mathfrak{F}$ -subgroup of  $U$  for each  $i$ , and  $U = U_{\mathfrak{F}}$  is an  $\mathfrak{F}$ -group. Moreover we have  $U \cap N_i A = W_i$  for every  $i \in \mathbb{N}$ . Let  $B$  be an ascendant subgroup of  $A\rho(G)$  and  $U_1$  be an  $\mathfrak{F}$ -subgroup of  $B$  containing  $U \cap B$ . For every  $i \in \mathbb{N}$ , as  $N_i A$  is a serial subgroup of  $A\rho(G)$ ,  $U_1 \cap N_i A$  is an  $\mathfrak{F}$ -group containing  $W_i \cap B$ . But  $W_i \cap B$  is a maximal  $\mathfrak{F}$ -subgroup of  $N_i A \cap B$  for every  $i \in \mathbb{N}$  since  $W_i$  is an  $\mathfrak{F}$ -a-injector of  $N_i A$ . So we have  $U_1 \cap N_i A = W_i \cap B$  for every  $i \in \mathbb{N}$ , and  $U_1 = \cup_{i \in \mathbb{N}} (U_1 \cap N_i A) = \cup_{i \in \mathbb{N}} (W_i \cap B) = U \cap B$ . Hence  $U$  is an  $\mathfrak{F}$ -a-injector of  $A\rho(G)$  containing  $W$ .

It follows from Fact 1.6 that any serial subgroup of  $G/\rho(G)$  is acendant [1, Lemma 7.2.11], so  $A\rho(G)$  is ascendant in  $G$ , and  $V \cap A\rho(G)$  is an  $\mathfrak{F}$ -a-injector of  $A\rho(G)$ . By Theorem 2.5, there is  $g \in A\rho(G)$  such that  $U^g = V \cap A\rho(G)$ . But there is  $k \in \mathbb{N}$  such that  $g \in N_k A$  and we have  $V \cap N_k A = (U \cap N_k A)^g = W_k^g$ . So  $V \cap N_k A$  is an  $\mathfrak{F}$ -a-injector of  $N_k A$ . Now, by successive applications of Lemma 3.6,  $V \cap A$  is an  $\mathfrak{F}$ -a-injector of  $A$ .  $\square$

Now we obtain Theorem 3.4 from Theorems 1.16, 2.5 and 3.7.

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