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# A STRICTLY STATIONARY $\beta$ -MIXING PROCESS SATISFYING THE CENTRAL LIMIT THEOREM BUT NOT THE WEAK INVARIANCE PRINCIPLE

DAVIDE GIRAUDD AND DALIBOR VOLNÝ

ABSTRACT. In 1983, N. Herrndorf proved that for a  $\phi$ -mixing sequence satisfying the central limit theorem and  $\liminf_{n \rightarrow \infty} \frac{\sigma_n^2}{n} > 0$ , the weak invariance principle takes place. The question whether for strictly stationary sequences with finite second moments and a weaker type  $(\alpha, \beta, \rho)$  of mixing the central limit theorem implies the weak invariance principle remained open.

We construct a strictly stationary  $\beta$ -mixing sequence with finite moments of any order and linear variance for which the central limit theorem takes place but not the weak invariance principle.

## 1. INTRODUCTION AND NOTATIONS

Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space. If  $T: \Omega \rightarrow \Omega$  is one-to-one, bi-measurable and measure preserving (in sense that  $\mu(T^{-1}(A)) = \mu(A)$  for all  $A \in \mathcal{F}$ ), then the sequence  $(f \circ T^k)_{k \in \mathbb{Z}}$  is strictly stationary for any measurable  $f: \Omega \rightarrow \mathbb{R}$ . Conversely, each strictly stationary sequence can be represented in this way.

For a zero mean square integrable  $f: \Omega \rightarrow \mathbb{R}$ , we define  $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$ ,

$\sigma_n^2(f) := \mathbb{E}(S_n(f)^2)$  and  $S_n^*(f, t) := S_{[nt]}(f) + (nt - [nt])f \circ T^{[nt]}$ , where  $[x]$  is the greatest integer which is smaller than  $x$ .

We say that  $(f \circ T^j)_{j \geq 1}$  satisfies the *central limit theorem with normalization*  $a_n$  if the sequence  $(a_n^{-1} S_n(f))_{n \geq 1}$  converges weakly to a standard normal distribution. Let  $C[0, 1]$  denote the space of continuous functions on the unit interval endowed with the norm  $\|g\|_\infty := \sup_{t \in [0, 1]} |g(t)|$ .

Let  $D[0, 1]$  be the space of real valued functions which have left limits and are continuous at each point of  $[0, 1]$ . We endow it with Skorohod metric (cf. [2]). We define  $S_n^{**}(f, t) := S_{[nt]}(f)$ , which gives a random element of  $D[0, 1]$ .

We shall say that the strictly stationary sequence  $(f \circ T^j)_{j \geq 0}$  satisfies the *weak invariance principle in  $C[0, 1]$  with normalization  $a_n$*  (respectively in  $D[0, 1]$ ) if the sequence of  $C[0, 1]$  (of  $D[0, 1]$ ) valued random variables  $(a_n^{-1} S_n^*(f, \cdot))_{n \geq 1}$  (resp.  $(a_n^{-1} S_n^{**}(f, \cdot))_{n \geq 1}$ ) weakly converges to a Brownian motion process in the corresponding space.

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two sub- $\sigma$ -algebras of  $\mathcal{F}$ , where  $(\Omega, \mathcal{F}, \mu)$  is a probability space. We define the  $\alpha$ -mixing coefficients as introduced by Rosenblatt in [12]:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup \{ |\mu(A \cap B) - \mu(A)\mu(B)|, A \in \mathcal{A}, B \in \mathcal{B} \},$$

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$\beta$ -mixing coefficients by

$$\beta(\mathcal{A}, \mathcal{B}) := \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|,$$

where the supremum is taken over the finite partitions  $\{A_i, 1 \leq i \leq I\}$  and  $\{B_j, 1 \leq j \leq J\}$  of  $\Omega$  of elements of  $\mathcal{A}$  (respectively of  $\mathcal{B}$ ). They were introduced by Volkonskii and Rozanov [14].

The  $\rho$ -mixing coefficients were introduced by Kolmogorov and Rozanov [10] and are defined by

$$\rho(\mathcal{A}, \mathcal{B}) := \sup \{|\text{Corr}(f, g)|, f \in \mathbb{L}^2(\mathcal{A}), g \in \mathbb{L}^2(\mathcal{B}), f \neq 0, g \neq 0\},$$

where  $\text{Corr}(f, g) := \frac{\mathbb{E}(fg) - \mathbb{E}(f)\mathbb{E}(g)}{\|f\|_{\mathbb{L}^2} \|g\|_{\mathbb{L}^2}}$ .

Ibragimov [7] introduced for the first time  $\phi$ -mixing coefficients, which are given by the formula

$$\phi(\mathcal{A}, \mathcal{B}) := \sup \{|\mu(B | A) - \mu(B)|, A \in \mathcal{A}, B \in \mathcal{B}, \mu(A) > 0\}.$$

The coefficients are related by the inequalities

$$(1) \quad \alpha(\mathcal{A}, \mathcal{B}) \leq 2\beta(\mathcal{A}, \mathcal{B}) \leq 2\phi(\mathcal{A}, \mathcal{B}), \quad \alpha(\mathcal{A}, \mathcal{B}) \leq \rho(\mathcal{A}, \mathcal{B}) \leq \phi(\mathcal{A}, \mathcal{B}).$$

For a strictly stationary sequence  $(X_k)_{k \in \mathbb{Z}}$  and  $n \geq 0$  we define  $\alpha_X(n) = \alpha(n) = \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty)$  where  $\mathcal{F}_u^v$  is the  $\sigma$ -algebra generated by  $X_k$  with  $u \leq k \leq v$  (if  $u = -\infty$  or  $v = \infty$ , the corresponding inequality is strict). In the same way we define coefficients  $\beta_X(n)$ ,  $\rho_X(n)$ ,  $\phi_X(n)$ .

We say that the sequence  $(X_k)_{k \in \mathbb{Z}}$  is  $\alpha$ -mixing if  $\lim_{n \rightarrow +\infty} \alpha_X(n) = 0$ , and similarly we define  $\beta$ ,  $\rho$  and  $\phi$ -mixing sequences. Inequalities (1) give a hierarchy between these classes of mixing sequences.

If  $(a_N)_{N \geq 1}$  and  $(b_N)_{N \geq 1}$  are two sequences of positive real numbers, we write  $a_N \asymp b_N$  if there exists a positive constant  $C$  such that for each  $N$ ,  $C^{-1}a_N \leq b_N \leq Ca_N$ .

The main results are

**Theorem A.** *Let  $\delta$  be a positive real number. There exists a strictly stationary real valued process  $Y = (Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0}$  satisfying the following conditions:*

- a) *the central limit theorem with normalization  $\sqrt{n}$  takes place;*
- b) *the weak invariance principle with normalization  $\sqrt{n}$  does not hold;*
- c) *we have  $\sigma_N(f)^2 \asymp N$ ;*
- d) *we have for some positive  $C$ ,  $\beta_Y(N) \leq \frac{C}{N^{1/2-\delta}}$ ;*
- e)  *$Y_0 \in \mathbb{L}^p$  for any  $p > 0$ .*

Alternatively, we can construct the process in order to have a control of the mixing coefficients on a subsequence.

**Theorem A'.** *Let  $(c_j)_{j \geq 0}$  be a decreasing sequence of positive numbers. Then there exists a strictly stationary real valued process  $Y = (Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0}$  satisfying conditions a), b), c), e), and:*

- d') *there is an increasing sequence  $(n_k)_{k \geq 1}$  of integers such that for each  $k$ ,  $\beta_Y(n_k) \leq c_{n_k}$ .*

*Remark 1.* Herrndorf proved ([6], Theorem 2.13) that if  $(\xi_n)$  is a strictly stationary  $\phi$ -mixing sequence for which  $\sigma_n \rightarrow \infty$ ,  $S_n/\sigma_n$  converges in distribution to a standard normal distribution and  $\sigma_n^{-1} \max_{1 \leq i \leq n} |\xi_i| \rightarrow 0$  in probability, then the weak invariance principle takes place. So Herrndorf's result does not extend to  $\beta$ -mixing sequences.

*Remark 2.* Rio *et al.* proved in [11] that the condition  $\int_0^1 \alpha^{-1}(u) Q^2(u) du < \infty$  implies the weak invariance principle, where  $\alpha^{-1}(u) := \inf \{k, \alpha(k) \leq u\}$  and  $Q$  is the right-continuous inverse of the quantile function  $t \mapsto \mu \{X_0 > t\}$ . If the process is strictly stationary, with finite moments of order  $2+r$ ,  $r > 0$ , the latter condition is satisfied whenever  $\sum_{n=1}^{\infty} (n+1)^{\frac{2}{r}} \alpha(n) < \infty$  (Ibragimov [8] found the condition  $\sum_{n=1}^{\infty} \alpha(n)^{1-\frac{2}{r}} < \infty$ ). Since  $Y_0 \in \mathbb{L}^p$  for all  $p < \infty$ , we have that  $\sum_N \alpha(N)^r = +\infty$  for any  $r < 1$ , hence in A' we can thus hardly get such a bound for the whole sequence.

Also, the mixing rates in Theorem A are at most  $\frac{\varepsilon_N}{N}$ , where  $(\varepsilon_N)_{N \geq 1}$  is bounded. Even if we had such a rate in Theorem A, the result of Theorem A' couldn't be deduced from this. Indeed, taking  $c_j := j^{-2}$ , we would not be able to get  $\beta_X(n_k) \leq c_{n_k}$  from the corresponding variant of d).

*Remark 3.* Ibragimov proved that for a strictly stationary  $\rho$ -mixing sequence with finite moments of order  $2+\delta$  for some positive  $\delta$ , the weak invariance principle holds, cf. [9]. In particular, this proves that our construction does not give a  $\rho$ -mixing process. Shao also showed in [13] that the condition  $\sum_n \rho(2^n) < \infty$  is sufficient in order to guarantee the weak invariance principle in  $D[0, 1]$  for stationary sequences having order two moments. So a potential  $\rho$ -mixing counter-example has to obey to restrictions on the moments as well as on the mixing rates.

### About the method of proof

In proving the result we will use properties of coboundaries  $h = g - g \circ T$  ( $g$  is called a transfer function). For a positive integer  $N$  and a measurable function  $v$ , we denote  $S_N(v) := \sum_{j=0}^{N-1} U^j v$ . Because  $S_n(g - g \circ T) = g - g \circ T^n$ , for any sequence  $a_n \rightarrow \infty$  we have  $\frac{1}{a_n} S_n(g - g \circ T) \rightarrow 0$  in probability hence adding a coboundary does not change validity of the central limit theorem. If, moreover,  $g \in \mathbb{L}^2$  then  $\frac{1}{\sqrt{n}} \|S_n^*(g - g \circ T)\|_{\infty} \rightarrow 0$  a.s. hence adding of such coboundary does not change validity of the invariance principle (if norming by  $\sqrt{n}$  or  $\sigma_n$  with  $\liminf_n \frac{\sigma_n^2}{n} > 0$ ), cf. [5]. On the other hand, if  $g \notin \mathbb{L}^2$ , adding a coboundary can spoil tightness even if  $g - g \circ T$  is square integrable, cf. [15]. A similar idea was used in [4]. In the proof of Theorem A and A' we will find a coboundary  $g - g \circ T$  which is  $\beta$ -mixing and spoils tightness. The coboundary has all finite moments but the transfer function is not integrable. We then add an  $m$  such that  $(m \circ T^i)_{i \in \mathbb{Z}}$  and  $(h \circ T^i)_{i \in \mathbb{Z}}$  are independent (enlarging the probability space), and  $m \circ T^i$  is i.i.d. with moments of any order (in particular, it satisfies the weak invariance principle).

In the proof it is used the fact that  $|\mu(A \cap B) - \mu(A)\mu(B)| \leq \mu(A)$ . The method does not seem to apply to processes which are  $\rho$ -mixing and for this kind of processes the problem remains open.

## 2. PROOF

**2.1. Construction of  $h$ .** Let us consider an increasing sequence of positive integers  $(n_k)_{k \geq 1}$ , and for each integer  $k \geq 1$  let  $A_k^-, A_k^+$  be measurable sets such that  $\mu(A_k^-) = \frac{1}{2n_k^2} = \mu(A_k^+)$ .

We define random variables  $e_k$  by

$$(2) \quad e_k(\omega) := \begin{cases} 1 & \text{if } \omega \in A_k^+, \\ -1 & \text{if } \omega \in A_k^-, \\ 0 & \text{otherwise,} \end{cases}$$

such that  $U^i e_k = e_k \circ T^i$  are mutually independent. We define  $A_k := A_k^+ \cup A_k^-$  and

$$(3) \quad h_k := \sum_{i=0}^{n_k-1} U^{-i} e_k - U^{-n_k} \sum_{i=0}^{n_k-1} U^{-i} e_k, \quad h := \sum_{k=1}^{+\infty} h_k.$$

Since  $\mu\{h_k \neq 0\} \leq \frac{1}{n_k}$ , the function  $h$  is almost everywhere well-defined (by Borel-Cantelli's lemma).

It will be useful to express, for  $N \geq n_k$ , the sum  $S_N(h_k)$  as a linear combination of  $U^j e_k$ . Denote  $s_k := \sum_{j=0}^{n_k-1} U^{-j} e_k$ . As  $N \geq n_k$  and  $h_k = s_k - U^{-n_k} s_k$ , we have

$$\begin{aligned} S_N(h_k) &= \sum_{j=0}^{N-1} (U^j s_k - U^{j-n_k} s_k) \\ &= \sum_{j=0}^{N-1} U^j s_k - \sum_{j=-n_k}^{N-n_k-1} U^j s_k \\ &= - \sum_{j=-n_k}^{-1} U^j s_k + U^N \sum_{j=-n_k}^{-1} U^j s_k. \end{aligned}$$

We also have

$$(4) \quad \begin{aligned} \sum_{j=-n_k}^{-1} U^j s_k &= \sum_{j=1}^{n_k} U^{-j} s_k \\ &= U^{-1} \sum_{i=0}^{n_k-1} \sum_{j=0}^{n_k-1} U^{-(i+j)} e_k \\ \sum_{j=-n_k}^{-1} U^j s_k &= U^{-2n_k} \left( \sum_{j=1}^{n_k} j U^j e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{n_k+j} e_k \right). \end{aligned}$$

The previous equations yield

$$(5) \quad \begin{aligned} S_N(h_k) &= \sum_{j=1}^{n_k} j U^{j+N-2n_k} e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{j+N-n_k} e_k \\ &\quad - \sum_{j=1}^{n_k} j U^{j-2n_k} e_k - \sum_{j=1}^{n_k-1} (n_k - j) U^{j-n_k} e_k. \end{aligned}$$

Each  $h_k$  is a coboundary, as if we define  $v_k := \sum_{j=0}^{n_k-1} U^{-j} s_k$ , then  $v_k - U^{-1}v_k =$

$s_k - U^{-n_k} s_k = h_k$  (so in this case the transfer function is  $-U^{-1}v_k$ ).

Since  $\mu\{v_k \neq 0\} \leq \frac{2}{n_k}$ , Borel-Cantelli's lemma shows that the function  $g :=$   
 $-\sum_{k=1}^{+\infty} U^{-1}v_k$  is almost everywhere well defined provided that  $\sum_k \frac{1}{n_k}$  is convergent.

Because  $h = g - Ug$ ,  $h$  is a coboundary.

**2.2. Mixing rates.** We show that the process  $(U^i f)_{i \in \mathbb{Z}}$  is  $\beta$ -mixing. In doing so we use the following proposition (cf. [3]).

**Proposition 4.** *Let  $(X_{k,i})_i, k = 1, 2, \dots$  be mutually independent strictly stationary processes with respective mixing coefficients  $\beta_k(n)$ , let  $X_i = \sum_{k=1}^{\infty} X_{k,i}$  converge. The process  $(X_i)_i$  is strictly stationary with mixing coefficients  $\beta(n) \leq \sum_{k=1}^{\infty} \beta_k(n)$ .*

This reduces the proof of  $\beta$ -mixing of  $(U^i f)_{i \in \mathbb{Z}}$  to those of  $(U^i h)_{i \in \mathbb{Z}}$ .

In the following text we denote by  $\beta_k(n)$  the mixing coefficients of the process  $(h_k \circ T^i)_{i \in \mathbb{Z}}$ .

**Lemma 5.** *For  $k \geq 1$ , we have the estimate  $\beta_k(0) < \frac{1}{n_k}$ .*

*Proof.* Let us estimate  $\beta_k(0)$ .

Denote  $\mathcal{F}_n^\infty$  by  $\mathcal{F}^n$  and  $\mathcal{F}_{-\infty}^n$  by  $\mathcal{F}_n$ . The  $\sigma$ -algebras  $\mathcal{F}_0$  and  $\mathcal{F}^{2n_k}$  are independent.

Let  $\mathcal{P} = \{P_i : i \in I\}$  be the partition generated by  $T^{-i}A'_k, T^{-i}A''_k, 0 \leq i \leq 2n_k - 1$  ( $\mathcal{P}$  has at most  $3^{2n_k}$  elements). Any element  $B \in \mathcal{F}^0$  can thus be represented as a union

$$B = \bigcup_{i \in I} (P_i \cap B_i)$$

with  $B_i \in \mathcal{F}^{2n_k}, i \in I$ .

Because  $\mu(A'_k), \mu(A''_k) \leq 1/(2n_k^2)$ , among the sets  $P_i$  there is one, denoted  $P_0 = \Omega \setminus \bigcup_{i=0}^{2n_k-1} (T^{-i}A'_k \cup T^{-i}A''_k)$ , the measure of which is greater than  $1 - \frac{1}{n_k}$ .

Notice that

$$B \Delta B_0 \subset P_0^c$$

and for any two measurable sets  $C, D$  it is  $|\mu(C) - \mu(D)| \leq \mu(C \Delta D)$ .

Let  $\{B^1, \dots, B^n\}$  be an  $\mathcal{F}^0$ -measurable partition of  $\Omega$ . For  $1 \leq i < j \leq n$  we have

$$(6) \quad B^i = \bigcup_{l \in I} (P_l \cap B_l^i), \quad B^j = \bigcup_{l \in I} (P_l \cap B_l^j).$$

From  $B^i \cap B^j = \emptyset$  it follows  $P_l \cap B_l^i \cap B_l^j = \emptyset$ ; we have  $B_l^i, B_l^j \in \mathcal{F}^{2n_k}$  and  $P_l \in \mathcal{F}_{2n_k-1}$  hence  $\mu(P_l \cap B_l^i \cap B_l^j) = \mu(P_l)\mu(B_l^i \cap B_l^j)$  hence  $\mu(B_l^i \cap B_l^j) = 0$ .

Let  $A \in \mathcal{F}_0, \{B^1, \dots, B^n\}$  be an  $\mathcal{F}^0$ -measurable partition of  $\Omega$ . We can estimate

$$\sum_{i=1}^n |\mu(A \cap B^i) - \mu(A \cap B_0^i)| \leq \sum_{i=1}^n \mu(A \cap (B^i \Delta B_0^i)) \leq \mu(A \cap P_0^c).$$

For an  $\mathcal{F}_0$ -measurable partition  $\{A^1, \dots, A^m\}$  of  $\Omega$  we thus get

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m |\mu(A^j \cap B^i) - \mu(A^j)\mu(B^i)| &\leq \sum_{i=1}^n \sum_{j=1}^m |\mu(A^j \cap B^i) - \mu(A^j \cap B_0^i)| \\ &+ \sum_{i=1}^n \sum_{j=1}^m |\mu(A^j \cap B_0^i) - \mu(A^j)\mu(B_0^i)| \\ &+ \sum_{i=1}^n \sum_{j=1}^m |\mu(A^j)\mu(B_0^i) - \mu(A^j)\mu(B^i)| \end{aligned}$$

By independence, the sum in the middle is zero and by the preceding calculation,

$$\sum_{i=1}^n \sum_{j=1}^m |\mu(A^j \cap B^i) - \mu(A^j \cap B_0^i)| \leq \sum_{j=1}^m \mu(A^j \cap (P_0^c)) \leq \mu(P_0^c)$$

and

$$\sum_{i=1}^n \sum_{j=1}^m |\mu(A^j)\mu(B_0^i) - \mu(A^j)\mu(B^i)| \leq \sum_{j=1}^m \mu(A^j)\mu(P_0^c) \leq \mu(P_0^c)$$

hence

$$\sum_{i=1}^n \sum_{j=1}^m |\mu(A^j \cap B^i) - \mu(A^j)\mu(B^i)| \leq 2\mu(P_0^c) \leq \frac{2}{n_k}.$$

□

Denoting by  $\beta(N)$  the mixing coefficients of the sequence  $(h \circ T^i)_{i \in \mathbb{Z}}$ , Proposition 4, Lemma 5 and  $\beta_k(N) = 0$  when  $N > 2n_k$  yield

**Corollary 6.** *For each integer  $k$ , we have*

$$(7) \quad \beta(N) \leq \sum_{j \geq 1} \beta_j(N) = \sum_{j: 2n_j \geq N} \frac{1}{n_j}$$

Now we can prove d) and A'. Let  $i(N)$  denote the unique integer such that  $n_{i(N)} \leq N < n_{i(N)+1}$  and for sequences  $(u_N)_{N \geq 1}$ ,  $(v_N)_{N \geq 1}$  of positive numbers,  $u_N \lesssim v_N$  means that there is  $C > 0$  such that for each  $N$ ,  $u_N \leq C \cdot v_N$ .

**Proposition 7.** *Let  $\delta > 0$ . With the choice  $n_k := \lfloor 16^{(2+\delta)^k} \rfloor$ , we have d).*

*Proof.* We deduce from (7)

$$\beta(2n_k) \leq \sum_{j \geq k} \frac{1}{n_j} \lesssim \sum_{j \geq 0} 16^{-(2+\delta)^k (2+\delta)^j} \lesssim 16^{-(2+\delta)^k}.$$

Consequently,

$$\beta(2N) \leq \beta(2n_{i(N)}) \lesssim \frac{1}{n_{i(N)}} \lesssim \frac{1}{n_{\frac{1}{2+\delta} i(N)+1}} \lesssim \frac{1}{N^{\frac{1}{2+\delta}}}.$$

□

Hence d) is fulfilled.

**Proposition 8.** *Given  $(c_k)_{k \geq 1}$  as in Theorem A', we can choose  $n_k \uparrow \infty$  such that A' holds.*

*Proof.* We can, without loss of generality, suppose that the sequence  $(c_n)$  is decreasing. For  $h_0 = e_0$  the process  $(h_0 \circ T^i)$  is a sequence of i.i.d. hence  $\beta_0(n) = 0$  for all  $n \geq 0$ .

Using Lemma 5 we find  $n_1$  such that

$$\beta_1(0) \leq \frac{1}{n_1} \leq \frac{c_0}{2}.$$

Then, in the same way, we find  $n_2$  such that

$$\beta_2(n_1) \leq \frac{1}{n_2} \leq \frac{c_{n_1}}{4}$$

and by induction we get a sequence of  $n_k$  such that

$$\beta_{k+1}(n_k) \leq \frac{c_{n_k}}{2^{k+1}}, \quad k \geq 0.$$

Indeed, we choose  $n_{k+1} \geq \frac{2^{k+1}}{n_k}$ .

For each  $k \geq 1$ , we have  $\beta_k(N) = 0$  as soon as  $N > 2n_{k-1}$  hence

$$(8) \quad \beta(2N) \leq \sum_{k=0}^{\infty} \beta_k(N) = \sum_{k: N \leq n_k} \beta_{k+1}(n_k) = \sum_{k: N \leq n_k} \frac{c_{n_k}}{2^{k+1}} \leq \frac{c_{n_j}}{2^j}$$

where  $j = \min\{k : n_k \geq N\}$ . In particular we get  $\beta(2n_k) \leq c_{n_k}$ .  $\square$

This proves A'.

*Remark 9.* If  $u: \mathbb{R} \rightarrow \mathbb{R}$  is an increasing function and if we choose at each step  $n_{k+1} := \max\left\{\frac{2^{k+1}}{n_k}, u(n_k)\right\}$ , we have  $\beta_X(n_k) \leq c_{n_k}$  for each  $k$ .

**2.3. Proof of non-tightness.** For a positive integer  $m$ , we denote  $[m] := \{1, \dots, m\}$ . If  $S$  is a finite subset of the set of integers,  $|S|$  denotes the number of elements of  $S$ .

**Lemma 10.** *There exists  $N_0$  such that*

$$(9) \quad \mu \left\{ \max_{2n_k \leq N \leq n_k^2 - 1} |S_N(h_k)| = n_k \right\} > 1/4$$

whenever  $n_k \geq N_0$ .

*Proof.* For  $2n_k \leq N \leq n_k^2$ , thanks to (5), we have

$$\{|S_N(h_k)| = n_k\} \supset \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j \in I} \{U^j e_k = 0\} \cap \bigcap_{j \in J_N} \{U^j e_k = 0\},$$

where  $I = [1-2n_k, -1] \cap \mathbb{Z}$  and  $J_N = ([N-2n_k+1, N-1-n_k] \cup [N+1-n_k, N-1]) \cap \mathbb{Z}$ . We define

$$\begin{aligned} B_{N,k} &:= \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j=1-2n_k}^{-1-n_k} \{U^{N+j} e_k = 0\} \cap \bigcap_{j=1-n_k}^{-1} \{U^{j+N} e_k = 0\} \\ &= \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j \in J_N} \{U^j e_k = 0\}. \end{aligned}$$



We have  $|S_N(h_k)| = n_k$  on  $\bigcap_{j \in I} \{U^j e_k = 0\} \cap B_{N,k}$  and the sets  $\bigcap_{j \in I} \{U^j e_k = 0\}$ ,  $\bigcup_{N=2n_k}^{n_k^2} B_{N,k}$  belong to independent  $\sigma$ -algebras. Therefore

$$(10) \quad \mu \left\{ \max_{N \in [n_k^2 - 1]} |S_N(h_k)| = n_k \right\} \geq \left(1 - \frac{1}{n_k^2}\right)^{2n_k} \mu \left( \bigcup_{N=2n_k}^{n_k^2} B_{N,k} \right).$$

Recall Bonferroni's inequality, which states that for any integer  $n$  and any events  $A_j, j \in [n]$ , we have

$$(11) \quad \mu \left( \bigcup_{j=1}^n A_j \right) \geq \sum_{j=1}^n \mu(A_j) - \sum_{1 \leq i < j \leq n} \mu(A_i \cap A_j).$$

It can be proved by induction. Notice that

$$\mu(B_{N,k}) = \frac{1}{n_k^2} \left(1 - \frac{1}{n_k^2}\right)^{2n_k - 2} \geq \frac{1}{n_k^2} \left(1 - \frac{2}{n_k}\right)$$

and for  $i \neq j$

$$\mu(B_{i+2n_k-1,k} \cap B_{j+2n_k-1,k}) \leq \mu \{ |U^{i+n_k-1} e_k| = 1 \} \mu \{ |U^{j+n_k-1} e_k| = 1 \} = \frac{1}{n_k^4}$$

hence

$$\begin{aligned} \mu \left( \bigcup_{N=2n_k}^{n_k^2} B_{N,k} \right) &= \mu \left( \bigcup_{N=1}^{(n_k-1)^2} B_{N+2n_k-1,k} \right) \\ &\geq \sum_{N=1}^{(n_k-1)^2} \frac{1}{n_k^2} \left(1 - \frac{2}{n_k}\right) - \sum_{1 \leq i < j \leq (n_k-1)^2} \mu(B_{i+2n_k-1,k} \cap B_{j+2n_k-1,k}) \\ &\geq \left(1 - \frac{2}{n_k}\right)^3 - \frac{1}{2} \end{aligned}$$

which together with (10) implies that

$$(12) \quad \mu \left\{ \max_{N \in [n_k^2 - 1]} |S_N(h_k)| = n_k \right\} \geq \frac{1}{4},$$

whenever  $n_k \geq N_0$ , where  $N_0$  is such that  $(1 - \frac{2}{n})^3 > \frac{3}{4}$  for  $n \geq N_0$ .  $\square$

**Lemma 11.** *Assume that the sequence  $(n_k)_{k \geq 1}$  satisfies the following two conditions of lacunarity:*

$$(13) \quad \text{for each } k, \quad 16 \sum_{j=1}^k n_j^2 \leq n_{k+1};$$

$$(14) \quad \text{for each } k, \quad n_{k+1} \geq k^2 n_k.$$

Then we have for  $k$  large enough

$$(15) \quad \mu \left\{ \frac{1}{n_k} \max_{N \in [n_k^2 - 1]} |S_N(h)| \geq 1/2 \right\} \geq 1/8.$$

*Proof.* Fix an integer  $k$ . For each  $j < k$  and  $2n_k \leq N \leq n_k^2 - 1$ , we have

$$(16) \quad \left| \frac{1}{n_k} \sum_{j=1}^{k-1} S_N(h_j) \right| \leq \frac{2}{n_k} \sum_{j=1}^{k-1} (n_j + 1)^2 \leq \frac{1}{2}.$$

Indeed, using (5), we can give an upper bound of  $S_N(h_j)$  (as  $N \geq 2n_k > 2n_j$ ) as

$$|S_N(h_j)| \leq 2 \sum_{l=1}^{n_j} l + 2 \sum_{l=1}^{n_j-1} l \leq 2(n_j + 1)^2,$$

and we conclude by (13).

Now fix  $j > k$ . Writing  $S_N(h_j) = \sum_{i=0}^{N-1} U^i s_j - \sum_{i=-n_j}^{N-n_j-1} U^i s_j$ , where  $s_j := \sum_{i=0}^{n_j-1} U^{-i} e_j$ ,

we can see that the support of  $S_N(h_j)$  is contained in  $\bigcup_{i=0}^{n_k^2-1} T^{-i} A_j \cup \bigcup_{i=-n_j}^{n_k^2-n_j-1} T^{-i} A_j =$

$\bigcup_{i=-n_j}^{n_k^2-1} T^{-i} A_j$ , hence using (14)

$$(17) \quad \mu \left( \bigcup_{N=2n_k}^{n_k^2-1} \{S_N(h_j) \neq 0\} \right) \leq \frac{n_k^2 + n_j}{n_j^2} \leq \frac{2n_k}{n_j} < 2j^{-2}.$$

Let  $E_k := \bigcup_{N=2n_k}^{n_k^2-1} \bigcup_{j \geq k+1} \{S_N(h_j) \neq 0\}$ . By (17), we have

$$(18) \quad \mu(E_k) \leq \sum_{j \geq k+1} 2j^{-2} \leq 2/k.$$

By (16),

$$\frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \geq k} h_j \right) \right| - \frac{1}{2},$$

hence

$$\begin{aligned} \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\} &\geq \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \geq k} h_j \right) \right| \geq 1 \right\} \\ &\geq \mu \left( \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left( \sum_{j \geq k} h_j \right) \right| \geq 1 \right\} \cap E_k^c \right) \\ &= \mu \left( \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \cap E_k^c \right) \\ &\geq \mu \left( \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \right) - \mu(E_k). \end{aligned}$$

The result follows from Lemma 10, (17) and (18).  $\square$

The previous lemma yields together with Theorems 8.1 and 15.1 of [2] and the convergence to 0 of the finite dimensional distributions of  $(N^{-1/2}S_N^*(h))_{N \geq 1}$  and  $(N^{-1/2}S_N^{**}(h))_{N \geq 1}$  the following corollary.

**Corollary 12.** *If  $(n_k)_{k \geq 1}$  satisfies (13) and (14), then the sequences  $(N^{-1/2}S_N^*(h, \cdot))_{N \geq 1}$  and  $(N^{-1/2}S_N^{**}(h, \cdot))_{N \geq 1}$  are not tight in their respective spaces.*

Let  $\delta > 0$ . Then the choice  $n_k := \lfloor 16^{(2+\delta)^k} \rfloor$  satisfies the conditions (13) and (14). Indeed, the inequalities  $n_{k+1} \geq n_k^{2+\delta} - 1 = n_k^2 n_k^\delta - 1$  and  $n_k^\delta \geq 16k$  for each  $k$  give (13) and (14) is satisfied because  $n_k^{1+\delta} \geq k^2$ . Hence b) is satisfied. By Remark 9, we can construct in Proposition 8 the sequence  $(n_k)_{k \geq 1}$  in order to satisfy (13) and (14), this shows b) in Theorem A'.

Let us denote by  $\sigma_N^2$  the variance of  $S_N(h)$ , that is,  $\mathbb{E}[S_N(h)^2]$ .

**Proposition 13.** *Under the conditions (13) and (14), we have  $\sigma_N^2 \lesssim N$ .*

*Proof.* From (5) and (13), we deduce that

$$(19) \quad \sum_{j=1}^{i(N)} \mathbb{E}[S_N(h_j)^2] \lesssim \sum_{j=1}^{i(N)} n_j \leq 2n_{i(N)} \lesssim N.$$

Recall that  $h_k = (I - U^{-n_k})s_k$  with  $s_k := \sum_{i=0}^{n_k-1} U^{-i}e_k$ . Therefore, when  $n_k \geq N$ , we have by a similar computation as for (4),

$$S_N(h_k) = (I - U^{-n_k}) \left( \sum_{j=1}^{N-1} j(U^{j-1-N} + U^{n_k+1-j})e_k \right) + N(I - U^{-n_k}) \left( \sum_{j=N}^{n_k-1} U^{j-N}e_k \right).$$

The first term has a variance of order  $\frac{N^3}{n_k^2}$  while those of the second one is  $\frac{(n_k-N)N^2}{n_k}$ .

We thus that for  $n_k \geq N$ ,  $\mathbb{E}[S_N(h_k)^2] \lesssim \frac{N^2}{n_k}$ , hence by (14),

$$(20) \quad \sum_{k \geq i(N)+1} \mathbb{E}[S_N(h_k)^2] \lesssim \sum_{k \geq i(N)+1} \frac{N^2}{n_k} \leq N + N^2 \sum_{j \geq i(N)+1} \frac{1}{j^2 n_j} \lesssim N.$$

Combining (19) and (20), and using for a fixed  $N$  the independence of the sequence  $(S_N(h_j))_{j \geq 1}$ , we conclude the result.  $\square$

When we add an independent sequence  $(m \circ T^i)_{i \in \mathbb{Z}}$  with moments of any order, the variance of  $((m+h) \circ T^i)_{i \geq 1}$  is bounded above and below by a quantity proportional to  $N$ , hence c) is satisfied in Theorems A and A'.

**2.4. Moments of the coboundary and the transfer function.** One can wonder to which  $\mathbb{L}^p$  space can  $g$  and  $g - g \circ T$  belong.

**Proposition 14.** *Under the conditions (13) and (14), we have  $g \in \mathbb{L}^p$  for  $0 < p < 1$  and  $g - g \circ T \in \mathbb{L}^p$  for each  $p > 0$ .*

*Proof.* Let  $g_k := U^{-1}v_k$ , where  $v_k = \sum_{j=0}^{n_k-1} U^{-j}s_k$  and  $s_k = \sum_{j=0}^{n_k-1} U^{-j}e_k$ . Recall that  $g = -\sum_{k=1}^{\infty} g_k$ . For  $0 < p < 1$  and any two non-negative real numbers  $a$  and  $b$ , we

have  $(a + b)^p \leq a^p + b^p$ . This gives, using (4),

$$\begin{aligned} \mathbb{E} |g_k|^p &\leq \left( \sum_{j=1}^{n_k} j^p \mathbb{E} |U^{-j} e_k| + \sum_{j=1}^{n_k-1} (n_k - j)^p \mathbb{E} |U^{-j+n_k} e_k| \right) \\ &= \frac{1}{n_k^2} \left( n_k^p + 2 \sum_{j=1}^{n_k-1} j^p \right) \\ &\leq \frac{1}{n_k^2} (n_k^p + 2n_k^{p+1}) \\ &\leq 3n_k^{p-1}. \end{aligned}$$

By (14), we have  $n_k \geq k! \cdot n_1$  hence the series  $\sum_{k \geq 1} \mathbb{E} |g_k|^p$  is convergent. This proves that  $g \in \mathbb{L}^p$  for  $0 < p < 1$ .

Corollary 2.4. in [1] states the following: given positive integers  $t$  and  $p$ ,  $X_1, \dots, X_t$  independent random variables such that  $\mu \{0 \leq X_j \leq 1\} = 1$  for each  $j \in [t]$ , then

$$(21) \quad \mathbb{E} (X)^p \leq B_p \cdot \max \{ \mathbb{E}(X), (\mathbb{E}(X))^p \},$$

where  $B_p$  is the  $p$ -th Bell's number (defined by the recursion relation  $B_{p+1} = \sum_{k=0}^p \binom{p}{k} B_k$  and  $B_0 = B_1 = 1$ ) and  $X := \sum_{j=1}^t X_j$ .

We shall show that the series  $\sum_{k \geq 1} \|h_k\|_p$  is convergent for any integer  $p$ . Fix  $k \geq 1$ , and let  $t := 2n_k$ ,  $X_j := |U^{j-2n_k} e_k|$ . Applying (21), we get

$$\|h_k\|_p^p \leq B_p \cdot \max \{ n_k^{-1}, n_k^{-p} \} = B_p \cdot n_k^{-1},$$

hence  $\|h_k\|_p \leq B_p^{1/p} \cdot 2^{-\frac{k^2}{2p}}$ . One could also use Rosenthal's inequality.  $\square$

Since the added process has moments of any order, Proposition 14 proves e) in Theorems A and A'.

**Proposition 15.** *The transfer function  $g$  does not belong to  $\mathbb{L}^1$ .*

*Proof.* Fix an integer  $k$ , and define for  $1 \leq j \leq n_k$ :

$$E_j := \{ |U^{-j-2n_k} e_k| = 1 \} \cap \bigcap_{i \in [2n_k-1] \setminus \{j\}} \{ U^{-i-2n_k} e_k = 0 \} \cap \bigcap_{l \neq k} \{ g_l = 0 \}.$$

Since these sets are pairwise disjoint and  $g = -U^{-1} \sum_{k \geq 1} \sum_{j=0}^{n_k-1} U^{-j} s_k$ , with  $s_k := \sum_{j=0}^{n_k-1} U^{-j} e_k$ , we have after having rearranged  $\sum_{j=0}^{n_k-1} U^{-j} s_k$  and used the fact that

$g_l(\omega) = 0$  if  $l \neq k$  and  $\omega \in \bigcup_{j=1}^{n_k} E_j$ :

$$\begin{aligned}
\left\| g\chi_{\bigcup_{j=1}^{n_k} E_j} \right\|_1 &= \left\| \sum_{j=1}^{n_k} \chi_{E_j} \left\| \sum_{i=1}^{n_k} iU^{-i-2n_k} e_k \right\| \right\|_1 \\
&= \sum_{j=1}^{n_k} \left\| \chi_{E_j} \left\| \sum_{i=1}^{n_k} iU^{-i-2n_k} e_k \right\| \right\|_1 \\
&= \sum_{j=1}^{n_k} j\mu(E_j) \\
&= \sum_{j=1}^{n_k} j \cdot \frac{1}{n_k^2} \left(1 - \frac{1}{n_k^2}\right)^{2n_k-2} \prod_{j \neq k} \left(1 - \frac{1}{n_j^2}\right)^{2n_j-1} \\
&\geq \frac{1}{2} \left(1 - \frac{1}{n_k^2}\right)^{-1} \prod_{j \geq 1} \left(1 - \frac{1}{n_j^2}\right)^{2n_j-1}.
\end{aligned}$$

As  $\left(1 - \frac{1}{n_k^2}\right)^{-1} \rightarrow 1$  as  $k \rightarrow +\infty$  and  $\prod_{j \geq 1} \left(1 - \frac{1}{n_j^2}\right)^{2n_j-1}$  is positive, we get  $\|g\chi_{F_k}\|_1 \geq c > 0$ , where  $F_k := \bigcup_{j=1}^{n_k} E_j$ . Since  $g\chi_{F_k} = g_k\chi_{F_k}$ , we get that  $\|g_k\|_1 > c$  for a  $c$  independent of  $k$ . Therefore, the series of independent random variables  $\sum_{k \geq 1} g_k$  cannot be convergent in  $\mathbb{L}^1$ .  $\square$

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