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# Nonlinear Randomized Urn Models: a Stochastic Approximation Viewpoint

SOPHIE LARUELLE \*      GILLES PAGÈS †

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## Abstract

This paper extends the link between stochastic approximation and randomized urn models investigated in [24] for application in clinical trials introduced in [2, 3, 4]. The idea is that the drawing rule is not necessarily uniform on the urn composition, but can be reinforced by a function  $f$ . Firstly, by considering that  $f$  is concave or convex and by reformulating the dynamics of the urn composition as a standard stochastic approximation ( $SA$ ) algorithm (with remainder), we derive the *a.s.* convergence and the asymptotic normality (Central Limit Theorem  $CLT$ ) of the normalized procedure by calling upon the  $ODE$  and  $SDE$  methods. An in-depth analysis of this reinforced drawing rule in dimension  $d = 2$  exhibits two different behaviours: either a single equilibrium point when  $f$  is concave, or a single, two or three ones when  $f$  is convex. The last setting is solved using results on traps for  $SA$  to remove the repulsive point and to deduce the *a.s.* towards one of the attractive points. Secondly the Pólya urn is investigated with the point of view of bandit algorithm. Finally, these results are applied to Finance for optimal allocation and to the case where  $f$  has regular variation.

**Keywords** *Stochastic approximation, extended Pólya urn models, reinforcement, nonhomogeneous generating matrix, strong consistency, asymptotic normality, bandit algorithm.*

*2010 AMS classification:* 62L20, 62E20, 62L05 secondary: 62F12, 62P10.

## 1 Introduction

The first aim of this paper is to illustrate the efficiency of Stochastic Approximation ( $SA$ ) Theory by applying it to generalized Pólya urn models with nonlinear drawing rules. The modeling appears as a generalization of a previous work (see [24]) on randomized urn models applied to clinical trials. In this paper, several recent results on the asymptotic behaviour (convergence weak rate) of the urn models (especially [2, 3, 4]) were revisited using  $SA$  techniques. Considering nonlinear drawing rules leads to dynamics for the (normalized) urn composition having several local attractors but also “parasitic” equilibrium points (where equilibrium point means zero of the mean function associated to the stochastic algorithm). This is a major difference with the linear case investigated in [24] since we will need to

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call upon the whole machinery of *SA*, in particular “second order” results about noisy non-attractive equilibrium (repellers, saddle points) also known as “traps” in the *SA* literature (see [11]). We will establish the *a.s.* convergence (strong consistency) and elucidate entirely the rate of convergence of the normalized urn composition, even in the presence of multiple attractors.

Moreover, we study the classical Pólya urn model, namely when one ball of the drawing color is added after each draw, with reinforced drawing rule. In this framework, the “second order” results collapse because the trap are noiseless (since they lie on the boundary of the simplex). To elucidate this problem, we go back over the idea introduced for the study of the bandit algorithm established in [23, 22] where the authors introduce a non-negative martingale and applied the inequality presented in Lemma 3.1. This result highlight the robustness of *SA* theory, even in presence of noiseless repulsive equilibrium points.

We retrieve existing results (see for examples [18, 27, 13]) for classical Pólya urn model and besides we extend them significantly with the addition of weak rates. Moreover we extend the nonlinear drawing rule to randomized urn models where one ball is added on conditional average (introduced in [3, 4] and studied with *SA* theory in [24]).

The generalized Pólya urn models (*GPU*) have been widely studied in the literature with different points of view: martingale method (see for example [17]), algebraic approach (see for example [25]), reinforcement process (see for example [25]), branching process (see for example [19]), stochastic approximation (see for example [7]). These models also have applications to many areas: biology, random walks, statistics, computer science, clinical trials, psychology, economics or finance for instance (see [27]).

In these adaptive models, the key point is the equation which governs the urn composition updating after each drawing. Basically, we will show that (a normalized version of) this urn composition can be formulated as a classical recursive stochastic algorithm with step  $\gamma_n = \frac{1}{n + \text{Tr}(Y_0)}$ . Doing so, we will be in position to first establish the *a.s.* convergence of the procedure by calling upon the so-called Ordinary Differential Equation Method (*ODE* method) toward a finite set of equilibrium points (but usually not reduced to a single point). As a second step, we will rely on non-convergence results toward traps for *SA* (see [11, 14]) and on multiple targets (see [6, 14, 16]). As a third step, we entirely elucidate the rate of convergence (namely a *CLT* or an *a.s.* rate) by using the Stochastic Differential Equation Method (*SDE* method, see *e.g.* [15, 8]). The three main theoretical results from *SA* are recalled in a self-contained form in the Appendix. They can be found in all classical textbooks on *SA* ([8, 14, 15, 21]) and go back to [20] and [10]. We will see that these general theorems are extremely efficient to solve these questions and spare tedious lengthy computations and somewhat repetitive proofs.

Let us be more precise of the urn model of interest. We consider an urn containing balls of  $d$  different types. All random variables involved in the model are supposed to be defined on the same probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ . Denote  $Y_0 = (Y_0^i)_{i=1, \dots, d} \in \mathbb{R}_+^d \setminus \{0\}$  the initial composition of the urn, where  $Y_0^i$  denotes the number of balls of type  $i \in \{1, \dots, d\}$  (of course a more realistic though not mandatory assumption would be  $Y_0 \in \mathbb{N}^d \setminus \{0\}$ ). The urn composition at draw  $n$  is denoted by  $Y_n = (Y_n^i)_{i=1, \dots, d}$ . At the  $n^{\text{th}}$  stage, one draws randomly (according to a law defined further on) a ball from the urn with instant replacement. The urn composition is updated by adding  $D_n^{ij}$  balls of type  $i \in \{1, \dots, d\}$ . The procedure is then iterated. The urn composition at stage  $n$ , modeled by an  $\mathbb{R}^d$ -valued vector  $Y_n$ , satisfies the following recursive procedure:

$$Y_n = Y_{n-1} + D_n X_n, \quad n \geq 1, \quad Y_0 \in \mathbb{R}_+^d \setminus \{0\}, \quad (1.1)$$

with  $D_n = (D_n^{ij})_{1 \leq i, j \leq d}$  is the addition rule matrix and  $X_n$  is the result of the  $n^{\text{th}}$  draw and  $X_n :$

$(\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \{e^1, \dots, e^d\}$  models the selected ball ( $\{e^1, \dots, e^d\}$  denotes the canonical basis of  $\mathbb{R}^d$  and  $e^j$  stands for ball of color  $j$ ). We assume that there is no extinction *i.e.*  $Y_n \in \mathbb{R}_+^d \setminus \{0\}$  *a.s.* for every  $n \geq 1$ : so is the case if all the entries  $D_n^{ij}$  are *a.s.* nonnegative (see [24]). Let  $\mathcal{F}_n = \sigma(Y_0, X_k, \xi_k, 1 \leq k \leq n)$  be the filtration of the procedure. Two types of drawing rules are considered and will be analyzed:

- *Normalized distorted empirical frequency (convex/concave distortion)*

$$\forall i \in \{1, \dots, d\}, \quad \mathbb{P}(X_{n+1} = e^i | \mathcal{F}_n) = \frac{f(Y_n^i / (n + \text{Tr}(Y_0)))}{\sum_{j=1}^d f(Y_n^j / (n + \text{Tr}(Y_0)))}, \quad n \geq 0, \quad (1.2)$$

where  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  will satisfy convexity property.

- *Normalized distorted colour distribution (distortion  $f$  with regular variations)*

$$\forall i \in \{1, \dots, d\}, \quad \mathbb{P}(X_{n+1} = e^i | \mathcal{F}_n) = \frac{f(Y_n^i)}{\sum_{j=1}^d f(Y_n^j)}, \quad n \geq 0, \quad (1.3)$$

where  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  will be assumed to have regular variations in the following sense:  $\forall t > 0$ ,  $\frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} t^\alpha$ , uniformly in  $t$ .

Let us remark that when  $f = \text{Id}_{[0,1]}$  both updating rules coincide. Moreover, the regular variation case can be deduced from the convex one by noticing that, if  $f$  has regular variation and is bounded on each interval  $(0, M]$ , then  $\frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} t^\alpha$  uniformly in  $t$  on each  $(0, b]$  ( $0 < b < \infty$ ) if  $\alpha > 0$  (see Theorem 1.5.2 p.22 in [9]), thus, if  $\frac{Y_n}{n + \sum_{i=1}^d Y_0^i}$  lies in a compact, then

$$\max_{1 \leq i \leq d} \left| \frac{f(Y_n^i)}{f(n + \sum_{i=1}^d Y_0^i)} - \left( \frac{Y_n}{n + \sum_{i=1}^d Y_0^i} \right)^\alpha \right| \xrightarrow{n \rightarrow \infty} 0.$$

Then we can apply the convex framework to the special function  $x \mapsto x^\alpha$ .

The *generating matrices* are defined as the  $\mathcal{F}_n$ -compensator of the additions rule sequence *i.e.*

$$H_n = (\mathbb{E} [D_n^{ij} | \mathcal{F}_{n-1}])_{1 \leq i, j \leq d}, \quad n \geq 1.$$

We will also assume that the sequence of generating matrices *a.s.* converges toward a limit generating matrix denoted by  $H$ .

Other fields of application can be considered for such procedures like the adaptive asset allocation by an asset manager or a trader. One may also consider this type of procedure as a strategy to update the composition of a portfolio or even a whole fund, based on the (recent) past performances of the assets (see Section 4 for more details). We will study randomized urn models (based on those introduced in [3]) with nonlinear drawing rules. We will consider that additional rule matrices are random and that the limit generating matrix is different from the identity. Indeed, if  $D_n = H = I_d$  (namely one adds a ball of the same color as the drawing one), this corresponds to the classical framework of Pólya urn and when one considers reinforced drawing rule, this leads to a “winner take all” behaviour closely connected to bandit algorithm (see [23]) where the aim is to find the best arm. The purpose here is more to classify according to their performances the different treatments without excluding one of

them of the game: this is related to cooperative/competitive systems. The case of bandit algorithms requires an independent study owing to the nature of the equilibria which lie on the boundary of the state space, hence being noiseless. This case will be investigated in Section 3.

Here we will consider two kinds of nonlinear drawing rules: either  $f$  satisfies a convexity property, or  $f$  has regular variations. In the first setting, an in-depth analysis of the zeros of the mean function associated to the algorithm when  $d = 2$  provides a new behaviour: namely, when  $f$  is concave, we have a unique target, and when  $f$  is convex, we may have one, two or three possible equilibrium points. By studying the attractiveness of the each equilibrium and the non-convergence of the algorithm towards a trap and by applying the *ODE* method, we establish the *a.s.* convergence of the normalized urn composition towards one of the attractive equilibrium points. Then we entirely elucidate the rate of convergence. In particular, we emphasize the existence of three different convergence rates depending on the spectrum of the differential matrix of the mean function of the algorithm at the equilibrium point. Finally, we show that when  $f$  has regular variations with index  $\alpha \in \mathbb{R}_+$ , then the procedure behaves like  $y \mapsto y^\alpha$ .

The paper is organized as follows. Section 2 presents the framework of randomized urn models with the required assumptions on both the addition rule matrices and the generating matrices. After rewriting the dynamics of the urn composition as a *SA* procedure, we analyze in Section 2.1 the equilibrium points and their stability for the associated *ODE* when  $d = 2$  and the reinforced drawing rule is convex/concave. We exhibit two kind of behaviours: either a unique stable equilibrium point when  $f$  is concave, or a single, two or three ones with two attractive and one repulsive points when  $f$  is convex. By calling upon result on traps in *SA*, we prove the *a.s.* convergence towards one of the attractive equilibrium points; then we derive from *SDE* method all the possible rates of convergence. In Section 3, we study the case of the classical Polya urn, namely where the addition rule matrix equals to identity. We use results derive from the bandit algorithm to prove the convergence towards the target and the non-convergence towards traps. Finally, Section 4 introduces applications of such recursive procedures to drawing rule with regular variation and to portfolio allocation.

NOTATIONS For  $u = (u^i)_{i=1,\dots,d} \in \mathbb{R}^d$ ,  $\|u\|$  denotes the canonical Euclidean norm of the column vector  $u$  on  $\mathbb{R}^d$ ,  $\text{Tr}(u) = \sum_{k=1}^d u^k$  denotes its “weight”,  $u^t$  denotes its transpose;  $\|A\|$  denotes the operator norm of the matrix  $A \in \mathcal{M}_{d,q}(\mathbb{R})$  with  $d$  rows and  $q$  columns with respect to canonical Euclidean norms. When  $d=q$ ,  $\text{Sp}(A)$  denotes the set of eigenvalues of  $A$ .  $\mathbf{1} = (1 \cdots 1)^t$  denotes the unit column vector in  $\mathbb{R}^d$ ,  $I_d$  denotes the  $d \times d$  identity matrix and  $\text{diag}(u) = [\delta_{ij} u_i]_{1 \leq i, j \leq d}$ , where  $\delta_{ij}$  is the Kronecker symbol.

## 2 Randomized urn models

With the notations and definitions described in the introduction, we then formulate the main assumptions to establish the *a.s.* convergence of the urn composition.

$$(\mathbf{A1}) \equiv \left\{ \begin{array}{l}
 (i) \quad \textit{Addition rule matrix:} \text{ For every } n \geq 1, \text{ the matrix } D_n \textit{ a.s. has nonnegative} \\
 \text{entries.} \\
 (ii) \quad \textit{Generating matrix:} \text{ For every } n \geq 1, \text{ the generating matrices} \\
 H_n = (H_n^{ij})_{1 \leq i, j \leq d} \textit{ a.s. satisfies} \\
 \qquad \qquad \qquad \forall j \in \{1, \dots, d\}, \quad \sum_{i=1}^d H_n^{ij} = c > 0. \\
 (iii) \quad \textit{Starting value:} \text{ The starting urn composition vector } Y_0 \in \mathbb{R}_+^d \setminus \{0\}.
 \end{array} \right.$$

The constant  $c$  is known as the balance of the urn. In fact, we may assume without loss of generality, up to a renormalization of  $Y_n$ , that  $c = 1$ : since  $\widehat{Y}_n = \frac{Y_n}{c}$  and  $\widehat{D}_{n+1} = \frac{D_{n+1}}{c}$ ,  $n \geq 0$ , formally satisfies the dynamics (1.1), namely

$$\widehat{Y}_n = \widehat{Y}_{n-1} + \widehat{D}_n X_n, \quad n \geq 1, \quad \widehat{Y}_0 \in \mathbb{R}_+^d \setminus \{0\}.$$

From now on, throughout the paper, we will consider this normalized balance version. Nevertheless, we will still denote by  $Y_n$  and  $D_n$  the normalized quantities and assume that  $c = 1$ .

**(A2)** The addition rule  $D_n$  is conditionally independent of the drawing procedure  $X_n$  given  $\mathcal{F}_{n-1}$  and satisfies

$$\forall 1 \leq j \leq d, \quad \sup_{n \geq 1} \mathbb{E} \left[ \|D_n^{\cdot j}\|^2 \mid \mathcal{F}_{n-1} \right] < +\infty \quad a.s. \quad (2.4)$$

where  $D_n^{\cdot j} = (D_n^{ij})_{i=1, \dots, d}$ .

**(A3)** Assume that there exists an irreducible  $d \times d$  matrix  $H$  (with nonnegative entries) such that

$$H_n \xrightarrow[n \rightarrow \infty]{a.s.} H. \quad (2.5)$$

and

$$\sum_{n \geq 1} \|H_n - H\|^2 < +\infty \quad a.s.. \quad (2.6)$$

$H$  is called the *limit generating matrix*.

## 2.1 Convex or concave function for the drawing rule

Let define the law of the drawing as follows

$$\forall i \in \{1, \dots, d\}, \quad \mathbb{P}(X_{n+1} = e^i \mid \mathcal{F}_n) = \frac{f(\widetilde{Y}_n^i)}{\sum_{j=1}^d f(\widetilde{Y}_n^j)}, \quad n \geq 1, \quad (2.7)$$

where  $\widetilde{Y}_n = \frac{Y_n}{n + \text{Tr}(Y_0)}$ ,  $n \geq 0$ , and  $f$  is a non-decreasing convex (or concave) function satisfying  $f(0) = 0$  and  $f(1) = 1$ . We use this renormalization since  $\mathbb{E}[\text{Tr}(Y_n)] = n + \text{Tr}(Y_0)$  and thus  $\widetilde{Y}_n$  lies on average in the simplex. Therefore this is a natural way to normalize the urn composition vector.

We can reformulate the dynamics (1.1)-(1.2) into a recursive stochastic algorithm to elucidate the asymptotic properties (*a.s.* convergence) of both the urn composition and the treatment allocation. We start from (1.1) with  $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$ . For  $n \geq 0$ ,

$$Y_{n+1} = Y_n + D_{n+1} X_{n+1} = Y_n + \mathbb{E}[D_{n+1} X_{n+1} \mid \mathcal{F}_n] + \Delta M_{n+1}, \quad (2.8)$$

where

$$\Delta M_{n+1} := D_{n+1} X_{n+1} - \mathbb{E}[D_{n+1} X_{n+1} \mid \mathcal{F}_n]$$

is an  $\mathcal{F}_n$ -martingale increment. By the definition of the generating matrix  $H_n$ , we have

$$\begin{aligned} \mathbb{E}[D_{n+1} X_{n+1} \mid \mathcal{F}_n] &= \sum_{i=1}^d \mathbb{E}[D_{n+1} \mathbb{1}_{\{X_{n+1}=e^i\}} e^i \mid \mathcal{F}_n] = \sum_{i=1}^d \mathbb{E}[D_{n+1} \mid \mathcal{F}_n] \mathbb{P}(X_{n+1} = e^i \mid \mathcal{F}_n) e^i \\ &= H_{n+1} \sum_{i=1}^d \frac{f(\widetilde{Y}_n^i)}{\text{Tr}(f(\widetilde{Y}_n))} e^i = H_{n+1} \frac{\widetilde{f}(\widetilde{Y}_n)}{\text{Tr}(\widetilde{f}(\widetilde{Y}_n))} \end{aligned}$$

where  $\tilde{f}((y^1, \dots, y^d)^t) = (f(y^i))_{1 \leq i \leq d} \in [0, 1]^d$  is a column vector, so that

$$Y_{n+1} = Y_n + H_{n+1} \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} + \Delta M_{n+1}.$$

Now we can derive a stochastic approximation for the normalized urn composition  $Y_n$ . First we have for every  $n \geq 0$ ,

$$\frac{Y_{n+1}}{n+1 + \text{Tr}(Y_0)} = \frac{Y_n}{n + \text{Tr}(Y_0)} + \frac{1}{n+1 + \text{Tr}(Y_0)} \left( H_{n+1} \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} - \frac{Y_n}{n + \text{Tr}(Y_0)} \right) + \frac{\Delta M_{n+1}}{n+1 + \text{Tr}(Y_0)}.$$

Consequently,  $\tilde{Y}_n = \frac{Y_n}{n + \text{Tr}(Y_0)}$ ,  $n \geq 0$ , satisfies a canonical recursive stochastic approximation procedure

$$\tilde{Y}_{n+1} = \tilde{Y}_n + \frac{1}{n+1 + \text{Tr}(Y_0)} \left( H_{n+1} \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} - \tilde{Y}_n \right) + \frac{1}{n+1 + \text{Tr}(Y_0)} \Delta M_{n+1} \quad (2.9)$$

$$= \tilde{Y}_n - \frac{1}{n+1 + \text{Tr}(Y_0)} \left( \tilde{Y}_n - H \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} \right) + \frac{1}{n+1 + \text{Tr}(Y_0)} (\Delta M_{n+1} + r_{n+1}) \quad (2.10)$$

with step  $\gamma_n = \frac{1}{n + \text{Tr}(Y_0)}$  and a remainder term given by

$$r_{n+1} := (H_{n+1} - H) \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))}. \quad (2.11)$$

Furthermore, in order to establish the *a.s.* boundedness of  $(\tilde{Y}_n)_{n \geq 0}$  we will rely on the following recursive equation satisfied by  $\text{Tr}(Y_n)$ :

$$\text{Tr}(Y_{n+1}) = \text{Tr}(Y_n) + \frac{\text{Tr}(H_{n+1} \tilde{f}(\tilde{Y}_n))}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} + \text{Tr}(\Delta M_{n+1}).$$

By the properties of the generating matrix  $H_{n+1}$ , we obtain

$$\text{Tr}(H_{n+1} \tilde{f}(\tilde{Y}_n)) = \sum_{i=1}^d (H_{n+1} \tilde{f}(\tilde{Y}_n))_i = \sum_{i=1}^d \sum_{j=1}^d H_{n+1}^{ij} f(\tilde{Y}_n^j) = \sum_{j=1}^d \left( \sum_{i=1}^d H_{n+1}^{ij} \right) f(\tilde{Y}_n^j) = \text{Tr}(\tilde{f}(\tilde{Y}_n)).$$

Consequently

$$\text{Tr}(Y_{n+1}) = \text{Tr}(Y_n) + 1 + \text{Tr}(\Delta M_{n+1}). \quad (2.12)$$

Set  $N_n := \sum_{k=1}^n X_k$ . Then we have

$$N_{n+1} = N_n + X_{n+1} = N_n + \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} + \Delta \tilde{M}_{n+1},$$

where  $\Delta \tilde{M}_{n+1} := X_{n+1} - \mathbb{E}[X_{n+1} | \mathcal{F}_n] = X_{n+1} - \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))}$ . Thus, for  $\tilde{N}_n := \frac{N_n}{n}$  we have, still for every  $n \geq 0$ ,

$$\tilde{N}_{n+1} = \tilde{N}_n - \frac{1}{n+1} \left( \tilde{N}_n - \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} \right) + \frac{1}{n+1} \Delta \tilde{M}_{n+1}.$$

**Proposition 2.1.** Let  $(Y_n)_{n \geq 0}$  be the urn composition sequence defined by (1.1)-(1.2).

(a) Under the assumptions **(A1)** and **(A2)**,

$$\frac{\text{Tr}(Y_n)}{n + \text{Tr}(Y_0)} \xrightarrow[n \rightarrow \infty]{a.s.} 1.$$

(b) If the addition rule matrices satisfy **(A1)**-(ii), then  $\text{Tr}(Y_n) = n + \text{Tr}(Y_0)$ , therefore the sequence  $(\tilde{Y}_n)_{n \geq 0}$  lies in the simplex.

**Proof.** (a) We have

$$D_{n+1}X_{n+1} = \sum_{j=1}^d D_{n+1}^j \mathbb{1}_{\{X_{n+1}=e^j\}}.$$

Therefore

$$\|D_{n+1}X_{n+1}\|^2 = \sum_{j=1}^d \|D_{n+1}^j\|^2 \mathbb{1}_{\{X_{n+1}=e^j\}},$$

$$\begin{aligned} \text{so that } \mathbb{E} \left[ \|D_{n+1}X_{n+1}\|^2 \mid \mathcal{F}_n \right] &= \sum_{j=1}^d \mathbb{E} \left[ \|D_{n+1}^j\|^2 \mid \mathcal{F}_n \right] \mathbb{P}(X_{n+1} = e^j \mid \mathcal{F}_n) \\ &\leq \sup_{n \geq 0} \sup_{1 \leq j \leq d} \mathbb{E} \left[ \|D_{n+1}^j\|^2 \mid \mathcal{F}_n \right] < +\infty \quad a.s. \end{aligned}$$

Consequently  $\sup_{n \geq 1} \mathbb{E} \left[ \|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n \right] < +\infty$  *a.s.*. Therefore thanks to the strong law of large numbers for conditionally  $L^2$ -bounded martingale increments, we have  $\frac{M_n}{n} \xrightarrow[n \rightarrow \infty]{} 0$  *a.s.*. Consequently it follows from (2.12) that

$$\frac{\text{Tr}(Y_n)}{n + \text{Tr}(Y_0)} = 1 + \frac{\text{Tr}(M_n)}{n + \text{Tr}(Y_0)} \xrightarrow[n \rightarrow \infty]{a.s.} 1.$$

(b) In this case  $\text{Tr}(M_n) = 0$ , consequently for every  $n \geq 0$ ,  $\text{Tr}(\tilde{Y}_n) = 1$ . □

The recursive procedure (2.10) is a zero search for the function  $h : \mathbb{R}^d \mapsto \mathbb{R}^d$  defined by

$$h(y) = \left( y - H \frac{\tilde{f}(y)}{\text{Tr}(\tilde{f}(y))} \right), \quad y \in \mathbb{R}^d. \quad (2.13)$$

Since the components of  $\tilde{Y}_n = \frac{Y_n}{n + \text{Tr}(Y_0)}$  are nonnegative and  $\text{Tr}(\tilde{Y}_n) = \frac{\text{Tr}(Y_n)}{n + \text{Tr}(Y_0)} \xrightarrow[n \rightarrow \infty]{} 1$  *a.s.*, it is clear that  $(\tilde{Y}_n)_{n \geq 0}$  is *a.s.* bounded and that *a.s.* the set  $\mathcal{Y}_\infty$  of all its limiting value is contained in the simplex

$$\mathcal{V} = \text{Tr}^{-1}\{1\} = \left\{ y \in \mathbb{R}_+^d \mid \text{Tr}(y) = 1 \right\}.$$

Consequently, we look for equilibrium points  $y \in \mathcal{V}$  such that  $h(y) = 0$ .

**Theorem 2.1.** The function  $h$  defined by (2.13) has at least one zero in  $\mathcal{V}$ .



**Proof.** Let  $\psi : \mathcal{V} \rightarrow \mathcal{V}$  be the function defined by

$$\begin{aligned} \psi : \mathcal{V} &\rightarrow \mathcal{V} \\ y &\mapsto H \frac{\tilde{f}(y)}{\text{Tr}(\tilde{f}(y))}. \end{aligned}$$

Notice that  $\psi$  is continuous since  $f$  is. Let  $y_0 \in \mathcal{V}$ . Then  $\psi(y_0) = H \frac{\tilde{f}(y_0)}{\text{Tr}(\tilde{f}(y_0))}$ . Using Assumptions **(A1)**-**(A3)** on the generating matrices, we have that the entries of  $H$  are non-negative and that  $\forall 1 \leq j \leq d$ ,  $\sum_{i=1}^d H^{ij} = 1$ . Consequently, as  $f$  is non-negative,  $\psi(y_0) \geq 0$  and

$$\sum_{i=1}^d \psi^i(y_0) = \sum_{i=1}^d \sum_{k=1}^d H^{ik} \frac{f(y_0^k)}{\text{Tr}(\tilde{f}(y_0))} = \sum_{k=1}^d \sum_{i=1}^d H^{ik} \frac{f(y_0^k)}{\text{Tr}(\tilde{f}(y_0))} = 1.$$

Therefore, the function  $\psi$  is defined on  $\mathcal{V}$  to itself. As  $\mathcal{V}$  is a convex set, by applying the Brouwer fixed-point theorem, we obtain that  $\psi$  has at least a fixed point in  $\mathcal{V}$ , *i.e.*  $h$  has at least a zero in  $\mathcal{V}$ .  $\square$

### ▷ First step: Search of equilibrium points

Let  $d = 2$ . Define the generating matrix  $H$  as follows

$$H = \begin{pmatrix} p_1 & 1 - p_2 \\ 1 - p_1 & p_2 \end{pmatrix}, \quad 0 < p_i < 1, \quad i = 1, 2.$$

Solving  $h(z) = 0$ ,  $z \in \mathbb{R}^2$ , on  $\mathcal{V}$  is therefore equivalent to solve  $h((y, 1-y)^t) = 0$ ,  $y \in [0, 1]$ . Consequently, we come down to an one-dimensional problem, namely solving the following equation

$$(p_1 - y)f(y) + (1 - p_2 - y)f(1 - y) = 0. \quad (2.14)$$

First we remark that if  $p_1 = 1 - p_2$ , then  $y = p_1$  is the unique solution of (2.14) because  $f > 0$  on  $(0, 1)$ .

Assume now that  $p_1 \neq 1 - p_2$ . Then it is obvious that the solutions of (2.14) are different from  $p_1$  or  $1 - p_2$ . Define the following two functions

$$g_1(y) = \frac{f(1 - y)}{p_1 - y}, \quad y \in [0, 1] \setminus \{p_1\}, \quad \text{and} \quad g_2(y) = \frac{f(y)}{y - 1 + p_2}, \quad y \in [0, 1] \setminus \{1 - p_2\}. \quad (2.15)$$

Consequently, solving (2.14) for  $p_1 \neq 1 - p_2$  is equivalent to solve  $g_1(y) = g_2(y)$  on  $\mathcal{S} = [0, 1] \setminus \{p_1, 1 - p_2\}$ . Let us compute the first and second derivatives of these functions: we obtain for  $y \in \mathcal{S}$ ,

$$g_1'(y) = \frac{f(1 - y) - f'(1 - y)(p_1 - y)}{(p_1 - y)^2}, \quad g_2'(y) = \frac{f'(y)(y - 1 + p_2) - f(y)}{(y - 1 + p_2)^2},$$

$$g_1''(y) = \frac{f''(1 - y)(p_1 - y)^2 + 2(f(1 - y) - f'(1 - y)(p_1 - y))}{(p_1 - y)^3} = \frac{f''(1 - y) + 2g_1'(y)}{p_1 - y}$$

and

$$g_2''(y) = \frac{f''(y)(y - (1 - p_2))^2 - 2(f'(y)(y - 1 + p_2) - f(y))}{(y - 1 + p_2)^3} = \frac{f''(y) - 2g_2'(y)}{y - 1 + p_2}.$$

**Proposition 2.2.** *1. If  $f$  is concave, then  $h$  has a unique zero in  $(p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$ .*

2. If  $f$  is strictly convex,  $1 - p_2 < p_1$  and there exist  $\xi_1, \xi_2 \in [0, 1]$  such that  $g'_1(\xi_1) = 0$ ,  $g'_2(\xi_2) = 0$ ,  $g_1(\xi_1) < g_2(\xi_1)$ ,  $g_2(\xi_2) < g_1(\xi_2)$ , then  $h$  has three zeros in  $(p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$ .
3. If  $f$  is strictly convex and 2. is not satisfied, then  $h$  has a unique zero of  $h$  in  $(p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$ .

**Proof.**

1. If  $f$  is concave, then  $g'_1 > 0$  and  $g'_2 < 0$ , and we have a unique solution  $y^* \in (p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$ .
3. (a) If  $f$  is strictly convex and  $p_1 \vee p_2 \leq \frac{1}{f'(1)}$ , then  $g'_1 \geq 0$  and  $g'_2 \leq 0$ , and we have a unique solution  $y^* \in (p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$ .  
 (b) If  $f$  is strictly convex,  $p_1 \leq \frac{1}{f'(1)}$  and  $p_2 > \frac{1}{f'(1)}$ , then  $g'_1 \geq 0$  and we have to study the variations of  $g_2$ . We have that  $g'_2 < 0$  on  $[0, 1 - p_2)$ ,  $g'_2((1 - p_2)_+) < 0$  and  $g'_2(1) > 0$ . As  $g'_2(y) = \frac{\varphi_2(y)}{(y-1+p_2)^2}$  with  $\varphi_2(y) = f'(y)(y-1+p_2) - f(y)$  and  $\varphi'_2(y) = f''(y)(y-1+p_2)$ , then  $\varphi_2$  is non-decreasing on  $(1 - p_2, 1]$  since  $f$  is convex. Therefore there exists  $\xi_2 \in ]1 - p_2, 1)$  such that  $\varphi_2(\xi_2) = 0$ , namely  $g'_2(\xi_2) = 0$ . As  $p_2 > p_1$ , then  $\frac{1}{p_1} > \frac{1}{p_2}$ , therefore there exists a unique solution  $y^* \in (p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$ .  
 (c) If  $f$  is strictly convex,  $p_1 > \frac{1}{f'(1)}$  and  $p_2 \leq \frac{1}{f'(1)}$ , then  $g'_2 \leq 0$  and we have to study the variations of  $g_1$ . We have that  $g'_1 > 0$  on  $(p_1, 1]$ ,  $g'_1((p_1)_-) > 0$  and  $g'_1(0) < 0$ . As  $g'_1(y) = \frac{\varphi_1(y)}{(p_1 - y)^2}$  with  $\varphi_1(y) = f(1 - y) - f'(1 - y)(p_1 - y)$  and  $\varphi'_1(y) = f''(1 - y)(p_1 - y)$ , then  $\varphi_1$  is non-decreasing on  $[0, p_1)$  since  $f$  is convex. Moreover  $y \mapsto (p_1 - y)^2$  is non-increasing and non-negative on  $[0, p_1)$ , consequently  $g'_1$  is non-decreasing on  $[0, p_1)$  and  $g_1$  is convex. Therefore there exists  $\xi_1 \in ]0, p_1)$  such that  $g'_1(\xi_1) = 0$ . As  $p_1 > p_2$ , then  $\frac{1}{p_1} < \frac{1}{p_2}$ , therefore there exists a unique solution  $y^* \in (p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$ .
2. If  $f$  is strictly convex and  $p_1 \wedge p_2 > \frac{1}{f'(1)}$ , then following the lines of the two previous cases, we have that there exist  $\xi_1 \in (0, p_1)$  and  $\xi_2 \in (1 - p_2, 1)$  such that  $g'_1(\xi_1) = 0$  and  $g'_2(\xi_2) = 0$ . If  $\xi_1 < p_1 < 1 - p_2 < \xi_2$ ,  $\xi_1 < 1 - p_2 < p_1 < \xi_2$ ,  $\xi_1 < 1 - p_2 < \xi_2 < p_1$ ,  $1 - p_2 < \xi_1 < p_1 < \xi_2$  or  $1 - p_2 < \xi_1 < \xi_2 < p_1$ , then there exists a unique equilibrium point  $y^* \in (p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$ .

If  $1 - p_2 < \xi_2 < \xi_1 < p_1$ , then we may have one, two or three equilibrium points.

In fact, we have the following conditions for each case:

- if  $g_1(\xi_1) < g_2(\xi_1)$  and  $g_2(\xi_2) < g_1(\xi_2)$ , then we have three equilibrium points in  $(1 - p_2, p_1)$  (one in  $(1 - p_2, \xi_2)$ , one in  $(\xi_2, \xi_1)$  and one in  $(\xi_1, p_1)$ ),
- if  $g_1(\xi_1) \geq g_2(\xi_1)$  and  $g_2(\xi_2) < g_1(\xi_2)$  (or  $g_2(\xi_2) \geq g_1(\xi_2)$  and  $g_1(\xi_1) < g_2(\xi_1)$  respectively), then we have a unique equilibrium point in  $(1 - p_2, \xi_2)$  (or in  $(\xi_1, p_1)$  resp.).  $\square$

**Remark.** By a change of variable  $z = 1 - y$ , we have that  $\tilde{g}_2(z) = g_2(y) = g_2(1 - z) = \frac{f(1-z)}{p_2 - z}$ , therefore as  $g_1$  is convex on  $[0, p_1)$ ,  $\tilde{g}_2$  is convex on  $[0, p_2)$  and as  $\tilde{g}_2''(z) = g_2''(y)$ , then  $g_2$  is convex on  $(1 - p_2, 1]$ . Using the fact that  $g_1$  and  $g_2$  are convex on  $(p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$ , then we deduce that there exist at most three zeros in  $(p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$ .

**Example of phase transitions**

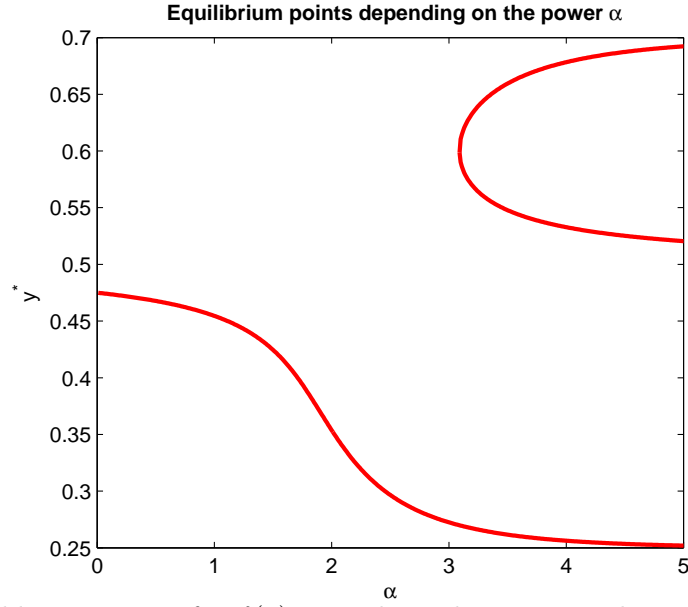


Figure 1: Equilibrium points for  $f(y) = y^\alpha$  depending on  $\alpha$  with  $p_1 = 0.7$  and  $p_2 = 0.75$ .

▷ **Second step: attractiveness**

After the study of the equilibrium points, we have to compute the eigenvalues of the differential matrix at the equilibrium point to deduce its attractiveness for  $ODE_h$ .

Indeed, as the set of limiting values of the algorithm is included in the simplex  $\mathcal{V}$ , we come down to a one dimensional problem, where all the equilibrium points belong to  $(p_1 \vee (1 - p_2), p_1 \wedge (1 - p_2))$ . This amounts to studying the zeros of  $h^1$ , the first component of  $h$ , namely  $h^1(y) = y - \frac{p_1 f(y) + (1 - p_2) f(1 - y)}{f(y) + f(1 - y)}$ . By simple computations, we obtain that, for every zeros  $y^*$  of  $h_1$ ,

$$(h^1)'(y^*) = \frac{f(y^*) + f'(y^*)(y^* - p_1) + f(1 - y^*) + f'(1 - y^*)(1 - p_2 - y^*)}{f(y^*) + f(1 - y^*)}, \quad y^* \in [0, 1], \quad h^1(y^*) = 0.$$

Therefore, for each zero  $y^*$  of  $h_1$ , we have three possibilities:

- $y^*$  is uniformly attractive ( $(h^1)'(y^*) > 0$ ),
- $y^*$  is repulsive ( $(h^1)'(y^*) < 0$ ),
- $y^*$  is undetermined ( $(h^1)'(y^*) = 0$ ).

**Theorem 2.2.** (i) *If  $h$  has a unique equilibrium point, then it is attractive.*

(ii) *If  $h$  has two equilibrium points, then the first equilibrium point is attractive (the smallest for  $h^1$ ) and the second is undetermined.*

(iii) *If  $h$  has three equilibrium points, then the first and the last (the smallest and the biggest for  $h^1$ ) are attractive and the one in the middle is repulsive.*

**Proof.** We have that  $h^1(0) = -(1 - p_2) < 0$  and  $h^1(1) = 1 - p_1 > 0$ . Then, if there exists a unique equilibrium point, the derivative at this point is positive, therefore the equilibrium point is attractive. Since there exists at most three equilibrium points, then:

- if  $h^1$  has two zeros, then the first equilibrium point is attractive and the second is undetermined;
- if  $h^1$  has three zeros, then we have two attractive equilibrium points (the first and the last) and one repulsive point (in the middle).  $\square$

### Examples of function $h^1$

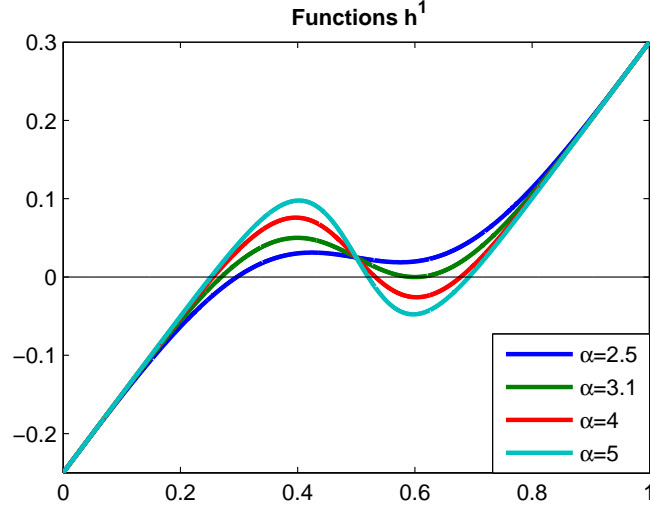


Figure 2: Function  $h^1$  for  $f(y) = y^3$  with  $p_1 = 0.75$  and  $p_2 = 0.75$ .

It remains to show that the algorithm does not converge towards the repulsive equilibrium point denoted by  $\hat{y}$ . To show that there is an excitation in the repulsive direction, we have to prove that assumption (A.27) (see Theorem A.2 in the appendix).

**Theorem 2.3.** *Let  $\hat{y}$  be a repulsive equilibrium point for  $h^1$ , namely  $(h^1)'(\hat{y}) < 0$ . Then*

$$\mathbb{P}(\tilde{Y}_n^1 \rightarrow \hat{y}) = 0.$$

**Proof.** As we consider the one-dimensional problem (namely the algorithm satisfied by the first component  $\tilde{Y}_n^1$ ), we have using the notations of Theorem A.2 in the appendix that

$$\Delta M_{n+1}^{(r)} = \Delta M_{n+1}^1.$$

Using Assumption (A1), we obtain

$$\Delta M_{n+1}^{(r)} = D_{n+1}^{11} X_{n+1}^1 + D_{n+1}^{12} X_{n+1}^2 - \frac{H_{n+1}^{11} f(\tilde{Y}_n^1) + H_{n+1}^{12} f(\tilde{Y}_n^2)}{\text{Tr}(f(\tilde{Y}_n))}.$$

Therefore

$$\begin{aligned} \mathbb{E} \left[ \left\| \Delta M_{n+1}^{(r)} \right\| \mid \mathcal{F}_n \right] &= \mathbb{E} \left[ \left\| \Delta M_{n+1}^1 \right\| \mid \mathcal{F}_n \right] \\ &= \frac{f(\tilde{Y}_n^1)}{\text{Tr}(f(\tilde{Y}_n))} \mathbb{E} \left[ \left\| D_{n+1}^{11} - \frac{H_{n+1}^{11} f(\tilde{Y}_n^1) + H_{n+1}^{12} f(\tilde{Y}_n^2)}{\text{Tr}(f(\tilde{Y}_n))} \right\| \mid \mathcal{F}_n \right] \\ &\quad + \frac{f(\tilde{Y}_n^2)}{\text{Tr}(f(\tilde{Y}_n))} \mathbb{E} \left[ \left\| D_{n+1}^{12} - \frac{H_{n+1}^{11} f(\tilde{Y}_n^1) + H_{n+1}^{12} f(\tilde{Y}_n^2)}{\text{Tr}(f(\tilde{Y}_n))} \right\| \mid \mathcal{F}_n \right]. \end{aligned}$$

By Jensen's inequality

$$\begin{aligned} \mathbb{E} \left[ \left\| \Delta M_{n+1}^{(r)} \right\| \mid \mathcal{F}_n \right] &\geq \frac{f(\tilde{Y}_n^1)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} \left| H_{n+1}^{11} - \frac{H_{n+1}^{11}f(\tilde{Y}_n^1) + H_{n+1}^{12}f(\tilde{Y}_n^2)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} \right| \\ &\quad + \frac{f(\tilde{Y}_n^2)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} \left| H_{n+1}^{12} - \frac{H_{n+1}^{11}f(\tilde{Y}_n^1) + H_{n+1}^{12}f(\tilde{Y}_n^2)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} \right|. \end{aligned}$$

Owing to **(A5)**,  $H_{n+1}^{ij} \xrightarrow[n \rightarrow \infty]{a.s.} H^{ij}$ , where  $H^{ii} = p_i$  and  $H^{ij} = 1 - p_j$ ,  $1 \leq i, j \leq 2$ . Furthermore, on  $\hat{\mathcal{Y}} = \left\{ \omega : \tilde{Y}_n^1(\omega) \rightarrow \hat{y} \right\}$ ,

$$\frac{H_{n+1}^{11}f(\tilde{Y}_n^1) + H_{n+1}^{12}f(\tilde{Y}_n^2)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} \xrightarrow[n \rightarrow \infty]{a.s.} \hat{y}.$$

Consequently,

$$\frac{f(\tilde{Y}_n^1)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} \left| H_{n+1}^{11} - \frac{H_{n+1}^{11}f(\tilde{Y}_n^1) + H_{n+1}^{12}f(\tilde{Y}_n^2)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} \right| \xrightarrow[n \rightarrow \infty]{a.s.} \frac{f(\hat{y})}{\text{Tr}(\tilde{f}(\hat{y}))} |p_1 - \hat{y}| > 0$$

and

$$\frac{f(\tilde{Y}_n^2)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} \left| H_{n+1}^{12} - \frac{H_{n+1}^{11}f(\tilde{Y}_n^1) + H_{n+1}^{12}f(\tilde{Y}_n^2)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))} \right| \xrightarrow[n \rightarrow \infty]{a.s.} \frac{f(\hat{y})}{\text{Tr}(\tilde{f}(\hat{y}))} |1 - p_2 - \hat{y}| > 0$$

because  $\hat{y} \in (1 - p_2, p_1)$ . Thus **(A.27)** is satisfied. Then, by using **(2.6)** and by applying Theorem **A.2** in the appendix (see **[11, 14]**),  $\mathbb{P}(\hat{\mathcal{Y}}) = 0$ .  $\square$

$\triangleright$  **Third step: a.s. convergence**

**Theorem 2.4.** *Let  $(Y_n)_{n \geq 0}$  be the urn composition sequence defined by **(1.1)**-**(1.2)**. Under the assumptions **(A1)**, **(A2)** and **(A3)**,*

$$(a) \quad \frac{Y_n}{\text{Tr}(Y_n)} \xrightarrow[n \rightarrow \infty]{a.s.} y^*.$$

$$(b) \quad \tilde{N}_n \xrightarrow[n \rightarrow \infty]{a.s.} \frac{\tilde{f}(y^*)}{\text{Tr}(\tilde{f}(y^*))}.$$

**Proof.** We will first prove that  $(a) \Rightarrow (b)$ , then we will prove  $(a)$ .

$(a) \Rightarrow (b)$ . We have

$$\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] = \sum_{i=1}^d \frac{f(\tilde{Y}_{n-1}^i)}{\text{Tr}(\tilde{f}(\tilde{Y}_{n-1}))} e^i = \frac{\tilde{f}(\tilde{Y}_{n-1})}{\text{Tr}(\tilde{f}(\tilde{Y}_{n-1}))}$$

and, by construction  $\|X_n\|^2 = 1$  so that  $\mathbb{E}[\|X_n\|^2 \mid \mathcal{F}_{n-1}] = 1$ . Hence the martingale

$$\tilde{M}_n = \sum_{k=1}^n \frac{X_k - \mathbb{E}[X_k \mid \mathcal{F}_{k-1}]}{k} \xrightarrow[n \rightarrow \infty]{a.s. \& L^2} \tilde{M}_\infty \in L^2,$$

and by the Kronecker Lemma we obtain

$$\frac{1}{n} \sum_{k=1}^n X_k - \frac{1}{n} \sum_{k=1}^n \frac{\tilde{f}(\tilde{Y}_{k-1})}{\text{Tr}(\tilde{f}(\tilde{Y}_{k-1}))} \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

This yields the announced implication owing to the Cesaro Lemma.

(a) Assumption **(A2)** implies that  $\sup_{n \geq n_0} \mathbb{E} \left[ \|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n \right] < +\infty$  *a.s.* and Assumption **(A3)** implies that  $r_n \xrightarrow[n \rightarrow \infty]{a.s.} 0$ .

- If  $h$  has a unique zero  $y^* \in \mathcal{V}$ , then the fundamental result derived from the *ODE* method (see Theorem A.1 in Appendix) yields

$$\tilde{Y}_n \xrightarrow[n \rightarrow \infty]{a.s.} y^*.$$

- If  $ODE_h$  has two attractive equilibrium points, the set of limiting value is *a.s.* compact connected and stable by the flow of  $ODE_h$  and the trap of the algorithm cannot be a limiting value (see the study of the attractiveness). Thus the stable chain recurrent sets are reduced to each of the two attractive equilibrium points (see [6]) and consequently, the stochastic algorithm converges towards one of the two attractive equilibrium points.  $\square$

#### ▷ Fourth step: rate of convergence

To establish a *CLT* for the sequence  $(\tilde{Y}_n)_{n \geq 0}$  we need to make the following additional assumptions:

**(A4)** The addition rules  $D_n$  *a.s.* satisfy

$$\forall 1 \leq j \leq d, \quad \begin{cases} \sup_{n \geq 1} \mathbb{E} \left[ \|D_n^j\|^{2+\delta} \mid \mathcal{F}_{n-1} \right] \leq C < \infty \quad \text{for a } \delta > 0, \\ \mathbb{E} \left[ D_n^j (D_n^j)^t \mid \mathcal{F}_{n-1} \right] \xrightarrow[n \rightarrow \infty]{} C^j, \end{cases}$$

where  $C^j = (C_{il}^j)_{1 \leq i, l \leq d}$ ,  $j = 1, \dots, d$ , are  $d \times d$  positive definite matrices.

Note that **(A4)**  $\Rightarrow$  **(A2)** since  $\mathbb{E} \left[ \|D_n^j\|^2 \mid \mathcal{F}_{n-1} \right] \leq \left( \mathbb{E} \left[ \|D_n^j\|^{2+\delta} \mid \mathcal{F}_{n-1} \right] \right)^{\frac{2}{2+\delta}}$ .

**(A5)** The matrix  $H$  satisfies

$$n \mathbb{E} \left[ \|H_n - H\|^2 \right] \xrightarrow[n \rightarrow \infty]{} 0. \quad (2.16)$$

**Theorem 2.5.** *Assume **(A1)**, **(A3)**, **(A4)** and **(A5)**.*

1. If  $p_1 < 1 - p_2$ , then

$$\sqrt{n} \left( \tilde{Y}_n - y^* \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Sigma) \quad \text{with} \quad \Sigma = \int_0^{+\infty} e^{u(Dh(y^*) - \frac{I}{2})} \Gamma e^{u(Dh(y^*) - \frac{I}{2})^t} du$$

$$\text{and} \quad \Gamma = \frac{f(y^{*1})C^1 + f(1 - y^{*1})C^2}{\text{Tr}(\tilde{f}(y^*))} - y^*(y^*)^t = a.s.-\lim_{n \rightarrow \infty} \mathbb{E} \left[ \Delta M_n \Delta M_n^t \mid \mathcal{F}_{n-1} \right]. \quad (2.17)$$

2. If  $1 - p_2 < p_1$ , we have three possible rate of convergence depending on the second eigenvalue:

(i) If

$$1 - \lambda := \frac{f'(y^{*1})(p_1 - y^{*1}) + f'(1 - y^{*1})(y^{*1} - (1 - p_2))}{f(y^{*1}) + f(1 - y^{*1})} < \frac{1}{2}, \quad (2.18)$$

then

$$\sqrt{n} \left( \tilde{Y}_n - y^* \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N} \left( 0, \frac{1}{2\lambda - 1} \Sigma \right).$$

(ii) If  $\lambda = \frac{1}{2}$ , then

$$\sqrt{\frac{n}{\log n}} \left( \tilde{Y}_n - y^* \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N} (0, \Sigma).$$

(iii) If  $\lambda < \frac{1}{2}$ , then  $n^\lambda \left( \tilde{Y}_n - y^* \right)$  a.s. converges as  $n \rightarrow +\infty$  towards a positive finite random variable.

**Remark.** • Condition (2.18) is satisfied as soon as

$$\begin{cases} \frac{(f'(1 - p_1) + f'(1 - p_2))(p_1 + p_2 - 1)}{f(1 - p_1) + f(1 - p_2)} < \frac{1}{2} & \text{if } f \text{ is concave,} \\ \frac{(f'(p_1) + f'(p_2))(p_1 + p_2 - 1)}{f(1 - p_1) + f(1 - p_2)} < \frac{1}{2} & \text{if } f \text{ is convex,} \end{cases}$$

by using the monotony of  $f$  and  $f'$  and that  $y^* \in (1 - p_2, p_1)$ .

• If  $f(y) = y$ , then the above criteria reads

$$\frac{2(p_1 + p_2 - 1)}{2 - p_1 - p_2} < \frac{1}{2} \quad \text{i.e.} \quad p_1 + p_2 < \frac{2}{5},$$

which ensures a CLT for the unique equilibrium point. This case has already been investigated (see [3, 4, 24]) and we can compute the equilibrium point and the second eigenvalue, namely

$$y^{*1} = \frac{1 - p_2}{2 - p_1 - p_2}, \quad y^{*2} = \frac{1 - p_1}{2 - p_1 - p_2} \quad \text{and} \quad \lambda_2 = 2 - p_1 - p_2.$$

Thus, if  $p_1 + p_2 < \frac{3}{2}$ , then the recursive procedure (2.10) satisfies a CLT; if  $p_1 + p_2 = \frac{3}{2}$ , (2.10) satisfies Theorem 2.5-2.(ii); and if  $p_1 + p_2 > \frac{3}{2}$ , (2.10) admits a a.s.-rate of convergence.

• In [12], they study some properties on the a.s. limit in claim (iii).

**Proof.** We will check the three assumptions of the CLT for SA algorithms recalled in the Appendix (Theorem A.3). Let us begin by computing the general differential matrix  $Dh(y)$ . We obtain

$$Dh(y) = \begin{pmatrix} 1 + \frac{f'(y^1)}{f(y^1) + f(y^2)} \left( \frac{p_1 f(y^1) + (1 - p_2) f(y^2)}{f(y^1) + f(y^2)} - p_1 \right) & \frac{f'(y^2)}{f(y^1) + f(y^2)} \left( \frac{p_1 f(y^1) + (1 - p_2) f(y^2)}{f(y^1) + f(y^2)} - (1 - p_2) \right) \\ \frac{f'(y^1)}{f(y^1) + f(y^2)} \left( \frac{(1 - p_1) f(y^1) + p_2 f(y^2)}{f(y^1) + f(y^2)} - (1 - p_1) \right) & 1 + \frac{f'(y^2)}{f(y^1) + f(y^2)} \left( \frac{(1 - p_1) f(y^1) + p_2 f(y^2)}{f(y^1) + f(y^2)} - p_2 \right) \end{pmatrix}.$$

As the equilibrium points  $y^*$  lie in the simplex  $\mathcal{V}$ , we have that  $y^{*2} = 1 - y^{*1}$ . Furthermore, using that  $h(y^*) = 0$ , we obtain

$$Dh(y^*)|_{\mathcal{V}} = \begin{pmatrix} 1 + \frac{f'(y^{*1})}{f(y^{*1})+f(1-y^{*1})} (y^{*1} - p_1) & \frac{f'(1-y^{*1})}{f(y^{*1})+f(1-y^{*1})} (y^{*1} - (1 - p_2)) \\ \frac{f'(y^{*1})}{f(y^{*1})+f(1-y^{*1})} (p_1 - y^{*1}) & 1 + \frac{f'(1-y^{*1})}{f(y^{*1})+f(1-y^{*1})} (1 - p_2 - y^{*1}) \end{pmatrix}.$$

Thus

$$\text{Sp}(Dh(y^*)|_{\mathcal{V}}) = \left\{ 1, \frac{f(y^{*1}) + f'(y^{*1})(y^{*1} - p_1) + f(1 - y^{*1}) + f'(1 - y^{*1})(1 - p_2 - y^{*1})}{f(y^{*1}) + f(1 - y^{*1})} \right\}.$$

The condition (A.30) on the spectrum of  $Dh(y^*)$  requested for algorithms with step  $\frac{1}{n}$  in Theorem A.3 reads  $\Re(\text{Sp}(Dh(y^*))) > \frac{1}{2}$ .

Secondly Assumption (A4) ensures that Condition (A.28) is satisfied since

$$\sup_{n \geq 1} \mathbb{E} \left[ \|\Delta M_n\|^{2+\delta} \mid \mathcal{F}_{n-1} \right] < +\infty \quad a.s. \quad \text{and} \quad \mathbb{E} [\Delta M_n \Delta M_n^t \mid \mathcal{F}_{n-1}] \xrightarrow[n \rightarrow \infty]{a.s.} \Gamma \quad \text{as} \quad n \rightarrow \infty,$$

where  $\Gamma$  is the symmetric nonnegative matrix given by

$$\begin{aligned} \mathbb{E} [\Delta M_{n+1} \Delta M_{n+1}^t \mid \mathcal{F}_n] &= \sum_{q=1}^2 \mathbb{P}(X_{n+1} = e^q \mid \mathcal{F}_n) (\mathbb{E} [D_{n+1}^q (D_{n+1}^q)^t \mid \mathcal{F}_n] \\ &\quad - \mathbb{E} [D_{n+1} X_{n+1} \mid \mathcal{F}_n] \mathbb{E} [D_{n+1} X_{n+1} \mid \mathcal{F}_n]^t) \\ &= \sum_{q=1}^2 \frac{f(\tilde{Y}_n^q)}{\text{Tr}(f(\tilde{Y}_n))} \mathbb{E} (D_{n+1}^q (D_{n+1}^q)^t \mid \mathcal{F}_n) - \left( H_{n+1} \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(f(\tilde{Y}_n))} \right) \left( H_{n+1} \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(f(\tilde{Y}_n))} \right)^t \\ &\xrightarrow[n \rightarrow \infty]{a.s.} \Gamma = \frac{f(y^{*1})C^1 + f(1 - y^{*1})C^2}{\text{Tr}(f(y^*))} - y^*(y^*)^t. \end{aligned}$$

Finally, using (A5), the remainder sequence  $(r_n)_{n \geq 1}$  satisfies (A.29).  $\square$

### 3 Pólya urn with reinforced drawing rule: a bandit approach

Assume that the drawing rule is given by (2.7), that  $D_n = I_d$ ,  $n \geq 1$ , and that the initial urn composition vector  $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$ . Then we normalize  $Y_n$  into  $\tilde{Y}_n := \frac{Y_n}{n + \text{Tr}(Y_0)}$ ,  $n \geq 0$ . The sequence  $(\tilde{Y}_n)_{n \geq 0}$  satisfies the following recursive stochastic algorithm

$$\tilde{Y}_{n+1} = \tilde{Y}_n - \frac{1}{n + 1 + \text{Tr}(Y_0)} \left( \tilde{Y}_n - \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(f(\tilde{Y}_n))} \right) + \frac{1}{n + 1 + \text{Tr}(Y_0)} \Delta M_{n+1}, \quad n \geq 1, \quad (3.19)$$

where

$$\Delta M_{n+1} := X_{n+1} - \mathbb{E} [X_{n+1} \mid \mathcal{F}_n]$$

is an  $(\mathcal{F}_n)_{n \geq 0}$ -martingale increment. Let us remark that in this case  $\frac{\text{Tr}(Y_n)}{n + \text{Tr}(Y_0)} = \frac{n + \text{Tr}(Y_0)}{n + \text{Tr}(Y_0)} = 1$ , so that the sequence  $(\tilde{Y}_n)_{n \geq 0}$  is bounded and lies in the simplex since it is a non-negative sequence.

The special case where  $f(x) = x$  follows from the following result



**Theorem 3.1.** *Let  $(Y_n)_{n \geq 0}$  be the urn composition sequence defined by (1.1)-(1.2). Under the assumption (2.7) with  $f(x) = x$ , there exists a random vector  $\tilde{Y}_\infty$  having values in the simplex such that*

$$\tilde{Y}_n = \frac{Y_n}{\text{Tr}(Y_n)} \xrightarrow[n \rightarrow \infty]{a.s.} \tilde{Y}_\infty \quad a.s.$$

Furthermore, if  $d = 2$ ,

(i) when  $Y_0^1 = Y_0^2 = 1$ ,  $\tilde{Y}_\infty^1$  has a uniform distribution on  $[0, 1]$ .

(ii) in general,  $\tilde{Y}_\infty^1$  has a beta distribution with parameter  $Y_0^1$  and  $Y_0^2$ .

**Proof.** Since the components of  $\tilde{Y}_n = \frac{Y_n}{n + \text{Tr}(Y_0)}$  are nonnegative and  $\text{Tr}(\tilde{Y}_n) = 1$ ,  $n \geq 0$ , it is clear that  $(\tilde{Y}_n)_{n \geq 0}$  is bounded and lies in the simplex and that the set  $\mathcal{Y}_\infty$  of all its limiting values is contained in

$$\mathcal{V} = \text{Tr}^{-1}\{1\} = \left\{ u \in \mathbb{R}_+^d \mid \text{Tr}(u) = 1 \right\}.$$

We can rewrite the recursive procedure (3.19) for  $f(x) = x$  in the following form

$$\tilde{Y}_{n+1} = \tilde{Y}_n + \frac{\Delta M_{n+1}}{n + 1 + \text{Tr}(Y_0)},$$

therefore  $\tilde{Y}_n$  is a non-negative bounded martingale. Moreover, the series  $\sum_{n \geq 1} \frac{\Delta M_n}{n}$  *a.s.* converges in  $\mathbb{R}^d$  since  $\sup_{n \geq 0} \mathbb{E}[\Delta M_{n+1}^2 \mid \mathcal{F}_n] \leq C < +\infty$  *a.s.*. Consequently  $\tilde{Y}_n \xrightarrow[n \rightarrow \infty]{a.s.} \tilde{Y}_\infty < +\infty$  *a.s.*

Claim (i) will be a consequence of (ii), so we will prove the second claim. We will use the moment method to prove that the law of  $\tilde{Y}_\infty^1$  is the beta law with parameter  $Y_0^1$  and  $Y_0^2$ . Indeed, by Lebesgue's theorem  $\lim_{n \rightarrow \infty} \mathbb{E}[(\tilde{Y}_n)^k] = \mathbb{E}[(\tilde{Y}_\infty)^k]$ .

Let us recall the moments of the beta law. Assume that a random variable  $X$  has the beta distribution with parameters  $\alpha$  and  $\beta$ . Then for every  $k \geq 1$ ,

$$\mathbb{E}[X^k] = \prod_{i=0}^{k-1} \frac{\alpha + i}{\alpha + \beta + i}.$$

We set, for  $n \geq 0$ ,

$$\tilde{M}_n = \frac{Y_n^1(Y_n^1 + 1) \cdots (Y_n^1 + k - 1)}{(n + \text{Tr}(Y_0))(n + \text{Tr}(Y_0) + 1) \cdots (n + \text{Tr}(Y_0) + k - 1)}.$$

Let show that  $(\tilde{M}_n)_{n \geq 0}$  is a  $(\mathcal{F}_n)_{n \geq 0}$ -martingale. We have *a.s.*

$$\begin{aligned} \mathbb{E}[\tilde{M}_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[\tilde{M}_{n+1} \mathbb{1}_{\{X_{n+1}^1=0\}} \mid \mathcal{F}_n] + \mathbb{E}[\tilde{M}_{n+1} \mathbb{1}_{\{X_{n+1}^1=1\}} \mid \mathcal{F}_n] \\ &= \frac{n + \text{Tr}(Y_0) - Y_n^1}{n + \text{Tr}(Y_0)} \frac{Y_n^1(Y_n^1 + 1) \cdots (Y_n^1 + k - 1)}{(n + \text{Tr}(Y_0) + 1)(n + \text{Tr}(Y_0) + 2) \cdots (n + \text{Tr}(Y_0) + k)} \\ &\quad + \frac{Y_n^1}{n + \text{Tr}(Y_0)} \frac{(Y_n^1 + 1)(Y_n^1 + 2) \cdots (Y_n^1 + k)}{(n + \text{Tr}(Y_0) + 1)(n + \text{Tr}(Y_0) + 2) \cdots (n + \text{Tr}(Y_0) + k)} \\ &= \frac{Y_n^1(Y_n^1 + 1) \cdots (Y_n^1 + k - 1)[(n + \text{Tr}(Y_0) - Y_n^1) + (Y_n^1 + k)]}{(n + \text{Tr}(Y_0))(n + \text{Tr}(Y_0) + 1) \cdots (n + \text{Tr}(Y_0) + k)} \\ &= \frac{Y_n^1(Y_n^1 + 1) \cdots (Y_n^1 + k - 1)}{(n + \text{Tr}(Y_0))(n + \text{Tr}(Y_0) + 1) \cdots (n + \text{Tr}(Y_0) + k - 1)} = \tilde{M}_n. \end{aligned}$$

Since  $\tilde{Y}_n^1 \xrightarrow[n \rightarrow \infty]{} \tilde{Y}_\infty^1$  a.s. then also  $\frac{Y_n^1 + r}{n + \text{Tr}(Y_0) + r} \xrightarrow[n \rightarrow \infty]{} \tilde{Y}_\infty^1$  a.s. for every fixed  $r$ , so that  $\tilde{M}_n \xrightarrow[n \rightarrow \infty]{} (\tilde{Y}_\infty^1)^k$  a.s. and, as  $0 \leq \tilde{M}_n \leq 1$ ,  $\lim_{n \rightarrow \infty} \mathbb{E}[\tilde{M}_n] = \mathbb{E}[(\tilde{Y}_\infty^1)^k]$ .  $(\tilde{M}_n)_{n \geq 1}$  being a martingale

$$\mathbb{E}[\tilde{M}_n] = \mathbb{E}[\tilde{M}_0] = \prod_{r=0}^{k-1} \frac{Y_0^1 + r}{\text{Tr}(Y_0) + r}. \quad \square$$

**Remark.** For arbitrary  $d \geq 2$ , it is known (see [1]) that  $\tilde{Y}_\infty$  has a Dirichlet distribution with parameter  $Y_0$ .

By the same kind of analysis as in Proposition 2.2, we can prove for  $d = 2$  that if  $f$  is strictly convex or strictly concave, the algorithm has exactly three equilibrium points, namely  $(1, 0)^t$ ,  $(\frac{1}{2}, \frac{1}{2})^t$  and  $(0, 1)^t$ . The attractiveness study of these equilibrium points leads to the following results

$$Dh((1, 0)^t)|_{\mathcal{V}} = \begin{pmatrix} 1 & f'_r(0) \\ 0 & 1 - f'_r(0) \end{pmatrix} \quad \text{so} \quad \text{Sp} \left( Dh((1, 0)^t)|_{\mathcal{V}} \right) = \{1, 1 - f'_r(0)\},$$

$$Dh \left( \left( \frac{1}{2}, \frac{1}{2} \right)^t \right) |_{\mathcal{V}} = \begin{pmatrix} 1 - \frac{f'(\frac{1}{2})}{4f(\frac{1}{2})} & \frac{f'(\frac{1}{2})}{4f(\frac{1}{2})} \\ \frac{f'(\frac{1}{2})}{4f(\frac{1}{2})} & 1 - \frac{f'(\frac{1}{2})}{4f(\frac{1}{2})} \end{pmatrix} \quad \text{so} \quad \text{Sp} \left( Dh \left( \left( \frac{1}{2}, \frac{1}{2} \right)^t \right) |_{\mathcal{V}} \right) = \left\{ 1, 1 - \frac{f'(\frac{1}{2})}{2f(\frac{1}{2})} \right\}$$

and

$$Dh((0, 1)^t)|_{\mathcal{V}} = \begin{pmatrix} 1 - f'_r(0) & 0 \\ f'_r(0) & 1 \end{pmatrix} \quad \text{so} \quad \text{Sp} \left( Dh((0, 1)^t)|_{\mathcal{V}} \right) = \{1, 1 - f'_r(0)\}.$$

Consequently, if  $f$  is strictly convex,  $(1, 0)^t$  and  $(0, 1)^t$  are attractive equilibrium points. Moreover  $(\frac{1}{2}, \frac{1}{2})^t$  is a “noisy” repulsive equilibrium point by the same argument as in Theorem 2.2. Still following the lines of Theorem 2.3, it follows that the recursive procedure (3.19) never converges toward  $(\frac{1}{2}, \frac{1}{2})^t$ . Then  $\frac{Y_n}{\text{Tr}(Y_n)} \xrightarrow[n \rightarrow \infty]{a.s.} \{(1, 0)^t, (0, 1)^t\}$ .

If  $f$  is strictly concave, then  $(\frac{1}{2}, \frac{1}{2})^t$  is an attractive equilibrium point and  $(1, 0)^t$  and  $(0, 1)^t$  are repulsive (for the ODE), but they are “noiseless” since they lie on the boundary of the limit set. Therefore, we have to analyze this case in another way, namely like for the two-armed bandit algorithm investigated in [23, 22, 26]. We will follow the approach introduced and developed in [22]: in particular we will prove that the algorithm never converges toward one of the traps.

**Theorem 3.2.** *If  $f$  satisfies  $1 < f'_r(0)$  or  $f'_r(0) = 1$  and  $f'_l(1) + \frac{f''_r(0)}{2} > 1$ , then for every deterministic initial value  $Y_0 \in \mathbb{R}^2 \setminus \{(1, 0)^t, (0, 1)^t\}$ ,*

$$\mathbb{P} \left( \tilde{Y}_\infty = \{(1, 0)^t, (0, 1)^t\} \right) = 0,$$

thus

$$\tilde{Y}_n \xrightarrow[n \rightarrow \infty]{a.s.} \left( \frac{1}{2}, \frac{1}{2} \right)^t.$$

**Remark.** Notice that in the previous theorem, no convexity property is requested on the function  $f$ . This result is indeed more general than the convex/concave framework that we study in this section.

**Proof.** We will prove that  $\mathbb{P}\left(\tilde{Y}_\infty = (0, 1)^t\right) = 0$  by considering the one-dimensional problem, *i.e.* we will prove that  $\mathbb{P}\left(\tilde{Y}_\infty^1 = 0\right) = 0$ . The other result, namely  $\mathbb{P}\left(\tilde{Y}_\infty = (1, 0)^t\right) = 0$ , can be deduced likewise by applying the method to the second component of the state vector (*i.e.* prove that  $\mathbb{P}\left(\tilde{Y}_\infty^2 = 0\right) = 0$ ).

Starting from the dynamics of  $\tilde{Y}_n^1$  given by (3.19), we have, for  $n \geq 0$ ,

$$\begin{aligned}\tilde{Y}_{n+1}^1 &= \tilde{Y}_n^1 - \frac{1}{n+1 + \text{Tr}(Y_0)} \left( \tilde{Y}_n^1 - \frac{f(\tilde{Y}_n^1)}{f(\tilde{Y}_n^1) + f(1 - \tilde{Y}_n^1)} \right) + \frac{1}{n+1 + \text{Tr}(Y_0)} \Delta M_{n+1}^1 \\ &= \tilde{Y}_n^1 \left( 1 - \frac{1}{n+1 + \text{Tr}(Y_0)} \left( 1 - \frac{f(\tilde{Y}_n^1)}{\tilde{Y}_n^1 (f(\tilde{Y}_n^1) + f(1 - \tilde{Y}_n^1))} \right) \right) + \frac{1}{n+1 + \text{Tr}(Y_0)} \Delta M_{n+1}^1.\end{aligned}$$

Set  $\tilde{h}(y) = 1 - \frac{f(y)}{y(f(y) + f(1-y))}$ . We have that  $\tilde{h}(y) < 1$  for  $y \in ]0, 1]$ , therefore  $\tilde{Y}_1^1 = Y_1^1 \neq 0$  implies that  $\tilde{Y}_n^1 \neq 0$  for every  $n$  because  $\tilde{Y}_n^1 \geq \tilde{Y}_1^1$ . We derive that

$$\tilde{M}_n := \frac{\tilde{Y}_n^1}{\prod_{k=1}^n \left( 1 - \frac{1}{k + \text{Tr}(Y_0)} \tilde{h}(\tilde{Y}_{k-1}^1) \right)}, \quad n \geq 0, \quad \tilde{M}_0 = 0,$$

is a positive martingale satisfying  $\tilde{M}_1 = \tilde{Y}_1$  and

$$\tilde{M}_{n+1} = \tilde{M}_n + \frac{1}{n+1 + \text{Tr}(Y_0)} \frac{\Delta M_{n+1}^1}{\prod_{k=1}^{n+1} \left( 1 - \frac{1}{k + \text{Tr}(Y_0)} \tilde{h}(\tilde{Y}_{k-1}^1) \right)}, \quad n \geq 0.$$

- If  $1 < f'_r(0)$ , then  $\tilde{h}(y) \xrightarrow{y \rightarrow 0} 1 - f'_d(0) =: \kappa < 0$ . Therefore, on  $\{Y_n^1 \rightarrow 0\}$ ,  $\tilde{h}(Y_{n-1}^1) \stackrel{a.s.}{\sim} \kappa < 0$ , so that  $\prod_{k=1}^n \left( 1 - \frac{1}{k + \text{Tr}(Y_0)} \tilde{h}(\tilde{Y}_{k-1}^1) \right) \xrightarrow{a.s.} +\infty$  *a.s.*  $\tilde{M}_n \xrightarrow{a.s.} 0$  on  $\{Y_n^1 \rightarrow 0\}$ . Consequently  $\{Y_n^1 \rightarrow 0\} \subset \{\tilde{M}_n \rightarrow 0\}$ .
- If  $f'_r(0) = 1$  and  $f'_l(1) + \frac{f''_r(0)}{2} > 1$ , then  $\tilde{h}(y) < 0$  for  $y$  in the neighborhood of 0. So we still have  $\{Y_n^1 \rightarrow 0\} \subset \{\tilde{M}_n \rightarrow 0\}$ .

**Remark.** If  $f(0) = 0$ ,  $f(1) = 1$ ,  $f'(0) = 1$  and  $f$  is convex or concave, then  $f = \text{Id}$  so the previous condition takes out of the convex framework.

Consequently

$$\mathbb{P}(Y_\infty^1 = 0 \mid \mathcal{F}_n) \leq \mathbb{P}(\tilde{M}_\infty = 0 \mid \mathcal{F}_n).$$

The end of the proof is based on the following lemma (see [23]) reproduced here for convenience.

**Lemma 3.1.** *Let  $(M_n)_{n \geq 0}$  be a non-negative martingale. Then*

$$\forall n \geq 0, \quad \mathbb{P}(M_\infty = 0 \mid \mathcal{F}_n) \leq \frac{\mathbb{E}[\Delta \langle M \rangle_{n+1}^\infty \mid \mathcal{F}_n]}{M_n^2}.$$

**Proof of Lemma 3.1.** It is sufficient to observe that, for every  $n \geq 0$ ,

$$\begin{aligned} \mathbb{P}(M_\infty = 0 \mid \mathcal{F}_n) &= \frac{\mathbb{E}[\mathbf{1}_{\{M_\infty=0\}} M_n^2 \mid \mathcal{F}_n]}{M_n^2} \leq \frac{\mathbb{E}[(M_\infty - M_n)^2 \mid \mathcal{F}_n]}{M_n^2} \\ &= \frac{\mathbb{E}[\Delta \langle M \rangle_{n+1}^\infty \mid \mathcal{F}_n]}{M_n^2}. \end{aligned} \quad \square$$

We have that

$$\begin{aligned} \mathbb{E}[(\Delta \widetilde{M}_{n+1})^2 \mid \mathcal{F}_n] &= \left( \frac{1}{n+1 + \text{Tr}(Y_0)} \right)^2 \frac{\mathbb{E}[(\Delta M_{n+1}^1)^2 \mid \mathcal{F}_n]}{\left( \prod_{k=1}^{n+1} \left( 1 - \frac{1}{k + \text{Tr}(Y_0)} \widetilde{h}(\widetilde{Y}_{k-1}^1) \right) \right)^2} \\ &= \frac{1}{(n+1 + \text{Tr}(Y_0))^2 \left( \prod_{k=1}^{n+1} \left( 1 - \frac{1}{k + \text{Tr}(Y_0)} \widetilde{h}(\widetilde{Y}_{k-1}^1) \right) \right)^2} \frac{f(\widetilde{Y}_n^1) f(1 - \widetilde{Y}_n^1)}{\left( f(\widetilde{Y}_n^1) + f(1 - \widetilde{Y}_n^1) \right)^2}. \end{aligned}$$

Then, by applying Lemma 3.1 to the positive martingale  $\widetilde{M}$ , we obtain

$$\begin{aligned} &\mathbb{P}(\widetilde{M}_\infty = 0 \mid \mathcal{F}_n) \\ &\leq \frac{\mathbb{E} \left[ \Delta \langle \widetilde{M} \rangle_{n+1}^\infty \mid \mathcal{F}_n \right]}{\widetilde{M}_n^2} \\ &= \frac{1}{\widetilde{M}_n^2} \mathbb{E} \left[ \sum_{k=n+1}^\infty \frac{1}{(k + \text{Tr}(Y_0))^2 \left( \prod_{\ell=1}^k \left( 1 - \frac{1}{\ell + \text{Tr}(Y_0)} \widetilde{h}(\widetilde{Y}_{\ell-1}^1) \right) \right)^2} \frac{f(\widetilde{Y}_{k-1}^1) f(1 - \widetilde{Y}_{k-1}^1)}{\left( f(\widetilde{Y}_{k-1}^1) + f(1 - \widetilde{Y}_{k-1}^1) \right)^2} \mid \mathcal{F}_n \right]. \end{aligned}$$

Besides the function  $F : y \mapsto \frac{f(y)f(1-y)}{(f(y)+f(1-y))^2}$  is continuous on  $(0, 1]$  and has  $f'_r(0)$  as a finite limit when  $y$  goes to 0. Therefore the function  $F$  is positive and bounded on  $[0, 1]$  by a constant  $\kappa_f$ . Consequently

$$\begin{aligned} &\mathbb{P}(\widetilde{M}_\infty = 0 \mid \mathcal{F}_n) \\ &\leq \frac{\kappa_f}{\widetilde{M}_n^2} \sum_{k=n+1}^\infty \frac{1}{(k + \text{Tr}(Y_0))^2} \mathbb{E} \left[ \underbrace{\frac{\widetilde{Y}_{k-1}^1}{\prod_{\ell=1}^{k-1} \left( 1 - \frac{1}{\ell + \text{Tr}(Y_0)} \widetilde{h}(\widetilde{Y}_{\ell-1}^1) \right)}}_{=\widetilde{M}_{k-1}} \frac{\left( 1 - \frac{1}{k + \text{Tr}(Y_0)} \widetilde{h}(\widetilde{Y}_{k-1}^1) \right)^{-1}}{\prod_{\ell=1}^k \left( 1 - \frac{1}{\ell + \text{Tr}(Y_0)} \widetilde{h}(\widetilde{Y}_{\ell-1}^1) \right)} \mid \mathcal{F}_n \right]. \end{aligned}$$

As  $\widetilde{h} \leq \widetilde{h}^+ := \max(\widetilde{h}, 0) \leq \|\widetilde{h}^+\|_\infty$  and  $\|\widetilde{h}^+\|_\infty < 1$  since  $\widetilde{h}^+(y) < 1$ ,  $y \in [0, 1]$ , since  $f(y) > 0$  and  $\widetilde{h}(y) \xrightarrow{y \rightarrow 0} 1 - f'_r(0) < 1$  since  $f'_r(0) > 0$ , we have

$$\left( 1 - \frac{1}{k + \text{Tr}(Y_0)} \widetilde{h}(\widetilde{Y}_{k-1}^1) \right)^{-1} \leq \left( 1 - \frac{1}{k + \text{Tr}(Y_0)} \|\widetilde{h}^+\|_\infty \right)^{-1}, \quad k \geq 1.$$

Then, as  $\mathbb{E} [\widetilde{M}_{k-1} | \mathcal{F}_n] = \widetilde{M}_n$  since  $\widetilde{M}$  is a  $(\mathbb{P}, \mathcal{F}_n)$ -martingale,

$$\begin{aligned}
& \mathbb{P}(\widetilde{M}_\infty = 0 | \mathcal{F}_n) \\
& \leq \frac{\kappa_f}{\widetilde{M}_n^2} \sum_{k=n+1}^{\infty} \frac{1}{(k + \text{Tr}(Y_0))^2} \frac{\widetilde{M}_n}{\left(1 - \frac{1}{k + \text{Tr}(Y_0)} \|\tilde{h}^+\|_\infty\right) \prod_{\ell=1}^n \left(1 - \frac{1}{\ell + \text{Tr}(Y_0)} \tilde{h}(\tilde{Y}_{\ell-1}^1)\right) \prod_{\ell=n+1}^k \left(1 - \frac{1}{\ell + \text{Tr}(Y_0)} \|\tilde{h}^+\|_\infty\right)} \\
& = \frac{\kappa_f}{\widetilde{M}_n \prod_{\ell=1}^n \left(1 - \frac{1}{\ell + \text{Tr}(Y_0)} \tilde{h}(\tilde{Y}_{\ell-1}^1)\right)} \sum_{k=n+1}^{\infty} \frac{1}{\left(1 - \frac{1}{k + \text{Tr}(Y_0)} \|\tilde{h}^+\|_\infty\right) (k + \text{Tr}(Y_0))^2 \prod_{\ell=n+1}^k \left(1 - \frac{1}{\ell + \text{Tr}(Y_0)} \|\tilde{h}^+\|_\infty\right)} \\
& = \frac{\kappa_f C_{n_0}}{\tilde{Y}_n^1} \sum_{k=n+1}^{\infty} \frac{1}{(k + \text{Tr}(Y_0))^2} \exp\left(-\sum_{\ell=n+1}^k \ln\left(1 - \frac{\|\tilde{h}^+\|_\infty}{\ell + \text{Tr}(Y_0)}\right)\right) \\
& \leq \frac{\kappa_f C_{n_0} C_{\tilde{h}}(n + \text{Tr}(Y_0))}{Y_n^1} \sum_{k=n+1}^{\infty} \frac{e^{\|\tilde{h}^+\|_\infty \ln \frac{k + \text{Tr}(Y_0)}{n + \text{Tr}(Y_0)}}}{(k + \text{Tr}(Y_0))^2} \quad \text{since } \tilde{Y}_n^1 = \frac{Y_n^1}{n + \text{Tr}(Y_0)} \\
& = \frac{\kappa_f C_{n_0} C_{\tilde{h}}(n + \text{Tr}(Y_0))}{Y_n^1} \sum_{k=n+1}^{\infty} \frac{(n + \text{Tr}(Y_0))^{-\|\tilde{h}^+\|_\infty}}{(k + \text{Tr}(Y_0))^{2 - \|\tilde{h}^+\|_\infty}} \\
& = \frac{\kappa_f C_{n_0} C_{\tilde{h}}(n + \text{Tr}(Y_0))^{1 - \|\tilde{h}^+\|_\infty}}{Y_n^1} \sum_{k=n+1}^{\infty} \frac{1}{(k + \text{Tr}(Y_0))^{2 - \|\tilde{h}^+\|_\infty}} \\
& \leq \frac{\kappa_f C_{n_0} C_{\tilde{h}}}{Y_n^1}.
\end{aligned}$$

Now we will prove that  $Y_n^1 \xrightarrow{a.s.} +\infty$ .

$$Y_{n+1}^1 = Y_n^1 + \frac{f(\tilde{Y}_n^1)}{f(\tilde{Y}_n^1) + f(1 - \tilde{Y}_n^1)} + \Delta M_{n+1}^1$$

where  $(\Delta M_n^1)_{n \geq 1}$  is a sequence of martingale increments satisfying  $\sup_n \mathbb{E} [|\Delta M_n^1|^2 | \mathcal{F}_{n-1}] < +\infty$  since  $M_n^1$  is bounded.

Moreover, one checks that

$$\{Y_\infty^1 < +\infty\} = \bigcup_{n \geq 0} \bigcap_{k > n} \left\{ U_k > \frac{f\left(\frac{Y_n^1}{k-1 + \text{Tr}(Y_0)}\right)}{f\left(\frac{Y_n^1}{k-1 + \text{Tr}(Y_0)}\right) + f\left(1 - \frac{Y_n^1}{k-1 + \text{Tr}(Y_0)}\right)} \right\},$$

then

$$\forall n \in \mathbb{N}, \quad \mathbb{P}(Y_\infty^1 < +\infty | Y_n^1 = y) = \prod_{k > n} \left(1 - \frac{f(y/(k-1 + \text{Tr}(Y_0)))}{f(y/(k-1 + \text{Tr}(Y_0))) + f(1 - y/(k-1 + \text{Tr}(Y_0)))}\right) = 0$$

since  $\sum_k \frac{f(y/k + \text{Tr}(Y_0))}{f(y/k + \text{Tr}(Y_0)) + f(1 - y/k + \text{Tr}(Y_0))} = +\infty$  because  $f'(0) > 0$ . Therefore  $Y_\infty^1 = \lim_n Y_n^1 = +\infty$  a.s.. Consequently, we obtain

$$\mathbb{P}(\tilde{Y}_\infty^1 = 0) = 0.$$

The second result follows from the same method applied to  $\tilde{Y}_n^2$  by setting as the positive martingale

$$\widehat{M}_n := \frac{\tilde{Y}_n^2}{\prod_{k=1}^n \left(1 - \frac{1}{k + \text{Tr}(Y_0)} \tilde{h}(\tilde{Y}_{k-1}^2)\right)}.$$

□

## 4 Applications

### 4.1 Function with regular variation for the drawing rule

Let define the law of the drawings as follows

$$\forall 1 \leq i \leq d, \quad \mathbb{P}(X_{n+1} = e^i | \mathcal{F}_n) = \frac{f(Y_n^i)}{\sum_{j=1}^d f(Y_n^j)}, \quad n \geq 0, \quad (4.20)$$

where  $f$  has regular variation with index  $\alpha > 0$ , namely  $\forall t > 0$ ,  $\frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} t^\alpha$  and  $f$  is bounded on each interval  $(0, M]$ . Then, by applying Theorem 1.5.2 p.22 in [9],  $\frac{f(tx)}{f(x)} \xrightarrow{x \rightarrow \infty} t^\alpha$  uniformly in  $t$  on each  $(0, b]$ ,  $0 < b < \infty$ .

We can reformulate the dynamics (1.1)-(1.2) into a recursive stochastic algorithm like in the Section 2.1, and we obtain the following recursive procedure satisfied by the sequence  $(\tilde{Y}_n)_{n \geq 0}$ , namely

$$\tilde{Y}_{n+1} = \tilde{Y}_n - \frac{1}{n + \text{Tr}(Y_0) + 1} \left( \tilde{Y}_n - H \frac{\tilde{Y}_n^\alpha}{\text{Tr}(\tilde{Y}_n^\alpha)} \right) + \frac{1}{n + \text{Tr}(Y_0) + 1} (\Delta M_{n+1} + \hat{r}_{n+1}) \quad (4.21)$$

where  $\tilde{Y}_n^\alpha = \left( (\tilde{Y}_n^i)^\alpha \right)_{1 \leq i \leq d}$  with step  $\gamma_n = \frac{1}{n + \text{Tr}(Y_0)}$  and a remainder term given by

$$\hat{r}_{n+1} := H_{n+1} \frac{\tilde{f}(Y_n)}{\text{Tr}(\tilde{f}(Y_n))} - H \frac{\tilde{Y}_n^\alpha}{\text{Tr}(\tilde{Y}_n^\alpha)} \in \mathcal{F}_n. \quad (4.22)$$

Notice that, in the convex case, the remainder term was  $r_{n+1} = (H_{n+1} - H) \frac{\tilde{f}(\tilde{Y}_n)}{\text{Tr}(\tilde{f}(\tilde{Y}_n))}$ , therefore assumption **(A3)** implied directly that  $r_n \xrightarrow[n \rightarrow \infty]{a.s.} 0$ . Here we have to use the uniform convergence of the regular variation to prove the required assumption on  $\hat{r}_{n+1}$ .

By the same arguments like in Section 2.1,  $\text{Tr}(Y_n)$  satisfies (2.12). Moreover, for the quantity  $\tilde{N}_n := \frac{1}{n} \sum_{k=1}^n X_k$ , we also devise a stochastic recursive procedure in the same way as before, namely

$$\tilde{N}_{n+1} = \tilde{N}_n - \frac{1}{n+1} \left( \tilde{N}_n - \frac{\tilde{Y}_n^\alpha}{\text{Tr}(\tilde{Y}_n^\alpha)} \right) + \frac{1}{n+1} (\Delta \tilde{M}_{n+1} + \tilde{r}_{n+1}),$$

where  $\tilde{r}_{n+1} = \frac{\tilde{f}(Y_n)}{\text{Tr}(\tilde{f}(Y_n))} - \frac{\tilde{Y}_n^\alpha}{\text{Tr}(\tilde{Y}_n^\alpha)}$ , thus  $\tilde{r}_{n+1} \in \mathcal{F}_n$ .

**Theorem 4.1.** Assume that **(A1)**, **(A2)** and **(A3)** hold.

1. If  $0 < \alpha \leq 1$ , then  $h$  has a unique zero  $y^* \in (p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$  and

$$\frac{\text{Tr}(Y_n)}{n + \text{Tr}(Y_0)} \xrightarrow[n \rightarrow \infty]{a.s.} 1, \quad \frac{Y_n}{\text{Tr}(Y_n)} \xrightarrow[n \rightarrow \infty]{a.s.} y^* \quad \text{and} \quad \tilde{N}_n \xrightarrow[n \rightarrow \infty]{a.s.} \frac{(y^*)^\alpha}{\text{Tr}((y^*)^\alpha)}.$$

2. If  $\alpha > 1$ , then  $h$  has a unique zero  $y^* \in (p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$  or  $ODE_h$  has two attractive equilibrium points in  $(p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$  (as we have established in Section 2.1). Thus, the stochastic recursive procedure a.s. converges to one of the possible limit values.

**Proof.** By the same arguments like in Section 2.1,  $\text{Tr}(Y_n)$  satisfies (2.12), therefore Proposition 2.1 holds. Consequently,  $\tilde{Y}_n$  lies in a compact of  $\mathbb{R}_+$ , thus

$$\max_{1 \leq i \leq d} \left| \frac{f(Y_n^i)}{f(n + \text{Tr}(Y_0))} - \left( \frac{Y_n}{n + \text{Tr}(Y_0)} \right)^\alpha \right| \xrightarrow[n \rightarrow \infty]{} 0.$$

Set  $a_n^i = \frac{f(Y_n^i)}{f(n + \text{Tr}(Y_0))}$  and  $b_n^i = (\tilde{Y}_n^i)^\alpha$ ,  $i \in \{1, \dots, d\}$ . Then, for every  $i \in \{1, \dots, d\}$ ,

$$\frac{a_n^i}{\text{Tr}(a_n)} - \frac{b_n^i}{\text{Tr}(b_n)} = \frac{a_n^i - b_n^i}{\text{Tr}(b_n)} + \frac{a_n^i}{\text{Tr}(a_n)} \left( 1 - \frac{\text{Tr}(a_n)}{\text{Tr}(b_n)} \right).$$

But

$$\text{Tr}(b_n) = \sum_{i=1}^d (\tilde{Y}_n^i)^\alpha \geq \begin{cases} \left( \sum_{i=1}^d \tilde{Y}_n^i \right)^\alpha = \text{Tr}(\tilde{Y}_n)^\alpha & \text{if } \alpha \in [0, 1] \\ d^{1-\alpha} \text{Tr}(\tilde{Y}_n)^\alpha & \text{if } \alpha > 1, \end{cases}$$

therefore

$$\text{Tr}(b_n) \geq \frac{\text{Tr}(\tilde{Y}_n)^\alpha}{d^{(\alpha-1)_+}} \underset{a.s.}{\sim} \frac{(n + \text{Tr}(\tilde{Y}_0))^\alpha}{d^{(\alpha-1)_+}}.$$

Consequently, for every  $i \in \{1, \dots, d\}$ ,

$$\frac{a_n^i}{\text{Tr}(a_n)} - \frac{b_n^i}{\text{Tr}(b_n)} \leq \frac{\max_{1 \leq i \leq d} |a_n^i - b_n^i| + \sum_{j=1}^d |a_n^j - b_n^j|}{\text{Tr}(b_n)}$$

i.e.

$$\max_{1 \leq i \leq d} \left| \frac{a_n^i}{\text{Tr}(a_n)} - \frac{b_n^i}{\text{Tr}(b_n)} \right| \leq \frac{d+1}{\text{Tr}(b_n)} \max_{1 \leq i \leq d} |a_n^i - b_n^i| \xrightarrow[n \rightarrow \infty]{a.s.} 0.$$

Thus

$$|\hat{r}_{n+1}| \leq \|H\| \max_{1 \leq i \leq d} \left| \frac{a_n^i}{\text{Tr}(a_n)} - \frac{b_n^i}{\text{Tr}(b_n)} \right| + \|H_{n+1} - H\| \xrightarrow[n \rightarrow \infty]{a.s.} 0$$

and in the same way  $\tilde{r}_{n+1} \xrightarrow[n \rightarrow \infty]{a.s.} 0$ . Consequently item 1. follows from Proposition 2.2-1. and Theorem 2.4.

2. We have to check the assumption on the remainder term to apply result on traps for SA. We have that

$$\max_{1 \leq i \leq d} \left| \frac{a_n^i}{\text{Tr}(a_n)} - \frac{b_n^i}{\text{Tr}(b_n)} \right| \lesssim \frac{(d+1)d^{(\alpha-1)_+}}{(n + \text{Tr}(Y_0))^\alpha} \max_{1 \leq i \leq d} |a_n^i - b_n^i| = o(n^{-\alpha}). \quad (4.23)$$

So, for  $\alpha > 1$ , under assumption **(A3)** on the generating matrices,

$$\sum_{n \geq 0} \|r_{n+1}\|^2 < +\infty.$$

The end of the proof follows from Proposition 2.2-2.-3. and Theorem 2.4.  $\square$

To establish a *CLT* for the sequence  $(\tilde{Y}_n)_{n \geq 0}$  we need that the remainder term  $(r_n)_{n \geq 1}$  satisfies (A.29). Then we will assume that the addition rule matrices  $(D_n)_{n \geq 1}$  satisfy **(A1)**-(ii) to ensure that  $(\tilde{Y}_n)_{n \geq 0}$  lies in the simplex (which implies that the rate in (4.23) is no more *a.s.*) and we assume also that  $\alpha > 1/2$ .

**Theorem 4.2.** *Assume that the index of regular variation  $\alpha > 1/2$ , that the addition rule matrices  $(D_n)_{n \geq 1}$  satisfy **(A1)**-(ii) and **(A1)**, **(A3)** and **(A4)** hold.*

1. If  $p_1 < 1 - p_2$ , then

$$\begin{aligned} \sqrt{n} \left( \tilde{Y}_n - y^* \right) &\xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Sigma) \quad \text{with} \quad \Sigma = \int_0^{+\infty} e^{u(Dh(y^*) - \frac{I}{2})} \Gamma e^{u(Dh(y^*) - \frac{I}{2})^t} du \\ \text{and} \quad \Gamma &= \frac{(y^{*1})^\alpha C^1 + (1 - y^{*1})^\alpha C^2}{\text{Tr}((y^*)^\alpha)} - y^*(y^*)^t = a.s.-\lim_{n \rightarrow \infty} \mathbb{E} [\Delta M_n \Delta M_n^t | \mathcal{F}_{n-1}]. \end{aligned} \quad (4.24)$$

2. If  $1 - p_2 < p_1$ , we have three possible rate of convergence depending on the second eigenvalue:

(i) If

$$1 - \lambda := \alpha \frac{(y^{*1})^{\alpha-1} (p_1 - y^{*1}) + (1 - y^{*1})^{\alpha-1} (y^{*1} - (1 - p_2))}{(y^{*1})^\alpha + (1 - y^{*1})^\alpha} < \frac{1}{2},$$

then

$$\sqrt{n} \left( \tilde{Y}_n - y^* \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N} \left( 0, \frac{1}{2\lambda - 1} \Sigma \right).$$

(ii) If  $\lambda = \frac{1}{2}$ , then

$$\sqrt{\frac{n}{\log n}} \left( \tilde{Y}_n - y^* \right) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Sigma).$$

(iii) If  $\lambda < \frac{1}{2}$ , then  $n^\lambda \left( \tilde{Y}_n - y^* \right)$  *a.s.* converges as  $n \rightarrow +\infty$  towards a positive finite random variable.

This result follows from Theorem A.3 and Theorem 2.5.

## 4.2 Application to Finance

Such urn based recursive procedures can be applied to adaptive portfolio allocation by an asset manager or a trader or to optimal split across liquidity pools. Indeed the first setting has already been done in [23] and successfully implemented with multi-armed bandit procedure. We develop in this section the adaptive portfolio allocation, but the optimal split across liquidity pools can be implemented in the same way, by considering that the different colors represent the different liquidity pools, and the trader want to optimally split a large volume of a single asset among the different possible destinations.



Imagine an asset manager who deals with a portfolio of  $d$  tradable assets. To optimize the yield of her portfolio, she can modify the proportions invested in each asset. She starts with the initial allocation vector  $Y_0$ . At stage  $n$ , she chooses a tradable asset according to the distribution (1.2) or (1.3) of  $X_n$ , then evaluates its performance over one time step and modifies the portfolio composition accordingly (most likely virtually) and proceeds. Thus the normalized urn composition  $\tilde{Y}_n$  represents the allocation vector among the assets and the addition rule matrices  $D_n$  model the successive reallocations depending on the past performances of the different assets. The evaluation of the asset performances can be carried out recursively with an estimator like with multi-arm clinical trials (see [4, 24]). In practice, it can be used to design the addition rule matrices  $D_n$ . For example, we may consider sequences of  $d$  independent  $[0, 1]$ -valued random variables  $(T_n^i)_{n \geq 1}$ ,  $i \in \{1, \dots, d\}$ , independent of the drawing  $X_n$ , such that

$$\mathbb{E}[T_n^i] = p_i, \quad 0 < p^i < 1, \quad i \in \{1, \dots, d\}.$$

If  $(T_n^i)_{n \geq 1}$ ,  $i \in \{1, \dots, d\}$ , is simply a *success indicator*, namely  $d$  independent sequences of i.i.d.  $\{0, 1\}$ -valued Bernoulli trials with respective parameter  $p_i$ , then the convention is to set  $T_n^i = 1$  if the return of the  $i^{\text{th}}$  asset in the  $n^{\text{th}}$  reallocation is positive and  $T_n^i = 0$  otherwise.

Let  $N_n^i := \sum_{k=1}^n X_k^i$  be the number of times the  $i^{\text{th}}$  asset is selected among the first  $n$  stages with  $N_0^i = 1$ ,  $i \in \{1, \dots, d\}$ , and let  $S_n$  be the  $d$  dimensional vector defined by

$$S_n^i = S_{n-1}^i + T_n^i X_n^i, \quad n \geq 1, \quad S_0^i = 1, \quad i \in \{1, \dots, d\},$$

denoting the number of successes of the  $i^{\text{th}}$  asset among these  $N_n^i$  reallocations. Define  $\Pi_n$  an estimator of the vector of success probabilities, namely  $\Pi_n^i = \frac{S_n^i}{N_n^i}$ ,  $i \in \{1, \dots, d\}$ . We can prove that  $\Pi_n \xrightarrow[n \rightarrow \infty]{a.s.} p := (p^1, \dots, p^d)^t$  (see [4, 24]). Then we build the following addition rule matrices

$$D_{n+1} = \begin{pmatrix} T_{n+1}^1 & \frac{\Pi_n^1(1-T_{n+1}^2)}{\sum_{j \neq 2} \Pi_n^j} & \dots & \frac{\Pi_n^1(1-T_{n+1}^d)}{\sum_{j \neq d} \Pi_n^j} \\ \frac{\Pi_n^2(1-T_{n+1}^1)}{\sum_{j \neq 1} \Pi_n^j} & T_{n+1}^2 & \dots & \frac{\Pi_n^2(1-T_{n+1}^d)}{\sum_{j \neq d} \Pi_n^j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Pi_n^d(1-T_{n+1}^1)}{\sum_{j \neq 1} \Pi_n^j} & \frac{\Pi_n^d(1-T_{n+1}^2)}{\sum_{j \neq 2} \Pi_n^j} & \dots & T_{n+1}^d \end{pmatrix}, \quad (4.25)$$

*i.e.* at stage  $n+1$ , if the return of the  $j^{\text{th}}$  asset is positive, then one ball of type  $j$  is added in the urn. Otherwise,  $\frac{\Pi_n^i}{\sum_{k \neq j} \Pi_n^k}$  (virtual) balls of type  $i$ ,  $i \neq j$ , are added. This addition rule matrix clearly satisfies **(A1)**-(i) and **(A2)**. Then, one easily checks that the generating matrices are given by

$$H_{n+1} = \mathbb{E}[D_{n+1} | \mathcal{F}_n] = \begin{pmatrix} p_1 & \frac{\Pi_n^1(1-p_2)}{\sum_{j \neq 2} \Pi_n^j} & \dots & \frac{\Pi_n^1(1-p_d)}{\sum_{j \neq d} \Pi_n^j} \\ \frac{\Pi_n^2(1-p_1)}{\sum_{j \neq 1} \Pi_n^j} & p_2 & \dots & \frac{\Pi_n^2(1-p_d)}{\sum_{j \neq d} \Pi_n^j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Pi_n^d(1-p_1)}{\sum_{j \neq 1} \Pi_n^j} & \frac{\Pi_n^d(1-p_2)}{\sum_{j \neq 2} \Pi_n^j} & \dots & p_d \end{pmatrix}$$

and satisfy **(A1)**-(ii). As soon as  $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$ ,  $H_n \xrightarrow{a.s.} H$  (see [4, 24]) where

$$H = \begin{pmatrix} p^1 & \frac{p^1(1-p^2)}{\sum_{j \neq 2} p^j} & \cdots & \frac{p^1(1-p^d)}{\sum_{j \neq d} p^j} \\ \frac{p^2(1-p^1)}{\sum_{j \neq 1} p^j} & p^2 & \cdots & \frac{p^2(1-p^d)}{\sum_{j \neq d} p^j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p^d(1-p^1)}{\sum_{j \neq 1} p^j} & \frac{p^d(1-p^2)}{\sum_{j \neq 2} p^j} & \cdots & p^d \end{pmatrix}.$$

Therefore, the number of each asset in the portfolio  $Y_n$  follows the dynamics (1.1) and the repartition of the portfolio in each asset follows the dynamics (2.10) or (4.21) depending on the drawing rule.

Here the components of the limit generating matrix  $H$  can be interpreted as constraints on the composition of the portfolio. Indeed, in presence of two assets (or colors), we prove that the first component of the allocation vector  $y^{*1}$  lies in  $(p_1 \vee (1-p_2), p_1 \wedge (1-p_2))$  (see Proposition 2.2), therefore the portfolio will contain at least  $p_1 \vee (1-p_2)\%$  and no more than  $p_1 \wedge (1-p_2)\%$  of the first asset. Such rules may be prescribed by the regulation, the bank policy or the bank customer, and our approach is a natural way to have them satisfied (at least asymptotically).

The idea of reinforcing the drawing rule (instead of considering the uniform drawing) like in (1.2) or (1.3) can be interpreted as a way to take into account the risk aversion of the trader or the customer. Indeed, if  $f$  is concave the equilibrium point will be in the middle of the simplex (see Theorem 2.2 and Theorem 2.3), so the trader prefers to have diversification in her portfolio. On the contrary, if  $f$  is convex, the equilibrium points will lie on the boundary of the set of constraints induced by the limit generating matrix  $H$ , so she prefers to take advantage of the most money-making asset (like in a “winner take all” or a “0-1” strategy).

**Numerical experiments.** We present some numerical experiment for the drawing rule defined by (1.2), firstly with a concave function  $f : y \mapsto \sqrt{y}$  and secondly with a convex function  $f : y \mapsto y^4$ . Therefore we have a unique equilibrium point in the first setting and two attractive targets in the second framework. We consider an asset manager who deal with a portfolio of 2 tradable assets. We model the addition rule matrices like in the multi-arm clinical trials, namely  $D_n$  is defined by (4.25). We use the same success probabilities, namely  $p_1 = 0.7$  and  $p_2 = 0.75$ , and the initial urn composition is chosen randomly in the simplex  $\mathcal{V}$ .

▷ *Convergence of the portfolio allocation with concave drawing rule* We have that  $y^{*1} \in (0.25, 0.7)$  and

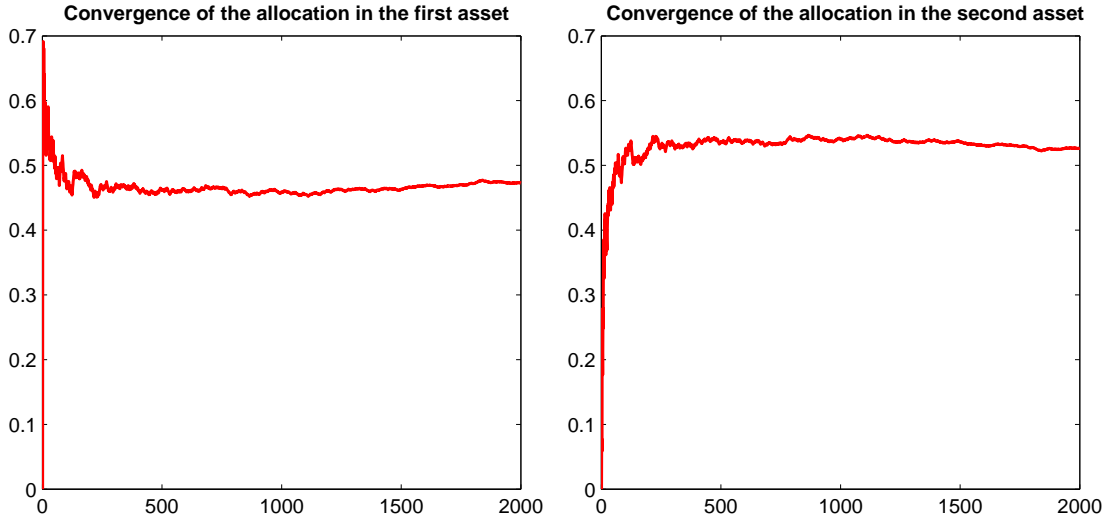


Figure 3: Convergence of  $\tilde{Y}_n$  toward  $y^*$  for  $f(y) = \sqrt{y}$  with  $p_1 = 0.7$  and  $p_2 = 0.75$ .

$y^{*1}$  and  $y^{*2}$  are close to  $\frac{1}{2}$ , so the portfolio is diversified because in this case the investor is risk adverse.

▷ *Convergence of the portfolio allocation with convex drawing rule*

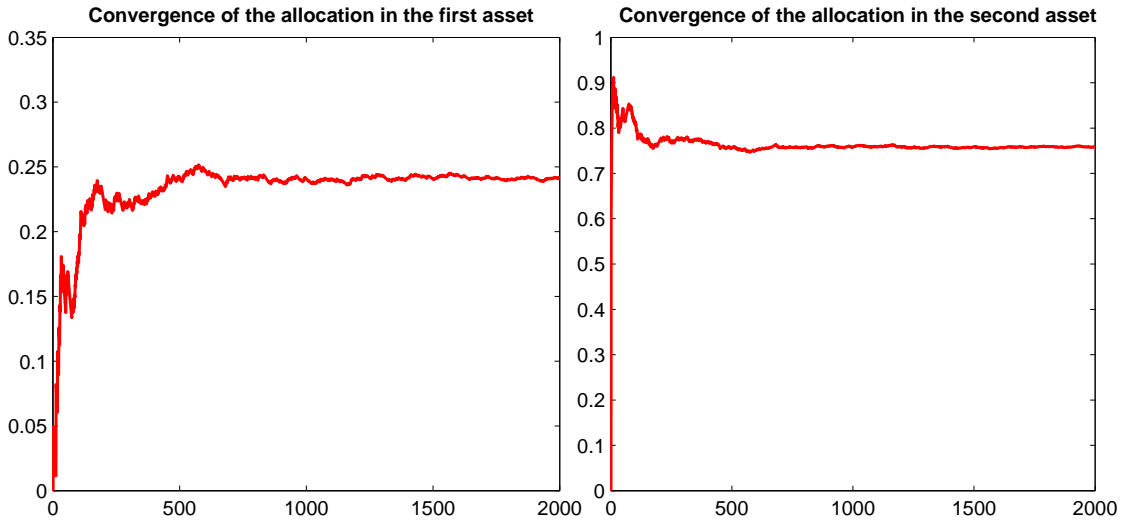


Figure 4: Convergence of  $\tilde{Y}_n$  toward  $y^*$  for  $f(y) = y^4$  with  $p_1 = 0.7$  and  $p_2 = 0.75$ .

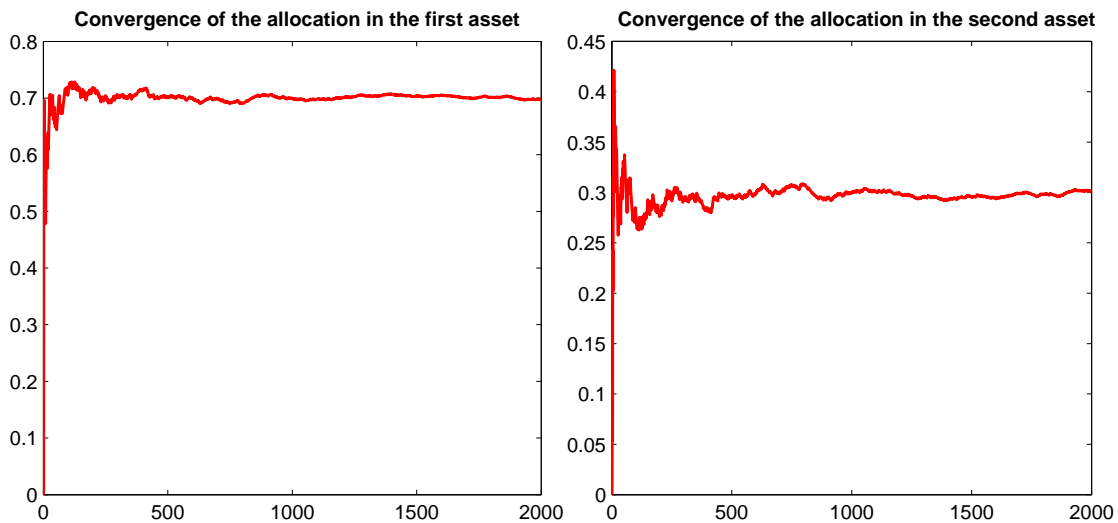


Figure 5: Convergence of  $\tilde{Y}_n$  toward  $y^*$  for  $f(y) = y^4$  with  $p_1 = 0.7$  and  $p_2 = 0.75$ .

In the convex framework, we have two possible strategies and they are close to the boundaries defined by regulation. Moreover the repartition of the portfolio between the two assets is more asymmetric, because the trader chooses to invest two times more in one asset than in the other.

**Acknowledgement.** We thank Frédéric Abergel (MAS Laboratory, ECP) for suggesting us to investigate the nonlinear case in randomized urn models and for helpful discussions in view of application to financial frameworks.

# Appendix

## A Basic tools of Stochastic Approximation

Consider the following recursive procedure defined on a filtered probability space  $(\Omega, \mathcal{A}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$

$$\forall n \geq n_0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) + \gamma_{n+1} (\Delta M_{n+1} + r_{n+1}), \quad (\text{A.26})$$

where  $h : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a locally Lipschitz continuous function,  $\theta_{n_0}$  an  $\mathcal{F}_{n_0}$ -measurable finite random vector and, for every  $n \geq n_0$ ,  $\Delta M_{n+1}$  is an  $\mathcal{F}_n$ -martingale increment and  $r_n$  is an  $\mathcal{F}_n$ -adapted remainder term.

**Theorem A.1.** (*A.s. convergence with ODE method, see e.g. [8, 15, 21, 16, 5]*). Assume that  $h$  is locally Lipschitz, that

$$r_n \xrightarrow[n \rightarrow \infty]{a.s.} 0 \quad \text{and} \quad \sup_{n \geq n_0} \mathbb{E} \left[ \|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n \right] < +\infty \quad a.s.,$$

and that  $(\gamma_n)_{n \geq 1}$  is a positive sequence satisfying

$$\sum_{n \geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n \geq 1} \gamma_n^2 < +\infty.$$

Then the set  $\Theta^\infty$  of its limiting values as  $n \rightarrow +\infty$  is a.s. a compact connected set, stable by the flow of

$$ODE_h \equiv \dot{\theta} = -h(\theta).$$

Let  $\Theta^*$  be the set of the uniformly stable equilibriums on  $\Theta^\infty$  of  $ODE_h$ . Then, the algorithm a.s. converges to one of the limiting values in  $\Theta^*$ , namely

$$\theta_n \xrightarrow[n \rightarrow \infty]{a.s.} \theta^* \in \Theta^*.$$

COMMENTS. By uniformly stable we mean that

$$\sup_{\theta \in \Theta^\infty} |\theta(\theta_0, t) - \theta^*| \longrightarrow 0 \quad \text{as} \quad t \rightarrow +\infty,$$

where  $\theta(\theta_0, t)_{\theta_0 \in \Theta^\infty, t \in \mathbb{R}_+}$  is the flow of  $ODE_h$  on  $\Theta^\infty$ .

**Theorem A.2.** (*Non-a.s. convergence toward a trap, see e.g. [11, 14]*). Assume that  $z^* \in \mathbb{R}^d$  is a trap for the stochastic algorithm (A.26), i.e.

- (i)  $h(z^*) = 0$ ,
- (ii) there exists a neighborhood  $V(z^*)$  of  $z^*$  in which  $h$  is differentiable with a Lipschitz differential,
- (iii) the eigenvalue of  $Dh(z^*)$  with the lowest real part, denoted by  $\lambda_{\min}$ , satisfies  $\Re(\lambda_{\min}) < 0$ .

Assume furthermore that a.s. on  $\Gamma(z^*) = \{\theta_n \xrightarrow{a.s.} z^*\}$ ,

$$\sum_{n \geq 1} \|r_n\|^2 < +\infty \quad \text{and} \quad \limsup_n \mathbb{E} \left[ \|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n \right] < +\infty.$$

Let  $K_+$  the subset of  $\mathbb{R}^d$  spanned by the eigenvectors whose associated eigenvalues have a non-negative real part and  $K_-$  the subset of  $\mathbb{R}^d$  spanned by the eigenvectors whose associated eigenvalues have a negative real part (then  $\mathbb{R}^d = K_+ \oplus K_-$ ). By setting  $\Delta M_{n+1}^{(r)}$  the projection of  $\Delta M_{n+1}$  on  $K_-$  alongside  $K_+$ , assume that a.s. on  $\Gamma(z^*)$

$$\liminf_n \mathbb{E} \left[ \|\Delta M_{n+1}^{(r)}\| \mid \mathcal{F}_n \right] > 0. \quad (\text{A.27})$$

Moreover, if the positive sequence  $(\gamma_n)_{n \geq 1}$  satisfies

$$\sum_{n \geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n \geq 1} \gamma_n^2 < +\infty,$$

then the event  $\Gamma(z^*)$  is negligible.

**Theorem A.3.** (Rate of convergence see [15] Theorem 3.III.14 p.131 (for CLT see also e.g. [8, 21])). Let  $\theta^*$  be an equilibrium point of  $\{h = 0\}$ . Assume that the function  $h$  is differentiable at  $\theta^*$  and all the eigenvalues of  $Dh(\theta^*)$  have positive real parts. Assume that for some  $\delta > 0$ ,

$$\sup_{n \geq n_0} \mathbb{E} \left[ \|\Delta M_{n+1}\|^{2+\delta} \mid \mathcal{F}_n \right] < +\infty \text{ a.s.}, \quad \mathbb{E} \left[ \Delta M_{n+1} \Delta M_{n+1}^t \mid \mathcal{F}_n \right] \xrightarrow[n \rightarrow \infty]{a.s.} \Gamma, \quad (\text{A.28})$$

where  $\Gamma$  is a deterministic symmetric definite positive matrix and for an  $\epsilon > 0$ ,

$$\mathbb{E} \left[ (n+1) \|r_{n+1}\|^2 \mathbf{1}_{\{\|\theta_n - \theta^*\| \leq \epsilon\}} \right] \xrightarrow[n \rightarrow \infty]{} 0. \quad (\text{A.29})$$

Specify the gain parameter sequence as follows

$$\forall n \geq 1, \quad \gamma_n = \frac{1}{n}. \quad (\text{A.30})$$

(a) If  $\Lambda := \Re(\lambda_{\min}) > \frac{1}{2}$ , where  $\lambda_{\min}$  denotes the eigenvalue of  $Dh(\theta^*)$  with the lowest real part, then, the above a.s. convergence is ruled on the convergence set  $\{\theta_n \rightarrow \theta^*\}$  by the following Central Limit Theorem

$$\sqrt{n}(\theta_n - \theta^*) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N} \left( 0, \frac{1}{2\Lambda - 1} \Sigma \right) \quad \text{with} \quad \Sigma := \int_0^{+\infty} \left( e^{-(Dh(\theta^*) - \frac{I_d}{2})u} \right)^t \Gamma e^{-(Dh(\theta^*) - \frac{I_d}{2})u} du.$$

(b) If  $\Lambda = \frac{1}{2}$ , then

$$\sqrt{\frac{n}{\log n}} (\theta_n - \theta^*) \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Sigma).$$

(c) If  $\Lambda < \frac{1}{2}$ , then  $n^\Lambda (\theta_n - \theta^*)$  a.s. converges as  $n \rightarrow +\infty$  towards a positive finite random variable.

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