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A STRICTLY STATIONARY β -MIXING PROCESS SATISFYING THE CENTRAL LIMIT THEOREM BUT NOT THE WEAK INVARIANCE PRINCIPLE

DAVIDE GIRAUDDO AND DALIBOR VOLNÝ

ABSTRACT. In 1983, N. Herrndorf proved that for a ϕ -mixing sequence satisfying the central limit theorem and $\liminf_{n \rightarrow \infty} \frac{\sigma_n^2}{n} > 0$, the weak invariance principle takes place. The question whether for strictly stationary sequences with finite second moments and a weaker type (α, β, ρ) of mixing the central limit theorem implies the weak invariance principle remained open.

We construct a strictly stationary β -mixing sequence with finite moments of any order and linear variance for which the central limit theorem takes place but not the weak invariance principle.

1. INTRODUCTION AND NOTATIONS

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. If $T: \Omega \rightarrow \Omega$ is one-to-one, bi-measurable and measure preserving (in sense that $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$), then the sequence $(f \circ T^k)_{k \in \mathbb{Z}}$ is strictly stationary for any measurable $f: \Omega \rightarrow \mathbb{R}$. Conversely, each strictly stationary sequence can be represented in this way.

For a zero mean square integrable $f: \Omega \rightarrow \mathbb{R}$, we define $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$,

$\sigma_n^2(f) := \mathbb{E}(S_n(f)^2)$ and $S_n^*(f, t) := S_{[nt]}(f) + (nt - [nt])f \circ T^{[nt]}$, where $[x]$ is the greatest integer which is smaller than x .

We say that $(f \circ T^j)_{j \geq 1}$ satisfies the *central limit theorem with normalization* a_n if the sequence $(a_n^{-1} S_n(f))_{n \geq 1}$ converges weakly to a standard normal distribution. Let $C[0, 1]$ denote the space of continuous functions on the unit interval endowed with the norm $\|g\|_\infty := \sup_{t \in [0, 1]} |g(t)|$.

Let $D[0, 1]$ be the space of real valued functions which have left limits and are continuous at each point of $[0, 1]$. We endow it with Skorohod metric (cf. [2]). We define $S_n^{**}(f, t) := S_{[nt]}(f)$, which gives a random element of $D[0, 1]$.

We shall say that the strictly stationary sequence $(f \circ T^j)_{j \geq 0}$ satisfies the *weak invariance principle in $C[0, 1]$ with normalization a_n* (respectively in $D[0, 1]$) if the sequence of $C[0, 1]$ (of $D[0, 1]$) valued random variables $(a_n^{-1} S_n^*(f, \cdot))_{n \geq 1}$ (resp. $(a_n^{-1} S_n^{**}(f, \cdot))_{n \geq 1}$) weakly converges to a Brownian motion process in the corresponding space.

Let \mathcal{A} and \mathcal{B} be two sub- σ -algebras of \mathcal{F} , where $(\Omega, \mathcal{F}, \mu)$ is a probability space. We define the α -mixing coefficients as introduced by Rosenblatt in [12]:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup \{ |\mu(A \cap B) - \mu(A)\mu(B)|, A \in \mathcal{A}, B \in \mathcal{B} \},$$

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β -mixing coefficients by

$$\beta(\mathcal{A}, \mathcal{B}) := \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|,$$

where the supremum is taken over the finite partitions $\{A_i, 1 \leq i \leq I\}$ and $\{B_j, 1 \leq j \leq J\}$ of Ω of elements of \mathcal{A} (respectively of \mathcal{B}). They were introduced by Volkonskii and Rozanov [14].

The ρ -mixing coefficients were introduced by Kolmogorov and Rozanov [10] and are defined by

$$\rho(\mathcal{A}, \mathcal{B}) := \sup \{|\text{Corr}(f, g)|, f \in \mathbb{L}^2(\mathcal{A}), g \in \mathbb{L}^2(\mathcal{B}), f \neq 0, g \neq 0\},$$

where $\text{Corr}(f, g) := \frac{\mathbb{E}(fg) - \mathbb{E}(f)\mathbb{E}(g)}{\|f\|_{\mathbb{L}^2} \|g\|_{\mathbb{L}^2}}$.

Ibragimov [7] introduced for the first time ϕ -mixing coefficients, which are given by the formula

$$\phi(\mathcal{A}, \mathcal{B}) := \sup \{|\mu(B | A) - \mu(B)|, A \in \mathcal{A}, B \in \mathcal{B}, \mu(A) > 0\}.$$

The coefficients are related by the inequalities

$$(1) \quad \alpha(\mathcal{A}, \mathcal{B}) \leq 2\beta(\mathcal{A}, \mathcal{B}) \leq 2\phi(\mathcal{A}, \mathcal{B}), \quad \alpha(\mathcal{A}, \mathcal{B}) \leq \rho(\mathcal{A}, \mathcal{B}) \leq \phi(\mathcal{A}, \mathcal{B}).$$

For a strictly stationary sequence $(X_k)_{k \in \mathbb{Z}}$ and $n \geq 0$ we define $\alpha_X(n) = \alpha(n) = \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty)$ where \mathcal{F}_u^v is the σ -algebra generated by X_k with $u \leq k \leq v$ (if $u = -\infty$ or $v = \infty$, the corresponding inequality is strict). In the same way we define coefficients $\beta_X(n)$, $\rho_X(n)$, $\phi_X(n)$.

We say that the sequence $(X_k)_{k \in \mathbb{Z}}$ is α -mixing if $\lim_{n \rightarrow +\infty} \alpha_X(n) = 0$, and similarly we define β , ρ and ϕ -mixing sequences. Inequalities (1) give a hierarchy between these classes of mixing sequences.

If $(a_N)_{N \geq 1}$ and $(b_N)_{N \geq 1}$ are two sequences of positive real numbers, we write $a_N \asymp b_N$ if there exists a positive constant C such that for each N , $C^{-1}a_N \leq b_N \leq Ca_N$.

The main results are

Theorem A. *Let δ be a positive real number. There exists a strictly stationary real valued process $Y = (Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0}$ satisfying the following conditions:*

- a) *the central limit theorem with normalization \sqrt{n} takes place;*
- b) *the weak invariance principle with normalization \sqrt{n} does not hold;*
- c) *we have $\sigma_N(f)^2 \asymp N$;*
- d) *we have for some positive C , $\beta_Y(N) \leq \frac{C}{N^{1/2-\delta}}$;*
- e) *$Y_0 \in \mathbb{L}^p$ for any $p > 0$.*

Alternatively, we can construct the process in order to have a control of the mixing coefficients on a subsequence.

Theorem A'. *Let $(c_j)_{j \geq 0}$ be a decreasing sequence of positive numbers. Then there exists a strictly stationary real valued process $Y = (Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0}$ satisfying conditions a), b), c), e), and:*

- d') *there is an increasing sequence $(n_k)_{k \geq 1}$ of integers such that for each k , $\beta_Y(n_k) \leq c_{n_k}$.*

Remark 1. Herrndorf proved ([6], Theorem 2.13) that if (ξ_n) is a strictly stationary ϕ -mixing sequence for which $\sigma_n \rightarrow \infty$, S_n/σ_n converges in distribution to a standard normal distribution and $\sigma_n^{-1} \max_{1 \leq i \leq n} |\xi_i| \rightarrow 0$ in probability, then the weak invariance principle takes place. So Herrndorf's result does not extend to β -mixing sequences.

Remark 2. Rio *et al.* proved in [11] that the condition $\int_0^1 \alpha^{-1}(u) Q^2(u) du < \infty$ implies the weak invariance principle, where $\alpha^{-1}(u) := \inf \{k, \alpha(k) \leq u\}$ and Q is the right-continuous inverse of the quantile function $t \mapsto \mu \{X_0 > t\}$. If the process is strictly stationary, with finite moments of order $2+r$, $r > 0$, the latter condition is satisfied whenever $\sum_{n=1}^{\infty} (n+1)^{\frac{2}{r}} \alpha(n) < \infty$ (Ibragimov [8] found the condition $\sum_{n=1}^{\infty} \alpha(n)^{1-\frac{2}{r}} < \infty$). Since $Y_0 \in \mathbb{L}^p$ for all $p < \infty$, we have that $\sum_N \alpha(N)^r = +\infty$ for any $r < 1$, hence in A' we can thus hardly get such a bound for the whole sequence.

Also, the mixing rates in Theorem A are at most $\frac{\varepsilon_N}{N}$, where $(\varepsilon_N)_{N \geq 1}$ is bounded. Even if we had such a rate in Theorem A, the result of Theorem A' couldn't be deduced from this. Indeed, taking $c_j := j^{-2}$, we would not be able to get $\beta_X(n_k) \leq c_{n_k}$ from the corresponding variant of d).

Remark 3. Ibragimov proved that for a strictly stationary ρ -mixing sequence with finite moments of order $2+\delta$ for some positive δ , the weak invariance principle holds, cf. [9]. In particular, this proves that our construction does not give a ρ -mixing process. Shao also showed in [13] that the condition $\sum_n \rho(2^n) < \infty$ is sufficient in order to guarantee the weak invariance principle in $D[0, 1]$ for stationary sequences having order two moments. So a potential ρ -mixing counter-example has to obey to restrictions on the moments as well as on the mixing rates.

About the method of proof

In proving the result we will use properties of coboundaries $h = g - g \circ T$ (g is called a transfer function). For a positive integer N and a measurable function v , we denote $S_N(v) := \sum_{j=0}^{N-1} U^j v$. Because $S_n(g - g \circ T) = g - g \circ T^n$, for any sequence $a_n \rightarrow \infty$ we have $\frac{1}{a_n} S_n(g - g \circ T) \rightarrow 0$ in probability hence adding a coboundary does not change validity of the central limit theorem. If, moreover, $g \in \mathbb{L}^2$ then $\frac{1}{\sqrt{n}} \|S_n^*(g - g \circ T)\|_{\infty} \rightarrow 0$ a.s. hence adding of such coboundary does not change validity of the invariance principle (if norming by \sqrt{n} or σ_n with $\liminf_n \frac{\sigma_n^2}{n} > 0$), cf. [5]. On the other hand, if $g \notin \mathbb{L}^2$, adding a coboundary can spoil tightness even if $g - g \circ T$ is square integrable, cf. [15]. A similar idea was used in [4]. In the proof of Theorem A and A' we will find a coboundary $g - g \circ T$ which is β -mixing and spoils tightness. The coboundary has all finite moments but the transfer function is not integrable. We then add an m such that $(m \circ T^i)_{i \in \mathbb{Z}}$ and $(h \circ T^i)_{i \in \mathbb{Z}}$ are independent (enlarging the probability space), and $m \circ T^i$ is i.i.d. with moments of any order (in particular, it satisfies the weak invariance principle).

In the proof it is used the fact that $|\mu(A \cap B) - \mu(A)\mu(B)| \leq \mu(A)$. The method does not seem to apply to processes which are ρ -mixing and for this kind of processes the problem remains open.

2. PROOF

2.1. Construction of h . Let us consider an increasing sequence of positive integers $(n_k)_{k \geq 1}$, and for each integer $k \geq 1$ let A_k^-, A_k^+ be measurable sets such that $\mu(A_k^-) = \frac{1}{2n_k^2} = \mu(A_k^+)$.

We define random variables e_k by

$$(2) \quad e_k(\omega) := \begin{cases} 1 & \text{if } \omega \in A_k^+, \\ -1 & \text{if } \omega \in A_k^-, \\ 0 & \text{otherwise,} \end{cases}$$

such that $U^i e_k = e_k \circ T^i$ are mutually independent. We define $A_k := A_k^+ \cup A_k^-$ and

$$(3) \quad h_k := \sum_{i=0}^{n_k-1} U^{-i} e_k - U^{-n_k} \sum_{i=0}^{n_k-1} U^{-i} e_k, \quad h := \sum_{k=1}^{+\infty} h_k.$$

Since $\mu\{h_k \neq 0\} \leq \frac{1}{n_k}$, the function h is almost everywhere well-defined (by Borel-Cantelli's lemma).

It will be useful to express, for $N \geq n_k$, the sum $S_N(h_k)$ as a linear combination of $U^j e_k$. Denote $s_k := \sum_{j=0}^{n_k-1} U^{-j} e_k$. As $N \geq n_k$ and $h_k = s_k - U^{-n_k} s_k$, we have

$$\begin{aligned} S_N(h_k) &= \sum_{j=0}^{N-1} (U^j s_k - U^{j-n_k} s_k) \\ &= \sum_{j=0}^{N-1} U^j s_k - \sum_{j=-n_k}^{N-n_k-1} U^j s_k \\ &= - \sum_{j=-n_k}^{-1} U^j s_k + U^N \sum_{j=-n_k}^{-1} U^j s_k. \end{aligned}$$

We also have

$$(4) \quad \begin{aligned} \sum_{j=-n_k}^{-1} U^j s_k &= \sum_{j=1}^{n_k} U^{-j} s_k \\ &= U^{-1} \sum_{i=0}^{n_k-1} \sum_{j=0}^{n_k-1} U^{-(i+j)} e_k \\ \sum_{j=-n_k}^{-1} U^j s_k &= U^{-2n_k} \left(\sum_{j=1}^{n_k} j U^j e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{n_k+j} e_k \right). \end{aligned}$$

The previous equations yield

$$(5) \quad \begin{aligned} S_N(h_k) &= \sum_{j=1}^{n_k} j U^{j+N-2n_k} e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{j+N-n_k} e_k \\ &\quad - \sum_{j=1}^{n_k} j U^{j-2n_k} e_k - \sum_{j=1}^{n_k-1} (n_k - j) U^{j-n_k} e_k. \end{aligned}$$

Each h_k is a coboundary, as if we define $v_k := \sum_{j=0}^{n_k-1} U^{-j} s_k$, then $v_k - U^{-1}v_k =$

$s_k - U^{-n_k} s_k = h_k$ (so in this case the transfer function is $-U^{-1}v_k$).

Since $\mu\{v_k \neq 0\} \leq \frac{2}{n_k}$, Borel-Cantelli's lemma shows that the function $g := -\sum_{k=1}^{+\infty} U^{-1}v_k$ is almost everywhere well defined provided that $\sum_k \frac{1}{n_k}$ is convergent.

Because $h = g - Ug$, h is a coboundary.

2.2. Mixing rates. We show that the process $(U^i f)_{i \in \mathbb{Z}}$ is β -mixing. In doing so we use the following proposition (cf. [3]).

Proposition 4. *Let $(X_{k,i})_i$, $k = 1, 2, \dots$ be mutually independent strictly stationary processes with respective mixing coefficients $\beta_k(n)$, let $X_i = \sum_{k=1}^{\infty} X_{k,i}$ converge. The process $(X_i)_i$ is strictly stationary with mixing coefficients $\beta(n) \leq \sum_{k=1}^{\infty} \beta_k(n)$.*

This reduces the proof of β -mixing of $(U^i f)_{i \in \mathbb{Z}}$ to those of $(U^i h)_{i \in \mathbb{Z}}$.

In the following text we denote by $\beta_k(n)$ the mixing coefficients of the process $(h_k \circ T^i)_{i \in \mathbb{Z}}$.

Lemma 5. *For $k \geq 1$, we have the estimate $\beta_k(0) < \frac{1}{n_k}$.*

Proof. Let us estimate $\beta_k(0)$.

Denote \mathcal{F}_n^∞ by \mathcal{F}^n and $\mathcal{F}_{-\infty}^n$ by \mathcal{F}_n . The σ -algebras \mathcal{F}_0 and \mathcal{F}^{2n_k} are independent.

Let $\mathcal{P} = \{P_i : i \in I\}$ be the partition generated by $T^{-i}A'_k, T^{-i}A''_k, 0 \leq i \leq 2n_k - 1$ (\mathcal{P} has at most 3^{2n_k} elements). Any element $B \in \mathcal{F}^0$ can thus be represented as a union

$$B = \bigcup_{i \in I} (P_i \cap B_i)$$

with $B_i \in \mathcal{F}^{2n_k}$, $i \in I$.

Because $\mu(A'_k), \mu(A''_k) \leq 1/(2n_k^2)$, among the sets P_i there is one, denoted $P_0 = \Omega \setminus \bigcup_{i=0}^{2n_k-1} (T^{-i}A'_k \cup T^{-i}A''_k)$, the measure of which is greater than $1 - \frac{1}{n_k}$.

Notice that

$$B \Delta B_0 \subset P_0^c$$

and for any two measurable sets C, D it is $|\mu(C) - \mu(D)| \leq \mu(C \Delta D)$.

Let $\{B^1, \dots, B^n\}$ be an \mathcal{F}^0 -measurable partition of Ω . For $1 \leq i < j \leq n$ we have

$$(6) \quad B^i = \bigcup_{l \in I} (P_l \cap B_l^i), \quad B^j = \bigcup_{l \in I} (P_l \cap B_l^j).$$

From $B^i \cap B^j = \emptyset$ it follows $P_l \cap B_l^i \cap B_l^j = \emptyset$; we have $B_l^i, B_l^j \in \mathcal{F}^{2n_k}$ and $P_l \in \mathcal{F}_{2n_k-1}$ hence $\mu(P_l \cap B_l^i \cap B_l^j) = \mu(P_l)\mu(B_l^i \cap B_l^j)$ hence $\mu(B_l^i \cap B_l^j) = 0$.

Let $A \in \mathcal{F}_0$, $\{B^1, \dots, B^n\}$ be an \mathcal{F}^0 -measurable partition of Ω . We can estimate

$$\sum_{i=1}^n |\mu(A \cap B^i) - \mu(A \cap B_0^i)| \leq \sum_{i=1}^n \mu(A \cap (B^i \Delta B_0^i)) \leq \mu(A \cap P_0^c).$$

For an \mathcal{F}_0 -measurable partition $\{A^1, \dots, A^m\}$ of Ω we thus get

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m |\mu(A^j \cap B^i) - \mu(A^j)\mu(B^i)| &\leq \sum_{i=1}^n \sum_{j=1}^m |\mu(A^j \cap B^i) - \mu(A^j \cap B_0^i)| \\ &+ \sum_{i=1}^n \sum_{j=1}^m |\mu(A^j \cap B_0^i) - \mu(A^j)\mu(B_0^i)| \\ &+ \sum_{i=1}^n \sum_{j=1}^m |\mu(A^j)\mu(B_0^i) - \mu(A^j)\mu(B^i)| \end{aligned}$$

By independence, the sum in the middle is zero and by the preceding calculation,

$$\sum_{i=1}^n \sum_{j=1}^m |\mu(A^j \cap B^i) - \mu(A^j \cap B_0^i)| \leq \sum_{j=1}^m \mu(A^j \cap (P_0^c)) \leq \mu(P_0^c)$$

and

$$\sum_{i=1}^n \sum_{j=1}^m |\mu(A^j)\mu(B_0^i) - \mu(A^j)\mu(B^i)| \leq \sum_{j=1}^m \mu(A^j)\mu(P_0^c) \leq \mu(P_0^c)$$

hence

$$\sum_{i=1}^n \sum_{j=1}^m |\mu(A^j \cap B^i) - \mu(A^j)\mu(B^i)| \leq 2\mu(P_0^c) \leq \frac{2}{n_k}.$$

□

Denoting by $\beta(N)$ the mixing coefficients of the sequence $(h \circ T^i)_{i \in \mathbb{Z}}$, Proposition 4, Lemma 5 and $\beta_k(N) = 0$ when $N > 2n_k$ yield

Corollary 6. *For each integer k , we have*

$$(7) \quad \beta(N) \leq \sum_{j \geq 1} \beta_j(N) = \sum_{j: 2n_j \geq N} \frac{1}{n_j}$$

Now we can prove d) and A'. Let $i(N)$ denote the unique integer such that $n_{i(N)} \leq N < n_{i(N)+1}$ and for sequences $(u_N)_{N \geq 1}$, $(v_N)_{N \geq 1}$ of positive numbers, $u_N \lesssim v_N$ means that there is $C > 0$ such that for each N , $u_N \leq C \cdot v_N$.

Proposition 7. *Let $\delta > 0$. With the choice $n_k := \lfloor 16^{(2+\delta)^k} \rfloor$, we have d).*

Proof. We deduce from (7)

$$\beta(2n_k) \leq \sum_{j \geq k} \frac{1}{n_j} \lesssim \sum_{j \geq 0} 16^{-(2+\delta)^k (2+\delta)^j} \lesssim 16^{-(2+\delta)^k}.$$

Consequently,

$$\beta(2N) \leq \beta(2n_{i(N)}) \lesssim \frac{1}{n_{i(N)}} \lesssim \frac{1}{n_{\frac{1}{2+\delta} i(N)+1}} \lesssim \frac{1}{N^{\frac{1}{2+\delta}}}.$$

□

Hence d) is fulfilled.

Proposition 8. *Given $(c_k)_{k \geq 1}$ as in Theorem A', we can choose $n_k \uparrow \infty$ such that A' holds.*

Proof. We can, without loss of generality, suppose that the sequence (c_n) is decreasing. For $h_0 = e_0$ the process $(h_0 \circ T^i)$ is a sequence of i.i.d. hence $\beta_0(n) = 0$ for all $n \geq 0$.

Using Lemma 5 we find n_1 such that

$$\beta_1(0) \leq \frac{1}{n_1} \leq \frac{c_0}{2}.$$

Then, in the same way, we find n_2 such that

$$\beta_2(n_1) \leq \frac{1}{n_2} \leq \frac{c_{n_1}}{4}$$

and by induction we get a sequence of n_k such that

$$\beta_{k+1}(n_k) \leq \frac{c_{n_k}}{2^{k+1}}, \quad k \geq 0.$$

Indeed, we choose $n_{k+1} \geq \frac{2^{k+1}}{n_k}$.

For each $k \geq 1$, we have $\beta_k(N) = 0$ as soon as $N > 2n_{k-1}$ hence

$$(8) \quad \beta(2N) \leq \sum_{k=0}^{\infty} \beta_k(N) = \sum_{k: N \leq n_k} \beta_{k+1}(n_k) = \sum_{k: N \leq n_k} \frac{c_{n_k}}{2^{k+1}} \leq \frac{c_{n_j}}{2^j}$$

where $j = \min\{k : n_k \geq N\}$. In particular we get $\beta(2n_k) \leq c_{n_k}$. \square

This proves A'.

Remark 9. If $u: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function and if we choose at each step $n_{k+1} := \max\left\{\frac{2^{k+1}}{n_k}, u(n_k)\right\}$, we have $\beta_X(n_k) \leq c_{n_k}$ for each k .

2.3. Proof of non-tightness. For a positive integer m , we denote $[m] := \{1, \dots, m\}$. If S is a finite subset of the set of integers, $|S|$ denotes the number of elements of S .

Lemma 10. *There exists N_0 such that*

$$(9) \quad \mu \left\{ \max_{2n_k \leq N \leq n_k^2 - 1} |S_N(h_k)| = n_k \right\} > 1/4$$

whenever $n_k \geq N_0$.

Proof. For $2n_k \leq N \leq n_k^2$, thanks to (5), we have

$$\{|S_N(h_k)| = n_k\} \supset \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j \in I} \{U^j e_k = 0\} \cap \bigcap_{j \in J_N} \{U^j e_k = 0\},$$

where $I = [1-2n_k, -1] \cap \mathbb{Z}$ and $J_N = ([N-2n_k+1, N-1-n_k] \cup [N+1-n_k, N-1]) \cap \mathbb{Z}$. We define

$$\begin{aligned} B_{N,k} &:= \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j=1-2n_k}^{-1-n_k} \{U^{N+j} e_k = 0\} \cap \bigcap_{j=1-n_k}^{-1} \{U^{j+N} e_k = 0\} \\ &= \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j \in J_N} \{U^j e_k = 0\}. \end{aligned}$$

We have $|S_N(h_k)| = n_k$ on $\bigcap_{j \in I} \{U^j e_k = 0\} \cap B_{N,k}$ and the sets $\bigcap_{j \in I} \{U^j e_k = 0\}$, $\bigcup_{N=2n_k}^{n_k^2} B_{N,k}$ belong to independent σ -algebras. Therefore

$$(10) \quad \mu \left\{ \max_{N \in [n_k^2 - 1]} |S_N(h_k)| = n_k \right\} \geq \left(1 - \frac{1}{n_k^2}\right)^{2n_k} \mu \left(\bigcup_{N=2n_k}^{n_k^2} B_{N,k} \right).$$

Recall Bonferroni's inequality, which states that for any integer n and any events $A_j, j \in [n]$, we have

$$(11) \quad \mu \left(\bigcup_{j=1}^n A_j \right) \geq \sum_{j=1}^n \mu(A_j) - \sum_{1 \leq i < j \leq n} \mu(A_i \cap A_j).$$

It can be proved by induction. Notice that

$$\mu(B_{N,k}) = \frac{1}{n_k^2} \left(1 - \frac{1}{n_k^2}\right)^{2n_k - 2} \geq \frac{1}{n_k^2} \left(1 - \frac{2}{n_k}\right)$$

and for $i \neq j$

$$\mu(B_{i+2n_k-1,k} \cap B_{j+2n_k-1,k}) \leq \mu \{ |U^{i+n_k-1} e_k| = 1 \} \mu \{ |U^{j+n_k-1} e_k| = 1 \} = \frac{1}{n_k^4}$$

hence

$$\begin{aligned} \mu \left(\bigcup_{N=2n_k}^{n_k^2} B_{N,k} \right) &= \mu \left(\bigcup_{N=1}^{(n_k-1)^2} B_{N+2n_k-1,k} \right) \\ &\geq \sum_{N=1}^{(n_k-1)^2} \frac{1}{n_k^2} \left(1 - \frac{2}{n_k}\right) - \sum_{1 \leq i < j \leq (n_k-1)^2} \mu(B_{i+2n_k-1,k} \cap B_{j+2n_k-1,k}) \\ &\geq \left(1 - \frac{2}{n_k}\right)^3 - \frac{1}{2} \end{aligned}$$

which together with (10) implies that

$$(12) \quad \mu \left\{ \max_{N \in [n_k^2 - 1]} |S_N(h_k)| = n_k \right\} \geq \frac{1}{4},$$

whenever $n_k \geq N_0$, where N_0 is such that $(1 - \frac{2}{n})^3 > \frac{3}{4}$ for $n \geq N_0$. \square

Lemma 11. *Assume that the sequence $(n_k)_{k \geq 1}$ satisfies the following two conditions of lacunarity:*

$$(13) \quad \text{for each } k, \quad 16 \sum_{j=1}^k n_j^2 \leq n_{k+1};$$

$$(14) \quad \text{for each } k, \quad n_{k+1} \geq k^2 n_k.$$

Then we have for k large enough

$$(15) \quad \mu \left\{ \frac{1}{n_k} \max_{N \in [n_k^2 - 1]} |S_N(h)| \geq 1/2 \right\} \geq 1/8.$$

Proof. Fix an integer k . For each $j < k$ and $2n_k \leq N \leq n_k^2 - 1$, we have

$$(16) \quad \left| \frac{1}{n_k} \sum_{j=1}^{k-1} S_N(h_j) \right| \leq \frac{2}{n_k} \sum_{j=1}^{k-1} (n_j + 1)^2 \leq \frac{1}{2}.$$

Indeed, using (5), we can give an upper bound of $S_N(h_j)$ (as $N \geq 2n_k > 2n_j$) as

$$|S_N(h_j)| \leq 2 \sum_{l=1}^{n_j} l + 2 \sum_{l=1}^{n_j-1} l \leq 2(n_j + 1)^2,$$

and we conclude by (13).

Now fix $j > k$. Writing $S_N(h_j) = \sum_{i=0}^{N-1} U^i s_j - \sum_{i=-n_j}^{N-n_j-1} U^i s_j$, where $s_j := \sum_{i=0}^{n_j-1} U^{-i} e_j$,

we can see that the support of $S_N(h_j)$ is contained in $\bigcup_{i=0}^{n_k^2-1} T^{-i} A_j \cup \bigcup_{i=-n_j}^{n_k^2-n_j-1} T^{-i} A_j =$

$\bigcup_{i=-n_j}^{n_k^2-1} T^{-i} A_j$, hence using (14)

$$(17) \quad \mu \left(\bigcup_{N=2n_k}^{n_k^2-1} \{S_N(h_j) \neq 0\} \right) \leq \frac{n_k^2 + n_j}{n_j^2} \leq \frac{2n_k}{n_j} < 2j^{-2}.$$

Let $E_k := \bigcup_{N=2n_k}^{n_k^2-1} \bigcup_{j \geq k+1} \{S_N(h_j) \neq 0\}$. By (17), we have

$$(18) \quad \mu(E_k) \leq \sum_{j \geq k+1} 2j^{-2} \leq 2/k.$$

By (16),

$$\frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left(\sum_{j \geq k} h_j \right) \right| - \frac{1}{2},$$

hence

$$\begin{aligned} \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\} &\geq \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left(\sum_{j \geq k} h_j \right) \right| \geq 1 \right\} \\ &\geq \mu \left(\left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left(\sum_{j \geq k} h_j \right) \right| \geq 1 \right\} \cap E_k^c \right) \\ &= \mu \left(\left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \cap E_k^c \right) \\ &\geq \mu \left(\left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \right) - \mu(E_k). \end{aligned}$$

The result follows from Lemma 10, (17) and (18). \square

The previous lemma yields together with Theorems 8.1 and 15.1 of [2] and the convergence to 0 of the finite dimensional distributions of $(N^{-1/2}S_N^*(h))_{N \geq 1}$ and $(N^{-1/2}S_N^{**}(h))_{N \geq 1}$ the following corollary.

Corollary 12. *If $(n_k)_{k \geq 1}$ satisfies (13) and (14), then the sequences $(N^{-1/2}S_N^*(h, \cdot))_{N \geq 1}$ and $(N^{-1/2}S_N^{**}(h, \cdot))_{N \geq 1}$ are not tight in their respective spaces.*

Let $\delta > 0$. Then the choice $n_k := \lfloor 16^{(2+\delta)^k} \rfloor$ satisfies the conditions (13) and (14). Indeed, the inequalities $n_{k+1} \geq n_k^{2+\delta} - 1 = n_k^2 n_k^\delta - 1$ and $n_k^\delta \geq 16k$ for each k give (13) and (14) is satisfied because $n_k^{1+\delta} \geq k^2$. Hence b) is satisfied. By Remark 9, we can construct in Proposition 8 the sequence $(n_k)_{k \geq 1}$ in order to satisfy (13) and (14), this shows b) in Theorem A'.

Let us denote by σ_N^2 the variance of $S_N(h)$, that is, $\mathbb{E}[S_N(h)^2]$.

Proposition 13. *Under the conditions (13) and (14), we have $\sigma_N^2 \lesssim N$.*

Proof. From (5) and (13), we deduce that

$$(19) \quad \sum_{j=1}^{i(N)} \mathbb{E}[S_N(h_j)^2] \lesssim \sum_{j=1}^{i(N)} n_j \leq 2n_{i(N)} \lesssim N.$$

Recall that $h_k = (I - U^{-n_k})s_k$ with $s_k := \sum_{i=0}^{n_k-1} U^{-i}e_k$. Therefore, when $n_k \geq N$, we have by a similar computation as for (4),

$$S_N(h_k) = (I - U^{-n_k}) \left(\sum_{j=1}^{N-1} j(U^{j-1-N} + U^{n_k+1-j})e_k \right) + N(I - U^{-n_k}) \left(\sum_{j=N}^{n_k-1} U^{j-N}e_k \right).$$

The first term has a variance of order $\frac{N^3}{n_k^2}$ while those of the second one is $\frac{(n_k-N)N^2}{n_k}$.

We thus that for $n_k \geq N$, $\mathbb{E}[S_N(h_k)^2] \lesssim \frac{N^2}{n_k}$, hence by (14),

$$(20) \quad \sum_{k \geq i(N)+1} \mathbb{E}[S_N(h_k)^2] \lesssim \sum_{k \geq i(N)+1} \frac{N^2}{n_k} \leq N + N^2 \sum_{j \geq i(N)+1} \frac{1}{j^2 n_j} \lesssim N.$$

Combining (19) and (20), and using for a fixed N the independence of the sequence $(S_N(h_j))_{j \geq 1}$, we conclude the result. \square

When we add an independent sequence $(m \circ T^i)_{i \in \mathbb{Z}}$ with moments of any order, the variance of $((m+h) \circ T^i)_{i \geq 1}$ is bounded above and below by a quantity proportional to N , hence c) is satisfied in Theorems A and A'.

2.4. Moments of the coboundary and the transfer function. One can wonder to which \mathbb{L}^p space can g and $g - g \circ T$ belong.

Proposition 14. *Under the conditions (13) and (14), we have $g \in \mathbb{L}^p$ for $0 < p < 1$ and $g - g \circ T \in \mathbb{L}^p$ for each $p > 0$.*

Proof. Let $g_k := U^{-1}v_k$, where $v_k = \sum_{j=0}^{n_k-1} U^{-j}s_k$ and $s_k = \sum_{j=0}^{n_k-1} U^{-j}e_k$. Recall that $g = -\sum_{k=1}^{\infty} g_k$. For $0 < p < 1$ and any two non-negative real numbers a and b , we

have $(a + b)^p \leq a^p + b^p$. This gives, using (4),

$$\begin{aligned} \mathbb{E} |g_k|^p &\leq \left(\sum_{j=1}^{n_k} j^p \mathbb{E} |U^{-j} e_k| + \sum_{j=1}^{n_k-1} (n_k - j)^p \mathbb{E} |U^{-j+n_k} e_k| \right) \\ &= \frac{1}{n_k^2} \left(n_k^p + 2 \sum_{j=1}^{n_k-1} j^p \right) \\ &\leq \frac{1}{n_k^2} (n_k^p + 2n_k^{p+1}) \\ &\leq 3n_k^{p-1}. \end{aligned}$$

By (14), we have $n_k \geq k! \cdot n_1$ hence the series $\sum_{k \geq 1} \mathbb{E} |g_k|^p$ is convergent. This proves that $g \in \mathbb{L}^p$ for $0 < p < 1$.

Corollary 2.4. in [1] states the following: given positive integers t and p , X_1, \dots, X_t independent random variables such that $\mu \{0 \leq X_j \leq 1\} = 1$ for each $j \in [t]$, then

$$(21) \quad \mathbb{E} (X)^p \leq B_p \cdot \max \{ \mathbb{E}(X), (\mathbb{E}(X))^p \},$$

where B_p is the p -th Bell's number (defined by the recursion relation $B_{p+1} = \sum_{k=0}^p \binom{p}{k} B_k$ and $B_0 = B_1 = 1$) and $X := \sum_{j=1}^t X_j$.

We shall show that the series $\sum_{k \geq 1} \|h_k\|_p$ is convergent for any integer p . Fix $k \geq 1$, and let $t := 2n_k$, $X_j := |U^{j-2n_k} e_k|$. Applying (21), we get

$$\|h_k\|_p^p \leq B_p \cdot \max \{ n_k^{-1}, n_k^{-p} \} = B_p \cdot n_k^{-1},$$

hence $\|h_k\|_p \leq B_p^{1/p} \cdot 2^{-\frac{k^2}{2p}}$. One could also use Rosenthal's inequality. \square

Since the added process has moments of any order, Proposition 14 proves e) in Theorems A and A'.

Proposition 15. *The transfer function g does not belong to \mathbb{L}^1 .*

Proof. Fix an integer k , and define for $1 \leq j \leq n_k$:

$$E_j := \{ |U^{-j-2n_k} e_k| = 1 \} \cap \bigcap_{i \in [2n_k-1] \setminus \{j\}} \{ U^{-i-2n_k} e_k = 0 \} \cap \bigcap_{l \neq k} \{ g_l = 0 \}.$$

Since these sets are pairwise disjoint and $g = -U^{-1} \sum_{k \geq 1} \sum_{j=0}^{n_k-1} U^{-j} s_k$, with $s_k := \sum_{j=0}^{n_k-1} U^{-j} e_k$, we have after having rearranged $\sum_{j=0}^{n_k-1} U^{-j} s_k$ and used the fact that

$g_l(\omega) = 0$ if $l \neq k$ and $\omega \in \bigcup_{j=1}^{n_k} E_j$:

$$\begin{aligned}
\left\| g\chi_{\bigcup_{j=1}^{n_k} E_j} \right\|_1 &= \left\| \sum_{j=1}^{n_k} \chi_{E_j} \left\| \sum_{i=1}^{n_k} iU^{-i-2n_k} e_k \right\| \right\|_1 \\
&= \sum_{j=1}^{n_k} \left\| \chi_{E_j} \left\| \sum_{i=1}^{n_k} iU^{-i-2n_k} e_k \right\| \right\|_1 \\
&= \sum_{j=1}^{n_k} j\mu(E_j) \\
&= \sum_{j=1}^{n_k} j \cdot \frac{1}{n_k^2} \left(1 - \frac{1}{n_k^2}\right)^{2n_k-2} \prod_{j \neq k} \left(1 - \frac{1}{n_j^2}\right)^{2n_j-1} \\
&\geq \frac{1}{2} \left(1 - \frac{1}{n_k^2}\right)^{-1} \prod_{j \geq 1} \left(1 - \frac{1}{n_j^2}\right)^{2n_j-1}.
\end{aligned}$$

As $\left(1 - \frac{1}{n_k^2}\right)^{-1} \rightarrow 1$ as $k \rightarrow +\infty$ and $\prod_{j \geq 1} \left(1 - \frac{1}{n_j^2}\right)^{2n_j-1}$ is positive, we get $\|g\chi_{F_k}\|_1 \geq c > 0$, where $F_k := \bigcup_{j=1}^{n_k} E_j$. Since $g\chi_{F_k} = g_k\chi_{F_k}$, we get that $\|g_k\|_1 > c$ for a c independent of k . Therefore, the series of independent random variables $\sum_{k \geq 1} g_k$ cannot be convergent in \mathbb{L}^1 . \square

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