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An Observer with Measurement-triggered Jumps for Linear Systems with Known Input ⋆

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Abstract: This paper deals with the estimation of the state of linear time invariant systems for which measurements of the output are available sporadically. An observer with jumps triggered by the arrival of such measurements is proposed and studied in a hybrid systems framework. The resulting system is written in estimation error coordinates and augmented with a timer variable that triggers the event of new measurements arriving. The design of the observer is performed to achieve uniform global asymptotic stability (UGAS) of a closed set including the points for which the state of the plant and its estimate coincide. Furthermore, a computationally tractable design procedure for the proposed observer is presented and illustrated in an example.

Keywords: Observers, hybrid systems, linear systems, linear matrix inequalities, networked control systems

1. INTRODUCTION

State observer design is undoubtedly a difficult problem, with high relevance in applications. Indeed, observers can be employed to obtain an estimation of certain state variables, which are not directly accessible or also to reduce the number of the sensors used in control systems. Many of the most interesting recent applications pertain to controlled systems linked together through data networks. The nature of such networks may often introduce time delays, asynchronism, packages drop-out, and communication channel limitations; see, for example, Lopez Hurtado et al. (2009). Moreover, in modern distributed systems, the communication mechanisms across the network are governed by logic statements, which aim at reducing the required bandwidth over the communication channel; see, for example, Wong and Brockett (1997). Such mechanisms lead to an intermittent availability of the measured variables. In this setting, the classical paradigm of continuously measured variables needs to be reconsidered to face the new challenges induced by data network constraints. Indeed, an observer can employ the measured output only at discrete-time instants, which are *a priori* unknown, that is the estimation algorithm is actually governed by an event-triggered mechanism (see Åström and Bernhardsson (2002) for further details). It is worthwhile to notice that for the periodic sampling case, several solutions are shown in the literature, (see for example Maroni et al. (2000)).

In this paper, we focus on the estimation problem for linear systems where the input injected into the plant is known and the measured output is gathered in an intermittent fashion. Building from the idea in Raff and Allgöwer (2007), we propose an open-loop observer along with a suitable event-triggered updating of the estimated state. Since the evolution of the considered observer exhibits both continuous-time behavior and instantaneous updating, we provide a hybrid model of the observer including the triggering logic. Then, using a Lyapunov function, we propose a condition that guarantees global uniform asymptotic stability of the estimation error as well as robustness with respect to bounded perturbations, in an input-to-state stability sense (see Sontag (1989) and Cai and Teel (2009)). To this end, by relaxing the input-to-state stability Lyapunov condition for hybrid systems proposed by Cai and Teel in Cai and Teel (2009), we exhibit a novel sufficient condition to prove input-to-state stability in presence of persistent jumps. Finally, the obtained condition is turned into a design algorithm for the proposed observer based on the solution of a set of linear matrix inequalities.

The proposed hybrid model allows us to effectively exploit the properties of the time domain of the solutions to the resulting hybrid system, in particular, the persistence of jumps. This feature not only provides a tighter understanding of the system behavior but also enables us to construct a more general Lyapunov function, so as to

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overcome the convexity issues induced by non-uniformity in sampling time, which are also pointed out in Raff and Allgöwer (2007), and, moreover, to characterize the effect of measurement noise via input-to-state stability.

The paper is organized as follows. Section II presents the system under consideration, the problem we intend to solve, and the hybrid modeling of the proposed observer. Section III is dedicated to the main results, which provide a solution to the stated estimation problem. Section IV is devoted to numerical issues and provides a convex design algorithm for the proposed observer. In Section V, the effectiveness of the approach is illustrated through a numerical example. Due to space limitations, proofs of the results will be published elsewhere.

Notation: The set $\mathbb{N}_0$ is the set of the positive integers including zero and $\mathbb{R}_{\geq 0}$ represents the set of the nonnegative real scalars. For every complex number ω, $\mathfrak{Re}(\omega)$ and $\mathfrak{Im}(\omega)$ stand respectively for the real and the imaginary part of ω. $\mathbf{I}$ denotes the identity matrix whereas $\mathbf{0}$ denotes the null matrix (equivalently the null vector) of appropriate dimensions. For a matrix $A \in \mathbb{R}^{n\times m}$, A' denotes the transpose of A and $\|A\|$ denotes the Euclidean induced norm. $\mathrm{He}(A) = A + A'$. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. In partitioned symmetric matrices, the symbol $\star$ stands for symmetric blocks. The matrix $\mathrm{diag}\{A_1; \ldots; A_n\}$ is the block-diagonal matrix having $A_1, \ldots, A_n$ as diagonal blocks. For a vector $x \in \mathbb{R}^n$, x' denotes the transpose of x, whereas $\|x\|$ denotes the Euclidean norm. For a function $s \in [0, +\infty) \to \mathbb{R}^n$, $\|s\|_t = \sup_{\tau\in[0,t]} \|s(\tau)\|$. Let X be a given set, $\mathrm{Co}\{X\}$ represents the convex hull of X. $\delta\mathbb{B}$ is the closed ball with radius δ of appropriate dimension in the Euclidean norm. A function $\alpha\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to the class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. A function $\alpha\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}_\infty$ if it belongs to the class $\mathcal{K}$ and is unbounded. A function $\beta\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, and $\lim_{s\to 0^+} \beta(s,t) = \lim_{t\to+\infty} \beta(s,t) = 0$. A function $\beta\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KLL}$ if, for each $r \in \mathbb{R}_{\geq 0}$, the functions $\beta(\cdot,\cdot,r)$ and $\beta(\cdot,r,\cdot)$ belong to class $\mathcal{KL}$.

2. PROBLEM STATEMENT

2.1 System description

Consider the following continuous-time linear system:

$$\begin{aligned} \dot{z} &= Az + Bu \\ y &= Mz \end{aligned} \tag{1}$$

where $z \in \mathbb{R}^n$, $y \in \mathbb{R}^q$ and $u \in \mathbb{R}^p$ are, respectively, the state, the measured output, and the input of the system, while A, B and M are constant matrices of appropriate dimensions. Assume also that the input u belongs to the class of the measurable and locally bounded functions $u\colon [0,\infty) \to \mathbb{R}^p$. We want to design an observer providing an estimate $\hat{z}$ of the state z when the output y is available only at some times t_k, for $k \in \mathbb{N}_0$, not known *a priori*. Figure 1 illustrates such a setting in the context of network control. Suppose that $\{t_k\}_0^{+\infty}$ is a strictly increasing unbounded real sequence of times. Furthermore, assume that there exist two positive real scalars T_1, T_2 with $T_1 < T_2$ such that [1]

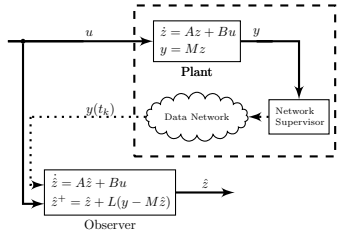

Fig. 1. State estimation for a linear system with output gathered through a data network.

$$T_1 \leq t_{k+1} - t_k \leq T_2. \tag{2}$$

Since the information on the output y is available in an impulsive fashion, motivated by the work of Raff and Allgöwer (2007), to solve the considered estimation problem, we design an observer with jumps in its state following the law:

$$\dot{\hat{z}} = A\hat{z} + Bu \quad \text{when } t \notin \{t_k\}_0^{+\infty} \tag{3a}$$

$$\hat{z}(t_k^+) = \hat{z}(t_k) + L(y(t_k) - M\hat{z}(t_k)) \quad \text{when } t \in \{t_k\}_0^{+\infty} \tag{3b}$$

where L is a real matrix of appropriate dimensions to be designed. It is worthwhile to point out that in Sur and Paden (1997) the same observer is adopted to state estimation in presence of quantized measurement.
Following the lines of Sanfelice and Praly (2012), the state estimation problem can be formulated as a set stabilization problem. Namely, define

$$\mathcal{A}_s = \left\{(z, \hat{z}) \in \mathbb{R}^{2n} \colon z = \hat{z}\right\} \tag{4}$$

our goal is to design the matrix L such that $\mathcal{A}_s$ is globally asymptotically stable for the plant (1) interconnected with the observer in (3a). At this stage, as usual in estimation problems, one considers the estimation error defined as $\varepsilon := z - \hat{z}$, so the error dynamics are given by the following dynamical system with jumps:

$$\dot{\varepsilon} = A\varepsilon \qquad \text{when } t \notin \{t_k\}_0^{+\infty} \tag{5a}$$

$$\varepsilon(t_k^+) = (\mathbf{I} - LM)\varepsilon(t_k) \qquad \text{when } t \in \{t_k\}_0^{+\infty}. \tag{5b}$$

Due to the linearity of the system (1), the estimation error dynamics and the dynamics of z are decoupled. Then, for the purpose of stabilizing the set $\mathcal{A}_s$, one can effectively just consider system (5).

Remark 1. Notice that assuming the knowledge of the input is not overly restrictive. Indeed, in many practical settings, all of the devices employed to control and supervise the plant may be embedded into the same system. This situation is depicted in Figure 1, where the dotted arrows denote impulsive data streams, while the solid arrows denote continuous data streams. Notice also that, often, the estimated state is part of a feedback controller (*e.g.* in linear observer-based controller architectures), in which case the input u is a static function of the estimated state that is perfectly known.

2.2 Hybrid modeling

The fact that the observer experiences jumps when a new measurement is available suggests that the updating

[1] Concerning this assumption, see Postoyan et al. (2011); Briat and Seuret (2012) and the references therein. Notice that, as pointed also in Hetel et al. (2012), condition (2) prevents the existence of accumulation points in the sequence $\{t_k\}_0^{+\infty}$, and, hence, it avoids the existence of Zeno behaviors, which is typically undesired in practice.

process of the error dynamics can be described via a hybrid system. Due to this, we represent the whole system composed by the plant (1), the observer (3), and the logic triggering jumps as a hybrid system (see Li and Sanfelice (2013) where similar techniques are adopted to model a finite time convergent observer).

Such a hybrid systems approach requires to model the hidden time-driven mechanism triggering the observer jumps. To this end, in this work, we augment the system state with an auxiliary timer variable τ, which keeps track of the duration of flows and triggers a jump whenever a certain condition is verified. This additional state allows to describe the time-driven jump triggering mechanism as a state-driven jump triggering mechanism, which leads to a model that can be efficiently represented by relying on the framework for hybrid systems proposed in Goebel et al. (2012). More precisely, we make τ to decrease as ordinary time t increases and, whenever it reaches zero, triggers a jump that makes a self reset of τ. In fact, after a jump occurs, τ is re-initialized to some value belonging to the interval $[T_1, T_2]$ and, after the reset, it flows again. Therefore, the whole system composed by the state ε and the timer variable τ can be represented by the following hybrid system:

$$\mathcal{H}_\varepsilon \begin{cases} \left.\begin{array}{l} \dot{\varepsilon} = A\varepsilon \\ \dot{\tau} = -1 \end{array}\right\} (\varepsilon, \tau) \in C \\ \\ \left.\begin{array}{l} \varepsilon^+ = (\mathbf{I} - LM)\varepsilon \\ \tau^+ \in [T_1, T_2] \end{array}\right\} (\varepsilon, \tau) \in D \end{cases} \tag{6a}$$

with the flow set and the jump set defined as

$$\begin{aligned} C &= \left\{(\varepsilon, \tau) \in \mathbb{R}^{n+1} \colon \tau \in [0, T_2]\right\} \\ D &= \left\{(\varepsilon, \tau) \in \mathbb{R}^{n+1} \colon \tau = 0\right\}. \end{aligned} \tag{6b}$$

For this system, we denote by $\tilde{x} = [\varepsilon' \ \tau']'$ the state and by f and G, respectively, the flow map and the jump map, *i.e.*,

$$f(\tilde{x}) = \begin{bmatrix} A\varepsilon \\ -1 \end{bmatrix} \tag{7a}$$

$$G(\tilde{x}) = \begin{bmatrix} (\mathbf{I} - LM)\varepsilon \\ [T_1, T_2] \end{bmatrix}. \tag{7b}$$

Notice that to make the hybrid system (6) an accurate description of the real time-triggered phenomenon, which governs the feedback update process, the variable τ needs to belong to the interval $[0, T_2]$, property that is guaranteed by the definition of C and D. Then, the stabilization objective can be formalized by introducing the set [2]

$$\mathcal{A} = \left\{(\varepsilon, \tau) \in \mathbb{R}^{n+1} \colon \varepsilon = 0, \tau \in [0, T_2]\right\}. \tag{8}$$

Then, the problem we intend to solve can be formulated as follows:

Problem 1. Given the matrices A, B, and M of appropriate dimensions and two positive scalars T_1 and T_2 such that $T_1 < T_2$, compute a matrix $L \in \mathbb{R}^{n \times q}$ such that the set $\mathcal{A}$ defined in (8) is Uniform Global Asymptotically Stable ($UGAS$) for the hybrid system (6).

[2] Since $\mathcal{A}$ is closed, given a vector $x \in \mathbb{R}^{n+1}$, the distance of x from $\mathcal{A}$ is defined as follows:

$$|x|_{\mathcal{A}} = \inf_{y \in \mathcal{A}} \|x - y\|.$$

It turns out that for every $\tilde{x} \in C \cup D \cup G(D)$, $|\tilde{x}|_{\mathcal{A}} = \|\varepsilon\|$.

About the notion of UGAS of a given set for a generic hybrid system $\mathcal{H}$, we consider the definition provided in (Goebel et al., 2012, Definition 3.6). Concerning the existence of solutions to system (6), relying on the concept of solution proposed in (Goebel et al., 2012, Definition 2.6), it is straightforward to check that for every initial condition $\tilde{x}(0,0) \in C \cup D$, every solution to $\mathcal{H}$ is complete. In addition, we can characterize the domain of these solutions. Indeed, the variable τ, acting as a timer, guarantees that for every initial condition $\tilde{x}(0,0) \in C \cup D$, at least for $j \geq 1$, $t_{j+1} - t_j \in [T_1, T_2]$. Therefore, the domain of a solution ϕ to $\mathcal{H}$ can be written as follows:

$$\operatorname{dom}\phi = ([t_0, t_1] \times \{0\}) \cup \left(\bigcup_{j \in \mathbb{N} \setminus \{0\}} ([t_j, t_{j+1}]) \times \{j\} \right)$$

$$\begin{aligned} &T_1 \leq t_{j+1} - t_j \leq T_2 \quad \forall j \in \mathbb{N} \setminus \{0\} \\ &0 \leq t_1 - t_0 \leq T_2 \end{aligned} \tag{9}$$

where $\operatorname{dom}\phi$ is the domain of ϕ, which is a hybrid time domain. It should be noticed that the structure of the foregoing hybrid time domain implies that

$$t \leq T_2(j+1) \ \ \forall (t, j) \in \operatorname{dom}\phi. \tag{10}$$

3. MAIN RESULTS

3.1 Conditions for Uniform Global Asymptotic Stability

The following result provides conditions for the UGAS of the set $\mathcal{A}$ defined in (8) for system (6). These conditions ensure that the assumptions of the Lyapunov result for hybrid systems presented in (Goebel et al., 2012, Proposition 3.24) hold.

Theorem 1. Given two positive scalars T_1 and T_2 such that $T_1 < T_2$, if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a matrix $L \in \mathbb{R}^{q \times n}$ such that

$$(\mathbf{I} - LM)' e^{A'v} P e^{Av} (\mathbf{I} - LM) - P < \mathbf{0}, \ \ \forall v \in [T_1, T_2], \tag{11}$$

then the set $\mathcal{A}$ defined in (8) is UGAS for the hybrid system (6).

Remark 2. Notice that assuming relation (11) to hold implies that the eigenvalues of $e^{Av}(\mathbf{I} - LM)$ are strictly contained in the unit circle for every v belonging to $[T_1, T_2]$. In Section 4, we provide a design procedure, including an algorithm.

3.2 Effect of measurement noise

Until now, the measured output y was assumed to be perfectly known at sampling times t_k. However, in real-world settings, the measured output is affected by measurement noise. Hence, having some insight on the robustness of hybrid system (6) with respect to a bounded measurement noise is undoubtedly useful.

To this end, denoting the measurement noise as $\eta : [0, +\infty) \to \delta\mathbb{B}$, with $\delta \geq 0$ the measured output is defined by

$$y = Mx + \eta.$$

Then, the hybrid system (6) is rewritten as follows:

$$\mathcal{H}_\eta \begin{cases} \left.\begin{aligned} \dot{\varepsilon} &= A\varepsilon \\ \dot{\tau} &= -1 \end{aligned}\right\} (\varepsilon,\tau) \in C \\ \left.\begin{aligned} \varepsilon^+ &= (\mathbf{I} - LM)\varepsilon - L\eta \\ \tau^+ &\in [T_1, T_2] \end{aligned}\right\} (\varepsilon,\tau) \in D \end{cases} \tag{12a}$$

with

$$\begin{aligned} C &= \left\{(\varepsilon,\tau) \in \mathbb{R}^{n+1} \colon \tau \in [0, T_2]\right\} \\ D &= \left\{(\varepsilon,\tau) \in \mathbb{R}^{n+1} \colon \tau = 0\right\}. \end{aligned} \tag{12b}$$

Thus, the flow map remains defined as in (7a) while the new jump map is given as

$$\widetilde{G}(\tilde{x},\eta) = \begin{bmatrix} (\mathbf{I} - LM)\varepsilon - L\eta \\ [T_1, T_2] \end{bmatrix} \tag{13}$$

To study the effect of the measurement noise, we consider the input-to-state-stability (*ISS*) concept introduced in Sontag (1989) for continuous-time nonlinear systems and recently extended to hybrid systems in Cai and Teel (2005, 2009). Notice that this extension of ISS to hybrid systems deals with hybrid signals as external perturbations, and for such class of signals, a suitable supremum norm is provided. However, in our case, the perturbation $t \mapsto \eta(t)$ is a purely continuous-time signal, so it needs to be transformed to a hybrid signal to fit in the framework proposed by Cai and Teel. To this end, as shown in Robles and Sanfelice (2011), given a solution ϕ to $\mathcal{H}_\eta$, the signal $t \mapsto \eta(t)$ can be represented as a hybrid signal $\eta_\mathcal{H}$ defined as

$$\eta_\mathcal{H}(t,j) = \eta(t) \ \forall (t,j) \in \operatorname{dom}\phi. \tag{14}$$

Now, if for the hybrid signal $\eta_\mathcal{H}$ we consider the (hybrid) supremum norm $\|\eta_\mathcal{H}\|_{(t,j)}$ in Cai and Teel (2009), due to (14), it turns out that for such signal one has $\|\eta_\mathcal{H}\|_{(t,j)} = \|\eta_\mathcal{H}\|_t$ for every $(t,j) \in \operatorname{dom}\phi$.

Notice that, although in Cai and Teel (2009) a condition for hybrid systems to be ISS is given, such a condition does not hold in our context, at least in general. Indeed, adopting the Lyapunov condition in Cai and Teel (2009) to our problem would require the existence of a Lyapunov function decreasing along the flow of the solutions to system (12), which requires the matrix A to be Hurwitz. On the other hand, since by Theorem 1 we exhibit the existence of a Lyapunov function which is not increasing along the flow of the solutions to system (12), by extending this result, we show that condition (11) actually suffices to guarantee the ISS property for the hybrid system (12).

Theorem 2. Given two positive scalars T_1, T_2 such that $T_1 < T_2$, if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a matrix $L \in \mathbb{R}^{q\times n}$ satisfying condition (11), then the set $\mathcal{A}$ defined in (8) is ISS with respect to η for the hybrid system (12).

Remark 3. The ISS property guaranteed by Theorem 2 only has perturbations on the jump map. On the other hand, due to unmodeled dynamics, perturbations may affect also the flow map. Thus, analyzing the behavior of the hybrid system $\mathcal{H}_\varepsilon$ in presence of a wider class of perturbation is a relevant matter. At this stage, one should notice that the way we adopted to model the hybrid system (6) leads to a hybrid system which is structurally robust with respect to bounded perturbations on the data; namely, the hybrid system (6) is well-posed in the sense defined in (Goebel et al., 2012, Definition 6.2). Thus, the UGAS property of the set $\mathcal{A}$ defined in (8) for the nominal system $\mathcal{H}_\varepsilon$ holds (semiglobally and practically) for the perturbed system as well. More specifically, provided that the set (8) is UGAS for the hybrid system $\mathcal{H}_\varepsilon$, then for each compact set $\mathcal{M}$ of the state space and each $\omega > 0$, there exists a function $\kappa \in \mathcal{KLL}$, and a scalar $\delta^* > 0$ such that for each $\delta \in [0, \delta^*]$, every solution ϕ_p to the perturbed system $\mathcal{H}_\varepsilon^p$ from $\mathcal{M}$ satisfies, for all $(t,j) \in \operatorname{dom}\phi_p$, $|\phi_p(t,j)|_\mathcal{A} \leq \kappa(\phi_p(t,j),t,j) + \omega$. It is worthwhile to remark that getting a hybrid system exhibiting the above mentioned well-posedness property may not be trivial and it actually derives from suitable choices done throughout the modeling stage.

4. NUMERICAL DESIGN PROCEDURE

In the previous section, a condition to establish the UGAS and ISS properties, respectively, for systems (6) and (12) was provided. However, due to its form, such a condition is not computationally tractable to obtain a solution to Problem 1. Indeed, from a numerical standpoint, condition (11) has two drawbacks: it is not convex in P and L, and it needs to be verified for infinitely many values of v. The relevance of the second drawback is evident at a first sight, while the lack of convexity is a severe constraint, since non-convex problems often lead to NP-hard problems; see, for example, Boyd et al. (1997). Thus, in order to make the problem numerically tractable, some manipulations are needed. To this end, the following result provides a first step toward a convex design procedure for the proposed observer.

Proposition 1. Let T_1 and T_2 be two given positive scalars such that $T_1 < T_2$. If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a matrix $J \in \mathbb{R}^{q\times n}$, and a matrix $F \in \mathbb{R}^{n\times n}$ such that for every $v \in [T_1, T_2]$

$$\begin{bmatrix} -\operatorname{He}(F) & F - JM & e^{A'v}P \\ \star & -P & \mathbf{0} \\ \star & \star & -P \end{bmatrix} < \mathbf{0} \tag{15}$$

then the matrices P and $L = F^{-1}J$ satisfy condition (11).

Remark 4. Notice that condition (15) is convex with respect to the unknown matrices F, L, and P.

To efficiently design the observer, one needs to avoid finding a solution to (15) for infinitely many values of v. To overcome this issue, we propose to embed the term e^{Av}, with v in the interval $[T_1, T_2]$, in a convex set, obtaining in this way a convex design procedure composed by a finite number of inequalities. This technique consists in finding some matrices $X_1, X_2, \ldots, X_\nu \in \mathbb{R}^{n\times n}$, such that $e^{Av} \in \operatorname{Co}\{X_1, X_2, \ldots, X_\nu\}$ whenever $v \in [T_1, T_2]$.
To this end, consider the following well known expression

$$\begin{aligned} e^{Av} = &\sum_{i=1}^{\sigma_r}\sum_{j=1}^{m_i^r} R_{ij} e^{\lambda_i v} \frac{v^{j-1}}{(j-1)!} + \\ &+ \sum_{i=1}^{\sigma_c}\sum_{j=1}^{m_i^c} 2e^{\mathfrak{Re}(\lambda_i)v}\Big(\mathfrak{Re}(R_{ij})\cos(\mathfrak{Im}(\lambda_i)v) \\ &- \mathfrak{Im}(R_{ij})\sin(\mathfrak{Im}(\lambda_i)v)\Big)\frac{v^{j-1}}{(j-1)!} \end{aligned} \tag{16}$$

where σ_r is the number of distinct eigenvalues, σ_c the number of distinct complex-conjugate eigenvalue pairs.

The constants m_i and m_c are, respectively, the multiplicity of the real eigenvalue λ_i and of the complex-conjugate eigenvalue pair λ_i, λ_i^* in the minimal polynomial of the matrix A. The matrices R_{ij} are real $n \times n$ matrices corresponding to the residuals associated to the partial fraction expansion of $(sI - A)^{-1}$. Notice that several methods can be adopted to compute such matrices. In this work, we rely on the procedure proposed in Leyva-Ramos (1993). Once the value of the residuals are known, to build a polytopic embedding of e^{Av} one can proceed in a similar manner of Heemels et al. (2010). Namely,

$$\{X_1, \dots, X_\nu\} = \left\{ \sum_{i=1}^{\sigma_r} \sum_{j=1}^{m_i^r} R_{ij}\beta_{ij} + \sum_{i=1}^{\sigma_c} \sum_{j=1}^{m_i^c} \gamma_{ij}\, \mathfrak{Re}(R_{ij}) + \gamma_{ij}^*\, \mathfrak{Im}(R_{ij}) \colon \beta_{ij} \in \{\overline{\beta_{ij}}, \underline{\beta_{ij}}\}, \gamma_{ij} \in \{\overline{\gamma_{ij}}, \underline{\gamma_{ij}}\}, \gamma_{ij}^* \in \{\overline{\gamma_{ij}^*}, \underline{\gamma_{ij}^*}\} \right\}, \tag{17}$$

where

$$\begin{aligned}
\overline{\beta_{ij}} &= \max_{v \in [T_1, T_2]} e^{\lambda_i v} \frac{v^{j-1}}{(j-1)!} \\
\underline{\beta_{ij}} &= \min_{v \in [T_1, T_2]} e^{\lambda_i v} \frac{v^{j-1}}{(j-1)!} \\
\overline{\gamma_{ij}} &= \max_{v \in [T_1, T_2]} 2e^{\mathfrak{Re}(\lambda_i)v} \cos(\mathfrak{Im}(\lambda_i)v) \frac{v^{j-1}}{(j-1)!} \\
\underline{\gamma_{ij}} &= \min_{v \in [T_1, T_2]} 2e^{\mathfrak{Re}(\lambda_i)v} \cos(\mathfrak{Im}(\lambda_i)v) \frac{v^{j-1}}{(j-1)!} \\
\overline{\gamma_{ij}^*} &= \max_{v \in [T_1, T_2]} -2e^{\mathfrak{Re}(\lambda_i)v} \sin(\mathfrak{Im}(\lambda_i)v) \frac{v^{j-1}}{(j-1)!} \\
\underline{\gamma_{ij}^*} &= \min_{v \in [T_1, T_2]} -2e^{\mathfrak{Re}(\lambda_i)v} \sin(\mathfrak{Im}(\lambda_i)v) \frac{v^{j-1}}{(j-1)!}
\end{aligned} \tag{18}$$

The proposed technique leads to the following result.

Corollary 1. Let T_1 and T_2 be two given positive scalars such that $T_1 < T_2$. Let $\{X_1, \dots, X_\nu\}$ be the matrices obtained by (17). If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, a matrix $J \in \mathbb{R}^{q \times n}$, and a matrix $F \in \mathbb{R}^{n \times n}$ such that, for every $i = 1, \dots, \nu$,

$$\begin{bmatrix} -\operatorname{He}(F) & F - JM & X_i P \\ \star & -P & \mathbf{0} \\ \star & \star & -P \end{bmatrix} < \mathbf{0} \tag{19}$$

then the matrices P and $L = F^{-1}J$ satisfy condition (11).

Corollary 1 represents an efficient solution to Problem 1, which finally can be solved by Algorithm 1, which is given below.

Algorithm 1 Observer design

1: Find the residual matrices R_{ij} in (16)
2: Compute the scalars $\overline{\beta_{ij}}, \underline{\beta_{ij}}, \overline{\gamma_{ij}}, \underline{\gamma_{ij}}, \overline{\gamma_{ij}^*}, \underline{\gamma_{ij}^*}$ as in (18)
3: Compute the matrices $\{X_1, \dots, X_\nu\}$ as in (17)
4: Solve (19) with respect to J, P and F
5: $L \leftarrow F^{-1}J$
6: **return** L

5. ILLUSTRATIVE EXAMPLE

Consider the mass-spring system proposed by Geromel and de Oliveira (2001), which is defined by the following data:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -1 & 0 \\ 2 & -2 & 0 & -2 \end{bmatrix}, \quad M = [1\ 0\ 0\ 0] \tag{20}$$
$$B' = [0\ 0\ 1\ 0]$$

consider also $u(t) = \sin(t)$. By fixing $T_1 = 0.2$ and $T_2 = 3$, Algorithm 1 yields:

$$P = \begin{bmatrix} 0.1180 & 0.2460 & 0.1889 & 0.1491 \\ 0.2460 & 1.1788 & 1.0392 & 0.9646 \\ 0.1889 & 1.0392 & 0.9407 & 0.8778 \\ 0.1491 & 0.9646 & 0.8778 & 0.8328 \end{bmatrix}, \quad L = \begin{bmatrix} 1.0000 \\ -0.9433 \\ -0.6773 \\ 1.6274 \end{bmatrix}. \tag{21}$$

Figure 2 depicts the projection onto ordinary time t of the states $z(t, j)$ and $\hat{z}$. In this simulation, the sampling instants are selected randomly in the interval $[T_1, T_2]$ according to a standard Gaussian distribution. Simulations show that the estimates appear to quickly converge toward the plant state z since the estimate $\hat{z}$ and the state z are nearly overlapped after three jumps.

6. CONCLUSION

This paper proposed a methodology to model and design, through a convex problem, an event-triggered observer to estimate the state of a linear plant whenever the output is measured in an impulsive fashion. Moreover, the proposed observer is shown to be ISS with respect to measurement noise and having a degree of robustness with respect to small enough bounded perturbations. The results in this paper suggest several directions of research on event-triggered observers. For example, the setting allows to consider a design problem for the updating logic of τ, in order to somehow schedule the sampling instants. Moreover, the design of an observer-based controller in the presence of impulsive output measurement represents certainly an interesting outlook.

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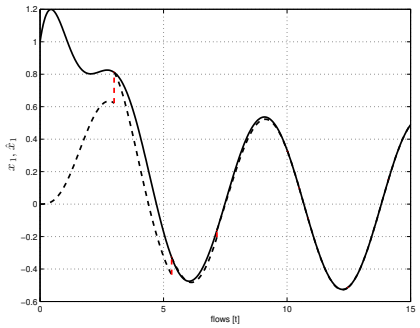
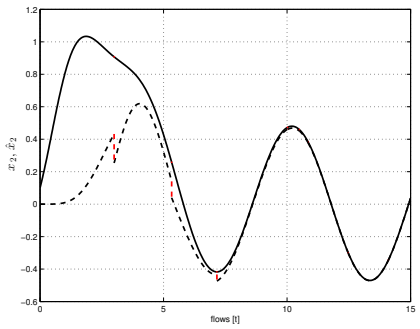
Fig. 2. The evolution of the states z and $\hat{z}$ projected onto ordinary time t.

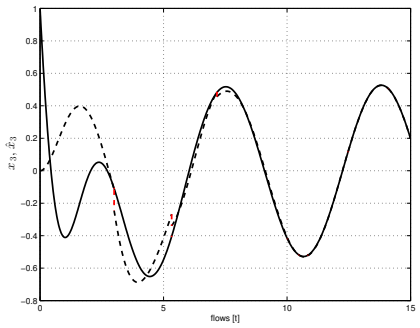
(a) Projection onto ordinary time t of $z_1(t,j)$ (solid) and $\hat{z}_1(t,j)$ (dashed).



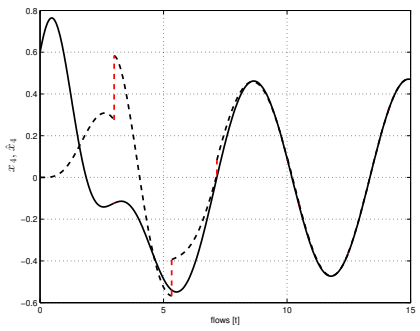
(b) Projection onto ordinary time t of $z_2(t,j)$ (solid) and $\hat{z}_2(t,j)$(dashed).



(c) Projection onto ordinary time t of $z_3(t,j)$ (solid) and $\hat{z}_3(t,j)$(dashed).



(d) Projection onto ordinary time t of $z_4(t,j)$ (solid) and $\hat{z}_4(t,j)$(dashed).



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