



HAL
open science

A strictly stationary β -mixing process satisfying the central limit theorem but not the weak invariance principle

Davide Giraud, Dalibor Volný

► **To cite this version:**

Davide Giraud, Dalibor Volný. A strictly stationary β -mixing process satisfying the central limit theorem but not the weak invariance principle. *Stochastic Processes and their Applications*, 2014, 124, pp.3769-3781. 10.1016/j.spa.2014.06.008 . hal-00911758v2

HAL Id: hal-00911758

<https://hal.science/hal-00911758v2>

Submitted on 14 Oct 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A STRICTLY STATIONARY β -MIXING PROCESS SATISFYING THE CENTRAL LIMIT THEOREM BUT NOT THE WEAK INVARIANCE PRINCIPLE

DAVIDE GIRAUDDO AND DALIBOR VOLNÝ

ABSTRACT. In 1983, N. Herrndorf proved that for a ϕ -mixing sequence satisfying the central limit theorem and $\liminf_{n \rightarrow \infty} \sigma_n^2/n > 0$, the weak invariance principle takes place. The question whether for strictly stationary sequences with finite second moments and a weaker type (α, β, ρ) of mixing the central limit theorem implies the weak invariance principle remained open.

We construct a strictly stationary β -mixing sequence with finite moments of any order and linear variance for which the central limit theorem takes place but not the weak invariance principle.

1. INTRODUCTION AND NOTATIONS

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. If $T: \Omega \rightarrow \Omega$ is one-to-one, bi-measurable and measure preserving (in sense that $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$), then the sequence $(f \circ T^k)_{k \in \mathbb{Z}}$ is strictly stationary for any measurable $f: \Omega \rightarrow \mathbb{R}$. Conversely, each strictly stationary sequence can be represented in this way.

For a zero mean square integrable $f: \Omega \rightarrow \mathbb{R}$, we define $S_n(f) := \sum_{j=0}^{n-1} f \circ T^j$, $\sigma_n^2(f) := \mathbb{E}(S_n(f)^2)$ and $S_n^*(f, t) := S_{[nt]}(f) + (nt - [nt])f \circ T^{[nt]}$, where $[x]$ is the greatest integer which is less than or equal to x .

We say that $(f \circ T^j)_{j \geq 1}$ satisfies the *central limit theorem with normalization* a_n if the sequence $(a_n^{-1}S_n(f))_{n \geq 1}$ converges weakly to a standard normal distribution. Let $C[0, 1]$ denote the space of continuous functions on the unit interval endowed with the norm $\|g\|_\infty := \sup_{t \in [0, 1]} |g(t)|$.

Let $D[0, 1]$ be the space of real valued functions which have left limits and are continuous-from-the-right at each point of $[0, 1]$. We endow it with Skorohod metric (cf. [2]). We define $S_n^{**}(f, t) := S_{[nt]}(f)$, which gives a random element of $D[0, 1]$.

We shall say that the strictly stationary sequence $(f \circ T^j)_{j \geq 0}$ *satisfies the weak invariance principle in* $C[0, 1]$ *with normalization* a_n (respectively *in* $D[0, 1]$) if the sequence of $C[0, 1]$ (of $D[0, 1]$) valued random variables $(a_n^{-1}S_n^*(f, \cdot))_{n \geq 1}$ (resp. $(a_n^{-1}S_n^{**}(f, \cdot))_{n \geq 1}$) weakly converges to a Brownian motion process in the corresponding space.

Let \mathcal{A} and \mathcal{B} be two sub- σ -algebras of \mathcal{F} , where $(\Omega, \mathcal{F}, \mu)$ is a probability space. We define the α -mixing coefficients as introduced by Rosenblatt in [14]:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup \{ |\mu(A \cap B) - \mu(A)\mu(B)|, A \in \mathcal{A}, B \in \mathcal{B} \}.$$

Date: September 2, 2014.

2000 Mathematics Subject Classification. 60F05; 60F17.

Key words and phrases. Central limit theorem, invariance principle, mixing conditions, strictly stationary process.

Define the β -mixing coefficients by

$$\beta(\mathcal{A}, \mathcal{B}) := \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |\mu(A_i \cap B_j) - \mu(A_i)\mu(B_j)|,$$

where the supremum is taken over the finite partitions $\{A_i, 1 \leq i \leq I\}$ and $\{B_j, 1 \leq j \leq J\}$ of Ω of elements of \mathcal{A} (respectively of \mathcal{B}). They were introduced by Volkonskii and Rozanov [16].

The ρ -mixing coefficients were introduced by Hirschfeld [8] and are defined by

$$\rho(\mathcal{A}, \mathcal{B}) := \sup \{|\text{Corr}(f, g)|, f \in \mathbb{L}^2(\mathcal{A}), g \in \mathbb{L}^2(\mathcal{B})\},$$

where $\text{Corr}(f, g) := [\mathbb{E}(fg) - \mathbb{E}(f)\mathbb{E}(g)] [\|f - \mathbb{E}(f)\|_{\mathbb{L}^2} \|g - \mathbb{E}(g)\|_{\mathbb{L}^2}]^{-1}$.

Ibragimov [9] introduced for the first time ϕ -mixing coefficients, which are given by the formula

$$\phi(\mathcal{A}, \mathcal{B}) := \sup \{|\mu(B | A) - \mu(B)|, A \in \mathcal{A}, B \in \mathcal{B}, \mu(A) > 0\}.$$

The coefficients are related by the inequalities

$$(1) \quad 2\alpha(\mathcal{A}, \mathcal{B}) \leq \beta(\mathcal{A}, \mathcal{B}) \leq \phi(\mathcal{A}, \mathcal{B}), \quad \alpha(\mathcal{A}, \mathcal{B}) \leq \rho(\mathcal{A}, \mathcal{B}) \leq 2\sqrt{\phi(\mathcal{A}, \mathcal{B})}.$$

For a strictly stationary sequence $(X_k)_{k \in \mathbb{Z}}$ and $n \geq 0$ we define $\alpha_X(n) = \alpha(n) = \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty)$ where \mathcal{F}_u^v is the σ -algebra generated by X_k with $u \leq k \leq v$ (if $u = -\infty$ or $v = \infty$, the corresponding inequality is strict). In the same way we define coefficients $\beta_X(n)$, $\rho_X(n)$, $\phi_X(n)$.

We say that the sequence $(X_k)_{k \in \mathbb{Z}}$ is α -mixing if $\lim_{n \rightarrow +\infty} \alpha_X(n) = 0$, and similarly we define β , ρ and ϕ -mixing sequences.

α , β and ϕ -mixing sequences were considered in the mentioned references, while ρ -mixing sequences first appeared in [12].

Inequalities (1) give a hierarchy between these classes of mixing sequences.

If $(a_N)_{N \geq 1}$ and $(b_N)_{N \geq 1}$ are two sequences of positive real numbers, we write $a_N \asymp b_N$ if there exists a positive constant C such that for each N , $C^{-1}a_N \leq b_N \leq Ca_N$.

The main results are

Theorem A. *Let δ be a positive real number. There exists a strictly stationary real valued process $Y = (Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0}$ satisfying the following conditions:*

- a) *the central limit theorem with normalization \sqrt{n} takes place;*
- b) *the weak invariance principle with normalization \sqrt{n} does not hold;*
- c) *we have $\sigma_N(f)^2 \asymp N$;*
- d) *we have for some positive C , $\beta_Y(N) \leq C/N^{1/2-\delta}$;*
- e) *$Y_0 \in \mathbb{L}^p$ for any $p > 0$.*

Alternatively, we can construct the process in order to have a control of the mixing coefficients on a subsequence.

Theorem A'. *Let $(c_j)_{j \geq 0}$ be a decreasing sequence of positive numbers. Then there exists a strictly stationary real valued process $Y = (Y_k)_{k \geq 0} = (f \circ T^k)_{k \geq 0}$ satisfying conditions a), b), c), e) in Theorem A, and:*

- d') *there is an increasing sequence $(m_k)_{k \geq 1}$ of integers such that for each k , $\beta_Y(m_k) \leq c_{m_k}$.*

Remark 1. Herrndorf proved ([7], Theorem 2.13) that if (ξ_n) is a strictly stationary ϕ -mixing sequence for which $\sigma_n \rightarrow \infty$, S_n/σ_n converges in distribution to a standard normal distribution and $\sigma_n^{-1} \max_{1 \leq i \leq n} |\xi_i| \rightarrow 0$ in probability, then the weak invariance principle takes place. So Herrndorf's result does not extend to β -mixing sequences.

Remark 2. Rio *et al.* proved in [13] that the condition $\int_0^1 \alpha^{-1}(u) Q^2(u) du < \infty$ implies the weak invariance principle, where $\alpha^{-1}(u) := \inf \{k, \alpha(k) \leq u\}$ and Q is the right-continuous inverse of the quantile function $t \mapsto \mu \{X_0 > t\}$. If the process is strictly stationary, with finite moments of order $2 + r$, $r > 0$, the latter condition is satisfied whenever $\sum_{n=1}^{\infty} (n+1)^{2/r} \alpha(n) < \infty$ (Ibragimov [10] found the condition $\sum_{n=1}^{\infty} \alpha(n)^{1-2/r} < \infty$). Since $Y_0 \in \mathbb{L}^p$ for all $p < \infty$, we have that $\sum_N \alpha(N)^r = +\infty$ for any $r < 1$, hence in Theorem A' we can thus hardly get such a bound as in d') for the whole sequence.

Remark 3. Ibragimov proved that for a strictly stationary ρ -mixing sequence with finite moments of order $2 + \delta$ for some positive δ , the weak invariance principle holds, cf. [11]. In particular, this proves that our construction does not give a ρ -mixing process. Shao also showed in [15] that the condition $\sum_n \rho(2^n) < \infty$ is sufficient in order to guarantee the weak invariance principle in $D[0, 1]$ for stationary sequences having order two moments. So a potential ρ -mixing counter-example has to adhere to restrictions on the moments as well as on the mixing rates.

About the method of proof

In proving the result we will use properties of coboundaries $h = g - g \circ T$ (g is called a transfer function). For a positive integer N and a measurable function v , we

denote $S_N(v) := \sum_{j=0}^{N-1} U^j v$ (Here and below, $U^j v := v \circ T^j$). Because $S_n(g - g \circ T) =$

$g - g \circ T^n$, for any sequence $a_n \rightarrow \infty$ we have $(a_n)^{-1} S_n(g - g \circ T) \rightarrow 0$ in probability hence adding a coboundary does not change validity of the central limit theorem. If, moreover, $g \in \mathbb{L}^2$ then $n^{-1/2} \|S_n^*(g - g \circ T)\|_{\infty} \rightarrow 0$ a.s. hence adding of such coboundary does not change validity of the invariance principle (if norming by \sqrt{n} or by σ_n with $\liminf_n \sigma_n^2/n > 0$), cf. [6], pages 140-141. On the other hand, if $g \notin \mathbb{L}^2$, adding a coboundary can spoil tightness even if $g - g \circ T$ is square integrable, cf. [17]. A similar idea was used in [5]. In the proof of Theorem A and A' we will find a coboundary $g - g \circ T$ which is β -mixing and spoils tightness. The coboundary has all finite moments but the transfer function is not integrable. We then add an m such that $(m \circ T^i)_{i \in \mathbb{Z}}$ and $(h \circ T^i)_{i \in \mathbb{Z}}$ are independent (enlarging the probability space), and $m \circ T^i$ is i.i.d. with moments of any order (in particular, it satisfies the weak invariance principle).

The proof uses the fact that $|\mu(A \cap B) - \mu(A)\mu(B)| \leq \mu(A)$. The method does not seem to apply to processes which are ρ -mixing and for this kind of processes the problem remains open.

2. PROOF

2.1. Construction of h . Let us consider an increasing sequence of positive integers $(n_k)_{k \geq 1}$ such that

$$n_1 \geq 2 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{n_k} < \infty,$$

and for each integer $k \geq 1$, let A_k^-, A_k^+ be disjoint measurable sets such that $\mu(A_k^-) = 1/(2n_k^2) = \mu(A_k^+)$.

Let the random variables e_k be defined by

$$(2) \quad e_k(\omega) := \begin{cases} 1 & \text{if } \omega \in A_k^+, \\ -1 & \text{if } \omega \in A_k^-, \\ 0 & \text{otherwise.} \end{cases}$$

We can choose the dynamical system $(\Omega, \mathcal{F}, \mu, T)$ and the sets A_k^+, A_k^- in such a way that the family $(e_k \circ T^i)_{k \geq 1, i \in \mathbb{Z}}$ is independent. We define $A_k := A_k^+ \cup A_k^-$ and

$$(3) \quad h_k := \sum_{i=0}^{n_k-1} U^{-i} e_k - U^{-n_k} \sum_{i=0}^{n_k-1} U^{-i} e_k, \quad h := \sum_{k=1}^{+\infty} h_k.$$

Since $\mu\{h_k \neq 0\} \leq 2/n_k$, the function h is almost everywhere well-defined (by the Borel-Cantelli lemma).

It will be useful to express, for $N \geq n_k$, the sum $S_N(h_k)$ as a linear combination of $U^j e_k$. Denote $s_k := \sum_{j=0}^{n_k-1} U^{-j} e_k$. As $N \geq n_k$ and $h_k = s_k - U^{-n_k} s_k$, we have

$$\begin{aligned} S_N(h_k) &= \sum_{j=0}^{N-1} (U^j s_k - U^{j-n_k} s_k) \\ &= \sum_{j=0}^{N-1} U^j s_k - \sum_{j=-n_k}^{N-n_k-1} U^j s_k \\ &= - \sum_{j=-n_k}^{-1} U^j s_k + U^N \sum_{j=-n_k}^{-1} U^j s_k. \end{aligned}$$

We also have

$$(4) \quad \begin{aligned} \sum_{j=-n_k}^{-1} U^j s_k &= \sum_{j=1}^{n_k} U^{-j} s_k \\ &= U^{-1} \sum_{i=0}^{n_k-1} \sum_{j=0}^{n_k-1} U^{-(i+j)} e_k \\ \sum_{j=-n_k}^{-1} U^j s_k &= U^{-2n_k} \left(\sum_{j=1}^{n_k} j U^j e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{n_k+j} e_k \right). \end{aligned}$$

The previous equations yield

$$(5) \quad \begin{aligned} S_N(h_k) &= \sum_{j=1}^{n_k} j U^{j+N-2n_k} e_k + \sum_{j=1}^{n_k-1} (n_k - j) U^{j+N-n_k} e_k \\ &\quad - \sum_{j=1}^{n_k} j U^{j-2n_k} e_k - \sum_{j=1}^{n_k-1} (n_k - j) U^{j-n_k} e_k. \end{aligned}$$

Each h_k is a coboundary, as if we define $v_k := \sum_{j=0}^{n_k-1} U^{-j} s_k$, then $v_k - U^{-1}v_k =$

$s_k - U^{-n_k} s_k = h_k$ (so in this case the transfer function is $-U^{-1}v_k$).

Since $\mu\{v_k \neq 0\} \leq 2/n_k$, Borel-Cantelli's lemma shows that the function $g := \sum_{k=1}^{+\infty} U^{-1}v_k$ is almost everywhere well defined under our assumption that $\sum_k 1/n_k$

is convergent. Because $h = g - Ug$, h is a coboundary.

2.2. Mixing rates. We show that the process $(U^i f)_{i \in \mathbb{Z}}$ is β -mixing. In doing so we use the following proposition (cf. [4], Theorem 6.2).

Proposition 4. *Let $(X_{k,i})_i$, $k = 1, 2, \dots$ be mutually independent strictly stationary processes with respective mixing coefficients $\beta_k(n)$, let $X_i = \sum_{k=1}^{\infty} X_{k,i}$ converge. The process $(X_i)_i$ is strictly stationary with mixing coefficients $\beta(n) \leq \sum_{k=1}^{\infty} \beta_k(n)$.*

This reduces the proof of β -mixing of $(U^i f)_{i \in \mathbb{Z}}$ (in Theorems A and A') to that of $(U^i h)_{i \in \mathbb{Z}}$ and thereby to that of $(U^i h_k)_{i \in \mathbb{Z}}$ for $k \geq 1$.

In the following text we denote by $\beta_k(n)$ the mixing coefficients of the process $(h_k \circ T^i)_{i \in \mathbb{Z}}$.

Lemma 5. *For $k \geq 1$, we have the estimate $\beta_k(0) \leq 4/n_k$.*

Proof. Suppose k is a positive integer. For $-\infty \leq j \leq l \leq \infty$, let \mathcal{H}_j^l denote the σ -field generated by $U^i h_k$, $j \leq i \leq l$, ($i \in \mathbb{Z}$), and let \mathcal{G}_j^l denote the σ -field generated by $U^i e_k$, $j \leq i \leq l$, ($i \in \mathbb{Z}$). Define the σ -fields $\mathcal{B}_1 := \mathcal{G}_{-\infty}^{-2n_k}$, $\mathcal{B}_2 := \mathcal{G}_{-2n_k+1}^0$ and $\mathcal{B}_3 := \mathcal{G}_1^{\infty}$. Now $\mathcal{H}_{-\infty}^0 \subset \mathcal{B}_1 \vee \mathcal{B}_2$ and $\mathcal{H}_0^{\infty} \subset \mathcal{B}_2 \vee \mathcal{B}_3$. Therefore $\beta_k(0) \leq \beta(\mathcal{B}_1 \vee \mathcal{B}_2, \mathcal{B}_2 \vee \mathcal{B}_3)$. The σ -fields $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$, are independent; hence the σ -fields $\mathcal{B}_1 \vee \mathcal{B}_3$ and $\mathcal{B}_2 \vee \mathcal{B}_2$ (with index 2 in both places) are independent; this implies by a result given e.g. in [4, Theorem 6.2],

$$\beta(\mathcal{B}_1 \vee \mathcal{B}_2, \mathcal{B}_2 \vee \mathcal{B}_3) \leq \beta(\mathcal{B}_1, \mathcal{B}_3) + \beta(\mathcal{B}_2, \mathcal{B}_2) = 0 + \beta(\mathcal{B}_2, \mathcal{B}_2).$$

Thus $\beta_k(0) \leq \beta(\mathcal{B}_2, \mathcal{B}_2)$. Also, the σ -field \mathcal{B}_2 has an atom $P_0 := \bigcap_{i=-2n_k+1}^0 \{U^i e_k = 0\}$ that satisfies $\mu(P_0) \geq 1 - 2/n_k$ (since $\mu(U^i e_k \neq 0) = 1/n_k^2$ for each i). By Lemma 2.2 of [3], if \mathcal{B} is a σ -field which has an atom D , then $\beta(\mathcal{B}, \mathcal{B}) \leq 2[1 - \mu(D)]$. Hence $\beta(\mathcal{B}_2, \mathcal{B}_2) \leq 2[1 - \mu(P_0)] \leq 4/n_k$. This finishes the proof of Lemma 5. \square

Denoting by $\beta(N)$ the mixing coefficients of the sequence $(h \circ T^i)_{i \in \mathbb{Z}}$, Proposition 4, Lemma 5 and the fact that $\beta_k(N) = 0$ when $N \geq 2n_k$ yield

Corollary 6. *For each integer k , we have*

$$(6) \quad \beta(N) \leq \sum_{j \geq 1} \beta_j(N) \leq \sum_{j: 2n_j \geq N} \frac{4}{n_j}.$$

Now we can prove d) and d'). Let $i(N)$ denote the unique integer such that $n_{i(N)} \leq N < n_{i(N)+1}$. For sequences $(u_N)_{N \geq 1}$, $(v_N)_{N \geq 1}$ of positive numbers, $u_N \lesssim v_N$ means that there is $C > 0$ such that for each N , $u_N \leq C \cdot v_N$.

Proposition 7. *Let $\delta > 0$. With the choice $n_k := \lfloor 16^{(2+\delta)^k} \rfloor$, we have d).*

Proof. We deduce from (6)

$$\beta(2n_k) \leq \sum_{j \geq k} \frac{4}{n_j} \lesssim \sum_{j \geq 0} 16^{-(2+\delta)^k (2+\delta)^j} \lesssim 16^{-(2+\delta)^k}.$$

Consequently,

$$\beta(2N) \leq \beta(2n_{i(N)}) \lesssim \frac{4}{n_{i(N)}} \lesssim \frac{1}{n_{i(N)+1}^{\frac{1}{2+\delta}}} \lesssim \frac{1}{N^{\frac{1}{2+\delta}}}.$$

□

Hence d) is fulfilled.

Proposition 8. *Given $(c_k)_{k \geq 1}$ as in Theorem A', one can recursively choose a sequence $(n_k)_{k \geq 1}$ growing to infinity arbitrarily fast, such that for the construction given above, one has that for each $k \geq 1$, $\beta(2n_k) \leq c_{2n_k}$.*

Proof. Suppose that the sequence $(n_k)_{k \geq 1}$ satisfies

$$n_{k+1} \geq \frac{8}{c_{2n_k}} \quad \text{and} \quad n_{k+1} \geq 2n_k, \quad k \geq 1.$$

Then

$$\beta(2n_k) \leq \sum_{j=k+1}^{\infty} \beta_j(2n_k) \leq \sum_{j=k+1}^{\infty} \beta_j(0),$$

and, by Lemma 5 and the condition $n_{j+l} \geq 2^l n_j$ for $j, l \geq 1$, we derive

$$\beta(2n_k) \leq \sum_{j=k+1}^{\infty} \frac{4}{n_j} \leq \frac{8}{n_{k+1}}.$$

The assumption $n_{k+1} \geq 8/c_{2n_k}$ yields $\beta(2n_k) \leq c_{2n_k}$ for each $k \geq 1$.

□

This proves d') with $m_k := 2n_k$.

Remark 9. The sequence of integers $(n_k)_{k \geq 1}$ can be chosen to meet all other conditions imposed in this paper.

2.3. Proof of non-tightness.

Lemma 10. *There exists N_0 such that*

$$(7) \quad \mu \left\{ \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq n_k \right\} > 1/4$$

whenever $n_k \geq N_0$.

Proof. For $2n_k \leq N \leq n_k^2$, thanks to (5), we have

$$\{|S_N(h_k)| = n_k\} \supset \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j \in I} \{U^j e_k = 0\} \cap \bigcap_{j \in J_N} \{U^j e_k = 0\},$$

where $I = [1-2n_k, -1] \cap \mathbb{Z}$ and $J_N = ([N-2n_k+1, N-1-n_k] \cup [N+1-n_k, N-1]) \cap \mathbb{Z}$.

We define

$$\begin{aligned} B_{N,k} &:= \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j=1-2n_k}^{-1-n_k} \{U^{N+j} e_k = 0\} \cap \bigcap_{j=1-n_k}^{-1} \{U^{j+N} e_k = 0\} \\ &= \{|U^{N-n_k} e_k| = 1\} \cap \bigcap_{j \in J_N} \{U^j e_k = 0\}. \end{aligned}$$

We have $|S_N(h_k)| = n_k$ on $\bigcap_{j \in I} \{U^j e_k = 0\} \cap B_{N,k}$ and the sets $\bigcap_{j \in I} \{U^j e_k = 0\}$, $\bigcup_{N=2n_k}^{n_k^2} B_{N,k}$ belong to independent σ -algebras. Therefore

$$(8) \quad \mu \left\{ \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq n_k \right\} \geq \left(1 - \frac{1}{n_k^2}\right)^{2n_k} \mu \left(\bigcup_{N=2n_k}^{n_k^2} B_{N,k} \right).$$

Recall Bonferroni's inequality, which states that for any integer n and any events $A_j, j \in \{1, \dots, n\}$, we have

$$(9) \quad \mu \left(\bigcup_{j=1}^n A_j \right) \geq \sum_{j=1}^n \mu(A_j) - \sum_{1 \leq i < j \leq n} \mu(A_i \cap A_j).$$

It can be proved by induction. Notice that

$$\mu(B_{N,k}) = \frac{1}{n_k^2} \left(1 - \frac{1}{n_k^2}\right)^{2n_k-2} \geq \frac{1}{n_k^2} \left(1 - \frac{2}{n_k}\right)$$

and for $i \neq j$

$$\mu(B_{i+2n_k-1,k} \cap B_{j+2n_k-1,k}) \leq \mu \{|U^{i+n_k-1} e_k| = 1\} \mu \{|U^{j+n_k-1} e_k| = 1\} = \frac{1}{n_k^4}$$

hence

$$\begin{aligned} \mu \left(\bigcup_{N=2n_k}^{n_k^2} B_{N,k} \right) &= \mu \left(\bigcup_{N=1}^{(n_k-1)^2} B_{N+2n_k-1,k} \right) \\ &\geq \sum_{N=1}^{(n_k-1)^2} \frac{1}{n_k^2} \left(1 - \frac{2}{n_k}\right) - \sum_{1 \leq i < j \leq (n_k-1)^2} \mu(B_{i+2n_k-1,k} \cap B_{j+2n_k-1,k}) \\ &\geq \left(1 - \frac{2}{n_k}\right)^3 - \frac{1}{2} \end{aligned}$$

which together with (8) and the inequality $(1 - 1/n_k^2)^{2n_k} \geq 1 - 2/n_k$ implies that

$$(10) \quad \mu \left\{ \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq n_k \right\} > \frac{1}{4},$$

whenever $n_k \geq N_0$, where $N_0 \geq 3$ is such that $(1 - 2/n) [(1 - 2/n)^3 - 1/2] > 1/4$ for $n \geq N_0$. \square

Lemma 11. *Assume that the sequence $(n_k)_{k \geq 1}$ satisfies the following two conditions of lacunarity:*

$$(11) \quad \text{for each } k \geq K, \quad 16 \sum_{j=1}^k n_j^2 \leq n_{k+1};$$

$$(12) \quad \text{for each } k \geq K, \quad n_{k+1} \geq (k+1)^2 n_k,$$

where $2 \leq K < \infty$. Then we have for k large enough

$$(13) \quad \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq 1/2 \right\} \geq 1/8.$$

Proof. Fix an integer $k \geq K$. For $2n_k \leq N \leq n_k^2$, we have

$$(14) \quad \left| \frac{1}{n_k} \sum_{j=1}^{k-1} S_N(h_j) \right| \leq \frac{2}{n_k} \sum_{j=1}^{k-1} (n_j + 1)^2 \leq \frac{1}{2}.$$

Indeed, using (5), for $j < k$, we can give an upper bound of $S_N(h_j)$ (as $N \geq 2n_k > 2n_j$) as

$$|S_N(h_j)| \leq 2 \sum_{l=1}^{n_j} l + 2 \sum_{l=1}^{n_j-1} l = 2n_j^2,$$

and (14) holds by (11).

Now fix $j > k$. Writing $S_N(h_j) = \sum_{i=0}^{N-1} U^i s_j - \sum_{i=-n_j}^{N-n_j-1} U^i s_j$, where $s_j := \sum_{i=0}^{n_j-1} U^{-i} e_j$,

we can see that

$$\bigcup_{N=2n_k}^{n_k^2} \{S_N(h_j) \neq 0\} \subset \bigcup_{i=-2n_j+1}^{n_k^2} T^{-i} A_j,$$

hence using (12)

$$(15) \quad \mu \left(\bigcup_{N=2n_k}^{n_k^2} \{S_N(h_j) \neq 0\} \right) \leq \frac{n_k^2 + 2n_j}{n_j^2} \leq \frac{2n_k}{n_j} \leq 2j^{-2}.$$

Let $E_k := \bigcup_{N=2n_k}^{n_k^2} \bigcup_{j \geq k+1} \{S_N(h_j) \neq 0\}$. By (15), we have

$$(16) \quad \mu(E_k) \leq \sum_{j \geq k+1} 2j^{-2} \leq 2/k.$$

By (14),

$$\frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left(\sum_{j \geq k} h_j \right) \right| - \frac{1}{2},$$

hence

$$\begin{aligned} \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h)| \geq \frac{1}{2} \right\} &\geq \mu \left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left(\sum_{j \geq k} h_j \right) \right| \geq 1 \right\} \\ &\geq \mu \left(\left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} \left| S_N \left(\sum_{j \geq k} h_j \right) \right| \geq 1 \right\} \cap E_k^c \right) \\ &= \mu \left(\left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \cap E_k^c \right) \\ &\geq \mu \left(\left\{ \frac{1}{n_k} \max_{2n_k \leq N \leq n_k^2} |S_N(h_k)| \geq 1 \right\} \right) - \mu(E_k). \end{aligned}$$

The result follows from Lemma 10 and (16). \square

The previous lemma yields together with Theorems 8.1 and 15.1 of [2] and the convergence to 0 of the finite dimensional distributions of $(N^{-1/2} S_N^*(h))_{N \geq 1}$ and $(N^{-1/2} S_N^{**}(h))_{N \geq 1}$ the following corollary.

Corollary 12. *If $(n_k)_{k \geq 1}$ satisfies (11) and (12), then the sequences $(N^{-1/2}S_N^*(h, \cdot))_{N \geq 1}$ and $(N^{-1/2}S_N^{**}(h, \cdot))_{N \geq 1}$ are not tight in their respective spaces.*

Let $\delta > 0$. Then the choice $n_k := \lfloor 16^{(2+\delta)^k} \rfloor$ satisfies the conditions (11) and (12).

Under assumptions of Proposition 7 (the choice of n_k) we get d) in A and because (11), (12) are satisfied, we get b) in A.

By Remark 9, we can construct in Proposition 8 the sequence $(n_k)_{k \geq 1}$ in such a way that it also satisfies (11) and (12); this yields b) in Theorem A', and of course from Proposition 8 itself, property d') in Theorem A' also holds.

2.4. Proof of a) and c). Let us denote by σ_N^2 the variance of $S_N(h)$, that is, $\mathbb{E}[S_N(h)^2]$.

Proposition 13. *Under the conditions (11) and (12), we have $\sigma_N^2 \lesssim N$.*

Proof. From (5) and (11), we deduce that

$$(17) \quad \sum_{j=1}^{i(N)} \mathbb{E}[S_N(h_j)^2] \lesssim \sum_{j=1}^{i(N)} n_j \leq 2n_{i(N)} \lesssim N.$$

Recall that $h_k = (I - U^{-n_k})s_k$ with $s_k := \sum_{i=0}^{n_k-1} U^{-i}e_k$. Therefore, when $n_k \geq N$, we have by a similar computation as for (4),

$$(18) \quad S_N(h_k) = (I - U^{-n_k}) \sum_{j=1}^{N-1} j(U^{j-n_k} + U^{N-j})e_k + N(I - U^{-n_k}) \sum_{j=N}^{n_k} U^{j-n_k}e_k.$$

The first term has a variance of order N^3/n_k^2 , and the variance of the second term is (at most) of order $N^2 n_k/n_k^2$. We thus have that for $n_k \geq N$, $\mathbb{E}[S_N(h_k)^2] \lesssim N^2/n_k$, hence by (12),

$$\begin{aligned} \sum_{k \geq i(N)+1} \mathbb{E}[S_N(h_k)^2] &\lesssim \sum_{k \geq i(N)+1} \frac{N^2}{n_k} \\ &= \frac{N^2}{n_{i(N)+1}} + \sum_{k \geq i(N)+2} \frac{N^2}{n_k} \leq \frac{N^2}{n_{i(N)+1}} \left(1 + \sum_{j \geq i(N)+2} \frac{1}{j^2} \right), \end{aligned}$$

therefore,

$$(19) \quad \sum_{k \geq i(N)+1} \mathbb{E}[S_N(h_k)^2] \lesssim N.$$

Combining (17) and (19), and using (for a fixed N) the independence of the sequence $(S_N(h_j))_{j \geq 1}$, we conclude that $\sigma_N^2(h) = \sigma_N^2(g - g \circ T) \lesssim N$.

When we add a mean-zero nondegenerate independent sequence $(m \circ T^i)_{i \in \mathbb{Z}}$ with moments of any order greater than 2, the variance of the N th partial sum of $((m + h) \circ T^i)_{i \geq 1}$ is bounded above and below by a quantity proportional to N , hence c) is satisfied in Theorems A and A'. By the observation in the paragraph "About the method of proof", a) holds. \square

2.5. Moments of the coboundary and the transfer function. One can wonder to which \mathbb{L}^p space can g and $g - g \circ T$ belong.

Proposition 14. *Under the conditions (11) and (12), we have $g \in \mathbb{L}^p$ for $0 < p < 1$ and $g - g \circ T \in \mathbb{L}^p$ for each $p > 0$.*

Proof. Let $g_k := U^{-1}v_k$, where $v_k = \sum_{j=0}^{n_k-1} U^{-j}s_k$ and $s_k = \sum_{j=0}^{n_k-1} U^{-j}e_k$. Recall that $g = -\sum_{k=1}^{\infty} g_k$. For $0 < p < 1$ and any two non-negative real numbers a and b , we have $(a+b)^p \leq a^p + b^p$. This gives, using (4),

$$\begin{aligned} \mathbb{E} |g_k|^p &\leq \left(\sum_{j=1}^{n_k} j^p \mathbb{E} |U^{-j}e_k| + \sum_{j=1}^{n_k-1} (n_k - j)^p \mathbb{E} |U^{-j+n_k}e_k| \right) \\ &= \frac{1}{n_k^2} \left(n_k^p + 2 \sum_{j=1}^{n_k-1} j^p \right) \\ &\leq \frac{1}{n_k^2} (n_k^p + 2n_k^{p+1}) \\ &\leq 3n_k^{p-1}. \end{aligned}$$

By (12), we have $n_k \geq k! \cdot n_1$ hence the series $\sum_{k \geq 1} \mathbb{E} |g_k|^p$ is convergent. This proves that $g \in \mathbb{L}^p$ for $0 < p < 1$.

Corollary 2.4. in [1] states the following: given positive integers t and p , X_1, \dots, X_t independent random variables such that $\mu \{0 \leq X_j \leq 1\} = 1$ for each $j \in [t]$, then

$$(20) \quad \mathbb{E} (X)^p \leq B_p \cdot \max \{ \mathbb{E}(X), (\mathbb{E}(X))^p \},$$

where B_p is the p -th Bell's number (defined by the recursion relation $B_{p+1} = \sum_{k=0}^p \binom{p}{k} B_k$ and $B_0 = B_1 = 1$) and $X := \sum_{j=1}^t X_j$.

We shall show that the series $\sum_{k \geq 1} \|h_k\|_p$ is convergent for any integer p . Fix $k \geq 1$, and let $t := 2n_k$, $X_j := |U^{j-2n_k}e_k|$. Applying (20), we get

$$\|h_k\|_p^p \leq B_p \cdot \max \{ 2n_k^{-1}, (2n_k^{-1})^p \} = 2B_p \cdot n_k^{-1},$$

hence $\|h_k\|_p \leq (2B_p)^{1/p} \cdot n_k^{-1/p}$ and condition (12) guarantees the convergence of the series $\sum_k n_k^{-1/p}$. One could also use Rosenthal's inequality. \square

Since the added process has moments of any order, Proposition 14 proves e) in Theorems A and A'.

Proposition 15. *The transfer function g does not belong to \mathbb{L}^1 .*

Proof. Fix an integer k , and define for $1 \leq j \leq n_k$:

$$E_j := \{ |U^{-j}e_k| = 1 \} \cap \bigcap_{i \in \{1, \dots, 2n_k-1\} \setminus \{j\}} \{ U^{-i}e_k = 0 \} \cap \bigcap_{l \neq k} \{ g_l = 0 \}.$$

Since these sets are pairwise disjoint, $g = \sum_{k \geq 1} g_k$, with $g_k := U^{-1} \sum_{i=0}^{n_k-1} U^{-i} \left[\sum_{h=0}^{n_k-1} U^{-h} e_k \right]$ and $g_l(\omega) = 0$ if $l \neq k$ and $\omega \in \bigcup_{j=1}^{n_k} E_j$, we have the equality of functions

$$\begin{aligned}
|g_k| \cdot \chi \left(\bigcup_{j=1}^{n_k} E_j \right) &= \sum_{j=1}^{n_k} |g_k| \cdot \chi(E_j) \\
&= \sum_{j=1}^{n_k} \chi(E_j) \cdot \left\| U^{-1} \sum_{i=0}^{n_k-1} U^{-i} \left[\sum_{h=0}^{n_k-1} U^{-h} e_k \right] \right\| \\
&= \sum_{j=1}^{n_k} \chi(E_j) \cdot \left\| \sum_{i=0}^{n_k-1} \sum_{h=0}^{n_k-1} U^{-1-(i+h)} e_k \right\| = \sum_{j=1}^{n_k} \chi(E_j) |j U^{-j} e_k| \\
&= \sum_{j=1}^{n_k} j \cdot \chi(E_j)
\end{aligned}$$

and hence

$$(21) \quad \left\| |g_k| \cdot \chi \left(\bigcup_{j=1}^{n_k} E_j \right) \right\|_1 = \sum_{j=1}^{n_k} j \cdot \mu(E_j).$$

As $(1 - 1/n_k^2)^{-1} \rightarrow 1$ for $k \rightarrow +\infty$ and $\prod_{j \geq 1} (1 - 1/n_j^2)^{2n_j}$ is positive, we get

$$(22) \quad \mu(E_j) \geq \frac{1}{n_k^2} \left(1 - \frac{1}{n_k^2}\right)^{2n_k-1} \prod_{l \neq k} \left(1 - \frac{1}{n_l^2}\right)^{2n_l} \geq \frac{c}{n_k^2}$$

for some positive constant c independent of k and j .

Let us define $F_k := \bigcup_{j=1}^{n_k} E_j$ for $k \geq 1$. Notice that the event F_k , $k \geq 1$ are pairwise disjoint because $F_k \subset \{g_k \neq 0\} \cap \bigcap_{l \neq k} \{g_l = 0\}$. Therefore, combining (21) and (22), we obtain for each $k \geq 1$,

$$\mathbb{E}[|g| \chi(F_k)] = \mathbb{E}[|g_k| \chi(F_k)] \geq c/2.$$

It then follows that

$$\mathbb{E}|g| \geq \sum_k \mathbb{E}[|g_k| \chi(F_k)] = \infty,$$

proving Proposition 15. □

Acknowledgements. The authors would like to thank an anonymous referee for a great number of helpful remarks and corrections. In particular, the referee suggested the present proof of Lemma 5 which is simpler and easier to read than the proof in the original version. During main part of the research the second author was visiting the Department of Statistics of the University of Michigan; he thanks for the hospitality of U-M. Both authors thank Professor Magda Peligrad for suggesting the topic and for encouragement.

REFERENCES

- [1] Daniel Berend and Tamir Tassa, *Improved bounds on Bell numbers and on moments of sums of random variables*, Probab. Math. Statist. **30** (2010), no. 2, 185–205. MR 2792580 (2012k:60058)
- [2] Patrick Billingsley, *Convergence of probability measures*, John Wiley & Sons Inc., New York, 1968. MR 0233396 (38 #1718)

- [3] Richard C. Bradley, *On the central limit question under absolute regularity*, Ann. Probab. **13** (1985), no. 4, 1314–1325. MR 806228 (87b:60031)
- [4] ———, *Introduction to strong mixing conditions. Vol. 1*, Kendrick Press, Heber City, UT, 2007. MR 2325294 (2009f:60002a)
- [5] Jérôme Dedecker, Florence Merlevède, and Dalibor Volný, *On the weak invariance principle for non-adapted sequences under projective criteria*, J. Theoret. Probab. **20** (2007), no. 4, 971–1004. MR 2359065 (2008g:60088)
- [6] P. Hall and C. C. Heyde, *Martingale limit theory and its application*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980, Probability and Mathematical Statistics. MR 624435 (83a:60001)
- [7] Norbert Herrndorf, *The invariance principle for φ -mixing sequences*, Z. Wahrsch. Verw. Gebiete **63** (1983), no. 1, 97–108. MR 699789 (84e:60040)
- [8] H. O. Hirschfeld, *A connection between correlation and contingency*, Mathematical Proceedings of the Cambridge Philosophical Society **31** (1935), 520–524.
- [9] I. A. Ibragimov, *Some limit theorems for stochastic processes stationary in the strict sense*, Dokl. Akad. Nauk SSSR **125** (1959), 711–714. MR 0105180 (21 #3923)
- [10] ———, *Some limit theorems for stationary processes*, Teor. Veroyatnost. i Primenen. **7** (1962), 361–392. MR 0148125 (26 #5634)
- [11] ———, *A remark on the central limit theorem for dependent random variables*, Teor. Veroyatnost. i Primenen. **20** (1975), 134–140. MR 0362448 (50 #14889)
- [12] A. N. Kolmogorov and Ju. A. Rozanov, *On a strong mixing condition for stationary Gaussian processes*, Teor. Veroyatnost. i Primenen. **5** (1960), 222–227. MR 0133175 (24 #A3009)
- [13] Emmanuel Rio, *Théorie asymptotique des processus aléatoires faiblement dépendants*, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 31, Springer-Verlag, Berlin, 2000. MR 2117923 (2005k:60001)
- [14] M. Rosenblatt, *A central limit theorem and a strong mixing condition*, Proc. Nat. Acad. Sci. U. S. A. **42** (1956), 43–47. MR 0074711 (17,635b)
- [15] Qi Man Shao, *On the invariance principle for ρ -mixing sequences of random variables*, Chinese Ann. Math. Ser. B **10** (1989), no. 4, 427–433, A Chinese summary appears in Chinese Ann. Math. Ser. A **10** (1989), no. 5, 640. MR 1038376 (91g:60034)
- [16] V. A. Volkonskiĭ and Yu. A. Rozanov, *Some limit theorems for random functions. I*, Teor. Veroyatnost. i Primenen **4** (1959), 186–207. MR 0105741 (21 #4477)
- [17] Dalibor Volný and Pavel Samek, *On the invariance principle and the law of iterated logarithm for stationary processes*, Mathematical physics and stochastic analysis (Lisbon, 1998), World Sci. Publ., River Edge, NJ, 2000, pp. 424–438. MR 1893125 (2003b:60049)

UNIVERSITÉ DE ROUEN, LMRS, AVENUE DE L'UNIVERSITÉ, BP 12 76801 SAINT-ÉTIENNE-DU-ROUVRAY CEDEX, FRANCE.

E-mail address: `davide.giraudol@univ-rouen.fr` and `dalibor.volny@univ-rouen.fr`