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Continuous-time limit of dynamic games with incomplete information and a more informed player.

Fabien Gensbittel*

November 28, 2013

Abstract

We study a model of two-player, zero-sum, dynamic game with incomplete information on one side in which the players receive exogenous information about the payoff-relevant state variable during the play. We assume that one of the players is always more informed than his opponent and that signals observed during the play correspond to the observation of some continuous-time Markov process at some fixed times on a finite grid. We show the existence of a limit value as the players play more and more frequently, and we provide two different characterizations for this value: through a stochastic optimization problem and through a variational inequality, related to some second-order Hamilton-Jacobi equation in some particular cases.

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*Toulouse School of Economics (GREMAQ, Université Toulouse 1 Capitole), Manufacture des Tabacs, MF213, 21, Allée de Brienne 31015 Toulouse Cedex 6. E-mail:fabien.gensbittel@tse-fr.eu

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1 Introduction.

This paper contributes to the literature on dynamic games with incomplete information, by introducing exogenous time-evolving constraints on information into the classical model of zero-sum repeated games with incomplete information.

A key feature appearing in recent contributions to the field of zero-sum dynamic games is the interplay between discrete-time and continuous-time dynamic models, as in Cardaliaguet-Laraki-Sorin [13], Neyman [44] or Cardaliaguet-Rainer-Rosenberg-Vieille [12], where the authors consider sequences of discrete-time dynamic games in which the players play more and more frequently. Such an analysis is often related to the study of a sequence of discretizations in time of a given continuous-time dynamic game. In the present work, we follow this approach in order to study a sequence of discrete-time zero-sum dynamic games where one player is always more informed than is opponent and where the inflow of information evolves according to an exogenous Markov process.

However, we depart from several papers (see e.g. Renault [45], Neyman [42], Gensbittel-Renault [23], Cardaliaguet-Rainer-Rosenberg-Vieille [12]) focusing on models with incomplete information and evolving types, i.e. where the unknown payoff-relevant parameters of the game are partially unknown and evolve over time (for example in [45], the stage payoffs depend on the current value of some Markov chain and one player observes this variable while his opponent only observes past actions). In the present work, we consider a model with only one payoff-relevant variable which is not evolving over time, but we introduce a different time-evolving feature by allowing players to learn information about this variable during the game through a sequence of signals. Precisely, we consider a sequence of discrete-time games with incomplete information and perfect observation of actions, where stages occur at times

$(t_q)_{q=0,\dots,n}$ for some finite grid $(t_q)_{q=0,\dots,n}$ in $[0, 1]$. At each stage, the more informed player (say player 1) observes a pair of signals (X_{t_q}, Y_{t_q}) while the less informed player (player 2) only observes Y_{t_q} , where $(X_t, Y_t)_{t \in [0,1]}$ is an exogenous continuous-time Markov process. The stage payoff function is assumed to depend on actions of both players, and on the final value of the Markov process (X_1, Y_1) . We assume that the stage payoffs are not observed and we analyze the limit of these games as the players play more and more frequently.

Let us briefly recall the exchange game introduced in De Meyer [16], which adapted to the present framework provides an interesting interpretation: Y_t is the price of some risky asset at time t , and the evolution of this price process depends on some auxiliary process X_t (we will analyze the case where Y is a real-valued Itô process whose drift component at time t depends on X_t). The informed player has access to the private signal X_t and to a public signal Y_t and is facing an opponent (which may also be seen as an aggregation of small players), having only access to the public signal Y_t . The decisions of the informed player are observed and therefore modify the evolution of public beliefs about the future evolution of the price process Y . The overall payoff of the game is the expected terminal value of the portfolio of the informed player, portfolio composed by shares of some risk neutral asset and shares of different European derivatives on R with maturity date $t = 1$. Each stage of the game results in an exchange of shares of these assets between the players, so that the expected benefit resulting from this exchange depends on the value of the risky asset at the maturity date $t = 1$, which explains why we assume that the realized stage payoffs are not observed (players can only compute their expected payoff conditionally on their information after each stage).

We provide two characterizations for the limit value of these games as the mesh of the partition of $[0, 1]$ induced by the grid $(t_q)_{q=0,\dots,n}$ goes to zero: The first one is a probabilistic representation formula, which was first introduced by De Meyer in [16] for a class discrete-time financial games and later extended to particular classes of differential games (see Cardaliaguet-Rainer [10, 11] and Grün [25, 24]), to repeated games games with infinite action spaces (see Gensbittel [21]), and more recently to continuous-time limits of Markov Games in Cardaliaguet-Rainer-Rosenberg-Vieille [12]. This representation formula is important as it provides a characterization of optimal processes of revelation (martingales of posteriors induced by optimal strategies). We first prove such a formula for the discrete-time games we analyze and then we pass to the limit by using a new result on the maximal norm-variation of measure-valued martingales. The problem of maximal norm-variation of martingales is intimately related with the asymptotic analysis of dynamic games with incomplete information on one side, and was already studied by Mertens and Zamir [40], De Meyer [15, 16], Neyman [43] and more recently Sandomirskiy [46]. We provide an upper bound for the maximal norm-variation of martingales taking values in measure over infinite spaces which differs from [46] in the sense that we consider the variation with respect to the Wasserstein norm instead of the total variation norm, obtaining thus stronger upper bounds at the cost of reducing the class of games for which this result applies (we ask for Lipschitz-continuity of the payoff function with respect to the state variable). The second characterization is a general variational characterization based on a dynamic programming principle in continuous-time. We apply this variational characterization to the particular case where Y is an Itô process whose drift parameter depends on some continuous-time Markov chain X_t . The limit value function is then proved to be the smallest concave supersolution of some PDE defined on a finite-dimensional simplex. This is to relate with the two equivalent PDE characterizations (dual solution or

solution of an obstacle problem) for the value function of differential games with incomplete information introduced by Cardaliaguet in [7, 8], and generalized in Cardaliaguet-Rainer [9], Grün [24] and Cardaliaguet-Rainer-Rosenberg-Vieille [12], as the the notion of dual solution is proved to be equivalent to the characterization as the smallest concave supersolution for a particular case in [11].

Finally, we also study a simple example in which the less informed player does not receive any signals, and for which we provide a more explicit formula for the limit value, based on the concavification operator. This formula relates our results to the classical $\text{Cav}(u)$ theorem of Aumann and Machler [3].

2 Main results.

Let us at first describe the continuous-time game we will approximate. This description is incomplete as we do not define strategies in continuous-time. Rather, we define below strategies in the different time-discretizations of this game adapted to finite partitions of the time interval $[0, 1]$. The notion of value for this game will therefore be the limit value along any sequence of discretizations when the mesh of the corresponding partitions go to zero. Precisely, we will prove that this value exists and is independent of the chosen sequence.

The following data are fixed throughout the paper, possible extensions (more general payoff functions, inhomogeneous processes...) are discussed in section 9.

- I, J are Polish spaces and denote the action sets of players 1 and 2 respectively.
- \mathcal{X} and \mathcal{Y} are Polish spaces, and $\mathcal{Z} := \mathcal{X} \times \mathcal{Y}$ is equipped with the complete metric d .
- $g(z, i, j) : \mathcal{Z} \times I \times J \rightarrow [0, 1]$ is a measurable payoff function.
- $(Z_s)_{s \geq t_0} = (X_s, Y_s)_{s \geq t_0}$ is an homogeneous \mathcal{Z} -valued Markov process with initial law ν and transition probabilities $(P_s)_{s \geq 0}$.

Notation 2.1. For any topological space E , $C_b(E)$ denotes the set of real-valued bounded continuous functions on E , $\Delta(E)$ denotes the set of Borel probability distributions on E endowed with the weak topology generated by $C_b(E)$ and the associated Borel σ -algebra. δ_x denotes the Dirac measure on $x \in E$. Finite sets are endowed with the discrete topology and Cartesian products with the product topology.

All the results are proved under the following assumption.

Hypothesis 2.2.

H1 g is Lipschitz with respect to $z \in \mathcal{Z}$, uniformly in $(i, j) \in I \times J$ (for convenience, the Lipschitz constant is assumed to be equal to 1).

Let us consider the following (heuristic) game, played on the time interval $[t_0, 1]$:

- Player 1 observes the trajectory of $Z = (X, Y)$.
- Player 2 observes only the trajectory of Y .

- They play the game $G(t_0, \nu)$ with total expected payoff:

$$\mathbb{E}\left[\int_{t_0}^1 g(Z_1, i_s, j_s) ds\right],$$

where i_s (resp. j_s) denote the action of player 1 at time s (resp. of player 2).

- Actions are observed during the game (and potentially convey relevant information).

We aim at studying how information is used by the informed player when playing optimally. In order to achieve this goal, we introduce a sequence of time-discretizations of this game. For simplicity, and without loss of generality, let us fix $t_0 = 0$ and consider for all $n \in \mathbb{N}^*$, the uniform partition of $[0, 1]$ of mesh $\frac{1}{n}$. The corresponding discrete-time game, denoted $G_n(\nu)$ proceeds as follows:

- $(Z_t)_{t \geq 0}$ is an homogeneous Markov process with initial law ν .
- The variable $Z_{\frac{q}{n}} = (X_{\frac{q}{n}}, Y_{\frac{q}{n}})$ is observed by player 1 before stage q for $q = 0, \dots, n-1$.
- The variable $Y_{\frac{q}{n}}$ is observed by player 2 before stage q for $q = 0, \dots, n-1$.
- At each stage, both players choose simultaneously a pair of actions $(i_q, j_q) \in I \times J$.
- Chosen actions are observed after each stage.
- Stage payoff of player 1 equals $g(Z_1, i_q, j_q)$ (realized stage payoffs are not observed).

The description of the game is common knowledge and we consider the game played in behavior strategies: at round q ($q = 0, \dots, n-1$), player 1 and player 2 select simultaneously and independently an action $i_q \in I$ for player 1 and $j_q \in J$ for player 2 using some lotteries depending on their information and past observations.

Formally, a behavior strategy σ for player 1 is a sequence $(\sigma_0, \dots, \sigma_{n-1})$ of transition probabilities:

$$\sigma_q : (\mathcal{Z} \times I \times J)^q \times \mathcal{Z} \rightarrow \Delta(I),$$

where $\sigma_q(Z_0, i_0, j_0, \dots, Z_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Z_{\frac{q}{n}})$ denotes the lottery used to select the action i_q played at round q by player 1 when past actions played during the game are $(i_0, j_0, \dots, i_{q-1}, j_{q-1})$ and the sequence of signals he received is $(Z_0, \dots, Z_{\frac{q}{n}})$. Let Σ_n denote the set of behavior strategies for player 1.

Similarly, a behavior strategy τ for player 2 is a sequence $(\tau_0, \dots, \tau_{n-1})$ of transition probabilities depending only of his past observations

$$\tau_q : (\mathcal{Y} \times I \times J)^q \rightarrow \Delta(J).$$

Let \mathcal{T}_n denote the set of behavior strategies for player 2.

A triplet (ν, σ, τ) induces by Tulcea's theorem (see e.g. theorem 5.17 in [30]) a unique probability $\mathbb{P}_{(\nu, \sigma, \tau)} \in \Delta(\mathcal{Z}^n \times I^n \times J^n)$. The payoff function in $G_n(\nu)$ is defined by

$$\gamma_n(\nu, \sigma, \tau) := \mathbb{E}_{\mathbb{P}_{(\nu, \sigma, \tau)}} \left[\frac{1}{n} \sum_{q=0}^{n-1} g(Z_1, i_q, j_q) \right].$$

The (lower) value of this game is defined by

$$V_n(\nu) := \sup_{\sigma \in \Sigma_n} \inf_{\tau \in \mathcal{T}_n} \gamma_n(\nu, \sigma, \tau).$$

Remark 2.3. *One may add ad hoc compactness and continuity assumptions in order for the value to exist (i.e. that the sup and inf commute in the above formula), and in this case all the following results may be expressed in terms of value functions. We chose to keep the formulation using the lower value since our results do not require the existence of the value. Moreover, the proof of the same kind of results for the upper value functions may rely on different tools as shown in [21]. Let us also mention that for this reason, optimal strategies of the informed player are understood as strategies guaranteeing the lower value.*

Let us now describe an important auxiliary one shot game denoted $\Gamma(\xi)$ where $\xi \in \Delta(\Delta(\mathcal{Z}))$. Consider a pair of random variables $(Z, \mu) \in \mathcal{Z} \times \Delta(\mathcal{Z})$ where μ has law ξ and the conditional law of Z given μ is precisely equal to μ . The game $\Gamma(\xi)$ proceeds as follows: player 1 observes the signal μ , player 2 does not observe any signal, both players choose simultaneously actions $(i, j) \in I \times J$, and the realized payoff of player 1 is $g(Z, i, j)$. Given a strategy $\sigma : \Delta(\mathcal{Z}) \rightarrow \Delta(I)$ for player 1 and a strategy $\tau \in \Delta(J)$ of player 2, the triplet (ξ, σ, τ) induces a probability distribution over $\mathcal{Z} \times \Delta(\mathcal{Z}) \times I \times J$ denoted $\mathbb{P}(\xi, \sigma, \tau)$. By construction, the expected payoff of player 1 is equal to:

$$\mathbb{E}_{\mathbb{P}(\xi, \sigma, \tau)}[g(Z, i, j)] = \mathbb{E}_{\mathbb{P}(\xi, \sigma, \tau)}[\mathbb{E}_{\mathbb{P}(\xi, \sigma, \tau)}[g(Z, i, j) | \mu, i, j]] = \mathbb{E}_{\mathbb{P}(\xi, \sigma, \tau)}\left[\int_{\mathcal{Z}} g(z, i, j) d\mu(z)\right].$$

Let us denote W the lower value of this game, defined on $\Delta(\Delta(\mathcal{Z}))$ by the relation

$$W(\xi) := \sup_{\sigma: \Delta(\mathcal{Z}) \rightarrow \Delta(I)} \inf_{\tau \in \Delta(J)} \mathbb{E}_{\mathbb{P}(\xi, \sigma, \tau)}\left[\int_{\mathcal{Z}} g(z, i, j) d\mu(z)\right],$$

where the supremum is taken over all the measurable maps σ .

We also need to consider the lower value function U of the non-revealing game, which is a modification of $\Gamma(\xi)$ in which player 1 cannot use his information. Precisely,

$$U(\xi) := \sup_{\sigma \in \Delta(I)} \inf_{\tau \in \Delta(J)} \mathbb{E}_{\mathbb{P}(\xi, \sigma, \tau)}\left[\int_{\mathcal{Z}} g(z, i, j) d\mu(z)\right].$$

Note that since σ does not depend on μ , the above expression depends on ξ only through the barycenter¹ of ξ denoted $\bar{\mu} := \text{Bar}(\xi)$. Precisely, it induces a function u defined on $\Delta(\mathcal{Z})$ by the relation

$$u(\bar{\mu}) := U(\xi).$$

This function coincides with the classical function u appearing in the well-known Cav(u) theorem of Aumann and Maschler and may also be seen as the instantaneous value function of the continuous-time game described above.

As we will manipulate conditional laws extensively, we introduce a specific notation in order to shorten several statements and proofs.

¹see Definition 3.1

Notation 2.4. Let E be a Polish space and A be an E -valued random variable defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

- $\llbracket A \rrbracket$ denotes the law of A .
- Given a σ -field $\mathcal{F} \subset \mathcal{A}$, $\llbracket A \mid \mathcal{F} \rrbracket$ denotes a version of the conditional law of A given \mathcal{F} , hence an \mathcal{F} -measurable random variable with values in $\Delta(E)$ (see e.g. [4] Proposition 7.26 for this last point).

In order to state the first main result, we need the following definitions.

Definition 2.5. Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space and $\mathcal{F}, \mathcal{G}, \mathcal{H}$ three sub σ -fields of \mathcal{A} . \mathcal{F} and \mathcal{G} are conditionally independent given \mathcal{H} if

$$\forall F \in \mathcal{F}, \forall G \in \mathcal{G}, \mathbb{P}(F \cap G \mid \mathcal{H}) = \mathbb{P}(F \mid \mathcal{H})\mathbb{P}(G \mid \mathcal{H}).$$

This relation is denoted $\mathcal{F} \amalg_{\mathcal{H}} \mathcal{G}$ and the definition extends to random variables by considering the σ -fields they generate.

Usual properties of conditional laws and conditional independence are recalled in section 4.

Definition 2.6. Given two filtrations $(\mathcal{F}_q, \mathcal{G}_q)_{q=0, \dots, n}$ on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we say that $(\mathcal{F}_q)_{q=0, \dots, n}$ is non-anticipative with respect to $(\mathcal{G}_q)_{q=0, \dots, n}$ if

$$\forall q = 0, \dots, n-1, \mathcal{F}_q \amalg_{\mathcal{G}_q} \mathcal{G}_n.$$

We may replace filtrations by random processes $(A_q, B_q)_{q=0, \dots, n}$ (with values in some Polish spaces) defined on $(\Omega, \mathcal{A}, \mathbb{P})$ in the definition by considering the natural filtrations they generate. For example, $(A_q)_{q=0, \dots, n}$ is non-anticipative with respect to $(B_q)_{q=0, \dots, n}$ if

$$\forall q = 0, \dots, n-1, (A_0, \dots, A_q) \amalg_{B_0, \dots, B_q} (B_0, \dots, B_n).$$

The following theorem is proved in section 4.

Theorem 2.7.

$$\forall \nu \in \Delta(\mathcal{Z}), \quad V_n(\nu) = \frac{1}{n} \sup_{(\tilde{Z}_q^n, \pi_q, \mathcal{F}_q)_{q=0, \dots, n} \in \mathfrak{B}_n(\nu)} \Psi_n((\tilde{Z}_q^n, \pi_q, \mathcal{F}_q)_{q=0, \dots, n}). \quad (2.1)$$

where

$$\tilde{Z}_q^n = (\tilde{X}_q^n, \tilde{Y}_q^n), \quad \Psi_n((\tilde{Z}_q^n, \pi_q, \mathcal{F}_q)_{q=0, \dots, n}) := \mathbb{E} \left[\sum_{q=0}^{n-1} W([\pi_q \mid \mathcal{F}_{q-1}, \tilde{Y}_q^n]) \right], \quad (2.2)$$

and the supremum is taken over the collection $\mathfrak{B}_n(\nu)$ of filtered probability spaces $(\Omega, \mathcal{A}, (\mathcal{F}_q)_{q=0, \dots, n}, \mathbb{P})$ and variables $(\tilde{Z}_q^n, \pi_q)_{q=0, \dots, n}$ defined on Ω such that:

- $(\tilde{Z}_q^n)_{q=0, \dots, n}$ has the same law as $(Z_{\frac{q}{n}})_{q=0, \dots, n}$ (Z denotes a Markov process with initial law ν),

- the filtration $(\mathcal{F}_q)_{q=0,\dots,n}$ is non-anticipative with respect to $(\tilde{Z}_q^n)_{q=0,\dots,n}$,
- for all $q = 0, \dots, n$, \tilde{Y}_q^n is \mathcal{F}_q -measurable and $\pi_q = \llbracket \tilde{Z}_q^n | \mathcal{F}_q \rrbracket$.

Let us comment briefly this result. We generalize here the idea introduced in [16] that the problem the informed player is facing can be decomposed into two parts: at first he may decide how information will be used during the whole game, and then maximize his payoff under this constraint. To apply this method of decomposition, we need to identify precisely the set $\mathfrak{B}_n(\nu)$ of achievable processes of revelations (martingales of posterior beliefs on Z_1 of the less informed player). The filtration $(\mathcal{F}_q)_{q=0,\dots,n}$ represents information of player 2. The condition of non-anticipativity reflects the fact that player 2 cannot receive any information on the process Z which is not known by player 1 (an upper bound on information) and the last condition simply says that player 2 observes the signal Y_q at stage q (a lower bound on information). Maximizers of the right-hand side of equation (2.1) represent optimal processes of revelation for the informed player.

The main results proved in sections 5 and 6 are two different characterizations for the limit of the sequence of value functions V_n . These results are prove under the following assumption.

Hypothesis 2.8.

H2 \mathcal{Z} is a locally compact metric space and Z is the canonical realization of a Feller process with transitions $(P_t)_{t \geq 0}$ on the space of \mathcal{Z} -valued càdlàg trajectories.

Let us now introduce some notations.

Notation 2.9.

- The natural filtration \mathcal{F}^A of a process $(A_t)_{t \in [0,1]}$ is defined by $\mathcal{F}_t^A = \sigma(A_s, s \leq t)$. The associated right-continuous filtration is denoted $\mathcal{F}^{A,+}$ with $\mathcal{F}_t^{A,+} := \cap_{s>t} \mathcal{F}_s^A$.
- For any time interval T and topological space E , $\mathbb{D}(T, E)$ denotes the set of E -valued càdlàg trajectories indexed by T .
- For all $\nu \in \Delta(\mathcal{Z})$, $\mathbb{P}_\nu \in \Delta(\mathbb{D}([0, +\infty), \mathcal{Z}))$ denotes the law of the process Z with initial law ν and \mathbb{E}_ν denotes the corresponding expectation.

Theorem 2.10. For all $\nu \in \Delta(\mathcal{Z})$,

$$V_n(\nu) \xrightarrow{n \rightarrow \infty} V(\nu) := \sup_{\llbracket (Z_t, \pi_t)_{t \in [0,1]} \rrbracket \in \mathcal{B}(\nu)} \mathbb{E} \left[\int_0^1 u(\pi_t) dt \right], \quad (2.3)$$

where $\mathcal{B}(\nu)$ denotes the subset of $\Delta(\mathbb{D}([0, 1], \mathcal{Z} \times \Delta(\mathcal{Z})))$ of laws of processes $(Z_t, \pi_t)_{t \in [0,1]}$ such that:

- $(Z_t)_{t \in [0,1]}$ is a Markov process with initial law ν .
- $(\pi_t)_{t \in [0,1]}$ is a càdlàg modification of $(\llbracket Z_1 | \mathcal{F}_t^{(Y, \pi), +} \rrbracket)_{t \in [0,1]}$.

- $(\pi_t)_{t \in [0,1]}$ is non-anticipative with respect to $(Z_t)_{t \in [0,1]}$, i.e.

$$\forall t \in [0, 1), \quad \mathcal{F}_t^\pi \prod_{\mathcal{F}_t^{Z,+}} \mathcal{F}_1^Z.$$

Theorem 2.10 is the continuous-time analog of Theorem 2.7 and is proved in section 5. The key tools to prove this result are: a general tightness criterion for laws of measure-valued martingales (see Theorem 5.8, which is an easy corollary of the main theorem of [34]), a criterion for stability of conditional independence (see Lemma 5.10), and a generalization of the classical bound of maximal norm-variation of martingales (see Proposition 5.11). This result provides a characterization of the limits of optimal martingales of revelation for the informed player as maximizers of the right-hand side of (2.3) and these optimal solutions induce asymptotically optimal strategies for the informed player in any sequence of discretizations (see Proposition 5.14). This result has to be compared with the results that were obtained for differential games with incomplete information (see [10] and [11]), in the sense that our control problem has the same objective function but a different set of admissible processes.

Section 6 is devoted to the proof of the variational characterization for the limit value function. Exactly as for the function $V(\nu)$, for all $t \in [0, 1]$, let $V(t, \nu)$ denote (with a slight abuse of notations) the limit value of any sequence of discrete-time games played on some sequence of partitions of the time interval $[t, 1]$, whose mesh is going to zero. Using the stationarity of the model, adapting the proof of Theorem 2.10 is straightforward.

Before stating our second characterization, we need the following result. For $t \geq 0$, define $\bar{\theta}_t := \llbracket Z_t | \mathcal{F}_t^{Y,+} \rrbracket$. The process $\bar{\theta}$ represents the beliefs of player 2 over Z_t given the observation of $\mathcal{F}_t^{Y,+}$. In Proposition 6.5, we prove that the process $(\bar{\theta}_t)_{t \geq 0}$ is a Markov process whose transition Q is given by the function

$$Q_t(\nu) \in \Delta(\Delta(\mathcal{Z})), \quad \forall \text{ Borel set } C, \quad Q_t(\nu, C) := \mathbb{P}_\nu \left(\mathbb{E}_\nu[\delta_{Z_t} | \mathcal{F}_t^{Y,+}] \in C \right).$$

Using this result, we obtain the following variational characterization.

Theorem 2.11. *V is the smallest upper semi-continuous function on $[0, 1] \times \Delta(\mathcal{Z})$ such that:*

- i) For all $\nu \in \Delta(\mathcal{Z})$, $V(1, \nu) = 0$.
- ii) For all $t \in [0, 1]$, $V(t, \cdot)$ is concave on $\Delta(\mathcal{Z})$.
- iii) For all $t \in [0, 1)$, for all $\nu \in \Delta(\mathcal{Z})$, for all $h \in (0, 1 - t]$,

$$\int_{\Delta(\mathcal{Z})} V(t+h, \mu) dQ_h(\nu, d\mu) - V(t, \nu) + \int_0^h \int_{\Delta(\mathcal{Z})} \tilde{u}(t+s, \mu) dQ_s(\nu, d\mu) ds \leq 0, \quad (2.4)$$

where for all $s \in [0, 1]$ and $\mu \in \Delta(\mathcal{Z})$, $\tilde{u}(s, \mu) := u(\mu \cdot P_{1-s})$ and $\mu \cdot P_{1-s}$ denotes the law of Z_{1-s} under \mathbb{P}_μ .

The proof of Theorem 2.11 is based on dynamic programming. Concavity is a typical feature of value function for games with incomplete information on one side (see e.g. chapter V.1

in [39]) while (2.4) corresponds to the following very intuitive fact: not using his private information on the interval $[t, t+h)$ is a sub-optimal behavior for player 1.

In section 7, we apply the preceding results to the particular case where X is a continuous-time Markov chain taking values in a finite set, and Y is a noisy observation of X . Precisely, assume that $(X_t)_{t \in [0,1]}$ is a continuous-time homogeneous Markov chain with finite state space \mathcal{X} , infinitesimal generator $\Lambda = (\Lambda_{\alpha,\beta})_{\alpha,\beta \in \mathcal{X}}$ and initial law $p \in \Delta(\mathcal{X})$. We identify \mathcal{X} with $\{1, \dots, N\}$ and $\Delta(\mathcal{X})$ with the canonical simplex Δ_N of \mathbb{R}^N defined by

$$\Delta_N := \{p \in \mathbb{R}^N \mid \forall i, p_i \geq 0, \sum_{i=1}^N p_i = 1\}.$$

Then, define the real-valued process $(Y_t)_{t \in [0,1]}$ as is the unique solution of the following stochastic differential equation (SDE)

$$Y_t = y + \int_0^t b(X_s, Y_s) ds + \int_0^t \sigma(Y_s) dW_t,$$

where $(W_t)_{t \in [0,1]}$ is a standard Brownian motion independent of X and $y \in \mathbb{R}$. The process Y may be seen as some noisy observation of the process X . In terms of the financial interpretation given in the introduction, this corresponds to the fact that player 1 knows the drift parameter of Y whereas player 2 can only form beliefs about this quantity based on the observation of Y .

We assume that the functions b and σ are bounded and Lipschitz, and that there exists $\epsilon > 0$ such that for all $y \in \mathbb{R}$, $\sigma(y) \geq \epsilon$. It's well known that $Z = (X, Y)$ is a well defined Markov process and fulfills the assumptions needed in order to apply Theorem 2.11. Moreover, using classical results in filtering theory, the semi-group Q is related to the semi-group of a diffusion process taking values in $\Delta_N \times \mathbb{R}$.

Precisely, let us define $b(y) := b(\alpha, y)_{\alpha=1, \dots, N} \in \mathbb{R}^N$ and $\chi_t(\alpha) := \mathbb{P}(X_t = \alpha \mid \mathcal{F}_t^{Y,+})$. Using Theorem 9.1 in [38] (see also the Note p.360 about the Markov property), the process $\theta_t := (\chi_t, Y_t) \in \mathbb{R}^{N+1}$ is a diffusion process satisfying the following SDE:

$$\theta_t = \theta_0 + \int_0^t c(\theta_t) dt + \int_0^t \kappa(\theta_t) d\bar{W}_t, \quad (2.5)$$

where \bar{W} is a standard $\mathcal{F}^{Y,+}$ -Brownian motion and the vectors $c(p, y)$ and $\kappa(p, y)$ in $\mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ are defined by

$$c(p, y) := ({}^T \Lambda p, \langle p, b(y) \rangle),$$

$$\kappa(p, y) := \left(\left(\frac{p_\alpha}{\sigma(y)} (b(\alpha, y) - \sum_{\beta=1}^N b(\beta, y) p_\beta) \right)_{\alpha=1, \dots, N}, \sigma(y) \right).$$

Let A denote the infinitesimal generator of the diffusion θ . From the particular structure of the process Z , we deduce that the study of the function V can be restricted to initial laws ν that can be written as $p \otimes \delta_y$ for some $(p, y) \in \Delta_N \times \mathbb{R}$ (see Lemma 7.6). This leads us to introduce the following notation:

Notation 2.12. For all $(t, p, y) \in [0, 1] \times \Delta_N \times \mathbb{R}$:

$$\tilde{u}(t, p, y) := \tilde{u}(t, p \otimes \delta_y), \quad V(t, p, y) := V(t, p \otimes \delta_y).$$

In order to state the main result of section 7, we need to define precisely the notion of weak solution we will use.

Definition 2.13. We say that a bounded lower semi-continuous function f is a (viscosity) supersolution of the equation

$$\frac{\partial f}{\partial t} + A(f)(t, p, y) + \tilde{u}(t, p, y) \leq 0 \tag{2.6}$$

on $[0, 1] \times \Delta_N \times \mathbb{R}$ if for any test function ϕ , C^2 in a neighborhood of $[0, 1] \times \Delta_N \times \mathbb{R}$ (in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}$) such that $\phi \leq f$ on $[0, 1] \times \Delta_N \times \mathbb{R}$ with equality in $(t, p, y) \in [0, 1] \times \Delta_N \times \mathbb{R}$, we have

$$\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial t}(t, p, y) + A(\phi)(t, p, y) + \tilde{u}(t, p, y) \leq 0.$$

Remark 2.14. Note that the function f is solution up to the boundary of Δ_N and not only in its relative interior. This is to relate with the recent results of Cardaliaguet-Rainer-Rosenberg-Vieille (see remark 5 p.19 in [12]).

Using Theorem 2.11 and classical regularization methods for viscosity solutions (see Proposition 7.13), we prove the following result.

Theorem 2.15. V is continuous and is the smallest bounded lower semi-continuous supersolution of (2.6) such that:

- for all $(p, y) \in \Delta_N \times \mathbb{R}$, $V(1, p, y) = 0$,
- for all $(t, y) \in [0, 1] \times \mathbb{R}$, $p \rightarrow V(t, p, y)$ is concave on Δ_N .

Section 8 is devoted to the analysis of a simple example, where player 2 does not receive signals (i.e., \mathcal{Y} is reduced to a single point) and where a more explicit formula can be given for the limit value V , related with the concavification operator appearing in the well-known Cav(u) theorem of Aumann and Maschler.

Note that all our results apply to the particular case where \mathcal{Y} is reduced to a single point, and that several proofs of sections 5 and 6 can be simplified in this case. In the following, we assume that $\mathcal{Y} = \{\emptyset\}$ and we identify \mathcal{Z} with \mathcal{X} so that V and u are defined on $\Delta(\mathcal{X})$.

The example we present here is inspired from an example given in [10]. It is actually a kind of dual example, since figure 2.1 below appears in both examples, related to the optimal martingale of beliefs in [10] and as a constraint here.

Consider two continuous functions $f, h : [0, 1] \rightarrow [0, 1]$ such that $f < h$, f is non-increasing, h is non-decreasing, $f(1) = 0$ and $h(1) = 1$. Define the process X as follows: $X_0 \in [0, f(0)] \cup [h(0), 1]$, X is a martingale which is almost surely constant until it reaches the set $\text{graph}(f) \cup \text{graph}(h)$ (where $\text{graph}(f)$ denotes the graph of the function f), and then stays in the set $\text{graph}(f) \cup \text{graph}(h)$ with probability 1 (see figure 2.1 below).

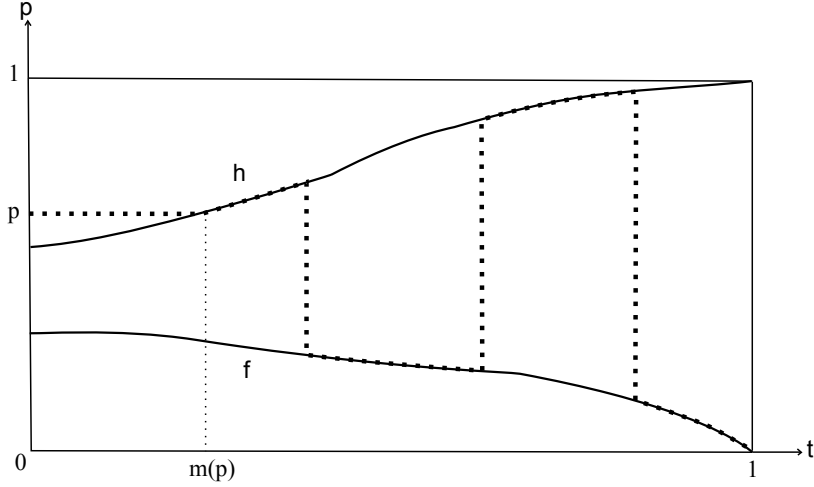


Figure 2.1: The process X (in dotted lines).

This description defines uniquely a martingale X and if we assume that for $t \geq 1$, $X_t = X_1$, $h(t) = 1$, and $f(t) = 0$, then (t, X_t) is easily seen to be a Feller process with state space $\{(t, x) \in \mathbb{R}_+ \times [0, 1] \mid x \geq h(t), \text{ or } x \leq f(t)\}$ (see Lemma 8.1), allowing us to apply Theorem 2.10 (see section 9 for the non-homogeneous case).

Let us denote $\bar{p} := \mathbb{E}_\nu[X_0]$. We assume that either $X_0 = \bar{p}$ almost surely for some $\bar{p} \in [0, f(0)] \cup [h(0), 1]$ or that $\nu = \lambda\delta_{f(0)} + (1 - \lambda)\delta_{h(0)}$ so that $\bar{p} \in [f(0), h(0)]$. Using that $X_1 \in \{0, 1\}$ almost surely, for any admissible process (X, π) in Theorem 2.10, the martingale π_t takes values in the set $\Delta(\{0, 1\})$, identified as a subset of $\Delta([0, 1])$. It is therefore sufficient to consider the martingale $\pi_t(\{1\}) \in [0, 1]$, and this reduces the problem to a one-dimensional problem. Define the function $\bar{u} : [0, 1] \rightarrow \mathbb{R}$ by

$$\bar{u}(p) := u(p\delta_1 + (1 - p)\delta_0).$$

A precise analysis of the maximization problem given by Theorem 2.10 in this case allows us to prove the following representation formula.

Proposition 2.16. *For any closed interval $I \subset [0, 1]$, define $\text{Cav}_I(\bar{u})$ as the smallest concave function on I which is above \bar{u} . Then, under the above assumptions on ν :*

$$V(\nu) = m(\bar{p})\bar{u}(\bar{p}) + \int_{m(\bar{p})}^1 \text{Cav}_{I(s)}(\bar{u})(\bar{p})ds, \quad (2.7)$$

where $I(s) := [f(s), h(s)]$ and $m(p) := \inf\{t \in [0, 1], p \in [f(t), h(t)]\}$. Moreover, if we define $\alpha(s) = \beta(s) = \bar{p}$ for $s \in [0, m(\bar{p})]$ and for $s \in (m(\bar{p}), 1]$:

$$\alpha(s) := \max\{x \in I(s) \mid x \leq \bar{p}, \bar{u}(x) = \text{Cav}_{I(s)}(\bar{u})(x)\},$$

$$\beta(s) := \min\{x \in I(s) \mid x \geq \bar{p}, \bar{u}(x) = \text{Cav}_{I(s)}(\bar{u})(x)\},$$

then α is non-increasing, β is non-decreasing and there exists an optimal martingale p^* taking values in the set $\text{graph}(\alpha^+) \cup \text{graph}(\beta^+)$ where α^+ and β^+ denote respectively the right-continuous versions of α and β .

Figure 2.2 below illustrates the behavior of the optimal martingale p^* .

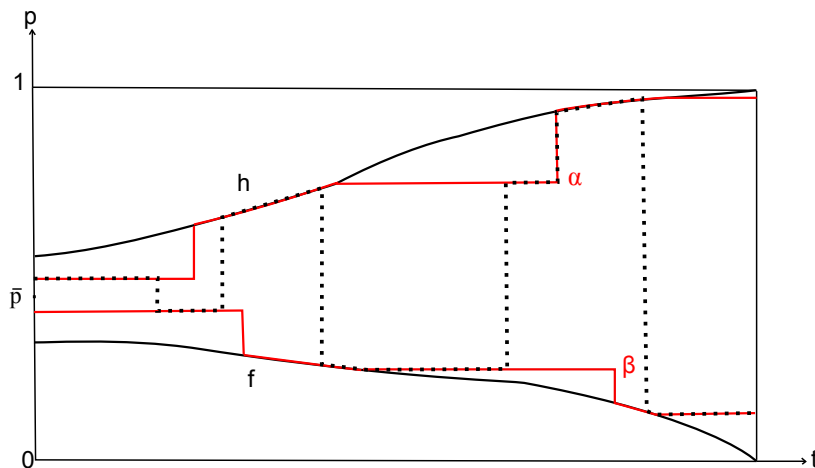


Figure 2.2: The optimal martingale p^* (in dotted lines).

3 The discrete-time games $G_n(\nu)$.

In this section, we study basic properties of the games $G_n(\nu)$, and of the auxiliary functions W and u introduced in section 2. We focus especially on the martingales of posterior beliefs of player 2.

3.1 The martingales of beliefs.

As usual when computing a best reply to some strategy of the informed player 1, we “assume” that player 2 knows which strategy player 1 is using and updates his beliefs on the state variable Z_1 given his observations using Bayes’ rules. This leads to introduce a sequence of virtual beliefs of player 2 about the state variable which depends on the strategy used by both players and is called the a posteriori martingale. Formally, this martingale is the sequence of conditional distributions of the state variable Z_1 given the information held by player 2 after each stage.

3.1.1 Preliminaries.

In the classical model of Aumann and Maschler, the state space \mathcal{Z} being finite, the sequence of conditional distribution is a random process taking values in a finite-dimensional space. In the present model, we allow the state variable to take values in any Polish space. The process of conditional distributions of the payoff-relevant variable Z_1 given past observations of player 2 is therefore a measure-valued process. For this reason, we need to recall definitions and basic notations about measure-valued martingales.

Definition 3.1. Let $\pi \in \Delta(\Delta(\mathcal{Z}))$, then $\bar{\mu} \in \Delta(\mathcal{Z})$ is a (weak) barycenter of π if

$$\forall \phi \in C_b(\mathcal{Z}), \int_K \phi d\bar{\mu} = \int_{\Delta(K)} \left(\int_K \phi d\mu \right) d\pi(\mu). \quad (3.1)$$

Lemma 3.2. Let $\pi \in \Delta(\Delta(\mathcal{Z}))$, then the (weak) barycenter of π is unique and denoted $\text{Bar}(\pi)$. Moreover, the map $\pi \rightarrow \text{Bar}(\pi)$ is continuous from $\Delta(\Delta(\mathcal{Z}))$ to $\Delta(\mathcal{Z})$.

Proof. See e.g. Winkler [48] Proposition 1.1.3 p 16, Corollaries 1.2.3-4 p 23-24, and Example 2.1 p 24. \square

Notation 3.3. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and \mathcal{F} a sub- σ -algebra of \mathcal{A} . Given a random variable X with values in $\Delta(\mathcal{Z})$, the conditional expectation $Y = \mathbb{E}[X|\mathcal{F}]$ denotes the variable $\text{Bar}(\mathcal{L})$ where \mathcal{L} is the conditional law of X given \mathcal{F} . It follows from the definitions that Y is the unique \mathcal{F} -measurable random variable (up to equality \mathbb{P} -almost surely) such that

$$\forall \phi \in C_b(\mathcal{Z}), \int_{\mathcal{Z}} \phi dY = \mathbb{E} \left[\int_{\mathcal{Z}} \phi dX | \mathcal{F} \right].$$

We have therefore the usual definition.

Definition 3.4. Let $(\Omega, \mathcal{A}, (\mathcal{F}_q)_{q \geq 0}, \mathbb{P})$ be a filtered probability space. A stochastic process $(X_q)_{q \geq 0}$ defined on Ω and taking values in $\Delta(\mathcal{Z})$ is an $(\mathcal{F}_q)_{q \geq 0}$ -martingale if

$$\forall q \geq 0, \mathbb{E}[X_{q+1} | \mathcal{F}_q] = X_q.$$

This definition extends to continuous-time martingales in an obvious way.

Remark 3.5. One deduces that usual properties such as the tower property of conditional expectations generalize directly. Note that we may weaken the definition of barycenter by requiring the equality 3.1 to hold only for a countable convergence determining class of functions (see e.g. [1] p106-107, this will be used implicitly in some proofs). One can also extend this equality to bounded (resp. non-negative) measurable functions as usual.

Remark 3.6. The preceding notations for barycenters and conditional barycenters is related to the notation for conditional laws introduced in section 2. Precisely, we have:

$$\llbracket A \rrbracket = \mathbb{E}[\delta_A], \quad \llbracket A | \mathcal{F} \rrbracket = \mathbb{E}[\delta_A | \mathcal{F}],$$

where δ_x denotes the Dirac mass at point x .

Let us now recall some properties of conditional independence.

Lemma 3.7. Let E be a Polish space and A, B be E -valued random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$. Let $\mathcal{F} \subset \mathcal{G}$ be two sub σ -fields of \mathcal{A} .

- If B is independent of (A, \mathcal{F}) , then $\llbracket A | \mathcal{F} \rrbracket = \llbracket A | \mathcal{F}, B \rrbracket$.
- If $\llbracket A | \mathcal{F} \rrbracket = \llbracket A | \mathcal{G} \rrbracket$, then for any σ -field \mathcal{H} such that $\mathcal{F} \subset \mathcal{H} \subset \mathcal{G}$, we have

$$\llbracket A | \mathcal{F} \rrbracket = \llbracket A | \mathcal{H} \rrbracket = \llbracket A | \mathcal{G} \rrbracket$$

Lemma 3.8.

- Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space and $\mathcal{F}, \mathcal{G}, \mathcal{H}$ three sub σ -fields of \mathcal{A} . \mathcal{F} and \mathcal{G} are conditionally independent given \mathcal{H} if

$$\forall F \in \mathcal{F}, \mathbb{P}(F|\mathcal{G}) = \mathbb{P}(F|\mathcal{G}, \mathcal{H}).$$

- Let A, B, C be three random variables (with values in some Polish space) defined on the same probability space. A is independent of B conditionally on C if and only if $\llbracket B|C \rrbracket = \llbracket B|C, A \rrbracket$.
- $A \perp\!\!\!\perp_C B$ if and only if there exists (on a possibly enlarged probability space) a random variable ξ uniform on $[0, 1]$ independent of (A, C) , and a measurable function f such that $B = f(C, \xi)$.

Proof. See Proposition 5.6 in [30] for the two first points, and Proposition 5.13 in [30] for the third point. \square

The next result is a variation on the third point of the above Lemma, and its proof is postponed to the appendix.

Lemma 3.9. *Given two random processes $(A_q, B_q)_{q=0, \dots, n}$ (with values in some Polish spaces), the process $(A_q)_{q=0, \dots, n}$ is non-anticipative with respect to $(B_q)_{q=0, \dots, n}$ if and only if there exists (on a possibly enlarged probability space) a sequence of independent random variables $(\xi_q)_{q=0, \dots, n}$ uniformly distributed on $[0, 1]$ and independent of $(B_q)_{q=0, \dots, n}$, and a sequence of measurable functions f_q (defined on appropriate spaces) such that for all $q = 0, \dots, n$*

$$A_q = f_q(B_p, \xi_p, p \leq q).$$

3.1.2 Beliefs of player 2.

The belief of player 2 after stage q , denoted π_q , is defined as the conditional law of Z_1 given the available information after stage q (i.e. before observing $Y_{\frac{q+1}{n}}$) which is represented by the σ -field $\mathcal{F}_{\mathbb{I}, q} = \sigma(Y_0, i_0, j_0, \dots, Y_{\frac{q}{n}}, i_q, j_q)$. This process is a martingale with respect to the filtration $\mathcal{F}_{\mathbb{I}, q}$ and we have

$$\pi_q = \llbracket Z_1 | \mathcal{F}_{\mathbb{I}, q} \rrbracket = \mathbb{E}[\delta_{Z_1} | \mathcal{F}_{\mathbb{I}, q}].$$

For convenience, define

$$\mathcal{F}_{\mathbb{I}, n} = \sigma(Y_0, i_0, j_0, \dots, Y_{\frac{n-1}{n}}, i_{n-1}, j_{n-1}, Y_1), \quad \text{and } \pi_n := \llbracket Z_1 | \mathcal{F}_{\mathbb{I}, n} \rrbracket.$$

In the next section, we will also consider the the information available to player 2 before playing stage q (i.e. after observing $Y_{\frac{q}{n}}$), corresponding to the σ -field $\mathcal{F}_{\mathbb{I}, q-1} \vee \sigma(Y_{\frac{q}{n}})$ with the convention $\mathcal{F}_{\mathbb{I}, -1} := \sigma(\emptyset)$.

As explained in the previous section, the main property of the sequence of beliefs of player 2 can be expressed informally as follows: player 2 cannot learn information on the state which is not known by player 1. Formally, it implies that the process of beliefs of player 2 is non-anticipative with respect to the signals of player 1 in the sense of definition 2.6.

Lemma 3.10. *For all pair of behavior strategies (σ, τ) , under the probability $\mathbb{P}_{\nu, \sigma, \tau}$, the filtration $(\mathcal{F}_{\mathbb{I}, q})_{q=0, \dots, n}$ is non-anticipative with respect to $(Z_{\underline{q}})_{q=0, \dots, n}$. Consequently, the process $(\pi_q)_{q=0, \dots, n}$ is non-anticipative with respect to $(Z_{\underline{q}})_{q=0, \dots, n}$.*

Proof. These assertions follows from the very definition of behavioral strategies. Indeed, we have

$$\llbracket Z_0, \dots, Z_1 | Z_0, \dots, Z_{\underline{q}} \rrbracket = \llbracket Z_0, \dots, Z_1 | Z_0, \dots, Z_{\underline{q}}, i_0, \dots, i_q, j_0, \dots, j_q \rrbracket,$$

since actions can be written as functions of observations and random variables independent of (Z_0, \dots, Z_1) , corresponding to lotteries used by the players. Using the last point of Lemma 3.8, this equality remains therefore true when conditioning with respect to all the intermediate σ -fields, which proves that $(\pi_q)_{q=0, \dots, n}$ is non-anticipative with respect to $(Z_{\underline{q}})_{q=0, \dots, n}$. \square

Remark 3.11. *Actually, any process $(Z_{\underline{q}}, \pi_q)_{q=0, \dots, n}$ meeting the above non-anticipative property and such that*

$$\begin{aligned} \forall q = 0, \dots, n-1, \pi_q &= \llbracket Z_1 | Y_0, \pi_0, \dots, Y_{\underline{q}}, \pi_q \rrbracket \\ \pi_n &= \llbracket Z_1 | Y_0, \pi_0, \dots, Y_{\underline{n-1}}, \pi_{n-1}, Y_1 \rrbracket \end{aligned}$$

can be achieved as a belief process in our model, assuming that player 1 has a large enough set of actions (for example, if we replace the action set I by $\tilde{I} = I \times [0, 1]$ without modifying the payoff function g , allowing thus player 1 to “send” messages to player 2).

3.2 The auxiliary one shot game $\Gamma(\xi)$.

Let us recall the description of $\Gamma(\xi)$ given in section 2 and the definitions of the functions W and u . In order to study the properties of these functions, we need to introduce the Wasserstein distance on spaces of probabilities.

In the following, we assume that the distance d on \mathcal{Z} is bounded by 1. This is without loss of generality since if g fulfills assumption **H1**, then **H1** also holds when replacing d by $d \wedge 1$. It follows that the Wasserstein space usually denoted $\Delta^1(\mathcal{Z})$ coincides here with $\Delta(\mathcal{Z})$.

Definition 3.12. *The Wasserstein distance on $\Delta(\mathcal{Z})$, denoted \mathbf{d} is defined by*

$$\mathbf{d}(\mu, \nu) = \inf_{\pi} \int_{\mathcal{Z} \times \mathcal{Z}} d(z, z') d\pi(z, z'),$$

where the supremum ranges over all probabilities π on $\mathcal{Z} \times \mathcal{Z}$ having (μ, ν) for marginals.

Let us recall some classical results (see Kantorovitch-Rubinstein [32]).

Theorem 3.13. *The space $(\Delta(\mathcal{Z}), \mathbf{d})$ is a complete separable metric space and \mathbf{d} metrizes the weak convergence. Moreover, for all $\mu, \nu \in \Delta(\mathcal{Z})$*

$$\mathbf{d}(\mu, \nu) = \sup_{f \in \text{Lip}_1(\mathcal{Z})} \int_{\mathcal{Z}} f d\mu - \int_{\mathcal{Z}} f d\nu,$$

where $\text{Lip}_1(\mathcal{Z})$ denotes the set of 1-Lipschitz functions on \mathcal{Z} .

Definition 3.14. The Wasserstein distance on $\Delta(\Delta(\mathcal{Z}))$, denoted $\bar{\mathbf{d}}$ is defined by

$$\bar{\mathbf{d}}(\xi, \xi') = \inf_{\pi} \int_{\mathcal{Z} \times \mathcal{Z}} \mathbf{d}(\mu, \mu') d\pi(z, z'),$$

where the supremum ranges over all probabilities π on $\Delta(\mathcal{Z}) \times \Delta(\mathcal{Z})$ having (ξ, ξ') for marginals.

The properties of Theorem 3.13 extend for the space $(\Delta(\Delta(\mathcal{Z})), \bar{\mathbf{d}})$. Let us now state some properties of the function W and of the value function of the non-revealing game U .

Lemma 3.15. The function W is bounded by 1, concave, and 1-Lipschitz with respect to $\bar{\mathbf{d}}$. Moreover, for all $\mu \in \Delta(\mathcal{Z})$, $W(\delta_{\mu}) = U(\delta_{\mu}) = u(\mu)$ and the function u is bounded and 1-Lipschitz on $(\Delta(\mathcal{Z}), \mathbf{d})$.

Proof. This is essentially the same proof as in Proposition 1.1 in [21].

The fact that W is bounded by 1 follows directly from the fact that g is bounded by 1.

Let us prove concavity using the splitting method (see e.g. Chapter V.1 in [39]). Let $\xi, \xi' \in \Delta(\Delta(\mathcal{Z}))$ and $\lambda \in [0, 1]$. Let μ, μ' be independent random variables of law ξ, ξ' . Construct an independent variable $L \in \{0, 1\}$ such that $\mathbb{P}(L = 0) = \lambda$. Define the variable μ_{λ} as equal to μ if $L = 0$ and to μ' if $L = 1$. Then μ_{λ} has law $\lambda\xi + (1 - \lambda)\xi'$. Using the conditional law of L given μ_{λ} , player 1 can generate (using some exogenous lottery) a variable L conditionally on μ_{λ} in the game $\Gamma(\lambda\xi + (1 - \lambda)\xi')$ having the property that the conditional law of μ_{λ} given L is equal to μ if $L = 0$ and to μ' if $L = 1$. Given an ε -optimal strategy σ in $\Gamma(\xi)$ and an ε -optimal strategy σ' in $\Gamma(\xi')$, we can construct then a strategy which plays $\sigma(\mu_{\lambda})$ when $L = 0$ and $\sigma'(\mu_{\lambda})$ when $L = 1$. Formally, let $\tilde{\sigma} : \Delta(\mathcal{Z}) \rightarrow \Delta(\{0, 1\} \times I)$ the transition corresponding to the above construction. For any strategy τ , the triplet $(\lambda\xi + (1 - \lambda)\xi', \tilde{\sigma}, \tau)$ induces a joint probability distribution $\mathbb{P}(\lambda\xi + (1 - \lambda)\xi', \tilde{\sigma}, \tau)$ for the variables (μ_{λ}, L, i, j) as described above. Moreover, the map $\tilde{\sigma}$ is equivalent to some strategy $\hat{\sigma}$. Indeed, the expected payoff depends only on the induced joint law of (μ_{λ}, i) , so that the strategy $\hat{\sigma}(\mu_{\lambda}) := \llbracket i | \mu_{\lambda} \rrbracket$ induces the same payoff as $\tilde{\sigma}$ against any strategy τ . We deduce that

$$\begin{aligned} W(\lambda\xi + (1 - \lambda)\xi') &\geq \inf_{\tau \in \Delta(J)} \mathbb{E}_{\mathbb{P}(\lambda\xi + (1 - \lambda)\xi', \tilde{\sigma}, \tau)} [g(z, i, j) d\mu_{\lambda}(z)] \\ &\geq \inf_{\tau \in \Delta(J)} \lambda \mathbb{E}_{\mathbb{P}(\lambda\xi + (1 - \lambda)\xi', \tilde{\sigma}, \tau)} [g(z, i, j) d\mu_{\lambda}(z) | L = 0] \\ &\quad + (1 - \lambda) \mathbb{E}_{\mathbb{P}(\lambda\xi + (1 - \lambda)\xi', \tilde{\sigma}, \tau)} [g(z, i, j) d\mu_{\lambda}(z) | L = 1] \\ &\geq \lambda \inf_{\tau \in \Delta(J)} \mathbb{E}_{\mathbb{P}(\xi, \sigma, \tau)} [g(z, i, j) d\mu(z)] + (1 - \lambda) \inf_{\tau \in \Delta(J)} \mathbb{E}_{\mathbb{P}(\xi', \sigma', \tau)} [g(z, i, j) d\mu'(z)] \\ &\geq \lambda W(\xi) + (1 - \lambda) W(\xi') - \varepsilon \end{aligned}$$

which proves that W is concave by sending ε to zero.

Let us prove the Lipschitz property. Given $\xi, \xi' \in \Delta(\Delta(\mathcal{Z}))$ and $\varepsilon > 0$, there exists a joint distribution π with marginals (ξ, ξ') such that for any pair (μ, μ') of $\Delta(\mathcal{Z})$ -valued random variables of law π

$$\mathbb{E}[\mathbf{d}(\mu, \mu')] \leq \bar{\mathbf{d}}(\xi, \xi') + \varepsilon.$$

Using the conditional law of μ' given μ under π , player 1 can generate some random variable μ' as a function of μ and some exogenous lottery (see Theorem A.3). Then, he can play some

strategy $\sigma(\mu')$ which is ε -optimal in the game $\Gamma(\xi')$. As above, this construction defines a transition $\tilde{\sigma} : \Delta(\mathcal{Z}) \rightarrow \Delta(\Delta(\mathcal{Z}) \times I)$ which is equivalent to a behavior strategy $\hat{\sigma}$ using that the expected payoff depends only on the induced law of (μ, i) . For any strategy τ , the triplet $(\xi, \tilde{\sigma}, \tau)$ induces a probability distribution $\mathbb{P}(\xi, \tilde{\sigma}, \tau)$ for the variables (μ, μ', i, j) and we have

$$\begin{aligned} W(\xi) &\geq \inf_{\tau \in \Delta(J)} \mathbb{E}_{\mathbb{P}(\xi, \tilde{\sigma}, \tau)}[g(z, i, j)d\mu(z)] \geq \inf_{\tau \in \Delta(J)} \mathbb{E}_{\mathbb{P}(\xi, \tilde{\sigma}, \tau)}[g(z, i, j)d\mu'(z) - \mathbf{d}(\mu, \mu')] \\ &\geq \inf_{\tau \in \Delta(J)} \mathbb{E}_{\mathbb{P}(\xi', \sigma, \tau)}[g(z, i, j)d\mu'(z)] - \bar{\mathbf{d}}(\xi, \xi') - \varepsilon \\ &\geq W(\xi') - \bar{\mathbf{d}}(\xi, \xi') - 2\varepsilon \end{aligned}$$

The conclusion follows by inverting the roles of ξ and ξ' and letting ε go to zero.

The properties of u follow directly from the definitions. \square

4 A representation formula for the value of $G_n(\nu)$.

In this section, we show that the value function of the n -stage game can be expressed as a stochastic optimization problem called maximal W -variation.

4.1 The problem of maximal W -variation.

Recall that $(Z_t)_{t \geq 0}$ is a Markov process with initial law ν , definition 2.6 and the notation $\mathfrak{B}_n(\nu)$ introduced in Theorem 2.7. Let us introduce some additional notations.

Notation 4.1.

- By convention, given any filtration $(\mathcal{F}_q)_{q=0, \dots, n}$, we always set $\mathcal{F}_{-1} = \sigma(\emptyset)$.
- The natural filtration induced by some process $(A_q)_{q=0, \dots, n}$ is denoted \mathcal{F}^A with

$$\mathcal{F}_q^A = \sigma(A_0, \dots, A_q) \text{ for all } q = 0, \dots, n.$$

- $\mathcal{B}_n(\nu) \subset \Delta((\mathcal{Z} \times \Delta(\mathcal{Z}))^{n+1})$: set of distributions of processes in $\mathfrak{B}_n(\nu)$

Definition 4.2. The W -variation of a triple $(Z_{\underline{q}}, \pi_q, \mathcal{F}_q)_{q=0, \dots, n}$ in $\mathfrak{B}_n(\nu)$ is defined by,

$$\Psi_n((\tilde{Z}_q^n, \pi_q, \mathcal{F}_q)_{q=0, \dots, n}) := \mathbb{E}\left[\sum_{q=0}^{n-1} W([\pi_q | \mathcal{F}_{q-1}, \tilde{Y}_q^n])\right]. \quad (4.1)$$

The problem of maximal W -variation given the initial $\nu \in \Delta(\mathcal{Z})$ is defined by:

$$\bar{\Psi}_n(\nu) := \sup_{(\tilde{Z}_q^n, \pi_q, \mathcal{F}_q)_{q=0, \dots, n} \in \mathfrak{B}_n(\nu)} \Psi_n((\tilde{Z}_q^n, \pi_q, \mathcal{F}_q)_{q=0, \dots, n}).$$

The following property results directly from the concavity of W and allows to replace the collection $\mathfrak{B}_n(\nu)$ by the set $\mathcal{B}_n(\nu)$ in the maximization problem $\bar{\Psi}_n(\nu)$.

Lemma 4.3. For all $\nu \in \Delta(\mathcal{Z})$, we have

$$\bar{\Psi}_n(\nu) = \sup_{\llbracket (\tilde{Z}_q^n, \pi_q)_{q=0, \dots, n} \rrbracket \in \mathfrak{B}_n(\nu)} \Psi_n((\tilde{Z}_q^n, \pi_q, \mathcal{F}_q^{\tilde{Y}^n, \pi})_{q=0, \dots, n}), \quad (4.2)$$

where $(\mathcal{F}_q^{\tilde{Y}^n, \pi})_{q=0, \dots, n}$ is the natural filtration of (\tilde{Y}^n, π) , i.e.

$$\mathcal{F}_q^{\tilde{Y}^n, \pi} = \sigma(\tilde{Y}_0^n, \dots, \tilde{Y}_q^n, \pi_0, \dots, \pi_q) \text{ for } q = 0, \dots, n \text{ and } \mathcal{F}_{-1}^{\tilde{Y}^n, \pi} = \sigma(\emptyset).$$

Proof. It is clearly sufficient to prove that $\bar{\Psi}_n(\nu)$ is not greater than the right-hand side of (4.2). Let $(\tilde{Z}_q^n, \pi_q, \mathcal{F}_q)_{q=0, \dots, n} \in \mathfrak{B}_n(\nu)$. Since W is concave and 1-Lipschitz, it follows from Jensen's inequality (see lemma A.9) that for all $q = 0, \dots, n-1$:

$$W(\llbracket \pi_q \mid \mathcal{F}_{q-1}, \tilde{Y}_q^n \rrbracket) \leq W(\llbracket \pi_q \mid \mathcal{F}_{q-1}^{\tilde{Y}^n, \pi}, \tilde{Y}_q^n \rrbracket).$$

The proof follows then by summation over q . \square

In order to prove Theorem 2.7, we will need the following two lemmas that follow from classical measurable selection results. The proof of these lemmas are standard and postponed to the appendix.

Lemma 4.4. For all $\varepsilon > 0$, there exists a measurable function

$$\varphi_\varepsilon : \Delta(\mathcal{Z}) \times \Delta(\Delta(\mathcal{Z})) \times [0, 1] \rightarrow I$$

such that the strategy induced by $i_1 = \varphi_\varepsilon(\theta, \xi, w)$ where w is some uniform random variable on $[0, 1]$ independent from θ , is ε -optimal in the game $\Gamma(\xi)$ for all $\xi \in \Delta(\Delta(\mathcal{Z}))$.

Let us denote $\Delta(I \times \Delta(\mathcal{Z}))$ the set of joint probability distributions on $I \times \Delta(\mathcal{Z})$. Note that a pair (σ, ξ) where σ is a strategy of player 1 in the game $\Gamma(\xi)$ defines naturally a probability in $\Delta(I \times \Delta(\mathcal{Z}))$ that will be denoted $\pi(\sigma, \xi)$.

Lemma 4.5. For all $\varepsilon > 0$, there exists a universally measurable function

$$\tau_\varepsilon : \Delta(I \times \Delta(\mathcal{Z})) \rightarrow \Delta(J)$$

such that for all (σ, ξ) with $\xi \in \Delta(\mathcal{Z})$

$$\mathbb{E}_{\mathbb{P}(\xi, \sigma, \tau_\varepsilon(\pi(\sigma, \xi)))} \left[\int_{\mathcal{Z}} g(z, i_1, j_1) d\mu(z) \right] \leq W(\xi) + \varepsilon.$$

Proof of Theorem 2.7. This proof extends the one appearing in [21].

Let us start with an ε -maximizer $(\tilde{Z}_q^n, \pi_q)_{q=0, \dots, n}$ for the problem $\bar{\Psi}_n(\nu)$. Without loss of generality, we may assume that $\tilde{Z}_q^n = Z_{\frac{q}{n}}$ for some Markov process Z with initial law ν defined on some possibly enlarged probability space. Using the conditional independence property (see Lemma 3.9), there exists a sequence $(\xi_q)_{q=0, \dots, n-1}$ of independent random variables uniformly distributed on $[0, 1]$ and independent from $(Z_{\frac{q}{n}})_{q=0, \dots, n}$, and a sequence of measurable functions $(f_q)_{q=0, \dots, n-1}$ such that

$$\pi_q = f_q((Z_{\frac{m}{n}}, \xi_m)_{m \leq q}) \text{ for } q = 0, \dots, n-1.$$

Using Lemma 4.4, we can define player 1's strategy σ as follows: given $\varepsilon > 0$, and a sequence $(\zeta_0, \dots, \zeta_{n-1})$ of independent random variables uniformly distributed on $[0, 1]$ that are independent from $(Z_{\frac{q}{n}})_{q=0, \dots, n}$ and $(\xi_q)_{q=0, \dots, n-1}$, the action i_q of player 1 at stage q is

$$i_q(Z_0, \dots, Z_{\frac{q}{n}}, \zeta_q) := \varphi_\varepsilon(\pi_q, \llbracket \pi_q \mid Y_0, \pi_0, \dots, Y_{\frac{q-1}{n}}, \pi_{q-1}, Y_{\frac{q}{n}} \rrbracket, \zeta_q).$$

This does not define a behavior strategy but this mixed strategy defines a joint law on $(Z_0, i_0, \dots, Z_{\frac{n-1}{n}}, i_{n-1}, Z_1)$ which can always be disintegrated in a behavior strategy that does not depend on player 2's actions since the process $(i_q)_{q=0, \dots, n-1}$ is by construction non-anticipative with respect to $(Z_{\frac{q}{n}})_{q=0, \dots, n}$ (using again Lemma 3.9). Without loss of generality, given a sequence $(\eta_0, \dots, \eta_{n-1})$ of independent random variables uniformly distributed on $[0, 1]$ and independent from $(Z_{\frac{q}{n}})_{q=0, \dots, n}$ and $(\xi_q, \zeta_q)_{q=0, \dots, n-1}$, we can assume that the strategy τ is given by a sequence g_0, \dots, g_{n-1} of measurable functions and that the action of player 2 at stage q is (see theorem A.3)

$$j_q = g_q(Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}}, \eta_q).$$

Let us define the filtration $(\mathcal{H}_q)_{q=0, \dots, n-1}$ by $\mathcal{H}_q = \sigma(Y_0, \pi_0, \zeta_0, \eta_0, \dots, Y_{\frac{q}{n}}, \pi_q, \zeta_q, \eta_q)$. With this notation, variables i_q, j_q are \mathcal{H}_q -measurable and we have by construction the following equalities for all $q = 0, \dots, n-1$

$$\llbracket Z_1 \mid \mathcal{H}_q \rrbracket = \llbracket Z_1 \mid Y_0, \pi_0, \dots, Y_{\frac{q}{n}}, \pi_q \rrbracket = \pi_q, \quad \llbracket \pi_q \mid \mathcal{H}_{q-1} \vee Y_{\frac{q}{n}} \rrbracket = \llbracket \pi_q \mid Y_0, \pi_0, \dots, Y_{\frac{q-1}{n}}, \pi_{q-1}, Y_{\frac{q}{n}} \rrbracket.$$

The conditional payoff at round q given the information of player 2 when playing round q is

$$\begin{aligned} & \mathbb{E}[g(Z_1, i_q, j_q) \mid Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}}] \\ &= \mathbb{E}[\mathbb{E}[g(Z_1, i_q, j_q) \mid \mathcal{H}_q] \mid Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}}] \\ &= \mathbb{E}\left[\int_{\mathcal{Z}} g(z, i_q, j_q) d\pi_q(z) \mid Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\int_{\mathcal{Z}} g(z, i_q, j_q) d\pi_q(z) \mid \mathcal{H}_{q-1} \vee Y_{\frac{q}{n}}\right] \mid Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}}\right]. \end{aligned}$$

Finally the conditional expectation given $\mathcal{H}_{q-1} \vee Y_{\frac{q}{n}}$ above is exactly the payoff in a one-shot game Γ where the law of the state variable (which corresponds to the beliefs of player 1) is $\llbracket \pi_q \mid Y_0, \pi_0, \dots, Y_{\frac{q-1}{n}}, \pi_{q-1}, Y_{\frac{q}{n}} \rrbracket$ and the joint conditional law of $\llbracket \pi_q, i_q \mid Y_0, \pi_0, \dots, Y_{\frac{q-1}{n}}, \pi_{q-1}, Y_{\frac{q}{n}} \rrbracket$ has been constructed so that

$$\mathbb{E}\left[\int g(z, i_q, j_q) d\pi_q(z) \mid \mathcal{H}_{q-1} \vee Y_{\frac{q}{n}}\right] \geq W(\llbracket \pi_q \mid Y_0, \pi_0, \dots, Y_{\frac{q-1}{n}}, \pi_{q-1}, Y_{\frac{q}{n}} \rrbracket) - \varepsilon.$$

By summing these inequalities, we deduce that $V_n(\nu) \geq \frac{1}{n} \overline{\Psi}_n(\nu) - 2\varepsilon$ and we obtain a first inequality by sending ε to zero.

It remains to prove the reverse inequality. Let us fix a pair (ν, σ) where σ is a behavior strategy for player 1 in $G_n(\nu)$ and some $\varepsilon > 0$. We will construct a strategy τ for player 2 by induction such that for all $q = 0, \dots, n-1$ the expected payoff at round q is not greater than

$$\mathbb{E}_{\mathbb{P}(\nu, \sigma, \tau)}[W(\llbracket \pi_q \mid Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}} \rrbracket)] + \varepsilon,$$

where π_q is defined by $\pi_q = \llbracket Z_1 \mid Y_0, i_0, j_0, \dots, Y_{\frac{q}{n}}, i_q, j_q \rrbracket$. Note that adding/omitting j_q in the conditional law defining π_q does not change its value, which does not depend on τ_q . Suppose that $(\tau_0, \dots, \tau_{q-1})$ is already constructed. Since $(\nu, \sigma, \tau_0, \dots, \tau_{q-1})$ defines a law for $(Z_0, \dots, Z_{\frac{q}{n}}, i_0, \dots, i_q, j_0, \dots, j_{q-1})$ that will be denoted \mathbb{P} , the variable π_q is well-defined and \mathbb{P} -almost surely equal to a Borel map

$$h_q(Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}}, i_q).$$

The disintegration theorem implies the existence of a $\Delta(I \times \Delta(\mathcal{Z}))$ -valued Borel map

$$M_q(Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}})$$

which is a version of the conditional law

$$\llbracket i_q, \pi_q \mid Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}} \rrbracket.$$

Using lemma 4.5, the mapping τ_ε being universally measurable, there exists a Borel map $\tilde{\tau}_\varepsilon$ which is almost surely equal to τ_ε with respect to the law of the random variable $\llbracket i_q, \pi_q \mid Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}} \rrbracket$ (which depends only on variables known by player 2). The strategy

$$\tau_q = \tilde{\tau}_\varepsilon(M_q(Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}}))$$

has then the required properties. The overall payoff of player 2 is therefore not greater than

$$\frac{1}{n} \sum_{q=0}^{n-1} \mathbb{E}_{\mathbb{P}(\nu, \sigma, \tau)} [W(\llbracket \pi_q \mid Y_0, i_0, j_0, \dots, Y_{\frac{q-1}{n}}, i_{q-1}, j_{q-1}, Y_{\frac{q}{n}} \rrbracket)] + \varepsilon \leq \frac{1}{n} \bar{\Psi}_n(\nu) + \varepsilon,$$

which concludes the proof since ε was arbitrary. \square

Remark 4.6. *Using the construction given in the first part of the proof, any ε -maximizer of $\bar{\Psi}_n$ induces a η -optimal strategy for player 1 for $\eta > \frac{\varepsilon}{n}$.*

5 The Continuous-Time Limit.

In reference to the paper of Meyer and Zheng [41], we will denote MZ the following topology on the set of càdlàg paths.

Notation 5.1. *For a separable metric space (E, d) , the MZ -topology on the set $\mathbb{D}([0, 1], E)$ of càdlàg functions is the topology of convergence in measure with respect to Lebesgue's measure together convergence of the value at time 1: a sequence y_n converges to y if*

$$\forall \varepsilon > 0, \int_0^1 \mathbb{1}_{d(y_n(t), y(t)) \geq \varepsilon} dt \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{and} \quad y_n(1) \xrightarrow[n \rightarrow \infty]{} y(1).$$

The associated weak topology over the set $\Delta(\mathbb{D}([0, 1], E))$ when $\mathbb{D}([0, 1], E)$ is endowed with the MZ -topology will be denoted $\mathcal{L}(MZ)$.

The usual Skorokhod topology on the set $\mathbb{D}([0, 1], E)$ will be denoted Sk (see e.g. [18] or [28] for the definition and usual properties). The associated weak topology over the set $\Delta(\mathbb{D}([0, 1], E))$ when $\mathbb{D}([0, 1], E)$ is endowed with the Sk -topology will be denoted $\mathcal{L}(Sk)$.

Remark 5.2. *In contrast to the Sk -topology, if $E = F \times F'$ is a product of separable metric spaces, the MZ topology is a product topology, i.e. (as topological spaces)*

$$(\mathbb{D}([0, 1], F \times F'), MZ) = (\mathbb{D}([0, 1], F), MZ) \times (\mathbb{D}([0, 1] \times F'), MZ).$$

The following remark will be used several times in the proofs.

Remark 5.3. *If E is a Polish space, the space $(\mathbb{D}([0, 1], E), MZ)$ is a separable metric space which is not topologically complete. However, its Borel σ -algebra is the same as the one generated by the Sk topology and its topology is weaker than the Sk topology for which the space is Polish, implying that all the probability measures are MZ -tight. Therefore, all the results about disintegration and measurable selection usually stated for Polish spaces and which depend only on the Borel structure apply to this space. Moreover, the direct part of Prohorov Theorem applies (i.e., uniform tightness implies relative compactness), and we have as a partial converse (Le Cam's Theorem) that convergent sequences of probabilities are tight (see e.g. Theorem 11.5.3 in Dudley [17]). The same remark also holds for the separable metric space $\Delta(\mathbb{D}([0, 1], E), \mathcal{L}(MZ))$ when comparing to the Polish space $\Delta(\mathbb{D}([0, 1], E), \mathcal{L}(Sk))$.*

Recall that the transition probabilities of Z are denoted $(P_t)_{t \geq 0}$, i.e. for any bounded measurable function ϕ on \mathcal{Z} , we have

$$\mathbb{E}_x[\phi(Z_t)] = \int_{\mathcal{Z}} \phi dP_t(x).$$

Assumption **H2** implies that Z fulfills the following properties (see e.g. [18]):

- Z is a càdlàg $(\mathcal{F}^{Z,+})$ -Markov process.
- The map $z \rightarrow \mathbb{P}_z$ is $\mathcal{L}(Sk)$ -continuous on \mathcal{Z} .
- The map $(t, z) \rightarrow P_t(z) \in \Delta(\mathcal{Z})$ is continuous on $[0, +\infty) \times \mathcal{Z}$.
- Z has no fixed point of discontinuity.

Remark 5.4. *One may replace **H2** by the above properties and obtain the same results.*

The following notation is slightly more general than notation $\mathcal{B}(\nu)$ (note that $\mathcal{B}(\nu) = \mathcal{B}(\mathbb{P}_\nu)$) introduced in theorem 2.10 and will be useful for its proof.

Notation 5.5. $\mathcal{B}(\mathbb{P})$ denotes the subset of $\Delta(\mathbb{D}([0, 1], \mathcal{Z} \times \Delta(\mathcal{Z})))$ of laws of processes $(Z_t, \pi_t)_{t \in [0, 1]}$ such that:

- The law of $(Z_t)_{t \in [0, 1]}$ is \mathbb{P} .
- $(\pi_t)_{t \in [0, 1]}$ is a càdlàg modification of $(\llbracket Z_1 | \mathcal{F}^{(Y, \pi), +} \rrbracket)_{t \in [0, 1]}$.
- $(\pi_t)_{t \in [0, 1]}$ is non-anticipative with respect to $(Z_t)_{t \in [0, 1]}$, i.e.

$$\forall t \in [0, 1), \quad \mathcal{F}_t^\pi \coprod_{\mathcal{F}_t^{Z,+}} \mathcal{F}_1^Z.$$

Let us add some useful facts about the non-anticipative condition, which follow easily from the Markov property.

Lemma 5.6. *Let (Z, π) be a process such that Z has law \mathbb{P}_ν for some $\nu \in \Delta(\mathcal{Z})$. Then, the condition*

$$\forall t \in [0, 1), \quad \mathcal{F}_t^\pi \prod_{\mathcal{F}_t^{Z,+}} \mathcal{F}_1^Z \quad (5.1)$$

is equivalent to the condition

$$\forall t \text{ in a dense subset of } (0, 1), \quad \mathcal{F}_t^\pi \prod_{\mathcal{F}_t^Z} (Z_s)_{s \in [t, 1]}. \quad (5.2)$$

If one of these conditions hold, we also have

$$\forall t \in [0, 1), \quad \mathcal{F}_t^{(Z, \pi), +} \prod_{\mathcal{F}_t^Z} \mathcal{F}_1^Z.$$

Proof. Recall at first that since Z is an $\mathcal{F}^{Z,+}$ Markov process, we have for all $t \in [0, 1)$:

$$\llbracket (Z_s)_{s \in [0, 1]} | \mathcal{F}_t^Z \rrbracket = \llbracket (Z_s)_{s \in [0, 1]} | \mathcal{F}_t^{Z,+} \rrbracket. \quad (5.3)$$

We prove the equivalence property. Assume that (5.1) holds. Together with (5.3), this implies

$$\llbracket (Z_s)_{s \in [t, 1]} | \mathcal{F}_t^Z \rrbracket = \llbracket (Z_s)_{s \in [t, 1]} | \mathcal{F}_t^{Z,+}, \mathcal{F}_t^\pi \rrbracket,$$

and we conclude using the second point of lemma 3.7 with the σ -field $\mathcal{F}_t^Z \vee \mathcal{F}_t^\pi$.

Let us prove the reverse implication. Assume that (5.2) holds for a dense subset T of $(0, 1)$. Let $t \in T$. Using the last point in Lemma 3.8, there exists (on a possibly enlarged probability space) a variable ξ uniformly distributed on $[0, 1]$ and independent from $(Z_s)_{0 \leq s \leq 1}$, and a map f such that

$$(\pi_s)_{0 \leq s \leq t} = f((Z_s)_{0 \leq s \leq t}, \xi).$$

Using the first point in Lemma 3.7, we deduce that

$$\llbracket (Z_s)_{s \in [0, 1]} | \mathcal{F}_t^{Z,+} \rrbracket = \llbracket (Z_s)_{s \in [0, 1]} | \mathcal{F}_t^{Z,+}, \xi \rrbracket.$$

We conclude that (5.1) holds for $t \in T$ using the second point of Lemma 3.7 with the sigma-field $\mathcal{F}_t^{Z,+} \vee \mathcal{F}_t^\pi$.

We now conclude the proof of equivalence and prove the last assertion at the same time. let $t < 1$ and $t_n \in T$ a sequence decreasing to t . We have for all n and all $B \in \mathcal{F}_1^Z$,

$$\mathbb{P}(B | \mathcal{F}_{t_n}^Z, \mathcal{F}_{t_n}^\pi) = \mathbb{P}(B | \mathcal{F}_{t_n}^Z).$$

Applying the backward martingale convergence theorem, we find

$$\mathbb{P}(B | \mathcal{F}_t^{(Z, \pi), +}) = \mathbb{P}(B | \mathcal{F}_t^{Z,+}) = \mathbb{P}(B | \mathcal{F}_t^Z),$$

which concludes the proof of the last assertion and also of the equivalence property using the second point of Lemma 3.7 with the sigma-field $\mathcal{F}_t^{Z,+} \vee \mathcal{F}_t^\pi$. \square

Let us now define the continuous-time limit of the functions $\bar{\Psi}_n$.

Notation 5.7. For all $\mathbb{P} \in \Delta(\mathbb{D}([0, 1], \mathcal{Z}))$, define

$$H(\mathbb{P}) := \sup_{\llbracket (Z_t, \pi_t)_{t \in [0, 1]} \rrbracket \in \mathcal{B}(\mathbb{P})} \mathbb{E} \left[\int_0^1 u(\pi_t) dt \right].$$

For all $\nu \in \Delta(\mathcal{Z})$, define

$$V(\nu) := H(\mathbb{P}_\nu).$$

5.1 Preliminary results.

We state here an easy corollary of the main results in Kurtz [34] which extends the classical result of Meyer-Zheng [41] to measure-valued processes.

Theorem 5.8 (easy consequence of Theorem 1.1 and Corollary 1.4 in [34]). *Let C a tight subset of $\Delta(\mathcal{Z})$ and $(X^n)_{t \in [0, 1]}$ be a sequence of $\Delta(\mathcal{Z})$ -valued càdlàg martingales such that $\mathbb{E}[X_1^n] \in C$. Then the set of laws of the processes $(X^n)_{n \geq 0}$ is $\mathcal{L}(MZ)$ -relatively compact.*

Proof. In order to fit in the framework of [34], one may extend the definition of the processes to $(X^n)_{t \in [0, +\infty]}$ by $X_t^n = X_1^n$ for $t \geq 1$. We have to verify the condition C1.1(i) (compactness containment) of Theorem 1.1 in [34], i.e. that for all $\varepsilon > 0$, for all T , there exists a compact $K \subset \Delta(\mathcal{Z})$ such that

$$\liminf_{n \rightarrow \infty} \mathbb{P}(X_t^n \in K, \forall t \in [0, T]) \geq 1 - \varepsilon.$$

Given $\varepsilon \in (0, 1)$, let $(K_i)_{i \geq 1}$ a sequence of compact subsets of \mathcal{Z} such that for all $\mu \in C$, $\mu(K_i) \geq 1 - \varepsilon^2 2^{-2i}$. For all $i \geq 1$ and n , the process $(X_t^n(\mathcal{Z} \setminus K_i))_{t \geq 0}$ is a martingale, and we may assume that for all $i \geq 1$, it has càdlàg paths by choosing a good version of X^n (see Theorem 2.5 in [26]). The martingale $(X_t^n(\mathcal{Z} \setminus K_i))_{t \geq 0}$ takes values in $[0, 1]$ and its expectation $\alpha_{n,i}$ belongs to $[0, \varepsilon^2 2^{-2i}]$. Let $\tau(n, i)$ denote the hitting time of the interval $(\varepsilon 2^{-i}, 1]$ by this martingale (hence a stopping time of the usual augmentation $\bar{\mathcal{F}}^{X^n, +}$). Then, applying the optional sampling theorem, we have

$$\mathbb{E}[X_{\tau(n,i)}^n(\mathcal{Z} \setminus K_i)] = \alpha_{n,i}.$$

Let us decompose this expression as follows:

$$X_{\tau(n,i)}^n(\mathcal{Z} \setminus K_i) = X_{\tau(n,i)}^n(\mathcal{Z} \setminus K_i) \mathbf{1}_{\tau(n,i) < \infty} + X_1^n(\mathcal{Z} \setminus K_i) \mathbf{1}_{\tau(n,i) = \infty}.$$

On the event $\{\tau(n, i) = \infty\}$, the trajectory of X^n belongs to the set $K'_i := \{\mu \in \Delta(\mathcal{Z}) \mid \mu(K_i) \geq \varepsilon 2^{-i}\}$. Moreover, the above equality implies

$$\varepsilon 2^{-i} \mathbb{P}(\tau(n, i) < \infty) \leq \mathbb{E}[X_{\tau(n,i)}^n(\mathcal{Z} \setminus K_i)] = \alpha_{n,i} \leq \varepsilon^2 2^{-2i}.$$

We deduce that:

$$\mathbb{P}(\tau(n, i) < \infty) \leq \varepsilon 2^{-i}.$$

For all i , we proved that $\mathbb{P}(\forall t \geq 0, X_t^n \in K'_i) \geq 1 - \varepsilon 2^{-i}$. Now, define $K := \bigcap_i K'_i$ and note that K is compact from Prokhorov theorem and that

$$\mathbb{P}(\forall t \geq 0, X_t^n \in K) \geq 1 - \varepsilon.$$

We verify now the condition of Corollary 1.4., i.e. that there exists a countable separating subset $\{f_i\}_{i \in I}$ in $C_b(\Delta(\mathcal{Z}))$ such that for all $i \in I$ and for all T

$$\sup_n \sup_{0=t_0 < \dots < t_k < \dots < t_N=T} \mathbb{E} \left[\sum_{k=0}^{N-1} \left| \mathbb{E}[f_i(X_{t_{k+1}}^n) - f_i(X_{t_k}^n) | \mathcal{F}_{t_k}^{X^n, +}] \right| \right] < +\infty$$

One may choose the maps $\mu \rightarrow \int \phi_i d\mu$ for $\{\phi_i\}_{i \in I} \subset C_b(\mathcal{Z})$ a countable convergence determining subset, so that the above quantity is equal to zero. The result of Kurtz implies (see point (b) of Theorem 1.1 having in mind that Lebesgue-almost sure convergence of trajectories implies convergence in measure) that there exist a process X and subsequence X^n such that X^n $\mathcal{L}(MZ)$ -converges to X which implies the result. \square

Remark 5.9. *A similar result for $\mathcal{Z} = \mathbb{R}^m$ and probabilities having a second order moment can be found in [11] with a proof based on the method of Meyer and Zheng [41].*

The second ingredient is the following stability result for conditional independence.

Lemma 5.10. *Let $(A_n, B_n, C_n)_{n \in \mathbb{N}}$ be a sequence of random variables with values in $E \times F \times G$ where E, F, G are separable metric spaces that are universally measurable (i.e. all the Borel probabilities are tight). Assume that:*

$$\begin{cases} \llbracket A_n, B_n, C_n \rrbracket \xrightarrow{n \rightarrow \infty} \llbracket A, B, C \rrbracket \\ \llbracket C_n | B_n \rrbracket \xrightarrow{n \rightarrow \infty} \llbracket C | B \rrbracket \end{cases}$$

Then, if for all n , $A_n \amalg_{B_n} C_n$, we have $A \amalg_B C$.

Proof. Let f be a bounded convex continuous function on $\Delta(G)$. Using Lemma A.8, we have

$$\mathbb{E}[f(\llbracket C | A, B \rrbracket)] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[f(\llbracket C_n | A_n, B_n \rrbracket)].$$

On the other hand, conditional independence implies $\llbracket C_n | A_n, B_n \rrbracket = \llbracket C_n | B_n \rrbracket$. Using the assumption of convergence of conditional laws, it follows that

$$\mathbb{E}[f(\llbracket C | A, B \rrbracket)] \leq \mathbb{E}[f(\llbracket C | B \rrbracket)].$$

Then, we deduce from Jensen's inequality that:

$$\mathbb{E}[f(\llbracket C | A, B \rrbracket) | B] \geq f(\llbracket C | B \rrbracket).$$

By taking expectation, we deduce that with probability 1:

$$\mathbb{E}[f(\llbracket C | A, B \rrbracket) | B] = f(\llbracket C | B \rrbracket).$$

Note finally that we can choose a countable family of bounded continuous convex functions on $\Delta(G)$ which is separating on $\Delta(\Delta(G))$ (see e.g. the first Lemma p.11 in [48]), which implies that $\llbracket C | A, B \rrbracket = \llbracket C | B \rrbracket$ with probability 1. \square

The proof of Theorem 2.10 is also based on the following proposition, which we state in a slightly more general framework as it can be interesting for its own sake. This proposition provides a bound for the \mathbf{d} -variation of a martingale taking values in a infinite dimensional space of measures. When the martingale is valued in a finite dimensional simplex, then this result is well-known (see e.g. chapter V.1 in [39]) and was recently improved by Neyman [43] and Sandomirskiy [46] for more general state spaces. Our estimations of speed of convergence rely on the bound given by Neyman [43], but the convergence result itself could be proven using only the bound given in [39].

Proposition 5.11. *Let (E, d) be a Polish space and let \mathbf{d} denote the Wasserstein distance of order 1 (see Definition 3.12) on the set $\Delta^1(E)$ of probabilities π such that*

$$\int_E d(x, x_0) d\pi(x) < \infty,$$

for some point $x_0 \in E$.

Then, for any martingale $(\pi_q)_{q=0, \dots, n-1}$ taking values in $\Delta(E)$ such that

$$\mathbb{E}\left[\int_E d(x, x_0) d\pi_0(x)\right] < \infty,$$

we have:

$$\frac{1}{n} \sum_{q=0}^{n-1} \mathbb{E}[\mathbf{d}(\pi_q, \pi_{q-1})] \xrightarrow{n \rightarrow \infty} 0.$$

with the convention that $\pi_{-1} := \mathbb{E}[\pi_0]$.

Moreover, if E is a compact subset of \mathbb{R}^m endowed with the Euclidean distance, there exists a constant C depending only on E such that

$$\frac{1}{n} \sum_{q=0}^{n-1} \mathbb{E}[\mathbf{d}(\pi_q, \pi_{q-1})] \leq C \sqrt{\frac{\ln(n)}{n}}.$$

Proof. Using Strassen's theorem, we can assume that a variable A with law $\mu := \pi_{-1}$ is defined on the same probability space and that $\pi_q = \mathbb{E}[\delta_A | \pi_0, \dots, \pi_q]$. Recall that $\mu \rightarrow d(\mu, \nu)$ is convex on $\Delta^1(E)$ for all $\nu \in \Delta^1(E)$. Using Jensen's inequality we have

$$\mathbb{E}[\mathbf{d}(\pi_q, \pi_{q-1})] \leq \mathbb{E}[\mathbf{d}(\delta_A, \pi_{q-1})] = \mathbb{E}\left[\int_E d(A, x) d\pi_{q-1}(x)\right] \leq 2 \int_E d(x, x_0) d\mu(x) < \infty,$$

Note then that \mathbf{d} is given by the following optimization problem

$$\mathbf{d}(\mu, \nu) = \sup_{F \in \mathcal{L}} \int_E F d(\mu - \nu), \text{ with } \mathcal{L} = \{F \in Lip_1(E) \mid F(x_0) = 0\}. \quad (5.4)$$

Note that \mathcal{L} is compact for the topology induced by the norm $\|F - G\|_* = \left\| \frac{F-G}{1+d(x_0, \cdot)} \right\|_\infty$ and that the application $(F, p_1, p_2) \rightarrow \int_E F d(p_1 - p_2)$ is jointly continuous on $\mathcal{L} \times \Delta^1(E) \times \Delta^1(E)$.

Using Proposition 7.33 in [4], there exists a measurable function H of $(x, p_1, p_2) \in E \times \Delta^1(E) \times \Delta^1(E)$ such that for all p_1, p_2 , $H(\cdot, p_1, p_2) \in \mathcal{L}$ and

$$\mathbf{d}(p_1, p_2) = \int_E H(\cdot, p_1, p_2) d(p_1 - p_2).$$

Define $F_q(x, \omega) = H(x, \pi_q, \pi_{q-1})$. Given $\varepsilon > 0$, there exists a compact set $C_\varepsilon \subset E$ containing x_0 such that

$$\int_{E \setminus C_\varepsilon} \mathbf{d}(x_0, x) d\mu(x) \leq \varepsilon.$$

Let $(B_i)_{i=1, \dots, N_\varepsilon}$ a covering of C_ε made by balls with center x_i and radius ε , so that N_ε is minimal. Let $(U_i)_{i=1, \dots, N_\varepsilon}$ a partition of C_ε such that $U_i \subset B_i$. It follows that

$$\begin{aligned} \sum_{q=0}^{n-1} \mathbb{E}[\mathbf{d}(\pi_q, \pi_{q-1})] &= \mathbb{E}\left[\sum_{q=0}^{n-1} \int_E F_q(\cdot, \omega) d(\pi_q - \pi_{q-1})\right] \\ &= \mathbb{E}\left[\int_E \sum_{i=1}^{N_\varepsilon} \mathbf{1}_{U_i}(x) \sum_{q=0}^{n-1} (F_q(x, \omega) - F_q(x_i, \omega)) d(\pi_q - \pi_{q-1})(\omega)[x]\right] \\ &\quad + \mathbb{E}\left[\sum_{i=1}^{N_\varepsilon} \sum_{q=0}^{n-1} F_q(x_i, \omega) (\pi_q - \pi_{q-1})(\omega)(U_i)\right] + \mathbb{E}\left[\sum_{q=0}^{n-1} \int_{E \setminus C_\varepsilon} \mathbf{1}_{E \setminus C_\varepsilon}(x) F_q(\cdot, \omega) d(\pi_q - \pi_{q-1})(\omega)\right] \\ &\leq 2n\varepsilon + \text{diam}(C_\varepsilon) \sum_{q=0}^{n-1} \mathbb{E}\left[\sum_{i=1}^{N_\varepsilon} |(\pi_q - \pi_{q-1})(\omega)(U_i)|\right] + 2n\varepsilon \end{aligned}$$

Consider now the vector-valued martingale $(M_q)_{q=0, \dots, n-1}$ with $M_q = (\pi_q(U_i))_{i=1, \dots, N_\varepsilon+1}$ with the convention $U_{N_\varepsilon+1} = E \setminus C_\varepsilon$. This martingale takes values in the canonical simplex of $\mathbb{R}^{N_\varepsilon+1}$, and using the bound on the L_1 -variation of martingales given by Neyman [43], we have

$$\sum_{q=0}^{n-1} \mathbb{E}\left[\sum_{i=1}^{N_\varepsilon} |(\pi_q - \pi_{q-1})(\omega)(U_i)|\right] \leq \sum_{q=0}^{n-1} \mathbb{E}[\|M_q - M_{q-1}\|_1] \leq \frac{\sqrt{2n \ln(N_\varepsilon + 1)}}{\sqrt{\ln(2)}}.$$

We conclude that

$$\sum_{q=0}^{n-1} \mathbb{E}[\mathbf{d}(\pi_q, \pi_{q-1})] \leq 4n\varepsilon + \text{diam}(C_\varepsilon) \frac{\sqrt{2n \ln(N_\varepsilon + 1)}}{\sqrt{\ln(2)}}.$$

To conclude, note that:

$$\inf_{\varepsilon > 0} 4\varepsilon + \text{diam}(C_\varepsilon) \frac{\sqrt{2 \ln(N_\varepsilon + 1)}}{\sqrt{n \ln(2)}} \xrightarrow{n \rightarrow \infty} 0.$$

If E is a compact subset of \mathbb{R}^m (endowed with the supremum norm), we can take $E = C_\varepsilon$ so that the above arguments lead (with obvious simplifications) to

$$\sum_{q=0}^{n-1} \mathbb{E}[\mathbf{d}(\pi_q, \pi_{q-1})] \leq 2n\varepsilon + \text{diam}(E) \frac{\sqrt{2n \ln(N_\varepsilon)}}{\sqrt{\ln(2)}}.$$

For all $\varepsilon > 0$, one may choose for $(U_i)_{i=1, \dots, N_\varepsilon}$ a covering of E made by adjacent cubes in \mathbb{R}^m with centre x_i and radius $\varepsilon/2$, so that $N_\varepsilon \leq (1 + \frac{\text{diam}(E)}{\varepsilon})^m$. It follows that for all $\varepsilon > 0$,

$$\frac{1}{n} \sum_{q=0}^{n-1} \mathbb{E}[\mathbf{d}(\pi_q, \pi_{q-1})] \leq 2\left(\varepsilon + \frac{\sqrt{m}M}{\sqrt{2n \ln(2)}} \ln^{\frac{1}{2}}\left(1 + \frac{M}{\varepsilon}\right)\right),$$

with $M = \text{diam}(E)$. Using that $\ln(1+x) \leq \ln(x) + \frac{1}{x}$ for $x > 0$, we find

$$\frac{1}{n} \sum_{q=0}^{n-1} \mathbb{E}[\mathbf{d}(\pi_q, \pi_{q-1})] \leq 2\left(\varepsilon + \frac{\sqrt{m}M}{\sqrt{2n \ln(2)}} (\ln(M) - \ln(\varepsilon) + \frac{\varepsilon}{M})^{\frac{1}{2}}\right).$$

Choosing $\varepsilon = n^{-\alpha}$ for some $\alpha > 0$, it follows that

$$\frac{1}{n} \sum_{q=0}^{n-1} \mathbb{E}[\mathbf{d}(\pi_q, \pi_{q-1})] \leq 2(n^{-\alpha} + \frac{\sqrt{m}M}{\sqrt{2n \ln(2)}} (\ln(M) + \alpha \ln(n) + \frac{n^{-\alpha}}{M})^{\frac{1}{2}})$$

With $\alpha = \frac{1}{2}$, we deduce that $\frac{1}{n} \sum_{q=0}^{n-1} \mathbb{E}[\mathbf{d}(\pi_q, \pi_{q-1})] = \mathcal{O}((\frac{\ln(n)}{n})^{\frac{1}{2}})$. □

5.2 Proof of Theorem 2.10

Let us introduce some notations for the discretizations in time used in the following proofs.

Notation 5.12. Given a process $(Z_t)_{t \in [0,1]}$ of law \mathbb{P} , we define $\widehat{\mathbb{P}}^n$ as the law of the process $(\widehat{Z}_t)_{t \in [0,1]} \in \mathbb{D}([0,1], \mathcal{Z})$ defined by

$$\forall t \in [0,1], \widehat{Z}_t := Z_{\lfloor nt \rfloor / n}$$

where $\lfloor a \rfloor$ denotes the greatest integer lower or equal to a .

Lemma 5.13.

- The set-valued map $\nu \rightarrow (\mathcal{B}(\nu), \mathcal{L}(MZ))$ has compact values.
- Any sequence in $\mathcal{B}(\widehat{\mathbb{P}}_\nu^n)$ admits an $\mathcal{L}(MZ)$ -convergent subsequence and all the limit points belong to $\mathcal{B}(\nu)$.

Proof. We only prove the second assertion, as the proof of the first one is simpler and relies on the same arguments (with obvious simplifications).

Let \mathbb{Q}_n denote a sequence of laws of processes $(Z_t^n, \pi_t^n)_{t \in [0,1]}$ in $\mathcal{B}(\widehat{\mathbb{P}}_\nu^n)$. It follows from Proposition VI.6.37 in [28] that $\widehat{\mathbb{P}}_\nu^n$ $\mathcal{L}(MZ)$ -converges to \mathbb{P}_ν (convergence actually holds for $\mathcal{L}(Sk)$). Therefore, Le Cam's Theorem (see Theorem 11.5.3 in [17] and remark 5.3) implies that the set $\{\mathbb{P}_\nu, \widehat{\mathbb{P}}_\nu^n, n \geq 1\}$ is MZ -tight. On the other hand, Theorem 5.8 implies that the set of possible laws for $(\pi_t^n)_{t \in [0,1]}$ is MZ -relatively sequentially compact since $\mathbb{E}[\pi_0^n] = \llbracket Z_1 \rrbracket$. Using again LeCam's theorem, up to extracting a convergent subsequence, we can assume that the set of laws of the processes π^n is tight. Finally, using Lemma A.2, we conclude that the set

of laws $\{\mathbb{Q}_n, n \geq 1\}$ is tight, and we may extract some convergent subsequence, which proves relative sequential compactness.

Let us now prove that the limit points belong to $\mathcal{B}(\nu)$. Assume (without loss of generality) that the sequence of processes $(Z_t^n, \pi_t^n)_{t \in [0,1]}$ (with laws in $\mathcal{B}(\widehat{\mathbb{P}}_\nu^n)$) $\mathcal{L}(MZ)$ -converges to $(Z_t, \pi_t)_{t \in [0,1]}$. Note at first that the law of $(Z_t)_{t \in [0,1]}$ is \mathbb{P}_ν since $\widehat{\mathbb{P}}_\nu^n$ $\mathcal{L}(MZ)$ -converges to \mathbb{P}_ν and using that the projection of the trajectories on the first coordinate is continuous (see Remark 5.2).

Using Skorokhod representation Theorem for separable metric spaces (see Theorem 11.7.31 in [17]), we can assume that the processes are defined on the same probability space and that $(Z_t^n, \pi_t^n)_{t \in [0,1]} \xrightarrow{MZ} (Z_t, \pi_t)_{t \in [0,1]}$ almost surely. Up to extracting a subsequence, we can also assume that there exists a subset I of full Lebesgue measure in $[0, 1]$ and containing 1 such that for all $t \in I$, $(Z_t^n, \pi_t^n) \rightarrow (Z_t, \pi_t)$ almost surely.

The fact that $(\pi_t)_{t \in [0,1]}$ is a càdlàg version of $\mathbb{E}[\delta_{Z_1} | \mathcal{F}_t^{(Y,\pi),+}]$ follows therefore from Theorem 11 in [41]. Indeed, for ϕ in a countable convergence determining subset of $C_b(\mathcal{Z})$, we may adapt the proof of this theorem to conclude that $\int \phi d\pi_t$ is an $\mathcal{F}^{Y,\pi}$ -martingale, and therefore also an $\mathcal{F}^{(Y,\pi),+}$ -martingale. This implies that the limit $(\pi_t)_{t \in [0,1]}$ is a $\mathcal{F}^{(Y,\pi),+}$ -martingale. Moreover, using the same argument, the equality $\mathbb{E}[\delta_{Z_1^n} | \mathcal{F}_1^{Y^n, \pi^n}] = \pi_1^n$ is also preserved in the limit.

It remains to prove the non-anticipative property.

Note that we have almost sure MZ -convergence of trajectories restricted to $[0, t]$ for all $t \in I$ in the space $\mathbb{D}([0, t], \mathcal{Z} \times \Delta(\mathcal{Z}))$ (the MZ -topology on this space is defined as above, replacing the convergence at time 1 by convergence at time t). Precisely, if $t \in I$, then

$$((Z_u^n)_{u \in [0,1]}, (Z_u^n)_{u \in [0,t]}, (\pi_u^n)_{u \in [0,t]}) \xrightarrow{\mathcal{L}(MZ)} ((Z_u)_{u \in [0,1]}, (Z_u)_{u \in [0,t]}, (\pi_u)_{u \in [0,t]}).$$

Since the law of $(Z_u^n)_{u \in [0,1]}$ is equal to $\widehat{\mathbb{P}}_\nu^n$, we have (assuming that trajectories of Z are defined on the whole time line for convenience),

$$\mathbb{P}[(Z_u^n)_{u \geq t} | (Z_u^n)_{u \leq t}] = M_n(\mathbb{P}_{Z_{\lfloor \frac{nt}{n} \rfloor}})$$

where $M_n(\mathbb{P})$ denotes the image probability of \mathbb{P} by the map

$$(x_u)_{u \in [0, \infty)} \rightarrow (x_{\lfloor \frac{n(\alpha_n + u)}{n} \rfloor})_{u \in [0, 1-t]}, \text{ with } \alpha_n = t - \frac{\lfloor nt \rfloor}{n}.$$

We now use that $\mathbb{P}_{Z_{\lfloor \frac{nt}{n} \rfloor}}$ converges $\mathcal{L}(Sk)$ -almost surely to \mathbb{P}_{Z_t} . We claim that this implies that $M_n(\mathbb{P}_{Z_{\lfloor \frac{nt}{n} \rfloor}})$ converges $\mathcal{L}(MZ)$ -almost surely to $M(\mathbb{P}_{Z_t})$ where $M(\mathbb{P})$ denotes the image probability of \mathbb{P} by the map

$$(x_u)_{u \in [0, \infty)} \rightarrow (x_u)_{u \in [0, 1-t]}.$$

Let us prove this claim. Fix ω such that the deterministic sequence $z_n = Z_{\lfloor \frac{nt}{n} \rfloor}(\omega)$ converges to $z = Z_t(\omega)$. Using Skorokhod's almost sure representation theorem, there exists a sequence of processes \widehat{Z}^n of law \mathbb{P}_{z_n} converging Sk -almost surely to \widehat{Z} of law \mathbb{P}_z . On one hand, using Proposition VI.6.37 in [28], we deduce that $(\widehat{Z}_{\lfloor \frac{nu}{n} \rfloor}^n)_{u \in [0, +\infty)}$ Sk -converges to \widehat{Z} . On the other

hand, since \widehat{Z} has no fixed points of discontinuity, the probability that \widehat{Z} is continuous at time $(1-t)$ is 1. In order to conclude, it is then sufficient to prove that $(x_{\alpha_n+u})_{u \in [0,1-t]}$ MZ -converges to $(x_u)_{u \in [0,1-t]}$ whenever x^n Sk -converges to x in $\mathbb{D}([0, \infty), \mathcal{Z})$ and x is continuous at time $(1-t)$. To see this, note at first that we have pointwise convergence at any point where x is continuous (see e.g. Proposition VI.2.1 in [28]), hence at time $(1-t)$. Since the set of discontinuities of x is at most countable, we have pointwise convergence Lebesgue almost everywhere on $[0, 1-t]$, and thus convergence in measure. The proof of the claim is complete.

Applying Lemma 5.10, we have that for all $t \in I$,

$$\mathcal{F}_t^\pi \prod_{\mathcal{F}_t^Z} \mathcal{F}_1^Z,$$

which concludes the proof by Lemma 5.6. □

Let us now turn to the proof Theorem 2.10.

Proof of Theorem 2.10: Using the first point of Lemma 5.13 and the fact that H is $\mathcal{L}(MZ)$ -continuous, there exists an optimal process $(Z_t, \pi_t)_{t \in [0,1]}$ with law in $\mathcal{B}(\nu)$ for the maximization problem $V(\nu)$. Let us prove at first that the discrete-time process $(Z_{\frac{q}{n}}, \pi_{\frac{q}{n}})_{q=0, \dots, n}$ has a law in $\mathcal{B}_n(\nu)$. The non-anticipative property of $(\pi_t)_{t \in [0,1]}$ at times $\frac{q}{n}$ for $q = 0, \dots, n$ implies (see Lemma 5.6)

$$\forall q = 0, \dots, n, \quad \mathcal{F}_{\frac{q}{n}}^{\pi_{\frac{q}{n}}} \prod_{\mathcal{F}_{\frac{q}{n}}^Z} \mathcal{F}_1^Z.$$

Therefore, using Lemma 3.9, there exists a sequence of uniform random variables ξ_0, \dots, ξ_n independent of $(Z_t)_{t \in [0,1]}$ and a sequence of measurable functions f_q such that

$$\pi_{\frac{q}{n}} = f_q((Z_u)_{u \leq \frac{q}{n}}, \xi_p, p \leq q).$$

We deduce therefore the required conditional independence property from Lemma 3.9. For the martingale property, we have $\pi_{\frac{q}{n}} = \llbracket Z_1 | \mathcal{F}_{\frac{q}{n}}^{Y, \pi} \rrbracket$ which implies $\pi_{\frac{q}{n}} = \llbracket Z_1 | \pi_0, Y_0, \dots, \pi_{\frac{q}{n}}, Y_{\frac{q}{n}} \rrbracket$ by taking conditional expectations. It follows that:

$$\begin{aligned}
V_n(\nu) &\geq \frac{1}{n} \mathbb{E} \left[\sum_{q=0}^{n-1} W(\llbracket \pi_{\frac{q}{n}} | (\pi_{\frac{m}{n}}, Y_{\frac{m}{n}}, m \leq q-1), Y_{\frac{q}{n}} \rrbracket) \right] \\
&\geq \frac{1}{n} \mathbb{E} \left[\sum_{q=0}^{n-1} W(\delta_{\pi_{\frac{q}{n}}}) \right] \\
&\geq \frac{1}{n} \mathbb{E} \left[\sum_{q=0}^{n-1} u(\pi_{\frac{q}{n}}) \right] \\
&\geq \mathbb{E} \left[\int_0^1 u(\pi_t) dt \right] + \mathbb{E} \left[\sum_{q=1}^n \int_{(q-1)/n}^{q/n} u(\pi_{\frac{(q-1)}{n}}) - u(\pi_t) dt \right] \\
&\geq \mathbb{E} \left[\int_0^1 u(\pi_t) dt \right] - \mathbb{E} \left[\sum_{q=1}^n \int_{(q-1)/n}^{q/n} \mathbf{d}(\pi_{\frac{(q-1)}{n}}, \pi_t) dt \right] \\
&= V(\nu) - \int_0^{1/n} \mathbb{E} \left[\sum_{q=1}^n \mathbf{d}(\pi_{\frac{(q-1)}{n}}, \pi_{\frac{(q-1)}{n}+t}) dt \right] \\
&\geq V(\nu) - \int_0^{1/n} \mathbb{E} \left[\sum_{q=1}^n \mathbf{d}(\pi_{\frac{(q-1)}{n}}, \pi_{\frac{q}{n}}) dt \right] \\
&\geq V(\nu) - \frac{1}{n} \mathbb{E} \left[\sum_{q=1}^n \mathbf{d}(\pi_{\frac{(q-1)}{n}}, \pi_{\frac{q}{n}}) \right]
\end{aligned}$$

where we used Jensen's inequality to obtain the second line, and then the Lipschitz property of u . Using Proposition 5.11, it follows that

$$\liminf_{n \rightarrow \infty} V_n(\nu) \geq V(\nu).$$

Let us prove the reverse inequality. Extract a subsequence denoted $m = m(n)$ such that

$$V_m(\nu) \xrightarrow{m \rightarrow +\infty} \limsup_{n \rightarrow \infty} V_n(\nu).$$

Let $(\tilde{Z}_q^n, \tilde{\pi}_q^n)_{q=0, \dots, n}$ be optimal for $V_n(\nu)$ and let (Z^n, π^n) denote the associated piecewise-continuous process on the uniform partition of $[0, 1]$. We verify easily that (Z^m, π^m) belongs to $\mathcal{B}(\widehat{\mathbb{P}}_\nu^m)$. Using Lemma 5.13, up to extracting again some subsequence, we can assume the sequence (Z^m, π^m) converges in law to some process (Z, π) which belongs to $\mathcal{B}(\nu)$. The Lipschitz property of W implies:

$$\forall \mu \in \Delta(\mathcal{Z}), \quad W(\mu) \leq U(\mu) + \mathbf{d}(\mu, \text{Bar}(\mu)). \quad (5.5)$$

We deduce that (with the convention that $\pi_{-\frac{1}{m}}^m := \mathbb{E}[\pi_0^m]$):

$$\begin{aligned}
V_m(\nu) &= \frac{1}{m} \mathbb{E} \left[\sum_{q=0}^{m-1} W(\llbracket \pi_{\frac{q}{m}}^m | (\pi_{\frac{p}{m}}^m, Y_{\frac{p}{m}}^m, p \leq q-1), Y_{\frac{q}{m}}^m \rrbracket) \right] \\
&\leq \frac{1}{m} \mathbb{E} \left[\sum_{q=0}^{m-1} W(\llbracket \pi_{\frac{q}{m}}^m | \pi_{\frac{p}{m}}^m, Y_{\frac{p}{m}}^m, p \leq q-1 \rrbracket) \right] \\
&\leq \frac{1}{m} \mathbb{E} \left[\sum_{q=0}^{m-1} u(\pi_{\frac{q-1}{m}}^m) \right] + \frac{1}{m} \mathbb{E} \left[\sum_{q=0}^{m-1} \mathbf{d}(\pi_{\frac{q}{m}}^m, \pi_{\frac{q-1}{m}}^m) \right] \\
&= H(\widehat{\mathbb{P}}_\nu^m) + \frac{1}{m} \mathbb{E} \left[\sum_{q=0}^{m-1} \mathbf{d}(\pi_{\frac{q}{m}}^m, \pi_{\frac{q-1}{m}}^m) \right],
\end{aligned}$$

where we used Jensen's inequality to obtain the second line and then inequality (5.5). Using that H is $\mathcal{L}(MZ)$ -continuous and Proposition 5.11, we conclude that

$$\limsup_{n \rightarrow \infty} V_n(\nu) \leq V(\nu).$$

□

5.3 Optimal martingales and asymptotically optimal strategies.

Maximizers of the limit problem induce asymptotically optimal strategies for player 1 in G_n .

Proposition 5.14. *If $\nu \in \Delta(\mathcal{Z})$, then any optimal process in the set*

$$\mathcal{P}_\infty(\nu) = \operatorname{argmax}_{\llbracket (Z_t, \pi_t)_{t \in [0,1]} \rrbracket \in \mathcal{B}(\nu)} \mathbb{E} \left[\int_0^1 u(\pi_t) dt \right]$$

induces by discretization a sequence of asymptotically optimal strategies for player 1 in $G_n(\nu)$.

Proof. Let $(Z_t, \pi_t)_{t \in [0,1]}$ be a process such that $\llbracket (Z_t, \pi_t)_{t \in [0,1]} \rrbracket \in \mathcal{P}_\infty(\nu)$. Using Remark 4.6, the discretization induces an asymptotically optimal strategy for the informed player in $G_n(\nu)$. Indeed, using the same arguments as for the proof of Theorem 2.10,

$$\frac{1}{n} \Psi_n((Z_{\frac{q}{n}}, \pi_{\frac{q}{n}})_{q=1, \dots, n}) \geq V(\nu) - \frac{1}{n} \mathbb{E} \left[\sum_{q=1}^n \mathbf{d}(\pi_{\frac{q-1}{n}}, \pi_{\frac{q}{n}}) \right].$$

The conclusion follows from Theorem 2.10 and Proposition 5.11. □

6 A variational formulation for the limit value.

Let us recall that

$$V(\nu) = \sup_{\llbracket (Z_t, \pi_t)_{t \in [0,1]} \rrbracket \in \mathcal{B}(\nu)} \mathbb{E} \left[\int_0^1 u(\pi_t) dt \right]$$

where $\mathcal{B}(\nu)$ is the set of laws of processes $(Z_t, \pi_t)_{t \in [0,1]}$ such that:

- Z has law \mathbb{P}_ν .
- π is a càdlàg modification of $(\llbracket Z_1 | \mathcal{F}_t^{(Y,\pi),+} \rrbracket)_{t \in [0,1]}$.
- \mathcal{F}^π is non-anticipative with respect to \mathcal{F}^Z .

and that $(P_t)_{t \geq 0}$ denote the transition probabilities of Z , i.e. for $t \geq 0$, P_t is a measurable map from \mathcal{Z} to $\Delta(\mathcal{Z})$ such that for all bounded measurable function on \mathcal{Z} and $0 \leq s \leq t$,

$$\mathbb{E}_\nu[\phi(Z_t) | \mathcal{F}_s^Z] = \int_{\mathcal{Z}} \phi dP_{t-s}(Z_s).$$

Notation 6.1. P_t induces a linear map from $\Delta(\mathcal{Z})$ to $\Delta(\mathcal{Z})$ denoted $\nu \rightarrow \nu \cdot P_t$ defined by

$$\forall \text{ Borel set } A \subset \mathcal{Z}, \nu \cdot P_t(A) = \int_{\mathcal{Z}} \left(\int_A dP_t(x) \right) d\nu(x).$$

Notation 6.2. Given some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, define \mathcal{N} as the set of null sets of \mathbb{P} in \mathcal{A} , and the usual augmentation of some filtration $(\mathcal{H}_t)_{t \in [0,1]}$ as $\overline{\mathcal{H}}_t = \mathcal{H}_t \vee \mathcal{N}$.

Let us fix $\nu \in \Delta(\mathcal{Z})$ and some process $(Z_t, \pi_t)_{t \in [0,1]}$ with law in $\mathcal{B}(\nu)$ defined on the canonical space denoted $(\Omega, \mathcal{A}, \mathbb{P})$.

It follows that:

$$\mathbb{E}_\nu[\delta_{Z_1} | \mathcal{F}_t^Z] = \delta_{Z_t} \cdot P_{1-t}.$$

Using that \mathcal{F}^π is non-anticipative with respect to \mathcal{F}^Z , we have that (see Lemma 5.6)

$$\mathbb{E}_\nu[\delta_{Z_1} | \mathcal{F}_t^{(Z,\pi),+}] = \mathbb{E}_\nu[\delta_{Z_1} | \mathcal{F}_t^{Z,+}] = \delta_{Z_t} \cdot P_{1-t}.$$

We deduce that:

$$\pi_t = \mathbb{E}_\nu[\delta_{Z_1} | \mathcal{F}_t^{(Y,\pi),+}] = \mathbb{E}_\nu[\delta_{Z_t} \cdot P_{1-t} | \mathcal{F}_t^{(Y,\pi),+}] = \mathbb{E}_\nu[\delta_{Z_t} | \mathcal{F}_t^{(Y,\pi),+}] \cdot P_{1-t}.$$

Let us define $\theta_t := \mathbb{E}[\delta_{Z_t} | \overline{\mathcal{F}}_t^{(Y,\pi),+}]$ (optional projection) and note that we have that for all t outside some countable set (see e.g. Lemma 1 in [36])

$$\theta_t = \mathbb{E}[\delta_{Z_t} | \overline{\mathcal{F}}_t^{Y,\pi}] = \mathbb{E}[\delta_{Z_t} | \mathcal{F}_t^{Y,\pi}].$$

The previous relation becomes

$$\forall t \in [0, 1], \quad \pi_t = \theta_t \cdot P_{1-t}.$$

Recall that the function \tilde{u} is defined on $[0, 1] \times \Delta(\mathcal{Z})$ by the relation

$$\tilde{u}(t, \mu) := u(\mu \cdot P_{1-t}).$$

Since Z is Feller, $(t, z) \rightarrow P_t(z)$ is continuous and therefore \tilde{u} is continuous. Moreover, we have

$$\mathbb{E} \left[\int_0^1 u(\pi_t) dt \right] = \mathbb{E} \left[\int_0^1 \tilde{u}(t, \theta_t) dt \right].$$

The previous construction implies that $\mathcal{F}^\pi \subset \overline{\mathcal{F}}^\theta$ and the definition of θ that $\mathcal{F}^\theta \subset \overline{\mathcal{F}}^{(Y,\pi),+}$. We deduce that $\overline{\mathcal{F}}^{Y,\pi} \subset \overline{\mathcal{F}}^{Y,\theta} \subset \overline{\mathcal{F}}^{(Y,\pi),+}$ and therefore $\overline{\mathcal{F}}^{(Y,\theta),+} = \overline{\mathcal{F}}^{(Y,\pi),+}$. It follows easily that \mathcal{F}^θ is non-anticipative with respect to \mathcal{F}^Z .

We can now state the following result which provides an alternative representation for the value function V .

Lemma 6.3.

$$V(\nu) = \max_{\llbracket (Z_t, \theta_t)_{t \in [0,1]} \rrbracket \in \mathcal{C}(\nu)} \mathbb{E} \left[\int_0^1 \tilde{u}(t, \theta_t) dt \right]$$

where $\mathcal{C}(\nu)$ is the set of laws of processes $(Z_t, \theta_t)_{t \in [0,1]}$ such that:

- Z has law \mathbb{P}_ν .
- θ is a càdlàg modification of $(\llbracket Z_t | \mathcal{F}_t^{(Y,\theta),+} \rrbracket)_{t \in [0,1]}$.
- \mathcal{F}^θ is non-anticipative with respect to \mathcal{F}^Z .

Proof. That V is lower or equal to the right-hand side of the equation follows directly from the above discussion. To prove the reverse inequality, given an admissible process (Z, θ) , one can define a process π by the relation $\pi_t = \theta_t \cdot P_{1-t}$. One checks easily that the law of (Z, π) belongs to $\mathcal{B}(\nu)$ and this proves the result. The existence of a maximum in $\mathcal{C}(\nu)$ follows from the existence of a maximum in $\mathcal{B}(\nu)$. \square

Remark 6.4. Note that by construction $\theta_t = \chi_t \otimes \delta_{Y_t}$ with $\chi_t = \mathbb{E}[\delta_{X_t} | \mathcal{F}_t^{(Y,\pi),+}]$.

6.1 The Markovian structure

The set of admissible processes (Z, θ) is more convenient than (Z, π) in order to obtain a variational formulation of our problem. In this subsection, we study the Markovian structure of belief processes θ .

Let us consider an admissible process $(Z_t, \theta_t)_{t \in [0,1]}$ with law in $\mathcal{C}(\nu)$ and $r \in [0, 1)$.

Define the process θ^r on the time interval $[0, 1]$ by (optional projection)

$$\theta_t^r := \mathbb{E}[\delta_{Z_t} | \overline{\mathcal{F}}_t^{(\theta, \wedge r, Y),+}].$$

Note that $\theta_t^r = \theta_t$ for $t \leq r$. and that for all t outside some countable set we have

$$\theta_t^r = \mathbb{E}[\delta_{Z_t} | \mathcal{F}_t^{(\theta, \wedge r, Y)}].$$

Define the right-continuous filtration \mathcal{G}^r by $\mathcal{G}_t^r = \overline{\mathcal{F}}_t^{(\theta, \wedge r, Y),+}$.

The following proposition is the main tool for proving an abstract variational characterization of the value. Note that similar general results already appear in [49] and [35].

Proposition 6.5. Using the above notations, $(\theta_t^r)_{t \in [r,1]}$ is a (\mathcal{G}^r) -Markov process whose transition Q is given by the function

$$Q_t(\nu) \in \Delta(\Delta(\mathcal{Z})), \quad \forall \text{ Borel set } C, \quad Q_t(\nu, C) := \mathbb{P}_\nu \left(\mathbb{E}_\nu[\delta_{Z_t} | \mathcal{F}_t^{Y,+}] \in C \right).$$

Proof. Note that the above formula does not necessarily define a transition function since we do not always have $Q_0(\nu) = \delta_\nu$ (for example if $X \in \mathbb{R}$ is almost surely constant and $Y \in \mathbb{R}$ is defined by $Y_t = Y_0 + \int_0^t X_0 dt$). The true state space of the Markov process is the set

$$\Delta^*(\mathcal{Z}) := \{\nu \in \Delta(\mathcal{Z}) \mid Q_0(\nu, \cdot) = \delta_\nu\}.$$

Note also that $\Delta^*(\mathcal{Z})$ is a subset of the set

$$\{p \otimes \delta_y \mid p \in \Delta(\mathcal{X}), y \in \mathcal{Y}\}.$$

Let us prove that this set is measurable and that the restriction of Q_t on $\Delta^*(\mathcal{Z})$ is a well-defined transition.

Let us define $\tilde{Q}_{t,n}(\nu) \in \Delta(\Delta(\mathcal{Z}))$ by

$$\tilde{Q}_{t,n}(\nu, C) := \mathbb{P}_\nu \left(\mathbb{E}_\nu[\delta_{Z_t} \mid \mathcal{F}_{t+\frac{1}{n}}^Y] \in C \right).$$

Using Lemma A.6, $\tilde{Q}_{t,n}$ is a measurable function of ν . Using the backward martingale convergence theorem (applied to a countable convergence determining subset of bounded continuous functions), we have almost sure convergence of

$$\mathbb{E}_\nu[\delta_{Z_t} \mid \mathcal{F}_{t+\frac{1}{n}}^Y] \xrightarrow[n \rightarrow \infty]{} \mathbb{E}_\nu[\delta_{Z_t} \mid \mathcal{F}_t^{Y,+}]$$

under \mathbb{P}_ν . This implies the convergence in law of $\tilde{Q}_{t,n}(\nu)$ to $Q_t(\nu)$ and thus the measurability with respect to ν . The same arguments also show that $t \rightarrow Q_t$ is right-continuous, and this implies joint measurability with respect to (t, ν) .

Let us now prove the Markov property and at the same time that $\Delta^*(\mathcal{Z})$ is stable by Q . Recall at first that θ^r is a càdlàg process and that for all t outside some countable set T we have

$$\theta_t^r = \mathbb{E}[\delta_{Z_t} \mid \theta_{\cdot \wedge r}, \mathcal{F}_t^Y].$$

Let $v \geq u \geq r$ and $v_n \notin T$ with $v_n \downarrow v$, then

$$\theta_v^r = \lim_n \theta_{v_n}^r = \lim_n \mathbb{E}[\delta_{Z_v} \mid \mathcal{G}_u^r, (Y_t)_{u \leq t \leq v_n}]$$

Define a process $(\tilde{Z}_t)_{t \in [0,1]}$ on an extension $\Omega \times \mathbb{D}([0,1], \mathcal{Z})$ of our probability space Ω such that the conditional law of \tilde{Z} given ω is $\mathbb{P}_{\theta_u^r}$. Using Lemma A.7, we deduce that

$$\mathbb{E}[\delta_{Z_v} \mid \mathcal{G}_u^r, (Y_t)_{u \leq t \leq v_n}] \mid \mathcal{G}_u^r = \mathbb{E}_{\theta_u^r}[\delta_{\tilde{Z}_{v-u}} \mid \mathcal{F}_{v_n-u}^{\tilde{Y}}] \mid \mathcal{G}_u^r.$$

Therefore, for any bounded continuous map ϕ we have (using Lebesgue's convergence theorem for conditional expectation)

$$\begin{aligned} \mathbb{E}[\phi(\theta_v^r) \mid \mathcal{G}_u^r] &= \lim_n \mathbb{E}[\phi(\mathbb{E}_{\theta_u^r}[\delta_{\tilde{Z}_{v_n-u}} \mid \mathcal{F}_{v_n-u}^{\tilde{Y}}]) \mid \mathcal{G}_u^r] \\ &= \mathbb{E}_{\theta_u^r}[\phi(\mathbb{E}_{\theta_u^r}[\delta_{\tilde{Z}_{v-u}} \mid \mathcal{F}_{v-u}^{\tilde{Y},+})]) \\ &= Q_{v-u}(\theta_u^r, \phi), \end{aligned}$$

which proves the Markov property. Note finally that the above construction also shows that $\theta_t^r \in \Delta^*(\mathcal{Z})$ almost surely by taking $v = u = t$. \square

Definition 6.6. Given an admissible process (Z, θ) with law in $\mathcal{C}(\nu)$, and a partition of $[0, 1]$ denoted $0 = t_0 < t_1 < \dots < t_m = 1$, the approximating process $(Z, \tilde{\theta})$ associated to the partition is defined by:

$$\tilde{\theta} = \theta^{t_i}, \quad \text{on the interval } [t_i, t_{i+1}).$$

Lemma 6.7. Given an admissible process (Z, θ) with law in $\mathcal{C}(\nu)$ and a sequence of partitions $(t_i^n)_{i=0, \dots, m_n}$ of $[0, 1]$ with $t_0^n = 0$ and $t_{m_n}^n = 1$ whose mesh π_n goes to zero, the sequence of approximation processes $\tilde{\theta}^n$, defined according to Definition 6.6, $\mathcal{L}(MZ)$ converges to θ as n goes to ∞ .

Proof. Assume that we are given an admissible process (Z, θ) defined on the canonical space denoted for simplicity $(\Omega, \mathcal{A}, \mathbb{P})$.

Let \mathcal{F}^n denote the sequence of filtrations defined by $\mathcal{F}_t^n = \mathcal{G}_t^{t_i^n}$ on the interval $[t_i^n, t_{i+1}^n)$ and $\mathcal{F}_t := \overline{\mathcal{F}_t^{(\theta, Y)}}^+$. Recall that for $t > 0$, $\mathcal{F}_{t-} := \bigvee_{s < t} \mathcal{F}_s$ and by convention define $\mathcal{F}_{0-} := \mathcal{G}_0^0$. We have $\mathcal{F}_t^n \subset \mathcal{F}_t$, and for all i , $\mathcal{F}_{t_i^n-} \subset \mathcal{F}_{t_i^n}^n$. Moreover, for all $t \in (0, 1]$,

$$\mathcal{F}_{t-} \subset \bigvee_{n \geq 1} \mathcal{F}_t^n \subset \mathcal{F}_t.$$

Note also that $\tilde{\theta}^n$ equals θ at times $\{t_0^n, \dots, t_{m_n}^n\}$.

It is sufficient to show that for all ϕ in a countable convergence determining subset of $C_b(\mathcal{Z})$, we have that $\int \phi d\tilde{\theta}_t^n$ converge in $dt \otimes d\mathbb{P}$ -measure to $\int \phi d\theta_t$. Indeed, if this property holds, for any subsequence, one may extract another subsequence (using diagonal method) to obtain a subsequence converging $dt \times d\mathbb{P}$ almost surely and convergence at time 1 is obvious.

At first, we show that for all $B \in L^\infty(\mathcal{F}_1)$, we have convergence in $dt \otimes d\mathbb{P}$ -measure of $(A_t^n)_{t \in [0, 1]} := (\mathbb{E}[B | \mathcal{F}_t^n])_{t \in [0, 1]}$ to $(A_t)_{t \in [0, 1]} := (\mathbb{E}[B | \mathcal{F}_t])_{t \in [0, 1]}$.

Let $\varepsilon > 0$. Note that $\mathbb{E}[B | \mathcal{F}_{t-}] = A_{t-}$ for $t \in (0, 1]$ and define $A_{0-} := \mathbb{E}[B | \mathcal{F}_{0-}]$ so that the same relation holds for $t = 0$. We deduce that

$$\begin{aligned} \mathbb{E} \left[\int_0^1 \mathbf{1}_{|A_t - A_t^n| > \varepsilon} dt \right] &\leq \int_0^1 \frac{1}{\varepsilon^2} \mathbb{E}[|A_t - A_t^n|^2] \\ &= \frac{1}{\varepsilon^2} \sum_{i=0}^{m_n-1} \int_{t_i^n}^{t_{i+1}^n} \mathbb{E}[A_t^2 - (A_t^n)^2] \\ &\leq \frac{1}{\varepsilon^2} \sum_{i=0}^{m_n-1} (t_{i+1}^n - t_i^n) \mathbb{E}[A_{t_{i+1}^n-}^2 - A_{t_i^n-}^2] \\ &\leq \frac{1}{\varepsilon^2} \pi_n \mathbb{E}[A_{1-}^2 - A_{0-}^2] \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

where we used Markov and Jensen inequalities and the martingale property.

Let us now assume that $(A_t)_{t \in [0, 1]}$ is a càdlàg bounded process. Define A^p as a sequence of càdlàg piecewise constant approximations of A on another sequence of partitions $(t_i^p)_{i=1, \dots, m_p}$ whose mesh is going to zero. It's well-known that A^p converges in $dt \otimes d\mathbb{P}$ -measure to A (see Jacod-Shiryaev Proposition VI.6.37). Note also that

$$A_t^p = \sum_{i=0}^{m_p} (A_{t_{i+1}^p} - A_{t_i^p}) \mathbf{1}_{t \geq t_{i+1}^p}.$$

Using the preceding property, we have that for all i , $(\mathbb{E}[(A_{t_{i+1}^p} - A_{t_i^p})|\mathcal{F}_t^n])_{t \in [0,1]}$ converges in $dt \otimes d\mathbb{P}$ -measure to $(\mathbb{E}[(A_{t_{i+1}^p} - A_{t_i^p})|\mathcal{F}_t])_{t \in [0,1]}$. It follows that

$$(\mathbb{E}[(A_{t_{i+1}^p} - A_{t_i^p})\mathbb{1}_{t \geq t_{i+1}^p}|\mathcal{F}_t^n])_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{dt \otimes d\mathbb{P}} (\mathbb{E}[(A_{t_{i+1}^p} - A_{t_i^p})\mathbb{1}_{t \geq t_{i+1}^p}|\mathcal{F}_t])_{t \in [0,1]}.$$

We deduce that for all p

$$(\mathbb{E}[A_t^p|\mathcal{F}_t^n])_{t \in [0,1]} \xrightarrow[n \rightarrow \infty]{dt \otimes d\mathbb{P}} (\mathbb{E}[A_t^p|\mathcal{F}_t])_{t \in [0,1]}.$$

On the other hand, for all $\varepsilon > 0$, Lebesgue convergence theorem implies,

$$\begin{aligned} \mathbb{E}\left[\int_0^1 \mathbb{1}_{|\mathbb{E}[A_t^p|\mathcal{F}_t] - \mathbb{E}[A_t|\mathcal{F}_t]} > \varepsilon} dt\right] &\leq \int_0^1 \frac{1}{\varepsilon} \mathbb{E}[|A_t - A_t^p|] \xrightarrow[p \rightarrow \infty]{} 0, \\ \mathbb{E}\left[\int_0^1 \mathbb{1}_{|\mathbb{E}[A_t^p|\mathcal{F}_t^n] - \mathbb{E}[A_t|\mathcal{F}_t^n]} > \varepsilon} dt\right] &\leq \int_0^1 \frac{1}{\varepsilon} \mathbb{E}[|A_t - A_t^p|] \xrightarrow[p \rightarrow \infty]{} 0. \end{aligned}$$

Finally, we deduce that for all p :

$$\limsup_{n \rightarrow \infty} \mathbb{E}\left[\int_0^1 \mathbb{1}_{|\mathbb{E}[A_t|\mathcal{F}_t] - \mathbb{E}[A_t|\mathcal{F}_t^n]} > 3\varepsilon} dt\right] \leq 2 \int_0^1 \frac{1}{\varepsilon} \mathbb{E}[|A_t - A_t^p|],$$

and the conclusion follows by sending p to ∞ . \square

6.2 A sub-dynamic programming principle.

Consider now the problem starting at $t \in [0, 1]$ with initial distribution $\nu \in \Delta(\mathcal{Z})$.

The filtration generated by some process A defined on the time interval $[t, 1]$ will be denoted

$$\mathcal{F}_{(t)}^A = (\mathcal{F}_{t,s}^A)_{s \in [t,1]} \text{ with } \mathcal{F}_{t,s}^A = \sigma(A_r, r \in [t, s]).$$

Let us define the following time-dependent extension of the value function V :

$$V(t, \nu) := \sup_{(Z, \theta) \in \mathcal{C}(t, \nu)} \mathbb{E}\left[\int_t^1 \tilde{u}(s, \theta_s) ds\right]$$

where $\mathcal{C}(t, \nu)$ is the set of laws of processes $(Z_s, \theta_s)_{s \in [t,1]}$ such that:

- Z is a Markov process starting at date t with initial law ν .
- θ is a càdlàg modification of $(\llbracket Z_t | \mathcal{F}_{t,s}^{(Y, \theta), +} \rrbracket)_{s \in [t,1]}$.
- $\mathcal{F}_{(t)}^\theta$ is non-anticipative with respect to $\mathcal{F}_{(t)}^Z$.

Similarly, adapting the proof of Lemma 6.3, we have:

$$V(t, \nu) := \sup_{(Z, \theta) \in \mathcal{B}(t, \nu)} \mathbb{E}\left[\int_t^1 u(\pi_s) ds\right]$$

where $\mathcal{B}(t, \nu)$ is the set of laws of processes $(Z_s, \pi_s)_{s \in [t,1]}$ such that:

- Z is a Markov process starting at date t with initial law ν .
- π is a càdlàg modification of $([Z_1 | \mathcal{F}_{t,s}^{(Y,\pi),+}])_{s \in [t,1]}$.
- $\mathcal{F}_{(t)}^\pi$ is non-anticipative with respect to $\mathcal{F}_{(t)}^Z$.

Note that $V(\nu) = V(0, \nu)$ and that all the properties proved for $V(\nu)$ translate automatically to $V(t, \nu)$. Precisely, Lemmas 5.13, 6.7 and 6.5 have obvious extensions. The next lemma provides an additional regularity property which will be useful for the next theorem.

Lemma 6.8. *Given a sequence (t_n, ν_n) converging to (t, ν) in $[0, 1] \times \Delta(\mathcal{Z})$ and any sequence $\mathbb{P}_n \in \mathcal{B}(t_n, \nu_n)$, there exists a subsequence also denoted \mathbb{P}_n and $\mathbb{P} \in \mathcal{B}(t, \nu)$ such that*

$$\mathbb{E}_{\mathbb{P}_n} \left[\int_{t_n}^1 u(\pi_s^n) ds \right] \rightarrow \mathbb{E}_{\mathbb{P}} \left[\int_t^1 u(\pi_s) ds \right].$$

Proof. At first, given processes (Z^n, π^n) of laws \mathbb{P}_n , we will replace these processes by processes defined on the same time interval $[0, 1]$. Define Z^n on the time interval $[t_n, t_n + 1]$ by $Z_s^n = Z_{s-t_n}^n$ for $s \geq 1$ and define $\pi_s^n = \pi_{s-t_n}^n$. Then, for all n , define the process $(\tilde{Z}_s^n, \tilde{\pi}_s^n)_{s \in [0,1]}$ by

$$\forall s \in [0, 1], (\tilde{Z}_s^n, \tilde{\pi}_s^n) := (Z_{t_n+s}^n, \pi_{t_n+s}^n).$$

Let $\tilde{\mathbb{P}}_n$ denote the law of $(\tilde{Z}^n, \tilde{\pi}^n)$ and $\tilde{\mathbb{Q}}_n$ the law of \tilde{Z}^n . By construction, $\tilde{\mathbb{P}}_n$ belongs to $\mathcal{B}(\tilde{\mathbb{Q}}^n)$. Then, we follow the same scheme as in Lemma 5.13. At first, note that $\tilde{\mathbb{P}}_n$ converge to the law $\tilde{\mathbb{P}}$ of the process $(Z_{s \wedge t})_{s \in [0,1]}$ where Z is a Markov process starting at time 0 with initial law ν (using that Z is continuous at time $1 - t$ with probability 1). Then, tightness of the laws of π^n follows from Theorem 5.8 since $\mathbb{E}[\pi_0^n] = \nu_n \cdot P_{1-t_n}$ and the set of laws $\{\nu_n \cdot P_{1-t_n}, n \geq 0\}$ is tight (as a convergent sequence).

Following the proof of Lemma 5.13 (with obvious modifications), for any limit point $\tilde{\pi}$, the process $(\pi_s)_{s \in [t,1]} := (\tilde{\pi}_{s-t})_{s \in [t,1]}$ has a law in $\mathcal{B}(t, \nu)$. The conclusion follows using the $\mathcal{L}(MZ)$ -continuity of H and that u is bounded. \square

We now prove theorem 2.11.

Proof of Theorem 2.11. The proof is split in two steps.

Step 1: We prove at that V is upper semi-continuous and fulfills the properties (i – iii).

That V is upper semi-continuous follows from Lemma 6.8.

i) is obvious.

ii) Concavity follows from the same method as the well-known splitting Lemma (see e.g. Chapter V.1 in [39]). Given ν_1, ν_2 and $\lambda \in [0, 1]$, $\mathbb{P}_1 \in C(t, \nu_1)$ and $\mathbb{P}_2 \in C(t, \nu_2)$, let us construct $\mathbb{P}_\lambda \in C(t, \lambda\nu_1 + (1 - \lambda)\nu_2)$ as follows. Assume that (Z^1, θ^1) and (Z^2, θ^2) are independent and of respective laws \mathbb{P}_1 and \mathbb{P}_2 . Let ξ be a random variable independent of (Z^1, θ^1) and (Z^2, θ^2) and such that $\mathbb{P}(\xi = 1) = \lambda$ and $\mathbb{P}(\xi = 2) = 1 - \lambda$. Define (Z, θ) as the process equal to (Z^i, θ^i) on $\{\xi = i\}$. It follows easily by conditioning on ξ that

$$\mathbb{E} \left[\int_t^1 \tilde{u}(s, \theta_s) ds \right] = \lambda \mathbb{E} \left[\int_t^1 \tilde{u}(s, \theta_s^1) ds \right] + (1 - \lambda) \mathbb{E} \left[\int_t^1 \tilde{u}(s, \theta_s^2) ds \right].$$

In order to conclude, it remains to prove that (Z, θ) has a law $\mathbb{P}_\lambda \in C(t, \lambda\nu_1 + (1-\lambda)\nu_2)$. That $(Z_{t+s})_{s \geq 0}$ has law $\mathbb{P}_{\lambda\nu_1 + (1-\lambda)\nu_2}$ follows directly from the definition. From the definition of (Z, θ) , we deduce that $\mathcal{F}_{t,s}^{(\theta, Y), +}$ is included in $\sigma(\xi) \vee \mathcal{F}_{t,s}^{(\theta^1, Y^1), +} \vee \mathcal{F}_{t,s}^{(\theta^2, Y^2), +}$. Using the definition of conditional laws, we verify easily that:

$$\begin{aligned} & \llbracket Z_s | \mathcal{F}_{t,s}^{(Y^1, \theta^1), +}, \mathcal{F}_{t,s}^{(Y^2, \theta^2), +}, \xi \rrbracket \\ &= \llbracket Z_s^1 | \mathcal{F}_{t,s}^{(Z^1, \theta^1), +}, \mathcal{F}_{t,s}^{(Z^2, \theta^2), +}, \xi \rrbracket \mathbf{1}_{\xi=1} + \llbracket Z_s^2 | \mathcal{F}_{t,s}^{(Z^1, \theta^1), +}, \mathcal{F}_{t,s}^{(Z^2, \theta^2), +}, \xi \rrbracket \mathbf{1}_{\xi=2} \\ &= \llbracket Z_s^1 | \mathcal{F}_{t,s}^{(Z^1, \theta^1), +} \rrbracket \mathbf{1}_{\xi=1} + \llbracket Z_s^2 | \mathcal{F}_{t,s}^{(Z^2, \theta^2), +} \rrbracket \mathbf{1}_{\xi=2} \\ &= \theta_s^1 \mathbf{1}_{\xi=1} + \theta_s^2 \mathbf{1}_{\xi=2} = \theta_s, \end{aligned}$$

where the second equality follows from Lemma 3.7. Using the tower property of conditional laws, we deduce that

$$\theta_s = \mathbb{E}[\delta_{Z_s} | \mathcal{F}_{t,s}^{(\theta, Y)}].$$

To prove the non-anticipative property, note at first that for $i = 1, 2$ and $s \in [t, 1]$, Lemma 3.7 implies that

$$\llbracket Z^i | \mathcal{F}_{t,s}^{Z^1, \theta^1}, \mathcal{F}_{t,s}^{Z^2, \theta^2}, \xi \rrbracket = \llbracket Z^i | \mathcal{F}_{t,s}^{Z^i} \rrbracket.$$

Note then that

$$\llbracket Z | \mathcal{F}_{t,s}^Z \rrbracket = \llbracket Z^1 | \mathcal{F}_{t,s}^{Z^1} \rrbracket \mathbf{1}_{\xi=1} + \llbracket Z^2 | \mathcal{F}_{t,s}^{Z^2} \rrbracket \mathbf{1}_{\xi=2}.$$

Similarly, we have

$$\llbracket Z | \mathcal{F}_{t,s}^{Z^1, \theta^1}, \mathcal{F}_{t,s}^{Z^2, \theta^2}, \xi \rrbracket = \llbracket Z^1 | \mathcal{F}_{t,s}^{Z^1, \theta^1}, \mathcal{F}_{t,s}^{Z^2, \theta^2}, \xi \rrbracket \mathbf{1}_{\xi=1} + \llbracket Z^2 | \mathcal{F}_{t,s}^{Z^1, \theta^1}, \mathcal{F}_{t,s}^{Z^2, \theta^2}, \xi \rrbracket \mathbf{1}_{\xi=2}.$$

From these we deduce that

$$\llbracket Z | \mathcal{F}_{t,s}^{Z^1, \theta^1}, \mathcal{F}_{t,s}^{Z^2, \theta^2}, \xi \rrbracket = \llbracket Z | \mathcal{F}_{t,s}^Z \rrbracket,$$

and we conclude using Lemma 3.7 with the intermediate σ -field $\mathcal{F}_{t,s}^{Z, \theta}$.

The inequality *iii*) follows easily from the following sub-dynamic programming principle: in words, player 1 can decide not to use his private information on the time interval $[t, t+h)$, and this behavior is sub-optimal. Precisely, given a process Z starting at time t with law ν , define θ on the interval $[t, t+h]$ by $\theta_s = \mathbb{E}[\delta_{Z_s} | \bar{\mathcal{F}}_{t,s}^{Y, +}]$ (optional projection). From Lemma 6.5, θ is a Markov process of transition Q , and θ_{t+h} has law $Q_h(\nu)$.

The function $\mathbb{P} \in C(t+h, \nu) \rightarrow J(\mathbb{P}) := \mathbb{E}[\int_{t+h}^1 \tilde{u}(s, \theta_s) ds]$ is $\mathcal{L}(MZ)$ -continuous, and the set-valued map $\mu \rightarrow C(t, \nu)$ is $\mathcal{L}(MZ)$ upper-semi-continuous. We deduce the the subset O of the space $\Delta(\mathcal{Z}) \times \Delta(\mathbb{D}([t+h, 1], \mathcal{Z} \times \Delta(\mathcal{Z})))$ defined by

$$O := \{(\nu, \mathbb{P}) | \mathbb{P} \in C(t+h, \nu), J(\mathbb{P}) \geq V(t+h, \nu)\}$$

is measurable. Moreover, Lemma 5.13 implies that for any ν , there exists some \mathbb{P} such that $(\nu, \mathbb{P}) \in O$. It follows therefore from Von Neumann's selection Theorem (see e.g. Proposition 7.49 in [4]) that there exists an optimal universally-measurable selection ϕ from $\Delta(\mathcal{Z})$ to $C(t+h, \nu)$ such that for all $\nu \in \Delta(\mathcal{Z})$, $\phi(\nu) \in O$.

By definition, ϕ is $Q_h(\nu)$ -almost surely equal to a Borel map ψ . Using Lemma A.4, we can construct a process (on some extension of the probability space) $\tilde{\theta}$ defined on the time interval $[t+h, 1]$ such that the conditional law of $(Z_s, \tilde{\theta}_s)_{s \in [t+h, 1]}$ given $\overline{\mathcal{F}}_{t, t+h}^{Y, +}$ is precisely $\psi(\theta_{t+h})$.

Let us define the process $(Z, \hat{\theta})$ where $\hat{\theta}$ is equal to θ on $[t, t+h]$ and to $\tilde{\theta}$ on $[t+h, 1]$. Using the preceding construction and assuming that the process $(Z, \hat{\theta})$ has a law in $\mathcal{C}(t, \nu)$, we deduce that:

$$\begin{aligned} V(t, \nu) &\geq \mathbb{E}\left[\int_t^1 \tilde{u}(s, \hat{\theta}_s) ds\right] = \mathbb{E}\left[\int_t^{t+h} \tilde{u}(s, \theta_s) ds\right] + \mathbb{E}\left[\mathbb{E}\left[\int_{t+h}^1 \tilde{u}(s, \tilde{\theta}_s) ds \mid \overline{\mathcal{F}}_{t, t+h}^{Y, +}\right]\right] \\ &\geq \int_0^h \int_{\Delta(\mathcal{Z})} \tilde{u}(t+s, \mu) dQ_s(\nu, d\mu) ds + \mathbb{E}[V(t+h, \theta_{t+h})] - \varepsilon \\ &= \int_0^h \int_{\Delta(\mathcal{Z})} \tilde{u}(t+s, \mu) dQ_s(\nu)(d\mu) ds + \int_{\Delta(\mathcal{Z})} V(t+h, \mu) dQ_h(\nu)(d\mu) - \varepsilon, \end{aligned}$$

which would conclude the proof of Step 1.

It remains therefore to prove that the process $(Z, \hat{\theta})$ has a law in $\mathcal{C}(t, \nu)$.

$(Z_s)_{s \geq t}$ is a Markov process starting at time t with law ν by assumption.

Let us prove that for all $s \in [t, 1]$, $\hat{\theta}_s = \mathbb{E}[\delta_{Z_s} | \mathcal{F}_{t, s}^{(Y, \hat{\theta}), +}]$. The result is obvious by construction for $s \in [t, t+h]$. Note that for $s \in [t+h, 1]$, we have (up to null sets) $\mathcal{F}_{t, s}^{Y, \hat{\theta}} = \mathcal{F}_{t, t+h}^Y \vee \mathcal{F}_{t+h, s}^{Y, \tilde{\theta}}$. Then for all rational $s \in [t+h, 1]$, for all $C \in \mathcal{F}_{t, t+h}^Y$ and $D \in \mathcal{F}_{t+h, s}^{Y, \tilde{\theta}}$ in some countable generating families, and for all $\phi \in C_b(\mathcal{Z})$ in some countable convergence determining subset, we have

$$\begin{aligned} &\mathbb{E}[\phi(Z_s) \mathbb{1}_D((Y_u, \tilde{\theta}_u)_{u \in [t+h, 1]}) \mathbb{1}_C((Y_u)_{u \in [t, t+h]})] \\ &= \mathbb{E}[\mathbb{E}[\phi(Z_s) \mathbb{1}_D((Y_u, \tilde{\theta}_u)_{u \in [t+h, 1]}) | \mathcal{F}_{t, t+h}^{Y, +}] \mathbb{1}_C((Y_u)_{u \in [t, t+h]})] \\ &= \mathbb{E}\left[\int_{\mathcal{Z}} \phi d\tilde{\theta}_s \mathbb{1}_D((Y_u, \tilde{\theta}_u)_{u \in [t+h, 1]}) \mathbb{1}_C((Y_u)_{u \in [t, t+h]})\right], \end{aligned}$$

which proves that $\hat{\theta}_s = \mathbb{E}[\delta_{Z_s} | \mathcal{F}_{t, s}^{(Y, \hat{\theta}), +}]$ for all rational $s \in [t+h, 1]$ and we conclude using that $(Z, \hat{\theta})$ has càdlàg trajectories and the backward martingale convergence theorem.

Let us prove the non-anticipative property. Note at first that it is sufficient to prove that for all $s \in [t, 1]$,

$$\llbracket (Z_u)_{u \geq s} | \mathcal{F}_{t, s}^{Z, \hat{\theta}} \rrbracket = \llbracket (Z_u)_{u \geq s} | \mathcal{F}_{t, s}^Z \rrbracket.$$

The property is obvious for $s \in [t, t+h]$. For each $s \in [t+h, 1]$, we will provide an alternative construction of the process $(Z, \hat{\theta})$ having the same law and for which the property will be easy to verify (we do not know how to prove this property in a simpler way, despite it is very intuitive). Before explaining the construction, note that the first equality in Lemma A.7 together with the fact that the conditional law of $(Z_s, \tilde{\theta}_s)_{s \in [t+h, 1]}$ given $\overline{\mathcal{F}}_{t, t+h}^{Y, +}$ verifies a non-anticipative property (since it belongs to $\mathcal{C}(t+h, \theta_{t+h})$) implies:

$$\llbracket (\tilde{\theta}_u)_{u \in [t+h, s]} | \overline{\mathcal{F}}_{t, t+h}^{Y, +}, (Z_u)_{u \in [t+h, 1]} \rrbracket = \llbracket (\tilde{\theta}_u)_{u \in [t+h, s]} | \overline{\mathcal{F}}_{t, t+h}^{Y, +}, (Z_u)_{u \in [t+h, s]} \rrbracket.$$

Let us now explain the construction. Using the remark following Lemma A.4, starting with the process Z defined on the interval $[t, 1]$, and with two independent variables ξ, ζ uniformly distributed on $[0, 1]$ and independent of Z , we may construct a process as follows. Construct at first $(\check{\theta}_u)_{u \in [t+h, s]}$ as a function of Z and ξ using the conditional law of $(\check{\theta}_u)_{u \in [t+h, s]}$ given $(\overline{\mathcal{F}}_{t, t+h}^{Y, +}, (Z_u)_{u \in [t+h, 1]})$. As explained above, this conditional law has the property that it is measurable with respect to $(\overline{\mathcal{F}}_{t, t+h}^{Y, +}, (Z_u)_{u \in [t+h, s]})$ so that $(\check{\theta}_u)_{u \in [t+h, s]}$ is measurable with respect to $(\overline{\mathcal{F}}_{t, t+h}^{Y, +}, (Z_u)_{u \in [t+h, s]}, \xi)$. Then, construct $(\check{\theta}_u)_{u \in [s, 1]}$ as a function of Z , $(\check{\theta}_u)_{u \in [t+h, s]}$ and ζ using the conditional law of $(\check{\theta}_u)_{u \in [s, 1]}$ given $(\overline{\mathcal{F}}_{t, t+h}^{Y, +}, (Z_u)_{u \in [t+h, 1]}, (\check{\theta}_u)_{u \in [t+h, s]})$. The process $(Z, \bar{\theta})$ where $\bar{\theta}$ equals θ on the interval $[t, t+h]$ and $\check{\theta}$ on the interval $[t+h, 1]$ has the the same law as $(Z, \hat{\theta})$ by construction. The non-anticipative property at time s is now obvious since

$$\mathbb{I}[(Z_u)_{u \geq s} | \mathcal{F}_{t, s}^Z, \zeta] = \mathbb{I}[(Z_u)_{u \geq s} | \mathcal{F}_{t, s}^Z],$$

and using the tower property of conditional laws.

Step 2: We prove that V is the smallest upper semi-continuous function fulfilling (i – iii).

Let Ψ be another u.s.c. function fulfilling these three properties. Let (Z, θ) be an admissible process starting at time r with initial law ν for Z and $r = t_0 < t_1 < \dots < t_m = 1$ a partition of $[r, 1]$. Define $\tilde{\theta}$ and $\tilde{\mathcal{F}}$ as in Lemma 6.7.

It follows from Lemma 6.5 that

$$\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \tilde{u}(s, \tilde{\theta}_s) | \mathcal{G}_{t_i}^{t_i}\right] = \int_0^{t_{i+1}-t_i} \int_{\Delta(\mathcal{Z})} \tilde{u}(t_i + s, \mu) dQ_{t_{i+1}-t_i}(\tilde{\theta}_{t_i}, d\mu) ds,$$

and

$$\mathbb{E}[\Psi(t_{i+1}, \theta_{t_{i+1}}^{t_i}) | \mathcal{G}_{t_i}^{t_i}] = \int_{\Delta(\mathcal{Z})} \Psi(t_{i+1}, \mu) dQ_{t_{i+1}-t_i}(\tilde{\theta}_{t_i}, d\mu).$$

Note that $\mathbb{E}[\tilde{\theta}_{t_{i+1}} | \mathcal{G}_{t_{i+1}}^{t_i}] = \theta_{t_{i+1}}^{t_i}$ so that using Jensen's inequality

$$\mathbb{E}[\Psi(t_{i+1}, \tilde{\theta}_{t_{i+1}}) | \mathcal{G}_{t_i}^{t_i}] \leq \int_{\Delta(\mathcal{Z})} \Psi(t_{i+1}, \mu) dQ_{t_{i+1}-t_i}(\tilde{\theta}_{t_i}, d\mu).$$

We deduce that

$$\mathbb{E}\left[\int_{t_i}^{t_{i+1}} \tilde{u}(s, \tilde{\theta}_s) | \mathcal{G}_{t_i}^{t_i}\right] \leq \Psi(t_i, \tilde{\theta}_{t_i}) - \mathbb{E}[\Psi(t_{i+1}, \tilde{\theta}_{t_{i+1}}) | \mathcal{G}_{t_i}^{t_i}].$$

And by summation

$$\mathbb{E}\left[\int_r^1 \tilde{u}(s, \tilde{\theta}_s)\right] \leq \mathbb{E}[\Psi(r, \tilde{\theta}_r)] \leq \Psi(r, \nu).$$

By choosing a sequence of partition whose mesh goes to zero, we obtain using Lemma 6.7

$$\mathbb{E}\left[\int_r^1 \tilde{u}(s, \theta_s)\right] \leq \Psi(r, \nu).$$

and therefore $V(r, \nu) \leq \Psi(r, \nu)$. □

Remark 6.9. *Note that in the above proof, the only time we used that Ψ is upper semi-continuous is when we used Jensen's inequality.*

The next lemma will be useful for the results of the next section.

Lemma 6.10. *We have for all $\nu \in \Delta(\mathcal{Z})$, $V(t, \nu) = \mathbb{E}[V(t, Q_0(\nu))]$.*

Proof. For simplicity of notations, and without loss of generality, we only write the proof for $t = 0$. Let (Z, θ) a process defined on the canonical space denoted $(\Omega, \mathcal{A}, \mathbb{P})$ with law in $\mathcal{C}(\nu)$ and which is optimal, meaning that

$$V(0, \nu) = \mathbb{E}\left[\int_0^1 \tilde{u}(t, \theta_t) dt\right] = \mathbb{E}\left[\mathbb{E}\left[\int_0^1 \tilde{u}(t, \theta_t) dt \mid \mathcal{F}_0^{Y,+}\right]\right].$$

Let us prove that the conditional law of $(Z_t, \theta_t)_{t \in [0,1]}$ given $\mathcal{F}_0^{Y,+}$ denoted $\tilde{\mathbb{P}}$ belongs almost surely to $\mathcal{C}(\llbracket Z_0 \mid \mathcal{F}_0^{Y,+} \rrbracket)$. Using that Z is an $\mathcal{F}^{Z,+}$ -Markov process, we deduce that the conditional law of Z given $\mathcal{F}_0^{Y,+}$ is $\mathbb{P}_{\llbracket Z_0 \mid \mathcal{F}_0^{Y,+} \rrbracket}$. The remaining properties follow easily by disintegration. In order to avoid ambiguous notations, let us enlarge the probability space by $\Omega \times \tilde{\Omega}$ where $\tilde{\Omega}$ is a copy of Ω and such that the law of the canonical process $(\tilde{Z}, \tilde{\theta})$ on $\tilde{\Omega}$ given (Z, θ) is equal to $\tilde{\mathbb{P}}$.

We prove at first that for all rational t , for all $\phi \in C_b(\mathcal{Z})$ in a countable convergence determining subset, for all $C \in \mathcal{F}_t^{\tilde{Y}, \tilde{\theta}}$ in a countable generating family, we have \mathbb{P} -almost surely

$$\mathbb{E}_{\tilde{\mathbb{P}}}[\phi(\tilde{Z}_t) \mathbf{1}_C(\tilde{Y}, \tilde{\theta})] = \mathbb{E}_{\tilde{\mathbb{P}}}[\int_{\mathcal{Z}} \phi d\tilde{\theta}_t \mathbf{1}_C(\tilde{Y}, \tilde{\theta})].$$

Due to the preceding construction, the above equality is equivalent to

$$\mathbb{E}_{\mathbb{P}}[\phi(Z_t) \mathbf{1}_C(Y, \theta) \mid \mathcal{F}_0^{Y,+}] = \mathbb{E}_{\mathbb{P}}[\int_{\mathcal{Z}} \phi d\theta_t \mathbf{1}_C(Y, \theta) \mid \mathcal{F}_0^{Y,+}].$$

This last equality follows easily by taking at first conditional expectations with respect to $\mathcal{F}_t^{(Y, \theta), +}$ and then with respect to $\mathcal{F}_0^{Y,+}$. We deduce that with probability 1, the law $\tilde{\mathbb{P}}$ is such that for all rational t ,

$$\mathbb{E}_{\tilde{\mathbb{P}}}[\delta_{\tilde{Z}_t} \mid \mathcal{F}_t^{(\tilde{Y}, \tilde{\theta})}] = \tilde{\theta}_t,$$

where the equality holds $\tilde{\mathbb{P}}$ -almost surely. The second property in the definition of $\mathcal{C}(\nu)$ follows now easily using that $(\tilde{Z}, \tilde{\theta})$ has càdlàg trajectories and the backward martingale convergence theorem.

The non-anticipative property follows by the same method. At first it is sufficient to prove that with probability 1, for all $t \in (0, 1]$, for all bounded MZ -continuous function $\phi : \mathbb{D}([0, 1-t], \mathcal{Z}) \rightarrow \mathbb{R}$ in a countable separating subset, for all $C \in \mathcal{F}_t^{\tilde{Z}, \tilde{\theta}}$ in a countable generating family, we have

$$\mathbb{E}_{\tilde{\mathbb{P}}}[\phi(\tilde{Z}_s)_{s \geq t} \mathbf{1}_C(\tilde{Z}, \tilde{\theta})] = \mathbb{E}_{\tilde{\mathbb{P}}}[\int_{\mathbb{D}([0, 1-t], \mathcal{Z})} \phi d\mathbb{P}_{\tilde{Z}_t} \mathbf{1}_C(\tilde{Z}, \tilde{\theta})].$$

This equality is equivalent to

$$\mathbb{E}_{\mathbb{P}}[\phi(Z_s)_{s \geq t} \mathbf{1}_C(Z, \theta) \mid \mathcal{F}_0^{Y,+}] = \mathbb{E}_{\mathbb{P}}[\int_{\mathbb{D}([0, 1-t], \mathcal{Z})} \phi d\mathbb{P}_{Z_t} \mathbf{1}_C(Z, \theta) \mid \mathcal{F}_0^{Y,+}].$$

And this last equality follows easily by taking successive conditional expectations with respect to $\mathcal{F}_t^{Z,\theta}$ and $\mathcal{F}_0^{Y,+}$.

We deduce from these properties that:

$$V(0, \nu) \leq \mathbb{E}[V(0, \llbracket Z_0 | \mathcal{F}_0^{Y,+} \rrbracket)] = \int V(0, \mu) dQ_0(\nu, d\mu).$$

Using that V is concave, we deduce that this inequality is an equality and this concludes the proof. \square

7 Filtering of continuous-time Markov chains.

Let us turn to a particular case where it is possible to obtain an infinitesimal version of the above variational inequality.

Let us recall briefly the framework and notations introduced in section 2. $\mathcal{X} = \{1, \dots, N\}$ is endowed with the distance $d_{\mathcal{X}}(\alpha, \alpha') = 2\mathbf{1}_{\alpha \neq \alpha'}$ and $\mathcal{Y} = \mathbb{R}$ with the Euclidean distance $d_{\mathcal{Y}}$. $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ is endowed with the distance $d = d_{\mathcal{X}} + d_{\mathcal{Y}}$. $\Delta(\mathcal{X})$ is identified with the canonical simplex Δ_N in \mathbb{R}^N .

In the following, we use the notation $|\cdot|$ for the Euclidean norm, and we will also use the notations $|\cdot|_1$, and $|\cdot|_{\infty}$ for the usual 1 and ∞ -norms in \mathbb{R}^m for $m \geq 1$.

$(X_t)_{t \in [0,1]}$ is a continuous-time homogeneous Markov chain with finite state space \mathcal{X} , infinitesimal generator $\Lambda = (\Lambda_{\alpha,\beta})_{\alpha,\beta \in \mathcal{X}}$ and initial law $p \in \Delta(\mathcal{X})$. If $\beta \neq \alpha$, $\Lambda_{\alpha,\beta}$ denotes the transition rate from state α to state β , and $\Lambda_{\alpha,\alpha} = -\sum_{\beta \neq \alpha} \Lambda_{\alpha,\beta}$.

The process Y_t is defined as the unique solution of the SDE

$$Y_t = y + \int_0^t b(X_s, Y_s) ds + \int_0^t \sigma(Y_s) dW_t,$$

where $(W_t)_{t \in [0,1]}$ is a standard Brownian motion independent of X and $y \in \mathbb{R}$ and b, σ are bounded, Lipschitz, and such that for all $y \in \mathbb{R}$, $\sigma(y) \geq \epsilon > 0$.

$Z = (X, Y)$ is a well defined homogeneous Feller Markov process and we may therefore apply Theorem 2.11. Recall that $b(y) = (b(\alpha, y)_{\alpha=1,\dots,N}) \in \mathbb{R}^N$ and $\chi_t(\alpha) := \mathbb{P}(X_t = \alpha | \mathcal{F}_t^{Y,+})$. Theorem 9.1 in [38] implies that the process $\theta_t := (\chi_t, Y_t) \in \mathbb{R}^{N+1}$ is a diffusion process² satisfying:

$$\theta_t = \theta_0 + \int_0^t c(\theta_t) dt + \int_0^t \kappa(\theta_t) d\bar{W}_t, \quad (7.1)$$

where \bar{W} is a standard $\mathcal{F}^{Y,+}$ -Brownian motion and the vectors $c(p, y)$ and $\kappa(p, y)$ in $\mathbb{R}^{N+1} = \mathbb{R}^N \times \mathbb{R}$ are defined by

$$c(p, y) = ({}^T \Lambda p, \langle p, b(y) \rangle)$$

$$\kappa(p, y) = \left(\left(\frac{p_{\alpha}}{\sigma(y)} (b(\alpha, y) - \sum_{\beta=1}^N b(\beta, y) p_{\beta}) \right)_{\alpha=1,\dots,N}, \sigma(y) \right).$$

²We use the same notation θ as in the previous section for a different object. Formally, the process θ_t introduced in section 6 should be equal to $\chi_t \otimes \delta_{Y_t}$.

where ${}^T\Lambda$ denotes the transpose of the matrix Λ . We deduce from standard properties of diffusion processes that for any function $f \in C^{1,2}([0, 1] \times \mathbb{R}^{N+1})$ with polynomial growth (say):

$$\forall 0 \leq s \leq t \leq 1, \quad \mathbb{E}[f(t, \theta_t) | \mathcal{F}_s^\theta] = f(s, \theta_s) + \mathbb{E}\left[\int_s^t A(f)(u, \theta_u) du | \mathcal{F}_s^\theta\right],$$

where $A(f)$ is the differential operator defined by

$$Af(t, \theta) = \frac{\partial f}{\partial t}(t, \theta) + \langle \nabla_\theta f(t, \theta), c(\theta) \rangle + \frac{1}{2} \langle \kappa(\theta), \nabla_\theta^2 f(t, \theta) \kappa(\theta) \rangle$$

Remark 7.1. Note that the process χ takes values in Δ_N , and that our assumptions on b and σ imply that the functions c and κ are Lipschitz continuous and bounded on $\Delta_N \times \mathbb{R}$. In the following, we will assume when needed that the functions c and κ are bounded and Lipschitz on the whole space \mathbb{R}^{N+1} (the above formula cannot be used directly since the resulting functions would be unbounded and only locally Lipschitz).

Let us recall some standard estimates for diffusion processes.

Lemma 7.2. Let \bar{W} denote a standard Brownian motion defined on the canonical space and let $\theta^{p,y}$ denote the unique strong solution of (7.1) starting at time 0 with the initial condition $\theta_0^{p,y} = (p, y)$. There exists a constant C such that for all $(p, y), (p', y') \in \Delta_N \times \mathbb{R}$ and for all $t, t' \in [0, 1]$:

$$\mathbb{E}[|\theta_t^{p,y} - \theta_{t'}^{p',y'}|] \leq C(|p - p'| + |y - y'| + |t - t'|^{1/2}).$$

Proof. These results are standard as the diffusion θ has bounded Lipschitz coefficients (see e.g. chapter 5 in [31]). \square

Given any probability $\mu \in \Delta(\mathcal{Z}) = \Delta(\{1, \dots, N\} \times \mathbb{R})$, we associate a probability $M(\mu) \in \Delta(\Delta_N \times \mathbb{R})$ as follows: given a pair of random variable (X, Y) of law μ , define the variable $p \in \Delta_N$ as the conditional law of X given Y , i.e.:

$$\forall \alpha = 1, \dots, N, \quad p_\alpha = \mathbb{E}[\mathbb{1}_{X=\alpha} | Y].$$

Then, define $M(\mu)$ as the law of (p, Y) . It is easily seen that this map is one-to-one. It is not onto as the image probability $M(\mu)$ is concentrated on the graph of some measurable function. However, we may define a right-inverse as follows: Given $\pi \in \Delta(\Delta_N \times \mathbb{R})$, and a variable (p, Y) of law π , construct a variable $X \in \{1, \dots, N\}$ whose conditional law given (θ, Y) is precisely θ . Define $M^{-1}(\pi)$ as the law of (X, Y) .

Define $\Delta^1(\{1, \dots, N\} \times \mathbb{R})$ as the set of probabilities μ such that

$$\int_{\mathcal{X} \times \mathbb{R}} |y| d\mu(x, y) < \infty.$$

This set is endowed with the Wasserstein distance \mathbf{d} induced by the distance d on \mathcal{Z} .

Define $\Delta^1(\Delta_N \times \mathbb{R})$ as the set of probabilities π such that

$$\int_{\Delta_N \times \mathbb{R}} |y| d\pi(p, y) < \infty.$$

This set is endowed with the Wasserstein distance of order 1, \mathbf{d}_W induced by the Euclidean distance. Note that the image of $\Delta^1(\Delta_N \times \mathbb{R})$ by M^{-1} is $\Delta^1(\{1, \dots, N\} \times \mathbb{R})$.

Lemma 7.3. *The function $\pi \rightarrow u(M^{-1}(\pi))$ is Lipschitz on $\Delta^1(\Delta_N \times \mathbb{R})$ with respect to \mathbf{d}_W .*

Proof. Note at first that Lemma 3.15 implies that u is 1-Lipschitz on $\Delta^1(\mathcal{Z})$ with respect to the Wasserstein distance \mathbf{d} induced by the distance d on \mathcal{Z} . Indeed, the result of Lemma 3.15 is stated with respect to a smaller distance induced by $d \wedge 1$ which implies directly the result. It is therefore sufficient to prove that for all $\pi, \pi' \in \Delta^1$, $\mathbf{d}(M^{-1}(\pi), M^{-1}(\pi')) \leq \mathbf{d}_W(\pi, \pi')$.

To prove this, recall the classical result that given two random variables $X, X' \in \{1, \dots, N\}$ of respective laws p, p' , then

$$\mathbb{E}[2\mathbf{1}_{X \neq X'}] \leq |p - p'|_1.$$

Then, given any coupling of variables (p, Y, p', Y') such that the law of (p, Y) is π and the law of (p', Y') is π' , construct two variables X, X' on a possibly enlarged probability space such that the law of X given (p, Y, p', Y') is equal to p and the law of X' given (p, Y, p', Y') is equal to p' . It follows that the law of (X, Y) is $M^{-1}(\pi)$ and that the law of (X', Y') is $M^{-1}(\pi')$. Using the above mentioned result, we have almost surely

$$\mathbb{E}[2\mathbf{1}_{X \neq X'} | p, y, p', Y'] \leq |p - p'|_1,$$

from which we deduce:

$$\begin{aligned} \mathbf{d}(M^{-1}(\pi), M^{-1}(\pi')) &\leq \mathbb{E}[2\mathbf{1}_{X, X'} + |Y - Y'|] \leq \mathbb{E}[|p - p'|_1 + |Y - Y'|] \\ &\leq \sqrt{N+1} \mathbb{E}[\sqrt{|p - p'|^2 + |Y - Y'|^2}], \end{aligned}$$

and we conclude by taking the infimum over all possible couplings of π, π' in the last expression. \square

We deduce from the two preceding lemmas more regularity for the functions \tilde{u} and V .

Lemma 7.4. *There exists a constant C' such that for all $(p, y), (p', y') \in \Delta_N \times \mathbb{R}$ and all $t, t' \in [0, 1]$, we have:*

$$|\tilde{u}(t, p \otimes \delta_y) - \tilde{u}(t', p' \otimes \delta_{y'})| \leq C'(|p - p'| + |y - y'| + |t - t'|^{\frac{1}{2}}), \quad |V(t, p \otimes \delta_y) - V(t', p \otimes \delta_y)| \leq C'|t - t'|^{\frac{1}{2}}$$

Proof. Let $t, t' \in [0, 1]$ and $(p, y) \in \Delta_N \times \mathbb{R}$. Using Lemma 7.2 (and the notations introduced there), we have:

$$\begin{aligned} |\tilde{u}(t, p \otimes \delta_y) - \tilde{u}(t', p' \otimes \delta_{y'})| &= |u(\llbracket \theta_{1-t}^{p,y} \rrbracket) - u(\llbracket \theta_{1-t'}^{p',y'} \rrbracket)| \\ &\leq \mathbf{d}_W(\llbracket \theta_{1-t}^{p,y} \rrbracket, \llbracket \theta_{1-t'}^{p',y'} \rrbracket) \\ &\leq \sqrt{N+1} \mathbb{E}[|\theta_{1-t}^{p,y} - \theta_{1-t'}^{p',y'}|] \\ &\leq \sqrt{N+1} C(|p - p'| + |y - y'| + |t - t'|^{\frac{1}{2}}). \end{aligned}$$

Let $t, t' \in [0, 1]$ and $\nu = p \otimes \delta_y \in \Delta(\mathcal{Z})$. Let us now go back to the notations of section 6. Let (Z, θ) be an optimal process with law in $\mathcal{C}(t, \nu)$ for the optimization problem given in Lemma 6.3 for $V(t, \nu)$ and note that θ takes values in the set $\{p \otimes \delta_y | (p, y) \in \Delta_N \times \mathbb{R}\}$. Let us construct a process θ' defined on $[t', 1 + (t - t')^+]$ as follows. First, assuming that Z_s is defined for all $s \geq t$, we extend the process θ on the time interval $[1, +\infty)$ by $\theta_s = \mathbb{E}[\delta_{Z_s} | \overline{\mathcal{F}}_1^\theta \vee \overline{\mathcal{F}}_s^{Y,+}]$.

Then, define $\theta'_s = \theta_{s+t-t'}$ and $Z'_s = Z_{s+t-t'}$. It follows easily that the restriction of (Z', θ') to $[t', 1]$ has a law in $\mathcal{C}(t', \nu)$ and we have

$$\begin{aligned} V(t, \nu) - V(t', \nu) &\leq \mathbb{E}\left[\int_t^{1-(t'-t)^+} (\tilde{u}(s, \theta_s) - \tilde{u}(s+t'-t, \theta'_{s+t'-t})) ds\right] + |t' - t| \\ &\leq \sqrt{N+1}C|t' - t|^{\frac{1}{2}} + |t' - t|, \end{aligned}$$

and we conclude the proof by inverting the roles of t and t' . \square

Remark 7.5. *Similarly to Remark 7.1, we will assume without loss of generality that \tilde{u} is uniformly continuous on the whole space $[0, 1] \times \mathbb{R}^{N+1}$.*

The following result follows directly from the specific structure of our problem and from the preceding lemma.

Lemma 7.6. *Using the notations of section 6, we have*

$$\Delta^*(\mathcal{Z}) = \{p \otimes \delta_y | p \in \Delta(\mathcal{X}), y \in \mathcal{Y}\}$$

and for all $(t, \mu) \in [0, 1] \times \Delta(\mathcal{Z})$, we have:

$$V(t, \mu) = \int V(t, \mu(x|y) \otimes \delta_y) d\mu(y),$$

where $\mu(x|y)$ denotes the conditional law of x given y induced by μ .

Proof. Note at first that by construction, the restriction of the map Q_t introduced in Lemma 6.5 to the set $\{p \otimes \delta_y | p \in \Delta(\mathcal{X}), y \in \mathcal{Y}\}$ coincides with the semi-group of the diffusion process (χ_t, Y_t) . This implies that for all $(p, y) \in \Delta_N \times \mathbb{R}$, $Q_0(p \otimes \delta_y) = p \otimes \delta_y$ and the first point follows since we always have the inclusion

$$\Delta^*(\mathcal{Z}) \subset \{p \otimes \delta_y | p \in \Delta(\mathcal{X}), y \in \mathcal{Y}\}.$$

The equality for V follows then directly from Lemma 6.10. \square

In view of the preceding Lemma, the function $V(t, \cdot)$ is characterized by its restrictions to $\Delta^*(\mathcal{Z})$. In the following, we will therefore use the following slightly abusive notations.

Notation 7.7. *For all $(t, p, y) \in [0, 1] \times \mathbb{R}^{N+1}$*

$$\tilde{u}(t, p, y) := \tilde{u}(t, p \otimes \delta_y), \quad V(t, p, y) := V(t, p \otimes \delta_y).$$

Notation 7.8. *In the following, we will use the notations $\mathbb{P}_{t,p,y}$ and $\mathbb{E}_{t,p,y}$ to denote the law (and the corresponding expectation) of the diffusion process θ starting at time t with initial position $\theta_t = (p, y)$.*

Let us now apply the characterization of Theorem 2.11 for distributions $\nu \in \Delta^*(\mathcal{Z})$ using the above notations.

Proposition 7.9. *V is upper semi-continuous and is the smallest bounded measurable function such that:*

- for all $(p, y) \in \Delta_N \times \mathbb{R}$, $V(1, p, y) = 0$,
- for all $(t, y) \in [0, 1) \times \mathbb{R}$, $p \rightarrow V(t, p, y)$ is concave on Δ_N .
- for all $(p, y) \in \Delta_N \times \mathbb{R}$, for all $t \in [0, 1)$:

$$\forall h \in (0, 1 - t], \quad \mathbb{E}_{t,p,y}[V(t+h, \theta_{t+h})] - V(t, p, y) + \mathbb{E}_{t,p,y}\left[\int_t^{t+h} \tilde{u}(s, \theta_s) ds\right] \leq 0. \quad (7.2)$$

Proof. At first, that V is upper semi-continuous and fulfills the three properties follows directly from Proposition 2.11 by considering the restriction of V to $[0, 1] \times \Delta^*(\mathcal{Z})$. The proof follows then exactly as in Proposition 2.11, using the identification $\Delta^*(\mathcal{Z}) \simeq \Delta_N \times \mathbb{R}$ as a finite dimensional space and replacing Ψ by any bounded measurable function f fulfilling the above three properties. Note that the upper semi-continuity assumption in the second part of the proof of Proposition 2.11 was only needed in order to apply Jensen's inequality. However, in finite dimensional spaces, Jensen's inequality holds for any measurable concave function. Rewriting the proof of Proposition 2.11 in the present framework, Jensen's inequality is only applied for conditional laws taking values in a finite dimensional simplex, and therefore any bounded measurable function f which is concave with respect to p and fulfills property (7.2) is greater than V . \square

Recall the definition of supersolution introduced in section 2. Depending on the generator Λ , some faces F of Δ_N may be invariant for the Markov process X , and in this case, one may define a projected equation on F .

Definition 7.10. *Given a subset S of \mathcal{X} , the face $F = \text{co}(\{e_m, m \in S\})$ of Δ_N is invariant if for all $\alpha \in S$ and all $\beta \in \mathcal{X} \setminus S$, $\Lambda_{\alpha,\beta} = 0$.*

It follows directly from this definition and the definition of the operator A that for any smooth function ϕ , the quantity $A(\phi)(t, p, y)$ for $(t, p, y) \in [0, 1) \times F \times \mathbb{R}$ depends only on the restriction of ϕ to the affine subspace spanned by F , denoted $\text{aff}(F)$.

Notation 7.11. *Let F be an invariant face of Δ_N . In the following, by a neighborhood of $[0, 1) \times F \times \mathbb{R}$, we always refer to a neighborhood in $\mathbb{R} \times \text{aff}(F) \times \mathbb{R}$.*

Given a smooth function ϕ defined on some neighborhood of $[0, 1) \times F \times \mathbb{R}$, we define the projected operator A^F by

$$A^F(\phi) := A(\hat{\phi})$$

for any smooth extension $\hat{\phi}$ of ϕ defined on a neighborhood of $[0, 1) \times \Delta_N \times \mathbb{R}$. Note that this restriction also applies to $F = \Delta_N$.

Lemma 7.12. *Assume that F is an invariant face of Δ_N , and that f is a bounded lower semi-continuous supersolution of (2.6) which is concave with respect to p . Then the restriction of f to F is a bounded supersolution of the projected equation*

$$\frac{\partial f}{\partial t} + A^F(f)(t, p, y) + \tilde{u}(t, p, y) \leq 0 \quad (7.3)$$

Proof. It is sufficient to prove that for any smooth test function ϕ defined on a neighborhood of $[0, 1) \times F \times \mathbb{R}$ such that $\phi \leq f$ on $[0, 1) \times F \times \mathbb{R}$, there exists an extension $\hat{\phi}$ of ϕ defined on a neighborhood of $[0, 1) \times \Delta_N \times \mathbb{R}$ such that $\hat{\phi} \leq f$ on $[0, 1) \times \Delta_N \times \mathbb{R}$. Proceeding inductively on the dimension, it is sufficient to prove this when F is a face of dimension $N - 1$, and we may assume without loss of generality that $F = \{p \in \Delta_N \mid p_1 = 0\}$ and that $N \geq 2$. The result follows easily from the concavity of f with respect to p . One may assume that ϕ is defined on a product $O \times O'$ where O is a neighborhood of $[0, 1) \times \mathbb{R}$ and O' a neighborhood of Δ_N . Let $g(t, y)$ be a smooth function defined on O such that

$$g(t, y) \geq \sup_{p \in F} |\nabla_p \phi(t, p, y)|_\infty$$

where ∇_p denotes the gradient of ϕ with respect to p (that we assume taking values in the subspace $(\text{aff}(F) - \text{aff}(F))$ of \mathbb{R}^N). Let $\{e_i, i = 1, \dots, N\}$ denote the canonical basis, define $p^* := (0, \frac{1}{N-1}, \dots, \frac{1}{N-1})$, and for any $p \in \Delta_N$ such that $p_1 \neq 0$, $h(p) := (0, \frac{p_2}{1-p_1}, \dots, \frac{p_N}{1-p_1}) \in D$. Define

$$\hat{\phi}(t, p, y) := \phi(t, p + p_1(p^* - e_1), y) - 2(\|f\|_\infty + g(t, y))p_1.$$

Note then that if $p_1 \neq 1$,

$$|p + p_1(p^* - e_1) - h(p)|_1 \leq \sum_{i=2}^N \left| \frac{p_i}{1-p_1} - p_i - \frac{p_1}{N-1} \right| \leq 2p_1.$$

It follows that

$$\begin{aligned} \hat{\phi}(t, p, y) &\leq \phi(t, h(p), y) + 2p_1 |\nabla_p \phi(t, h(p), y)|_\infty - 2(\|f\|_\infty + g(t, y))p_1 \\ &\leq (1 - p_1)\phi(t, h(p), y) - \|f\|_\infty p_1 \\ &\leq (1 - p_1)f(t, h(p), y) + p_1 f(t, e_1, y) \\ &\leq f(t, (1 - p_1)h(p) + p_1 e_1, y) = f(t, p, y). \end{aligned}$$

Note that $\hat{\phi}(t, e_1, y) = \phi(t, p^*, y) - 2(\|f\|_\infty + g(t, y)) \leq -\|f\|_\infty \leq f(t, e_1, y)$ which proves that $\hat{\phi} \leq f$ on $[0, 1) \times \Delta_N \times \mathbb{R}$. Finally, the formula defining $\hat{\phi}$ is meaningful in a neighborhood of this set which concludes the proof. \square

The proof of Theorem 2.15 is based on Proposition 7.13 below, whose proof is rather long and technical and relies on classical approximations arguments for viscosity solutions by inf-convolutions and on approximations in law for diffusion processes.

Proposition 7.13. *Let f be a bounded lower semi-continuous supersolution of (2.6) fulfilling the conditions of Theorem 2.15, then f fulfills property (7.2) of Proposition 7.9.*

Proof of theorem 2.15. The proof is divided in two parts.

part 1: We prove that the lower semicontinuous envelope of V , denoted V_* , is a supersolution of (2.6).

Let ϕ be any smooth test function such that $\phi \leq V_*$ with equality in $(t, p, y) \in [0, 1) \times \Delta_N \times \mathbb{R}$. We may assume without loss of generality that ϕ is bounded. Consider a sequence

$(t_n, p_n, y_n) \rightarrow (t, p, y)$ such that $V(t_n, p_n, y_n) \rightarrow V_*(t, p, y)$. From property (7.2), we deduce that

$$\phi(t_n, p_n, y_n) - V(t_n, p_n, y_n) + \mathbb{E}_{t_n, p_n, y_n} \left[\int_{t_n}^{t_n+h} \left(\frac{\partial \phi}{\partial t} + A(\phi) \right)(s, \theta_s) ds \right] + \mathbb{E}_{t_n, p_n, y_n} \left[\int_{t_n}^{t_n+h} \tilde{u}(s, \theta_s) ds \right] \leq 0.$$

Letting $n \rightarrow \infty$, we obtain that

$$\mathbb{E}_{t, p, y} \left[\int_t^{t+h} \left(\frac{\partial \phi}{\partial t} + A(\phi) \right)(s, \theta_s) ds \right] + \mathbb{E}_{t, p, y} \left[\int_t^{t+h} \tilde{u}(s, \theta_s) ds \right] \leq 0.$$

Letting then $h \rightarrow 0$, it follows from usual arguments that

$$\frac{\partial \phi}{\partial t}(t, p, y) + A(\phi)(t, p, y) + \tilde{u}(t, p, y) \leq 0,$$

which proves that V_* is a bounded supersolution of $\frac{\partial f}{\partial t} + A(f) + \tilde{u} \leq 0$.

Note also that $V_*(1, p, y) = 0$ (using the Hölder continuity with respect to t proved in Lemma 7.4).

Let us prove that V_* is concave with respect to p . Let $(t, y) \in [0, 1] \times \mathbb{R}$ and $p = \lambda p_1 + (1 - \lambda) p_2$ for some $p, p_1, p_2 \in \Delta_N$ and $\lambda \in [0, 1]$. Let (t^n, p^n, y^n) a sequence converging to (t, p, y) such that $V(t^n, p^n, y^n) \rightarrow V_*(t, p, y)$. Then, there exists $p_1^n, p_2^n \in \Delta_N$ such that $p^n = \lambda p_1^n + (1 - \lambda) p_2^n$ and $(p_1^n, p_2^n) \rightarrow (p_1, p_2)$ (it is for example a consequence of Lemma 8.2 in [37]). It follows that

$$V(t^n, p^n, y^n) \geq \lambda V(t^n, p_1^n, y^n) + (1 - \lambda) V(t^n, p_2^n, y^n).$$

By letting $n \rightarrow \infty$ and using the definition of V_* , we deduce that

$$V_*(t, p, y) \geq \lambda V_*(t, p_1, y) + (1 - \lambda) V_*(t, p_2, y),$$

which concludes the proof.

part 2: Any function G fulfilling the assumptions of the theorem is greater or equal to V using Propositions 7.9 and 7.13. This also proves that $V \leq V_*$ and therefore that V is continuous. \square

Let us now prove Proposition 7.13.

Proof of Proposition 7.13. The proof of this proposition will be divided in 6 steps.

step 1: Dealing with possible absorbing faces of Δ_N for the process $(\chi_t)_{t \in [0, 1]}$.

Recall that a subset S of \mathcal{X} is invariant for the Markov process (X_t) , if $\forall \alpha \in S, \forall \beta \notin S, \Lambda_{\alpha, \beta} = 0$. In this case, it follows directly from the definition of χ that for all $\beta \notin S$, and for all $t \geq 0$, $\mathbb{P}(\chi_t(\beta) = 0) = 1$. Moreover, the projected operator A^F on the face F of Δ_N of probabilities supported by S is well-defined.

We claim that for all $t > 0$, χ_t belongs with probability 1 to the relative interior of the largest face of Δ_N which contains the support of χ_0 and is invariant for the Markov process X_t . Despite the fact that this result is well-known and very intuitive, we provide a short formal proof in the appendix (see Lemma A.1).

step 2: A first reduction.

Recall that f is bounded and lower semi-continuous on $[0, 1] \times \Delta_N \times \mathbb{R}$. We aim at proving that f satisfies property (7.2). We will assume that $(\bar{t}, \bar{p}, \bar{y}) \in [0, 1] \times \Delta_N \times \mathbb{R}$ is fixed and such that the smallest invariant face F of Δ_N containing p is equal to Δ_N . This is without loss of generality since all the proof is still valid by replacing Δ_N by F in what follows (and adapting notations).

For simplicity, we will also consider that f is defined on $[0, 1] \times D \times \mathbb{R} \subset \mathbb{R}^{N+1}$ where D denoted the image of Δ_N by the projection on the first $N - 1$ coordinates. Note that this does not modify any of the properties of the corresponding definition of supersolution of (2.6) since we only have to replace p_N by $1 - \sum_{i=1}^{N-1} p_i$ in the definition of A , and that the supersolution property depends only on the restriction of the test functions to the affine subspace generated by Δ_N . Moreover, the last coordinate of the process χ_t can also be deduced from the $N - 1$ first coordinates. In the following, the process χ and the operator A correspond implicitly to these restrictions without modifying the notations.

step 3: Properties of inf-convolution.

Let $\|f\|_\infty := \sup_{(t,p,y) \in [0,1] \times D \times \mathbb{R}} |f(t,p,y)|$. For all $\varepsilon > 0$, let us define the function f^ε on $\mathbb{R} \times \mathbb{R}^{N-1} \times \mathbb{R}$ by

$$f^\varepsilon(t,p,y) = \min_{(t',p',y') \in [0,1] \times D \times \mathbb{R}} f(t',p',y') + \frac{1}{2\varepsilon}(|t' - t|^2 + |p' - p|^2 + |y' - y|^2)$$

Recall that $f^\varepsilon \uparrow f$ when $\varepsilon \rightarrow 0$ on the domain of f and converges to $+\infty$ otherwise. Note that for any $(t,p,y) \in [0, 1] \times D \times \mathbb{R}$, the minimum is attained for (t', p', y') in a closed ball of center (t, p, y) and radius $\gamma_\varepsilon = \sqrt{4\varepsilon\|f\|_\infty}$.

f^ε is continuous and $\frac{1}{2\varepsilon}$ semi-concave. Precisely, define the function

$$\hat{f}^\varepsilon(t,p,y) := f^\varepsilon(t,p,y) - \frac{1}{2\varepsilon}(|t|^2 + |p|^2 + |y|^2).$$

Then

$$\hat{f}^\varepsilon(t,p,y) = \min_{(t',p',y') \in [0,1] \times D \times \mathbb{R}} f(t',p',y') + \frac{1}{2\varepsilon}(|t'|^2 + |p'|^2 + |y'|^2) - \frac{1}{\varepsilon}(t't + \langle p', p \rangle + y'y)$$

is concave (as an infimum of linear functions). We also have

$$-\|f\|_\infty - \frac{1}{\varepsilon}(|t| + |p|) - \frac{1}{2\varepsilon}|y|^2 \leq \hat{f}^\varepsilon(t,p,y) \leq \|f\|_\infty.$$

From this inequality and the concavity, we deduce easily the following growth condition on the supergradients of \hat{f}^ε :

$$\partial^+ \hat{f}^\varepsilon(t,p,y) \subset B(0, C_\varepsilon(1 + |y|)),$$

for some constant $C_\varepsilon > 0$.

It is well-known that the inf-convolution f^ε are approximate supersolutions equation (2.6). The result we use here is a slight modification of Theorem 3 in Ishii [27].

Let us introduce some notations. Define $\mathcal{L} : \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}^N \times S^N \rightarrow \mathbb{R}$ by

$$\mathcal{L}(p,y,v,M) = \langle c(p,y), v \rangle + \frac{1}{2} \langle \kappa(p,y), M \kappa(p,y) \rangle.$$

so that for any smooth function ϕ defined on $[0, 1] \times \mathbb{R}^{N-1} \times \mathbb{R}$:

$$A(\phi)(t, p, y) = \mathcal{L}(p, y, \nabla\phi(t, p, y), \nabla^2\phi(t, p, y)).$$

With our assumptions on c and κ , it is well known (see e.g. [14], p.19) that there exists $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that for any $\eta > 1$, $(t, p, y), (t', p', y') \in [0, 1] \times D \times \mathbb{R}$ and matrices $M, M' \in S^N$ with

$$-3\eta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix} \leq 3\eta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

then, denoting $v = (p, y)$ and $v' = (p', y')$, we have:

$$\begin{aligned} \mathcal{L}(p', y', \eta(v - v'), -M') + \tilde{u}(t', p', y') &\leq \mathcal{L}(p, y, \eta(v - v'), M) + \tilde{u}(t, p, y) \\ &\quad + \omega(\eta(|v - v'|^2 + |t - t'|^2) + \sqrt{|v - v'|^2 + |t - t'|^2}). \end{aligned}$$

Let us also denote $C_\varepsilon(t, p, y)$ the set of minimizers in $[0, 1] \times D \times \mathbb{R}$ of:

$$(t', p', y') \rightarrow f(t', p', y') + \frac{1}{2\varepsilon}(|t - t'|^2 + |p - p'|^2 + |y - y'|^2).$$

Using that f is lower semi-continuous and bounded on $[0, 1] \times D \times \mathbb{R}$, we deduce that the correspondence $(t, p, y) \rightarrow C_\varepsilon(t, p, y)$ has a closed graph with compact values on $[0, 1] \times D \times \mathbb{R}$.

Then, adapting the result of Ishii, f^ε is supersolution of

$$\frac{\partial\phi}{\partial t} + A(\phi) + \tilde{u} \leq R_\varepsilon \tag{7.4}$$

on $[0, 1] \times D \times \mathbb{R}$ where

$$R_\varepsilon(t, p, y) := \max_{(t', p', y') \in C_\varepsilon(t, p, y)} \omega\left(\frac{1}{2\varepsilon}(|t - t'|^2 + |p - p'|^2 + |y - y'|^2) + \sqrt{|t - t'|^2 + |p - p'|^2 + |y - y'|^2}\right).$$

Note that R_ε is upper semi-continuous and bounded by $\omega(\gamma_\varepsilon + 2\|f\|_\infty)$ and that $R_\varepsilon \rightarrow 0$ when $\varepsilon \rightarrow 0$. Let us briefly explain how to adapt the proof of Ishii to our framework. At first, a crucial point is that f is a supersolution on a closed set (i.e. up to the boundary) which allows us to obtain the supersolution property for f^ε on the whole set and not only in a subset. Then, since we only assume lower semi-continuity for f and not uniform continuity, we have to replace all the estimates based on the modulus of continuity of f by R_ε . This last point explains why we do not control uniformly the error in the right-hand side of (7.4).

step 4: Approximation by non-degenerate diffusion processes.

Recall that $(\bar{t}, \bar{p}, \bar{y}) \in [0, 1] \times D \times \mathbb{R}$ is fixed. Let $(\theta_s)_{s \in [\bar{t}, 1]}$ be a diffusion process of generator A and such that $\theta_{\bar{t}} = (\bar{p}, \bar{y})$. Define a sequence of diffusion processes $(\theta_s^n)_{s \in [\bar{t}, 1]}$ with values in \mathbb{R}^N such that $\theta_{\bar{t}}^n = (\bar{p}, \bar{y})$, and with generator $A^n(\phi) = A(\phi) + \frac{1}{n}Tr(\nabla^2\phi)$. Such processes exist using Theorem 10.1.4 p.251 in [47] (and Theorem 10.2.2 for the criterium of non-explosion), their transition functions admit a density with respect to Lebesgue's measure, and according to Theorem 11.1.4 p.264 in [47], the sequence of processes θ^n converge in law to θ . Moreover, using standard estimates for diffusion processes, for all $m \geq 1$, we have that

$$\sup_{n \geq 1} \sup_{t \in [\bar{t}, 1]} \mathbb{E}_{\bar{t}, \bar{p}, \bar{y}}[|\theta_t^n|^m] < \infty.$$

In the following steps, the symbols \mathbb{P}, \mathbb{E} stand respectively for $\mathbb{P}_{\bar{t}, \bar{p}, \bar{y}}, \mathbb{E}_{\bar{t}, \bar{p}, \bar{y}}$ and correspond to the laws of the processes θ or θ^n depending on the context.

step 5: A variational inequality for semi-concave functions.

Let us recall the classical Jensen's Lemma (see Lemma 3.3 in [29]). \hat{f}^ε being concave, it is twice differentiable almost everywhere, and its second order derivative $D^2 \hat{f}^\varepsilon$ in the sense of distributions is a Radon measure. At first, recall that the first order derivatives are locally bounded functions with at most linear growth denoted $\frac{\partial \hat{f}^\varepsilon}{\partial t}$ and $\nabla \hat{f}^\varepsilon$. Moreover, there exists two matrices $U^\varepsilon = (u_{i,j}^\varepsilon)_{i,j=1,\dots,N+1}$ and $V^\varepsilon = (v_{i,j}^\varepsilon)_{i,j=1,\dots,N+1}$, where $u_{i,j}^\varepsilon$ are locally integrable functions and $v_{i,j}^\varepsilon$ are Radon measures such that :

$$D^2(\hat{f}^\varepsilon) = U^\varepsilon + V^\varepsilon$$

in the sense of distributions, U^ε takes values in the set of negative semi-definite matrices, V^ε is singular with respect to the Lebesgue's measure and for any Borel set B , $V^\varepsilon(B) \leq 0$ in the sense of positive semi-definite matrices. Moreover, for Lebesgue almost every (t, p, y) , $(\frac{\partial \hat{f}^\varepsilon}{\partial t}(t, p, y), \nabla \hat{f}^\varepsilon(t, p, y), \tilde{U}^\varepsilon(t, p, y)) \in J^2 f^\varepsilon$ where J^2 denotes the set of parabolic semijets of f^ε and \tilde{U}^ε denotes the restricted matrix $(u_{i,j}^\varepsilon)_{i,j=2,\dots,N+1}$ (we refer to [14] for the definitions of semijets). Using the notations of step 4, we claim that for all n , for all $\varepsilon > 0$, for all $h \in (0, 1 - \bar{t}]$:

$$\begin{aligned} & \mathbb{E}[f^\varepsilon(\bar{t} + h, \theta_{\bar{t}+h}^n)] - f^\varepsilon(\bar{t}, \bar{p}, \bar{y}) \\ & \leq \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \left(\frac{\partial f^\varepsilon}{\partial s}(s, \theta_s^n) + \mathcal{L}(s, \theta_s^n, \nabla f^\varepsilon(s, \theta_s^n), \frac{1}{\varepsilon}\text{Id} + \tilde{U}^\varepsilon(s, \theta_s^n)) + \frac{1}{n\varepsilon}\right) ds\right]. \end{aligned} \quad (7.5)$$

Applying Itô's formula to the process $\frac{1}{2\varepsilon}(|s|^2 + |\theta_s^n|^2)$, taking expectations and subtracting to the above inequality, we see that it is sufficient to prove that

$$\mathbb{E}[\hat{f}^\varepsilon(\bar{t} + h, \theta_{\bar{t}+h}^n)] - \hat{f}^\varepsilon(\bar{t}, \bar{p}, \bar{y}) \leq \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \left(\frac{\partial \hat{f}^\varepsilon}{\partial s}(s, \theta_s^n) + \mathcal{L}(s, \theta_s^n, \nabla \hat{f}^\varepsilon(s, \theta_s^n), \tilde{U}^\varepsilon(s, \theta_s^n))\right) ds\right].$$

For $\eta > 0$, let us introduce the functions

$$\hat{f}_\eta^\varepsilon := (\hat{f}^\varepsilon * \rho_\eta)(t, p, y)$$

where $*$ denotes the convolution and $\rho_\eta(w) := \frac{1}{\eta} \rho(\frac{1}{\eta} w)$ is a family of smooth centered densities with compact support. Precisely, we choose $\rho \geq 0$ such that $\rho \in C_c^\infty(\mathbb{R}^{N+1})$, ρ is supported by the unit ball of \mathbb{R}^{N+1} , $\int_{\mathbb{R}^{N+1}} w d\rho(w) = 0$, $\int_{\mathbb{R}^{N+1}} \rho(w) dw = 1$, and there exists a smooth non-increasing function h defined on $[0, +\infty)$ such that $\rho(w) = h(|w|)$.

Note that \hat{f}_η^ε is C^2 and that using Jensen's inequality, \hat{f}_η^ε is concave and $\hat{f}_\eta^\varepsilon \uparrow \hat{f}^\varepsilon$ when $\eta \rightarrow 0$. We deduce from Jensen's lemma that

$$\begin{aligned} \frac{\partial \hat{f}_\eta^\varepsilon}{\partial t}(w) &= \int_{\mathbb{R}^{N+1}} \frac{\partial \hat{f}^\varepsilon}{\partial t}(w') \rho_\eta(w' - w) dw', \quad \nabla \hat{f}_\eta^\varepsilon(w) = \int_{\mathbb{R}^{N+1}} \nabla \hat{f}^\varepsilon(w') \rho_\eta(w' - w) dw', \\ D^2 \hat{f}_\eta^\varepsilon(w) &= \int_{\mathbb{R}^{N+1}} \rho_\eta(w' - w) dD^2 \hat{f}^\varepsilon(w') = \int_{\mathbb{R}^{N+1}} \rho_\eta(w' - w) U^\varepsilon(w') dw' + \int_{\mathbb{R}^{N+1}} \rho_\eta(w' - w) dV^\varepsilon(w'). \end{aligned}$$

We deduce from step 3 that $|\nabla \hat{f}_\eta^\varepsilon(w)| \leq C_\varepsilon(1 + \eta + |w|)$.

It follows easily from Lebesgue's differentiation theorem (see e.g. theorem 1 p.43 in [19]) that we have Lebesgue almost everywhere convergence of

$$\begin{aligned} \int_{\mathbb{R}^{N+1}} \frac{\partial \hat{f}_\eta^\varepsilon}{\partial t}(w') \rho_\eta(w' - w) dw' &\xrightarrow{\eta \rightarrow 0} \frac{\partial \hat{f}_\eta^\varepsilon}{\partial t}(w) \\ \int_{\mathbb{R}^{N+1}} \nabla \hat{f}_\eta^\varepsilon(w') \rho_\eta(w' - w) dw' &\xrightarrow{\eta \rightarrow 0} \nabla \hat{f}_\eta^\varepsilon(w) \\ \int_{\mathbb{R}^{N+1}} \rho_\eta(w' - w) U^\varepsilon(w') dw' &\xrightarrow{\eta \rightarrow 0} U^\varepsilon(w) \end{aligned}$$

To prove this claim, note that given any function $\psi(w)$ which is real-valued and locally integrable (w.r.t. Lebesgue's measure), we have

$$\int_{\mathbb{R}^{N+1}} \psi(w') \rho_\eta(w' - w) dw' = \frac{1}{\eta} \int_0^\eta (-h'(\frac{r}{\eta})) \left(\int_{B(0,r)} \psi(w + w') dw' \right) dr.$$

It follows that the above expression converge to $\psi(w)$ at all Lebesgue points of ψ which proves the claim.

For all $\eta > 0$, it follows from Itô's formula that:

$$\begin{aligned} \mathbb{E}[\hat{f}_\eta^\varepsilon(\bar{t} + h, \theta_{\bar{t}+h}^n)] - \hat{f}_\eta^\varepsilon(\bar{t}, \bar{p}, \bar{y}) \\ = \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \left(\frac{\partial \hat{f}_\eta^\varepsilon}{\partial s}(s, \theta_s^n) + \mathcal{L}(s, \theta_s^n, \nabla \hat{f}_\eta^\varepsilon(s, \theta_s^n), \nabla^2 \hat{f}_\eta^\varepsilon(s, \theta_s^n)) + \frac{1}{n} \text{Tr}(\nabla^2 \hat{f}_\eta^\varepsilon(s, \theta_s^n)) \right) ds \right]. \end{aligned}$$

Applying Lebesgue's dominated convergence theorem, we deduce that

$$\begin{aligned} \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \left(\frac{\partial \hat{f}_\eta^\varepsilon}{\partial s}(s, \theta_s^n) + \langle c(s, \theta_s^n), \nabla \hat{f}_\eta^\varepsilon(s, \theta_s^n) \rangle \right) ds \right] &\xrightarrow{\eta \rightarrow 0} \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \left(\frac{\partial \hat{f}_\eta^\varepsilon}{\partial s}(s, \theta_s^n) + \langle c(s, \theta_s^n), \nabla \hat{f}_\eta^\varepsilon(s, \theta_s^n) \rangle \right) ds \right], \\ \mathbb{E}[\hat{f}_\eta^\varepsilon(\bar{t} + h, \theta_{\bar{t}+h}^n)] - \hat{f}_\eta^\varepsilon(\bar{t}, \bar{p}, \bar{y}) &\xrightarrow{\eta \rightarrow 0} \mathbb{E}[\hat{f}_\eta^\varepsilon(\bar{t} + h, \theta_{\bar{t}+h}^n)] - \hat{f}_\eta^\varepsilon(\bar{t}, p, y). \end{aligned}$$

Since \hat{f}_η^ε is concave,

$$\mathbb{E}\left[\frac{1}{n} \int_{\bar{t}}^{\bar{t}+h} \text{Tr}(\nabla^2 \hat{f}_\eta^\varepsilon(s, \theta_s^n)) ds \right] \leq 0,$$

and

$$\frac{1}{2} \langle \nabla^2 \hat{f}_\eta^\varepsilon(s, \theta_s^n) \kappa(s, \theta_s^n), \kappa(s, \theta_s^n) \rangle \leq \frac{1}{2} \langle \kappa(s, \theta_s^n), \int_{\mathbb{R}^{N+1}} \rho_\eta(w' - (s, \theta_s^n)) \tilde{U}^\varepsilon(w') dw' \kappa(s, \theta_s^n) \rangle \leq 0.$$

Using Fatou Lemma, we deduce that

$$\limsup_{\eta \rightarrow 0} \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \frac{1}{2} \langle \kappa(s, \theta_s^n), \nabla^2 \hat{f}_\eta^\varepsilon(s, \theta_s^n) \kappa(s, \theta_s^n) \rangle ds \right] \leq \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \frac{1}{2} \langle \kappa(s, \theta_s^n), \tilde{U}^\varepsilon(s, \theta_s^n) \kappa(s, \theta_s^n) \rangle ds \right].$$

The proof of (7.5) is now complete.

step 6: Putting pieces together.

In the following, we will use the notation $z = (p, y)$. Let us bound the right-hand side of inequality (7.5). Using the characterization of the viscosity supersolution property in terms of semijets, we have the following inequality for Lebesgue almost every (s, z) in $[0, 1] \times D \times \mathbb{R}$:

$$\left(\frac{\partial f^\varepsilon}{\partial s}(s, z) + \mathcal{L}(s, z, \nabla f^\varepsilon(s, z), \frac{1}{\varepsilon}I + \tilde{U}^\varepsilon(s, z))\right) \leq -\tilde{u}(s, z) + R_\varepsilon(s, z).$$

For $(s, z) \notin [0, 1] \times D \times \mathbb{R}$, using that $\tilde{U}^\varepsilon(s, z)$ is negative semi-definite, the Lipschitz conditions on c and κ and the linear growth of the first order derivatives of f^ε , there exists a constant $C'_\varepsilon > 0$ such that:

$$\left(\frac{\partial f^\varepsilon}{\partial s}(s, z) + \mathcal{L}(s, z, \nabla f^\varepsilon(s, z), \frac{1}{\varepsilon}I + \tilde{U}^\varepsilon(s, z))\right) \leq C'_\varepsilon(1 + |z| + |z|^2)$$

Using that for all $s > \bar{t}$, $\mathbb{P}(\theta_s \in \partial D \times \mathbb{R}) = 0$, we have that

$$\begin{aligned} \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \mathbf{1}_{\theta_s^n \in D \times \mathbb{R}} \tilde{u}(s, \theta_s^n) ds\right] &\xrightarrow{n \rightarrow \infty} \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \tilde{u}(s, \theta_s) ds\right], \\ \limsup_{n \rightarrow \infty} \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \mathbf{1}_{\theta_s^n \in D \times \mathbb{R}} R_\varepsilon(s, \theta_s^n) ds\right] &\leq \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} R_\varepsilon(s, \theta_s) ds\right], \end{aligned}$$

and $\mathbb{P}(\theta_s^n \notin D \times \mathbb{R}) \rightarrow 0$ when $n \rightarrow \infty$. It follows that

$$\begin{aligned} \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \mathbf{1}_{\theta_s^n \notin D \times \mathbb{R}} \left(\frac{\partial f^\varepsilon}{\partial s}(s, \theta_s^n) + \mathcal{L}(s, \theta_s^n, \nabla f^\varepsilon(s, \theta_s^n), \frac{1}{\varepsilon}I + \tilde{U}^\varepsilon(s, \theta_s^n)) + \frac{1}{n\varepsilon}\right) ds\right] \\ \leq C'_\varepsilon \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \mathbf{1}_{\theta_s^n \notin D \times \mathbb{R}} (1 + |\theta_s^n| + |\theta_s^n|^2) ds\right] \\ \leq C'_\varepsilon \int_{\bar{t}}^{\bar{t}+h} (\mathbb{P}(\theta_s^n \notin D \times \mathbb{R}))^{\frac{1}{2}} (\mathbb{E}[(1 + |\theta_s^n| + |\theta_s^n|^2)^2])^{\frac{1}{2}} ds \\ \leq C'_\varepsilon C'' \int_{\bar{t}}^{\bar{t}+h} (\mathbb{P}(\theta_s^n \notin D \times \mathbb{R}))^{\frac{1}{2}} ds \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

where we used Cauchy-Schwarz inequality, the constant C'' is obtained from the estimates of step 4 on the moments of θ^n , and bounded convergence. Using that f^ε has at most quadratic growth and convergence in law, we have

$$\mathbb{E}[f^\varepsilon(\bar{t} + h, \theta_{\bar{t}+h}^n)] \xrightarrow{n \rightarrow \infty} \mathbb{E}[f^\varepsilon(\bar{t} + h, \theta_{\bar{t}+h})].$$

Putting the pieces together, we obtain that

$$\mathbb{E}[f^\varepsilon(\bar{t} + h, \theta_{\bar{t}+h})] - f^\varepsilon(\bar{t}, \bar{p}, \bar{y}) \leq -\mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \tilde{u}(s, \theta_s) ds\right] + \mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} R_\varepsilon(s, \theta_s) ds\right].$$

We can now conclude the proof by letting $\varepsilon \rightarrow 0$. Using monotone convergence on the left-hand side and bounded convergence for the second term in the right-hand side, we obtain:

$$\mathbb{E}[f(\bar{t} + h, \theta_{\bar{t}+h})] - f(\bar{t}, \bar{p}, \bar{y}) \leq -\mathbb{E}\left[\int_{\bar{t}}^{\bar{t}+h} \tilde{u}(s, \theta_s) ds\right].$$

□

8 A simple example linked with the concavification operator.

In this section, we consider the case where \mathcal{Y} is reduced to a single point ($\mathcal{Y} = \{\emptyset\}$) and we identify \mathcal{Z} with \mathcal{X} .

Let us as first state a simplified expression for the function V appearing in Theorem 2.10 in the case where $\mathcal{Y} = \{\emptyset\}$. Assume that X is a Feller process with transitions $(P_t)_{t \geq 0}$. Define the set $\mathcal{B}(\nu)$ as the set of laws of processes $(X, \pi) \in \mathcal{X} \times \Delta(\mathcal{X})$ such that

- $(X_t)_{t \in [0,1]}$ is a Markov process of initial law ν .
- $(\pi_t)_{t \in [0,1]}$ is a càdlàg modification of $(\llbracket X_1 | \mathcal{F}_t^{\pi,+} \rrbracket)_{t \in [0,1]}$.
- π is non-anticipative with respect to X .

The function u is defined on $\Delta(\mathcal{X})$ and we have:

$$V(\nu) = \sup_{\llbracket X, \pi \rrbracket \in \mathcal{B}(\nu)} \mathbb{E} \left[\int_0^1 u(\pi_s) ds \right].$$

If we define $\Pi(u, \xi) := \sup_{\zeta \preceq \xi} \int u(\mu) d\zeta(\mu)$, where \preceq denotes the convex order, we deduce easily the following upper bound

$$V(\nu) \leq \mathbb{E} \left[\int_0^1 \Pi(u, \llbracket \delta_{X_t} \cdot P_{1-t} \rrbracket) dt \right].$$

Indeed, the martingale property $\pi_t = \mathbb{E}[\delta_{X_t} \cdot P_{1-t} | \mathcal{F}_t^{\pi,+}]$ implies that the law of π_t is smaller than the law of $\delta_{X_t} \cdot P_{1-t}$ for the convex order.

Let us now consider the very specific example presented in section 2 in which this upper bound is attained and for which the value is related to the concavification operator. In the following $\mathcal{X} = [0, 1]$.

Recall that $f, h : [0, 1] \rightarrow [0, 1]$ are continuous, such that $f < h$, f is non-increasing, h is non-decreasing, $f(1) = 0$ and $h(1) = 1$. Then, define the process X as follows: $X_0 \in [0, f(0)] \cup [h(0), 1]$, X is a martingale which is almost surely constant until it reaches the set $\text{graph}(f) \cup \text{graph}(h)$, and then stays in the set $\text{graph}(f) \cup \text{graph}(h)$ with probability 1.

Lemma 8.1. *The above description defines uniquely a martingale X and if we assume that $X_t = X_1$, $h(t) = 1$, and $f(t) = 0$ for $t \geq 1$, (t, X_t) is easily seen to be a Feller process with state space $\{(t, x) \in \mathbb{R}_+ \times [0, 1] | x \geq h(t), \text{ or } x \leq f(t)\}$.*

Proof. For $t > s$, the conditional law of X_t given X_s is completely determined by the martingale condition if $X_s \in \{f(s), h(s)\}$. For example, we have:

$$\mathbb{P}(X_t \in \{f(t), h(t)\} | X_s = f(s)) = 1, \quad \mathbb{P}(X_t = f(t) | X_s = f(s)) = \frac{f(s) - f(t)}{h(t) - f(t)}.$$

Define $h^{-1}(x) = \inf\{t | h(t) \geq x\}$ for $x \in [h(0), 1]$ and $f^{-1}(x) = \inf\{t | f(t) \leq x\}$ for $x \in [0, f(0)]$. Then, if $x \in [h(0), 1]$, we have $\mathbb{P}(X_t = X_s | X_s = x) = 1$ if $t \leq h^{-1}(x)$ and $\mathbb{P}(X_t = f(t) | X_s = x) = \frac{f(h^{-1}(x)) - f(t)}{h(t) - f(t)}$ if $t > h^{-1}(x)$. The other cases are similar, and we verify easily that this defines a Feller semi-group for (t, X_t) . The existence of such a process follows therefore from standard theorems (see e.g. [18]). \square

We may therefore apply Theorem 2.10 (see section 9 for the non-homogeneous case).

Recall that $\bar{p} := \mathbb{E}_\nu[X_0]$ and that we assume that either $X_0 = \bar{p}$ almost surely for some $\bar{p} \in [0, f(0)] \cup [h(0), 1]$ or that $\nu = \lambda\delta_{f(0)} + (1 - \lambda)\delta_{h(0)}$ so that $\bar{p} \in [f(0), h(0)]$.

Lemma 8.2. *Under the above assumptions, we have:*

$$V(\nu) = \sup_{\{p_s\}_{s \in [0,1]} \in \mathcal{B}'(\nu)} \mathbb{E}\left[\int_0^1 \bar{u}(p_s) ds\right], \quad (8.1)$$

where the set $\mathcal{B}'(\nu)$ is the set of all martingales such that $p_t = \bar{p}$ for $t \leq m(\bar{p})$ and for all $t > m(\bar{p})$, $p_t \in [f(t), h(t)]$ almost surely.

Proof. Indeed, starting with any admissible process (X, π) , we can define the martingale $p_t := \pi_t(\{1\})$ so that

$$\mathbb{E}\left[\int_0^1 u(\pi_s) ds\right] = \mathbb{E}\left[\int_0^1 \bar{u}(p_t) dt\right].$$

and the martingale p_t has a law in $\mathcal{B}'(\nu)$ since $\mathbb{P}(X_1 = 1 | X_t) = X_t$ and $p_t = \mathbb{E}[X_t | \mathcal{F}_t^{\pi,+}]$. This proves that V is not greater than the right-hand side of (8.1). It remains to prove the reverse inequality. As the function $(p_t)_{t \in [0,1]} \rightarrow \int_0^1 \bar{u}(p_t)$ is bounded and MZ -continuous, using a density argument, it is sufficient to prove that for any martingale p which is piecewise-constant on some finite partition $0 = t_0 < \dots < t_n = 1$ of $[0, 1]$, there exists an admissible process (X, π) such that $\pi_t(\{1\})$ has the same law as p_t .

Let us choose some process $(p_t)_{t \in [0,1]}$ defined on some probability space Ω . Assume that the process X and a family of independent random variables $(U_i)_{i=0, \dots, n}$ uniformly distributed on $[0, 1]$ and independent from (X, p) are defined on the same probability space, we will construct a process \tilde{p} having the same law as p such that (X, π) is admissible with $\pi_t := \tilde{p}_t \delta_1 + (1 - \tilde{p}_t) \delta_0$.

At first, assume that $X_0 \in \{f(0), h(0)\}$ and let us construct a variable $\tilde{X}_0 \in \{f(0), h(0)\}$ conditionally on p_0 such that $\mathbb{E}[\tilde{X}_0 | p_0] = p_0$. Then construct \tilde{p}_0 as a function of (X_0, U_0) such that the conditional law of \tilde{p}_0 given X_0 is the same as the conditional law of p_0 given \tilde{X}_0 . It follows that $\mathbb{P}(X_1 = 1 | \tilde{p}_0) = \tilde{p}_0$. Assume now, that the variables \tilde{p}_{t_i} are constructed up to $i = m - 1$ and such that \tilde{p}_{t_i} is measurable with respect to $(X_{t_0}, \dots, X_{t_i}, U_0, \dots, U_i)$ and $\mathbb{P}(X_1 = 1 | \tilde{p}_{t_0}, \dots, \tilde{p}_{t_i}) = \tilde{p}_{t_i}$. Let us construct a variable $\tilde{X}_{t_m} \in \{f(t_m), h(t_m)\}$ conditionally on $(p_{t_0}, \dots, p_{t_{m-1}})$ such that $\mathbb{E}[\tilde{X}_{t_m} | p_{t_0}, \dots, p_{t_{m-1}}] = p_{t_m}$. Then construct \tilde{p}_{t_m} as a function of $(X_{t_m}, \tilde{p}_{t_0}, \dots, \tilde{p}_{t_{m-1}}, U_m)$ such that the conditional law of \tilde{p}_{t_m} given $(X_{t_m}, \tilde{p}_{t_0}, \dots, \tilde{p}_{t_{m-1}})$ is the same as the conditional law of p_{t_m} given $(\tilde{X}_{t_m}, p_{t_0}, \dots, p_{t_{m-1}})$. It follows that $\mathbb{P}(X_1 = 1 | \tilde{p}_{t_0}, \dots, \tilde{p}_{t_m}) = \tilde{p}_{t_m}$ and this concludes the proof by induction.

The case $X_0 = \bar{p}$ is similar, and we only need to discretize the interval $[m(\bar{p}), 1]$ since any admissible martingale is constant on the interval $[0, m(\bar{p})]$. \square

Now, we prove the main result of this section.

Proof of proposition 2.16. As for the previous proof, we assume that $X_0 \in \{f(0), h(0)\}$, the proof being similar for the general case.

Given a process p in $\mathcal{B}'(\nu)$, we deduce easily from Jensen inequality that:

$$\mathbb{E}[\bar{u}(p_t)] \leq \mathbb{E}[\text{Cav}_{I(t)}(\bar{u})(p_t)] \leq \text{Cav}_{I(t)}(\bar{u})(\bar{p}).$$

It follows that $V(\nu)$ is not greater than the right-hand side of (2.7). Let us now prove that this bound is attained.

For all $s \in [0, 1]$, using that \bar{u} is continuous, it is well-known that α and β are well-defined, and that either $\bar{p} = \alpha(s) = \beta(s)$ and $\bar{u}(\bar{p}) =: \text{Cav}_{I(s)}(\bar{u})(\bar{p})$ or that $\alpha(s) < \bar{p} < \beta(s)$ and

$$\text{Cav}_{I(s)}(\bar{u})(\bar{p}) = \frac{\bar{p} - \alpha(s)}{\beta(s) - \alpha(s)} \bar{u}(\beta(s)) + \frac{\beta(s) - \bar{p}}{\beta(s) - \alpha(s)} \bar{u}(\alpha(s)). \quad (8.2)$$

As $s \rightarrow I(s)$ is non-decreasing, it follows that $s \rightarrow \text{Cav}_{I(s)}(\bar{u})(\bar{p})$ is also non-decreasing. We deduce easily that β is non-decreasing and α non-increasing. Indeed, if $s < t$, and $x \in I(s)$, note that $\bar{u}(x) < \text{Cav}_{I(s)}(\bar{u})(x)$ implies $\bar{u}(x) < \text{Cav}_{I(s)}(\bar{u})(x)$. Then, given a sequence s_n decreasing to s and assuming that $\alpha^+(s) < \beta^+(s)$, by taking the limit in the right-hand side of (8.2), we deduce that

$$\text{Cav}_{I(s)}(\bar{u})(\bar{p}) \leq \frac{\bar{p} - \alpha^+(s)}{\beta^+(s) - \alpha^+(s)} \bar{u}(\beta^+(s)) + \frac{\beta^+(s) - \bar{p}}{\beta^+(s) - \alpha^+(s)} \bar{u}(\alpha^+(s)), \quad (8.3)$$

which proves that this is an equality.

It is now easy to construct an optimal martingale p^* . At first, we construct a sequence p^n of piecewise-constant martingales on the dyadic partitions: $[t_k^n, t_{k+1}^n) := [k2^{-n}, (k+1)2^{-n})$ for $k = 0, \dots, 2^n - 1$ and $\{1\}$. Define p_0^n as the unique variable taking values in $\{\alpha^+(0), \beta^+(0)\}$ and centered in \bar{p} . Then, given $p_{t_k}^n$, construct $p_{t_{k+1}}^n \in \{\alpha^+(t_{k+1}), \beta^+(t_{k+1})\}$ such that the process is a martingale. The sequence of laws of the martingales p^n is $\mathcal{L}(MZ)$ -tight, and we may extract an $\mathcal{L}(MZ)$ -convergent subsequence p^{n_k} . We may assume that the processes p^{n_k} are defined on the same probability space and converge MZ -almost surely to some limit process p^* . Up to extracting again some subsequence, there exists an interval I of full Lebesgue measure containing 1 such that for all $t \in I$, $p_t^{n_k}$ converges almost surely to p_t^* . From Theorem 11 in [41], the process p^* is a martingale. For any fixed $t \in I$, the variable $p_t^{n_k}$ belongs to $\{\alpha^+(\frac{\lfloor 2^{n_k} t \rfloor}{2^{n_k}}), \beta^+(\frac{\lfloor 2^{n_k} t \rfloor}{2^{n_k}})\}$ and is centered in \bar{p} . It follows easily that the sequence $p_t^{n_k}$ converges in law to a variable taking values in $\{\alpha^-(t), \beta^-(t)\}$ if t is not dyadic (where $\alpha^-(t), \beta^-(t)$ denote the left-limits in t of α, β) or in $\{\alpha^+(t), \beta^+(t)\}$ if t is dyadic. As α, β have at most a countable number of points of discontinuity, we deduce that $p_t^* \in \{\alpha^+(t), \beta^+(t)\}$ for t in a dense subset containing 1 and since p^* has càdlàg paths, this property extends for all $t \in [0, 1]$. Finally, that p^* is optimal follows from equation (8.3). \square

9 Extensions and Open questions.

9.1 Extensions.

9.1.1 More general discretizations.

We already mentioned in section 2 that the choice of the time-discretization does not affect our results. We let the reader check by himself that all our proofs can be adapted without any change except notations to any sequence of finite partitions of $[0, 1]$ whose mesh goes to zero.

9.1.2 Extension to non-homogeneous processes.

Let us now explain how to adapt these results to the case of non-homogeneous Markov processes $(Z_t)_{t \in [0,1]}$ with transition probabilities $(P_{t,t+s})_{t,s \geq 0}$.

At first, define $Z_s := Z_1$ for $s \geq 1$ and consider the process (Z_t, t) taking values in $\mathcal{Z}' = \mathcal{Z} \times \mathbb{R}_+$ (endow \mathbb{R}_+ with the usual distance and \mathcal{Z}' with a product metric). Define $\mathcal{X}' = \mathcal{X}$, $\mathcal{Y}' = \mathcal{Y} \times \mathbb{R}_+$ and $Z'_t := (X'_t, Y'_t)$ with $X'_t := X_t$ and $Y'_t := (Y_t, t)$. Then, we can apply Theorem 2.11 by replacing the process Z by Z' and the payoff function g by g' defined on $\mathcal{Z}' \times I \times J$ by $g'(z', i, j) := g(z, i, j)$ for $z' = (z, t)$. All the associated functions and processes are denoted V', π', θ', \dots etc...

For an initial law $\nu' = \nu \otimes \delta_0 \in \Delta(\mathcal{Z} \times \mathbb{R}_+)$, note that the value functions of $G'_n(\nu')$ equals the value function of $G_n(\nu)$, since allowing both players to observe the time-variable does not modify the value (the initial time of the game is known!). Recall that the function u is defined on $\Delta(\mathcal{Z})$, and note that for any $\nu' \in \Delta(\mathcal{Z}')$, we have $u'(\nu') = u(\nu)$ where ν denotes the marginal of ν' on \mathcal{Z} .

Then, we may apply Theorems 2.10 and 2.11, and restrict the analysis on the subset of initial laws at time t in the set $\nu' = \nu \otimes \delta_t$.

When applying Theorem 2.10 (starting at time 0, say), note that for any admissible process (Z', π') , we have for all $s \in [0, 1]$, $\pi'_s = \pi_s \otimes \delta_1$ for some process $\pi_s \in \Delta(\mathcal{Z})$. The optimization problem can therefore be easily reduced to a maximum over processes (Z, π) and we conclude that exactly the same result holds true in the non-homogeneous case.

For Theorem 2.11, note at first that by composition, the function

$$(t, \nu) \in [0, 1] \times \Delta(\mathcal{Z}) \rightarrow V'(t, \nu \otimes \delta_t)$$

is upper semi-continuous. Define $V(t, \nu) := V'(t, \nu \otimes \delta_t)$. That $V(1, \cdot) = 0$ is obvious and concavity follows easily from the fact that for all $t \in [0, 1]$, the set of laws $\{\nu \otimes \delta_t, \nu \in \Delta(\mathcal{Z})\}$ is convex. Then, define the map $Q_{t,t+s}(\nu) := \Pi_{\mathcal{Z}}(Q'_s(\nu \otimes \delta_t))$ where $\Pi_{\mathcal{Z}}$ denotes the marginal projection on \mathcal{Z} . Define $\tilde{u}(t, \nu) := u'((\nu \otimes \delta_t) \cdot P'_{1-t}) = u(\nu \cdot P_{t,1})$.

Then, given some initial law $\nu' = \nu \otimes \delta_t$, we have

$$\int_{\Delta(\mathcal{Z}')} V'(t+h, \mu) dQ'_h(\nu \otimes \delta_t, d\mu) - V(t, \nu) + \int_0^h \int_{\Delta(\mathcal{Z}')} \tilde{u}'(t+s, \mu) dQ'_s(\nu \otimes \delta_t, d\mu) ds \leq 0, \quad (9.1)$$

which can be written as

$$\int_{\Delta(\mathcal{Z})} V(t+h, \mu) dQ_{t,t+h}(\nu, d\mu) - V(t, \nu) + \int_0^h \int_{\Delta(\mathcal{Z})} \tilde{u}(t+s, \mu) dQ_{t,t+s}(\nu, d\mu) ds \leq 0. \quad (9.2)$$

To conclude, we let the reader verify that the proof of the second step of Theorem 2.11 can be adapted without change to this new framework. Actually, one has just to notice that the function Ψ has only to be defined on the subset $\{\nu \otimes \delta_t | (t, \nu) \in [0, 1] \times \Delta(\mathcal{Z})\}$ of $\Delta(\mathcal{Z}')$, and extend the function by the value $-\infty$ outside this set.

The results of section 7 can also be easily extended to the case where Y_t is the solution of some non-homogeneous SDE, allowing for time-dependent coefficients b and σ being globally Lipschitz.

9.2 Open problems.

9.3 A general PDE characterization.

The results of section 7 lead to the following question: is it possible to write an Hamilton-Jacobi equation in the general case of a Feller process Z taking values in $\mathbb{R}^m \times \mathbb{R}^p$? Note that such an equation would be stated in an infinite dimensional space of probability measures.

9.4 Models with time-varying types.

Most of the techniques introduced in this work can be easily extended to model with time-varying types, i.e. for example when replacing our payoff function by a payoff at time t of the form $g(Z_t, i_t, j_t)$. However, note that the tightness criterion given in Theorem 5.8 and Proposition 5.11 do not cover this case and have to be adapted.

A Technical Proofs and auxiliary tools.

A.1 Proofs of Lemmas 3.9,4.4,4.5 and A.1

Proof of Lemma 3.9. The “if” part is obvious. Let us prove the “only if” part. For $q = 0$, this is just Lemma 3.8. However, we need to be more precise on how to construct this variable. We assume that there exists a family of independent variables $(\zeta_0, \dots, \zeta_n)$ uniformly distributed on $[0, 1]$ and independent of $(A_0, B_0, \dots, A_n, B_n)$. Then, the variable ξ_0 given by Lemma 3.8 can be constructed as a function of (A_0, B_0, ζ_0) . Let us now proceed by induction and assume the above property is true for $p \leq q$ and that ξ_p is measurable with respect to $(A_0, B_0, \zeta_0, \dots, A_p, B_p, \zeta_p)$. Since

$$(A_0, \dots, A_{q+1}) \coprod_{(B_0, \dots, B_{q+1})} (B_0, \dots, B_n),$$

we have

$$\llbracket B_0, \dots, B_n | B_0, \dots, B_{q+1}, A_0, \dots, A_{q+1} \rrbracket = \llbracket B_0, \dots, B_n | B_0, \dots, B_{q+1} \rrbracket.$$

Using the second point of Lemma 3.7, we deduce that

$$\llbracket B_0, \dots, B_n | B_0, \dots, B_{q+1}, A_{q+1} \rrbracket = \llbracket B_0, \dots, B_n | B_0, \dots, B_{q+1} \rrbracket.$$

Using now the induction hypothesis and the first point of Lemma 3.7, we also have

$$\begin{aligned} \llbracket B_0, \dots, B_n | B_0, \dots, B_{q+1}, \xi_0, \dots, \xi_q \rrbracket &= \llbracket B_0, \dots, B_n | B_0, \dots, B_{q+1} \rrbracket, \\ \llbracket B_0, \dots, B_n | B_0, \dots, B_{q+1}, \xi_0, \dots, \xi_q, A_{q+1} \rrbracket &= \llbracket B_0, \dots, B_n | B_0, \dots, B_{q+1}, A_{q+1} \rrbracket. \end{aligned}$$

Finally, we deduce that $A_{q+1} \coprod_{(\xi_0, \dots, \xi_q, B_0, \dots, B_{q+1})} (B_0, \dots, B_n)$ and the result follows then by applying Lemma 3.8. \square

Proof of Lemma 4.4. We prove here that for all $\varepsilon > 0$, there exists a measurable function

$$\varphi_\varepsilon : \Delta(\mathcal{Z}) \times \Delta(\Delta(\mathcal{Z})) \times [0, 1] \rightarrow I$$

such that the strategy induced by $i_1 = \varphi_\varepsilon(\theta, \xi, w)$ where w is some uniform random variable on $[0, 1]$ independent from θ , is ε -optimal in the game $\Gamma(\xi)$ for all $\xi \in \Delta(\Delta(\mathcal{Z}))$. At first, this statement is equivalent to the existence of a measurable function

$$H_\varepsilon : \Delta(\mathcal{Z}) \times \Delta(\Delta(\mathcal{Z})) \rightarrow \Delta(I)$$

having the property that the strategy $\mu \rightarrow H_\varepsilon(\mu, \xi)$ is ε -optimal in $\Gamma(\xi)$ using the correspondence $\phi_\varepsilon(\mu, \xi, U) = \Phi(H_\varepsilon(\mu, \xi), U)$ where Φ is given by Theorem A.3. We can proceed by a standard discretization method. Let $(\Omega_q)_{q \geq 0}$ be a countable measurable partition of $\Delta(\Delta(\mathcal{Z}))$ of mesh smaller than $\varepsilon/2$ (with respect to $\bar{\mathbf{d}}$) and $(\xi_q)_{q \geq 0}$ a sequence of points such that $\xi_q \in \Omega_q$ for all q . It is sufficient to construct the map H_ε on the set Ω_q for all q . Let us fix q and consider the set-valued map $\xi \rightarrow \Pi_\varepsilon(\xi_q, \xi)$ where $\Pi_\varepsilon(\xi, \xi_q)$ is the set of probabilities having marginals (ξ, ξ_q) and such that

$$\int \mathbf{d}(\mu, \mu') d\pi(\mu, \mu') \leq \bar{\mathbf{d}}(\xi, \xi_q) - \varepsilon/2.$$

This map having closed values and a closed graph, it admits a measurable selection (see [33]) denoted $\pi(\xi)$. Let σ_q be an $\varepsilon/2$ -optimal strategy in $\Gamma(\xi_q)$. Define the map $\tilde{\sigma}_q$ from $\Delta(\mathcal{Z}) \times \Delta(\mathcal{Z})$ to $\Delta(I)$ by

$$\tilde{\sigma}_q(\mu, \mu') = \sigma(\mu')$$

and the map $\kappa(\xi)$ by

$$\kappa(\xi) := \pi(\xi) \otimes \tilde{\sigma}.$$

The map κ being measurable, it follows from Lemma A.6 that there exists a measurable map of (ξ, μ) which is a version of the conditional law of i given μ whenever (μ, μ', i) has law $\kappa(\xi)$ denoted $H_\varepsilon(\xi, \mu)$. Using the same argument as in Lemma 3.15, the strategy $H_\varepsilon(\xi, \cdot)$ is ε -optimal in $\Gamma(\xi)$ for any ξ in Ω_q . \square

Proof of Lemma 4.5. Endow the sets $\Delta(I \times \Delta(\mathcal{Z}))$ and $\Delta(J)$ with the usual weak topology. The application

$$(\pi, \tau) \rightarrow \int \left(\int g(z, i, j) dx(z) \right) d\pi(i, x) \otimes \tau(j)$$

is jointly measurable. If ξ denotes the marginal law on $\Delta(\mathcal{Z})$ induced by π , the set

$$\{(\pi, \tau) : \int \left(\int g(z, i, j) dx(z) \right) d\pi(i, x) \otimes \tau(j) \leq W(\xi) + \varepsilon\}$$

is therefore a Borel subset of $\Delta(I \times \Delta(\mathcal{Z})) \times \Delta(J)$. The existence of an ε -optimal universally measurable selection τ_ε follows therefore from Von Neumann's selection theorem (see e.g. proposition 7.49 in [4]). \square

Lemma A.1. *Using the notations of section 7, for all $t > 0$, χ_t belongs with probability 1 to the relative interior of the largest face of Δ_N which contains the support of $\chi_0 \in \Delta_N$ and is invariant for the Markov process X_t .*

Proof. Define $\phi_n(x) = e^{-nx}$ for $x \in \mathbb{R}$. Then, $\phi_n'(x) = -ne^{-nx}$ and $\phi_n''(x) = n^2e^{-nx}$. The process χ_t being bounded, we may apply Itô's formula and deduce that for all $\alpha = 1, \dots, N$ and all $t \geq 0$

$$\begin{aligned} \mathbb{E}[\phi_n(\chi_t(\alpha))] &= \phi_n(\chi_0(\alpha)) - \mathbb{E}\left[\int_0^t ne^{-n\chi_s(\alpha)} ({}^T\Lambda\chi_s)(\alpha) ds\right] \\ &\quad + \frac{1}{2}\mathbb{E}\left[\int_0^t n^2\chi_s(\alpha)^2 e^{-n\chi_s(\alpha)} \frac{1}{\sigma(Y_s)^2} (b(\alpha, Y_s) - \sum_{\beta=1}^N b(\beta, Y_s)\chi_s(\beta))^2 ds\right]. \end{aligned}$$

Using Lebesgue convergence theorem, the last term goes to zero when n goes to infinity. The second term requires a more precise analysis. recall that

$$({}^T\Lambda p)(\alpha) = \sum_{\beta=1}^N \Lambda_{\beta,\alpha} p(\beta)$$

Using again Lebesgue's theorem,

$$\mathbb{E}\left[\int_0^t ne^{-n\chi_s(\alpha)} \Lambda_{\alpha,\alpha} \chi_s(\alpha) ds\right] \xrightarrow{n \rightarrow \infty} 0.$$

Using that $\sum_{\beta \neq \alpha} \Lambda_{\beta,\alpha} p(\beta) \geq 0$, we obtain the following inequality

$$\begin{aligned} \mathbb{E}[\phi_n(\chi_t(\alpha))] &\leq \phi_n(\chi_0(\alpha)) - \mathbb{E}\left[\int_0^t ne^{-n\chi_s(\alpha)} \Lambda_{\alpha,\alpha} \chi_s(\alpha) ds\right] \\ &\quad + \frac{1}{2}\mathbb{E}\left[\int_0^t n^2\chi_s(\alpha)^2 e^{-n\chi_s(\alpha)} \frac{1}{\sigma(Y_s)^2} (b(\alpha, Y_s) - \sum_{\beta=1}^N b(\beta, Y_s)\chi_s(\beta))^2 ds\right]. \end{aligned}$$

Taking the limit when n goes to $+\infty$, we deduce that $\mathbb{P}(\chi_t(\alpha) = 0) \leq \mathbf{1}_{\chi_0(\alpha)=0}$, from which we deduce that $\chi_0(\alpha) > 0$ implies $\mathbb{P}(\chi_t(\alpha) = 0) = 0$ for all $t \geq 0$.

Let us now assume that α, β are such that $\chi_0(\alpha) = 0$, $\chi_0(\beta) > 0$ and $\Lambda_{\beta,\alpha} > 0$. We deduce from the above inequalities that:

$$\begin{aligned} 0 \leq \mathbb{E}[\phi_n(\chi_t(\alpha))] &\leq \phi_n(\chi_0(\alpha)) - \mathbb{E}\left[\int_0^t ne^{-n\chi_s(\alpha)} \Lambda_{\alpha,\alpha} \chi_s(\alpha) ds\right] \\ &\quad - \mathbb{E}\left[\int_0^t ne^{-n\chi_s(\alpha)} \mathbf{1}_{\chi_s(\alpha)=0} \Lambda_{\beta,\alpha} \chi_s(\beta) ds\right] \\ &\quad + \frac{1}{2}\mathbb{E}\left[\int_0^t n^2\chi_s(\alpha)^2 e^{-n\chi_s(\alpha)} \frac{1}{\sigma(Y_s)^2} (b(\alpha, Y_s) - \sum_{\beta=1}^N b(\beta, Y_s)\chi_s(\beta))^2 ds\right] \end{aligned}$$

We deduce from this inequality that

$$\mathbb{E}\left[\int_0^t \mathbf{1}_{\chi_s(\alpha)=0} \Lambda_{\beta,\alpha} \chi_s(\beta) ds\right] = 0$$

otherwise the preceding expression would converge to $-\infty$ when n goes to $+\infty$. We deduce easily from this equality and the preceding properties that for all $t > 0$, $\mathbb{P}(\chi_t(\alpha) = 0) = 0$. This concludes the proof. \square

A.2 Auxiliary Tools

The following lemma is classical.

Lemma A.2. *Let E, E' be two separable metric spaces and A, A' two tight (resp. closed, convex) subsets of $\Delta(E)$ and $\Delta(E')$. Then the set $\mathcal{P}(A, A')$ of probabilities on $E \times E'$ having marginals in the sets A and A' is itself tight (resp. closed, convex).*

Proof. Let us prove the tightness property. Let $\mu \in A$, $\nu \in A'$ and $\pi \in \mathcal{P}(\mu, \nu)$. By assumption, for any $\varepsilon > 0$ there is a compact K_ε of E , independent of the choice of μ in A , such that $\mu(E/K_\varepsilon) \leq \varepsilon$, and a compact K'_ε , independent of the choice of ν in A' such that $\nu(E'/K'_\varepsilon) \leq \varepsilon$. Then for any pair of random variables (U, V) of law π :

$$\mathbb{P}[(U, V) \notin K_\varepsilon \times L_\varepsilon] \leq \mathbb{P}[U \notin K_\varepsilon] + \mathbb{P}[V \notin L_\varepsilon] \leq 2\varepsilon$$

The closed and convex properties follow directly from the continuity and linearity of the application mapping π to its marginals. \square

The following theorem is well-known and allows to construct variables with prescribed conditional laws.

Theorem A.3. *(Blackwell-Dubins [5])*

Let E be a polish space with $\Delta(E)$ the set of Borel probabilities on E , and $([0, 1], \mathcal{B}([0, 1]), \lambda)$ the unit interval equipped with Lebesgue's measure. There exists a measurable mapping

$$\Phi : \Delta(E) \times [0, 1] \longrightarrow E$$

such that for all $\mu \in \Delta(E)$, the law of $\Phi(\mu, U)$ is μ where U is the canonical element in $[0, 1]$.

In the proofs of section 6, we use indirectly this result together with the a disintegration theorem. Precisely:

Lemma A.4. *Let E, F be Polish spaces, $(\Omega, \mathcal{A}, \mathbb{P})$ be some probability space, Y be an E -valued random variable defined on Ω , and \mathcal{F} a sub- σ -field of \mathcal{A} . Assume that f is an \mathcal{F} -measurable map from Ω to $\Delta(E \times F)$ such that the marginal $f_1(x) \in \Delta(E)$ of $f(x)$ on the first coordinate is a version of the conditional law of Y given \mathcal{F} . Then, (up to enlarging the probability space, there exists a random variable Z such that $f(\omega)$ is a version of the conditional law of (Y, Z) given \mathcal{F} .*

Proof. Up to enlarging the probability space, we may assume that there exists some random variable U uniformly distributed on $[0, 1]$ and independent of (Y, \mathcal{F}) . One can define using Theorem A.3 a variable $(\tilde{Y}, \tilde{Z}) = \Phi(f(\omega), U)$ having the property that $f_1(\omega)$ is a version of the conditional law of \tilde{Y} given \mathcal{F} . Let $g(\omega, \tilde{Y})$ be a version of the conditional law of \tilde{Z} given (\mathcal{F}, \tilde{Y}) , it follows easily that $Z = \Phi(g(\omega, Y), U)$ fulfills the required properties. \square

Remark A.5. *In case of a product, we can have a more precise construction (which is useful to prove some conditional independence properties). Given three (or more) Polish spaces E, F, G , $(\Omega, \mathcal{A}, \mathbb{P})$ some probability space, Y an E -valued random variable defined on Ω , and \mathcal{F} a sub- σ -field of \mathcal{A} . Assume that f is an \mathcal{F} -measurable map from Ω to $\Delta(E \times F \times G)$ such that the marginal $f_1(x) \in \Delta(E)$ of $f(x)$ on the first coordinate is a version of the conditional*

law of Y given \mathcal{F} . Then, we may assume that there exist some independent random variables U, V uniformly distributed on $[0, 1]$ and independent of (Y, \mathcal{F}) . One can define using Theorem A.3 a variable $(\tilde{Y}, \tilde{Z}, \tilde{T}) = \Phi(f(\omega), U)$ having the property that $f_1(\omega)$ is a version of the conditional law of \tilde{Y} given \mathcal{F} . Let $g(\omega, \tilde{Y})$ be a version of the conditional law of \tilde{Z} given (\mathcal{F}, \tilde{Y}) , and $h(\omega, \tilde{Y}, \tilde{Z})$ a version of the conditional law of \tilde{T} given $(\mathcal{F}, \tilde{Y}, \tilde{Z})$ it follows easily that $Z = \Phi(g(\omega, Y), U)$ and $T = \Phi(h(\omega, Y, Z), V)$ fulfill the required properties.

Lemma A.6. *Let E, F be Polish spaces, there exists a measurable map $\mathcal{D} : \Delta(E \times F) \times F \rightarrow \Delta(E)$ such that for all $\mu \in \Delta(E \times F)$, $f \rightarrow \mathcal{D}(\mu, f)$ is a version of the conditional law of e given f whenever the pair of random variables (e, f) has law μ .*

Proof. See Corollary 7.27.1 in [4]. □

Lemma A.7. *Let (A, B, A', B') be random variables defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ taking values in some Polish spaces and \mathcal{F} a sub σ -field of \mathcal{A} . Assume that $\llbracket A, B | \mathcal{F} \rrbracket = \llbracket A', B' | \mathcal{F} \rrbracket$ almost surely, then*

$$\llbracket \llbracket A | B, \mathcal{F} \rrbracket | \mathcal{F} \rrbracket = \llbracket \llbracket A' | B', \mathcal{F} \rrbracket | \mathcal{F} \rrbracket.$$

Proof. Using the notations of preceding lemma, it follows from the definition of conditional laws that

$$\llbracket A | B, \mathcal{F} \rrbracket = \mathcal{D}(\llbracket A, B | \mathcal{F} \rrbracket, B).$$

Using that the same relation holds with (A', B') and that $\llbracket A, B | \mathcal{F} \rrbracket = \llbracket A', B' | \mathcal{F} \rrbracket$ almost surely, the conclusion follows. □

Let E, F be separable metric spaces that are universally measurable (i.e. all the Borel probabilities are tight) so that the disintegration theorem applies. Define the application:

$$T : \Delta(E \times F) \rightarrow \Delta(\Delta(E))$$

which associates to the law π of a pair (X, Y) of $E \times F$ -valued random variables, the law of the conditional law $\llbracket X | Y \rrbracket$. Formally $T(\pi)$ is defined as the unique probability such that for every bounded continuous function f on $\Delta(E)$

$$\int_{\Delta(E)} f(u) dT(\pi)(u) = \mathbb{E}[f(\llbracket X | Y \rrbracket)] = \int_F f(\pi(\cdot | y)) d\pi_F(y)$$

where $\pi(\cdot | y)$ denotes a version of the conditional law of X given Y under π and π_F the marginal of π on F .

The following Lemma is a slight modification of Lemma 6.2 in [21].

Lemma A.8. *For any convex lower semi-continuous function $f : \Delta(E) \rightarrow \mathbb{R} \cup \{+\infty\}$, the application*

$$\pi \rightarrow \int_{\Delta(E)} f(u) dT(\pi)(u)$$

is convex and lower semi-continuous on $\Delta(E \times F)$.

Proof. We can assume that f is non-negative up to adding a continuous linear function. From usual arguments, there exists a countable family $\phi_n \in C_b(E)$ such that

$$f(\mu) = \sup_{n \in \mathbb{N}} \tilde{\phi}_n(\mu) = \sup_{n \in \mathbb{N}} \int_E \phi_n(t) d\mu(t)$$

Define f_p for $p \in \mathbb{N}$ as

$$f_p(\mu) = \sup_{n \in \mathbb{N}} \tilde{\phi}_{n,p}(\mu)$$

where $\phi_{n,p} = \phi_n$ if $\forall x \in E, \phi_n(x) \geq -p$ and $\phi_{n,p} \equiv 0$ otherwise. As f is non-negative, we have:

$$f(\mu) = \sup_{n \in \mathbb{N}} \sup_{p \in \mathbb{N}} \tilde{\phi}_{n,p}(\mu) = \sup_p f_p(\mu).$$

Note that the sequence f_p is non-decreasing and $f_0 \geq 0$. Using monotone convergence, it is then sufficient to prove that for any fixed p , the function

$$\pi \rightarrow \int f_p(u) dT(\pi)(u)$$

is convex and lower semi-continuous. As the functions $\phi_{n,p}$ for $n \geq 0$ are bounded from below by $-p$, we may assume that they are nonnegative. Applying lemma 3.11 p125 in Buttazzo [6],

$$\begin{aligned} \int_F f_p(\pi(\cdot | y)) d\pi_F(y) &= \sup_{(B_i)_{i=1, \dots, i_p}} \sum_i \int_{B_i} \tilde{\phi}_{i,p}(\pi(\cdot | y)) d\pi_F(y) \\ &= \sup_{(B_i)_{i=1, \dots, i_p}} \sum_i \int_{B_i} \int_E \phi_{i,p}(x) d\pi(x|y) d\pi_F(y) \end{aligned}$$

where the supremum is over the finite measurable partitions of the space F . For a fixed pair (n, p) , consider the application defined on the Borel σ -field of F , $\mathcal{B}(F)$:

$$B \in \mathcal{B}(F) \rightarrow \mathcal{F}_{n,p}(\pi, B) = \int_B \int_E \phi_{n,p}(x) d\pi(x | y) d\pi_F(y) = \int_{E \times F} \phi_{n,p}(x) \mathbb{1}_B(y) d\pi(x, y).$$

This is clearly a bounded positive measure over F . All bounded positive measures on F are inner regular, i.e.

$$\mathcal{F}_{n,p}(\pi, B) = \sup\{\mathcal{F}_{n,p}(\pi, K) : K \subset B, K \text{ compact}\}.$$

One can then replace

$$\sup_{(B_i)_{i=1, \dots, i_p}} \sum_i \mathcal{F}_{i,p}(\pi, B_i) = \sup_{(K_i)_{i=1, \dots, i_p}} \sum_i \mathcal{F}_{i,p}(\pi, K_i) = \sup_{(O_i)_{i=1, \dots, i_p}} \sum_i \mathcal{F}_{i,p}(\pi, O_i)$$

where the second supremum is over finite families of pairwise disjoint compact subsets and the third over finite families of pairwise disjoint open sets. Finally, we just have to remark that for a fixed family of open sets, the mapping

$$\pi \rightarrow \sum_i \mathcal{F}_{i,p}(\pi, O_i)$$

is linear and lower semi-continuous, which implies that our functional is convex and lower semi-continuous. \square

Lemma A.9. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space, $\mathcal{G} \subset \mathcal{F}$ two sub σ -algebra of \mathcal{A} , and f a concave bounded upper semi-continuous mapping from $\Delta(\mathcal{Z})$ to \mathbb{R} . Then, for all \mathcal{Z} -valued random variable X :*

- $f(\llbracket X \rrbracket) \geq \mathbb{E}[f(\llbracket X \rrbracket | \mathcal{F})]$
- $f(\llbracket X \rrbracket | \mathcal{G}) \geq \mathbb{E}[f(\llbracket X \rrbracket | \mathcal{F}) | \mathcal{G}]$ almost surely.

Proof. This is the usual Jensen's inequality. It is sufficient to note that $\llbracket X \rrbracket$ is the barycenter (with respect to the dual $C_b(\mathcal{Z})$) of $\llbracket \llbracket X \rrbracket | \mathcal{F} \rrbracket$ by definition of conditional laws. For the second assertion, it is sufficient to prove that $\llbracket X \rrbracket | \mathcal{G}$ is almost surely the barycenter of the random variable

$$\Psi := \llbracket \llbracket X \rrbracket | \mathcal{F} \rrbracket | \mathcal{G}.$$

For a well-chosen countable subset \mathcal{C}_0 of $C_b(\mathcal{Z})$ and by using the definitions of conditional laws and conditional expectations, we have with probability one

$$\forall h \in \mathcal{C}_0, \quad \int \langle h, \nu \rangle d\Psi(\nu) = \mathbb{E}[\mathbb{E}[h(X) | \mathcal{F}] | \mathcal{G}] = \mathbb{E}[h(X) | \mathcal{G}] \quad (\text{A.1})$$

Now \mathcal{C}_0 can be taken as a countable convergence determining subset of $C_b(\mathcal{Z})$ (see e.g. [1] p106-107). The property (A.1) can therefore be extended to all $h \in C_b(\mathcal{Z})$ and this implies

$$f(\llbracket X \rrbracket | \mathcal{G}) \geq \mathbb{E}[f(\llbracket X \rrbracket | \mathcal{F}) | \mathcal{G}]$$

with probability one. □

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