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► **To cite this version:**

V. Chernousov, Philippe Gille, Arturo Pianzola. Generalized Onsager algebras and Grothendieck's dessins d'enfants. 2013. hal-00910221v1

HAL Id: hal-00910221

<https://hal.science/hal-00910221v1>

Preprint submitted on 27 Nov 2013 (v1), last revised 13 Oct 2014 (v2)

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GENERALIZED ONSAGER ALGEBRAS AND GROTHENDIECK'S DESSINS D'ENFANTS

V. CHERNOUSOV, P. GILLE, AND A. PIANZOLA

ABSTRACT. We define and classify the analogues of the affine Kac-Moody Lie algebras for the ring corresponding to the complex projective line minus three points. The classification is given in terms of Grothendieck's dessins d'enfants. We also study the question of conjugacy of Cartan subalgebras for these algebras.

Keywords: Reductive group scheme, dessins d'enfants, torsor, multiloop algebra, affine and three pointed affine Lie algebras, Cartan subalgebra.

MSC 2000 17B67, 11E72, 14L30, 14E20.

1. INTRODUCTION

If X is a manifold and G a Lie group, the set of morphisms of manifolds

$$(1.1) \quad \text{Mor}(X, G)$$

has a natural group structure (with pointwise multiplication). In the case when X is the real circle S^1 , this type of infinite dimensional Lie groups have played an illuminating role in modern theoretical Physics. Natural mathematical and physical considerations have lead to the study, within the above picture, of manifolds other than S^1 . The resulting theory is an insightful combination of geometry and analysis. (See for example [EF] and [PS] for details and further references). From an algebraic point of view – one which we will hereon take – the “groups” as in (1.1) are quite natural. To every algebraic group \mathbf{G} over a field k corresponds a k -group scheme (also denoted \mathbf{G}). Thus for every k -scheme \mathbf{X} the set

$$\text{Mor}_{k\text{-sch}}(\mathbf{X}, \mathbf{G}) := \mathbf{G}(\mathbf{X})$$

has a natural group structure. As is customary, much about the representations of $\mathbf{G}(\mathbf{X})$ (which are of interest to Physics) can be learned by looking at the corresponding Lie algebras, or more precisely the \mathbf{X} -points of the Lie algebra \mathfrak{g} of \mathbf{G} .

V. Chernousov was partially supported by the Canada Research Chairs Program and an NSERC research grant.

P. Gille was supported by the IDEI Romanian project PN-II-ID-PCE-2012-4.

A. Pianzola wishes to thank NSERC and CONICET for their continuous support.

There are two very important cases (related to the circle S^1 and algebraic variations thereof) of $\mathbf{X} = \text{Spec}(R)$ that have been extensively studied:

$$(i) R = k[t^{\pm 1}] \text{ and } (ii) R = k[t^{\pm 1}]_{(t-1)}.$$

The first case appears in the study of untwisted affine Kac-Moody Lie algebras. As for case (ii), it is related to the Onsager algebra used in the resolution of the Ising model [HT]. (See also [B] and [BT]).

It is interesting to observe that there are (twisted) affine Kac-Moody Lie algebras which are not of the form $\mathfrak{g} \otimes k[t^{\pm 1}]$. By contrast, no such algebras have been considered for $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, namely for $R = k[t^{\pm 1}]_{(t-1)}$ as above. One of the objectives of this paper is to give a definition and classification of the analogues of the (twisted) affine Kac-Moody algebras in this case. To some extent, this can be viewed as a generalized definition of Onsager algebras, one that fits within a framework analogous to that of the affine Kac-Moody Lie algebras. The resulting classification, which in the affine case is given in term of extended Coxeter-Dynkin diagrams, will in our new situation be given by Grothendieck's designs d'enfants. We also address the problem of conjugacy of Cartan subalgebras of these new algebras, which is of importance for their (future) representation theory.

By looking at the analogue situation in the affine Kac-Moody case, we expect our work to be relevant to possible generalizations of the Wess-Zumino-Witten models and the theory of open strings (where the twisted algebras enter into the picture). Other connections to Physics as well as relevant references can be found in [Sch].

2. NOTATION AND CONVENTIONS

Throughout k will denote an algebraically closed field of characteristic 0, and \mathfrak{g} a finite dimensional simple Lie algebra over k .

We fix a (necessarily split) Cartan subalgebra \mathfrak{h} of \mathfrak{g} , and a base Δ of the root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$. The simply connected (resp. adjoint) Chevalley-Demazure group over k corresponding to \mathfrak{g} will be denoted by \mathbf{G}^{sc} (resp. \mathbf{G}^{ad}). We then have the exact sequence of algebraic k -groups

$$1 \rightarrow \boldsymbol{\mu} \rightarrow \mathbf{G}^{\text{sc}} \xrightarrow{\text{Ad}} \mathbf{G}^{\text{ad}} \rightarrow 1$$

where $\boldsymbol{\mu} = Z(\mathbf{G}^{\text{sc}})$ is the centre of \mathbf{G}^{sc} . The split torus of \mathbf{G}^{sc} (resp. \mathbf{G}^{ad}) corresponding to \mathfrak{h} will be denoted by \mathbf{T}^{sc} (resp. \mathbf{T}^{ad}).

If $\mathbf{Aut}(\mathfrak{g})$ denotes the (linear algebraic) k -group of automorphisms of \mathfrak{g} , then $\mathbf{Aut}(\mathfrak{g})^0 \simeq \mathbf{G}^{\text{ad}}$ via the adjoint representation. We have a split exact sequence

$$(2.1) \quad 1 \rightarrow \mathbf{G}^{\text{ad}} \rightarrow \mathbf{Aut}(\mathfrak{g}) \rightarrow \mathbf{Out}(\mathfrak{g}) \rightarrow 1$$

of algebraic k -groups. Here $\mathbf{Out}(\mathfrak{g})$ is the finite constant k -group corresponding to the finite (abstract) group of symmetries of the Dynkin diagram of \mathfrak{g} .

In what follows we will denote the abstract groups $\mathbf{Aut}(\mathfrak{g})(k)$ and $\mathbf{Out}(\mathfrak{g})(k)$ by $\text{Aut}_k(\mathfrak{g})$ and $\text{Out}_k(\mathfrak{g})$ respectively.

If \mathbf{X} is a k -scheme and \mathfrak{G} a group scheme over \mathbf{X} , we will denote throughout $H_{fppf}^1(\mathbf{X}, \mathfrak{G})$ by $H^1(\mathbf{X}, \mathfrak{G})$. Note that if \mathfrak{G} is smooth then we can replace the fppf-topology on \mathbf{X} by the étale topology.

3. APPLICATIONS OF A RESULT BY HARDER

Proposition 3.1. *Let C be a smooth connected affine curve over k and let \mathfrak{G}/C be a semisimple group scheme. Then $H^1(C, \mathfrak{G}) = 1$.*

Proof. Since $H^1(k(C), \mathfrak{G}) = 1$ by Steinberg's theorem [St, Theorem 11.2], all \mathfrak{G} -torsors over C are rationally trivial.

If \mathfrak{G} is simply connected, we have

$$\ker\left(H^1(C, \mathfrak{G}) \rightarrow H^1(k(C), \mathfrak{G})\right) = 1$$

by [H, Satz 3.3], hence $H^1(C, \mathfrak{G}) = 1$. For the general case, we consider the simply connected covering $f : \tilde{\mathfrak{G}} \rightarrow \mathfrak{G}$. Its kernel μ is a finite étale C -group scheme and is central in $\tilde{\mathfrak{G}}$. We consider the following exact sequence of pointed sets [Gi, IV.4.2.10]

$$H^1(C, \tilde{\mathfrak{G}}) \rightarrow H^1(C, \mathfrak{G}) \rightarrow H^2(C, \mu).$$

But $H^1(C, \tilde{\mathfrak{G}}) = 1$ by the first case, while $H^2(C, \mu) = 0$ by [M, Theorem 15.1] given that $\text{cd}(C) \leq \dim(C) = 1$. It follows that $H^1(C, \mathfrak{G}) = 1$. \square

Corollary 3.2. *Let $C = \text{Spec}(R)$ be a smooth connected affine curve over k . Let \mathbf{G} be a semisimple split k -group scheme. Then the map $p : \mathbf{Aut}(\mathbf{G}) \rightarrow \mathbf{Out}(\mathbf{G})$ induces a bijection of pointed sets*

$$p_* : H^1(C, \mathbf{Aut}(\mathbf{G})) \xrightarrow{\sim} H^1(C, \mathbf{Out}(\mathbf{G})).$$

Furthermore all C -forms of $\mathbf{G}_C = \mathbf{G} \times_k \text{Spec}(C)$ are quasi-split. Similar considerations apply if we replace \mathbf{G} by \mathfrak{g} .

Proof. We have the split exact sequence of k (or C , by base change)-groups

$$(3.1) \quad 1 \longrightarrow \mathbf{G}^{\text{ad}} \xrightarrow{\text{int}} \mathbf{Aut}(\mathbf{G}) \xrightarrow{p} \mathbf{Out}(\mathbf{G}) \longrightarrow 1.$$

We consider a pinning (épinglage) of \mathbf{G} , this includes the choice of a Killing couple (\mathbf{B}, \mathbf{T}) of \mathbf{G} and defines a section

$$h : \mathbf{Out}(\mathbf{G}) \rightarrow \mathbf{Aut}(\mathbf{G}, \mathbf{B}, \mathbf{T}) \subset \mathbf{Aut}(\mathbf{G})$$

(see [SGA3, XXIV.1]).

Let \mathbf{F}/C be a representative of a class of $H^1(C, \mathbf{Out}(\mathbf{G}))$. Consider $\mathbf{G}_{h_*\mathbf{F}}^{\text{ad}}$, namely the semisimple C -group obtained by twisting \mathbf{G}^{ad} by the $\mathbf{Aut}(\mathbf{G})$ -torsor $h_*\mathbf{F}$. Note that $\mathbf{G}_{h_*\mathbf{F}}^{\text{ad}}$ is quasi-split since $h_*\mathbf{F}$ stabilizes \mathbf{B} , so that the second assertion follows from the first one (or, alternatively, see [H, Satz 3.1]).

It is well known (see [DG] II.4.5.1) that the classes in $H^1(C, \mathbf{Aut}(\mathbf{G}))$ which map to the class of \mathbf{F} under the map

$$p_* : H^1(C, \mathbf{Aut}(\mathbf{G})) \rightarrow H^1(C, \mathbf{Out}(\mathbf{G}))$$

are in one-to-one correspondence with elements in the image of

$$H^1(C, \mathbf{G}_{h_*\mathbf{F}}^{ad}) \rightarrow H^1(C, \mathbf{Aut}(\mathbf{G}_{h_*\mathbf{F}}^{ad})).$$

But $H^1(C, \mathbf{G}_{h_*\mathbf{F}}^{ad})$ vanishes by the Proposition. This shows that p_* is injective. It is surjective because (3.1) is split.

Finally, if \mathbf{G} is of simply connected type then $\mathbf{Aut}(\mathbf{G}) \simeq \mathbf{Aut}(\mathfrak{g})$ by [SGA3, Exp XXIV §7.3]. \square

4. AFFINE KAC-MOODY ANALOGUES OF THE 3-POINTED ALGEBRAS

Kac-Moody algebras are defined by generators and relations (à la Chevalley-Serre). In the affine case, which is the most interesting outside the classical finite dimensional setup, the algebras can be “realized” in terms of loop algebras (a construction that we will describe shortly). Let $R = k[t^{\pm 1}]$ and $R' = k[t^{\pm 1}]_{(t-1)}$.

The simplest case is that of the untwisted affine algebras. These correspond to the k -algebras $\mathfrak{g} \otimes_k R$. The elements of $\mathfrak{g} \otimes_k R$ can be thought as morphisms (in the algebraic sense) of $\mathbb{A}_k^1 \setminus \{0\} \rightarrow \mathbb{A}_k^{\dim_k(\mathfrak{g})}$ which has a natural appeal to many current constructions in Physics. It seems natural to replace $\mathbb{A}_k^1 \setminus \{0\} = \mathbb{P}_k^1 \setminus \{0, \infty\}$ by $\mathbb{P}_k^1 \setminus \{0, 1, \infty\}$ (or indeed replace even \mathbb{P}_k^1 by an arbitrary smooth connected projective curve). This is the case of the “untwisted” k -algebras $\mathfrak{g} \otimes_k R'$. This raises an inevitable question: What are the analogues, in the case of R' , of the affine Kac-Moody Lie algebras which are not of the form $\mathfrak{g} \otimes_k R$?

Our point of view is that the affine Kac-Moody algebras (derived modulo their centres) can be thought as twisted forms of some $\mathfrak{g} \otimes_k R$ in the étale topology of $\mathrm{Spec}(R)$ as we will explain in Theorem 4.2 below.

We now review the realization of the affine algebras as loop algebras.

Fix once and for all a set $(\zeta_n)_{n>0}$ of compatible primitive n -roots of unity (i.e. $\zeta_{nh}^h = \zeta_n$). Having done this, the ingredient for defining loop algebras based on \mathfrak{g} is a finite order automorphism σ of \mathfrak{g} . For each $i \in \mathbb{Z}$ consider the eigenspace

$$\mathfrak{g}_i := \{x \in \mathfrak{g} : \sigma(x) = \zeta_m^i x\}$$

where m is a (fixed) period of the σ i.e. $\sigma^m = 1$. Let

$$R_m = k[t^{\pm \frac{1}{m}}], \text{ and } R_\infty = \varinjlim_m R_m.$$

The loop algebra $L(\mathfrak{g}, \sigma)$ is then defined as follows

$$L(\mathfrak{g}, \sigma) = \bigoplus \mathfrak{g}_i \otimes t^{\frac{i}{m}} \subset \mathfrak{g} \otimes_k R_m \subset \mathfrak{g} \otimes_k R_\infty.$$

Remark 4.1. $L(\mathfrak{g}, \sigma)$ is a Lie subalgebra of $\mathfrak{g} \otimes_k R_\infty$ which does not depend on the choice of period m of σ . Besides being an (infinite dimensional) Lie algebra over k , the multiloop algebra $L(\mathfrak{g}, \sigma)$ is also naturally a Lie algebra over $R = k[t^{\pm 1}]$.

These are the loop algebras appearing in affine Kac-Moody theory. To connect then to non-abelian cohomology the crucial observation, which is easy to verify, is that

$$(4.1) \quad L(\mathfrak{g}, \sigma) \otimes_R R_m \simeq \mathfrak{g} \otimes_k R_m \simeq (\mathfrak{g} \otimes_k R) \otimes_R R_m$$

where the isomorphism is of R_m -Lie algebras. Thus loop algebras are a particular kind of twisted forms of the R -Lie algebra $\mathfrak{g} \otimes_k R$. Their isomorphism classes correspond to a subset $H_{loop}^1(R, \mathbf{Aut}(\mathfrak{g}))$ of $H^1(R, \mathbf{Aut}(\mathfrak{g}))$ where this last H^1 classifies *all* twisted forms of $\mathfrak{g} \otimes_k R$. It is known that in this case $H_{loop}^1 = H^1$, namely that all twisted forms are loop algebras (see below). This fails, however, for forms of algebras over Laurent polynomial rings in more than one variable (see [GP]).

Let \mathcal{L} be a Lie algebra over k . Recall that the centroid $C(\mathcal{L})$ of \mathcal{L} is defined by

$$C(\mathcal{L}) = \{\theta \in \text{End}_k(\mathcal{L}) : \theta[x, y] = [x, \theta(y)] = [\theta(x), y] \text{ for all } x, y \in \mathcal{L}\}.$$

If \mathcal{L} is perfect, then $C(\mathcal{L})$ is commutative and we can view \mathcal{L} naturally as a Lie algebra over the (commutative and unital) ring $C(\mathcal{L})$.

Theorem 4.2. *For a Lie algebra \mathcal{L} over k the following conditions are equivalent.*

- (i): \mathcal{L} is the derived algebra modulo its centre of an affine Kac-Moody Lie algebra.
- (ii): $\mathcal{L} \simeq L(\mathfrak{g}, \sigma)$ for some \mathfrak{g} (unique up to isomorphism) and some $\sigma \in \text{Aut}_k(\mathfrak{g})$ of finite order.
- (ii)(bis): As in (ii) but with $\sigma \in \text{Out}_k(\mathfrak{g}) \subset \text{Aut}_k(\mathfrak{g})$.
- (iii): $C(\mathcal{L}) \simeq R$, and there exists an étale extension S of $C(\mathcal{L})$ such that

$$\mathcal{L} \otimes_{C(\mathcal{L})} S \simeq \mathfrak{g} \otimes_k S.$$

- (iii)(bis): As in (iii) but with S Galois.

Proof. Except for uniqueness, the equivalence of (i), (ii) and (ii)(bis) is the realization theorem of the affine algebras as described in Ch. 6 and 7 of [Kac]. That (ii) implies (iii)(bis) is easy in view of (4.1) and the fact that R_m/R is Galois. The rest of the equivalences and uniqueness hinges on the vanishing of $H^1(R, \mathfrak{G})$ for any reductive group scheme \mathfrak{G} over R as explained in [P]. \square

We now explain how cohomology enters into the picture, and how it looks if we replace $R = k[t^{\pm 1}] = k[t]_t$ by $R' = k[t^{\pm 1}]_{t-1} = k[t]_{t(t-1)}$. In other words, the Lie algebra $\mathfrak{g} \otimes_k R'$ plays the role of the untwisted affine algebra. The “rest” of the so-called 3-pointed algebras (or generalized Onsager algebras, as we are calling them) are those whose centroids coincide with R' , and which look, locally for the étale topology on $\text{Spec}(R')$, like $\mathfrak{g} \otimes_k R'$. Because of Theorem 4.2, this definition is in perfect analogy with the situation that one encounters in the case of affine Kac-Moody Lie algebras.

One knows that, as algebras over R' , generalized Onsager Lie algebras are then classified by the pointed set $H^1(R', \mathbf{Aut}(\mathfrak{g}))$. By considering the split exact sequence of R' -groups corresponding to (2.1) and passing to cohomology we obtain

$$(4.2) \quad H^1(R', \mathbf{G}^{\text{ad}}) \rightarrow H^1(R', \mathbf{Aut}(\mathfrak{g})) \rightarrow H^1(R', \mathbf{Out}(\mathfrak{g})) \rightarrow 1.$$

By Corollary 3.2 we in fact have a bijection of pointed sets

$$(4.3) \quad H^1(R', \mathbf{Aut}(\mathfrak{g})) \rightarrow H^1(R', \mathbf{Out}(\mathfrak{g})).$$

Moreover by [SGA1]

$$(4.4) \quad H^1(R', \mathbf{Out}(\mathfrak{g})) \simeq \text{Hom}_{\text{cont}}(\pi_1(R'), \text{Out}_k(\mathfrak{g}))/\text{conjugation}$$

where $\pi_1(R')$ is the algebraic fundamental group at the geometric point

$$\bar{x} = \text{Spec } \overline{k(t)} \rightarrow \text{Spec } (R')$$

and $\text{Out}(\mathfrak{g})$ acts on $\text{Hom}_{\text{cont}}(\pi_1(R'), \text{Out}(\mathfrak{g}))$ by conjugation.

Remark 4.3. By [Sz] 4.6.12 or the Comparison Theorem $\pi_1(R') \simeq \widehat{\mathbb{Z} * \mathbb{Z}}$. If we fix generators e_1 and e_2 for the two copies of \mathbb{Z} , then $H^1(R', \text{Out}_k(\mathfrak{g}))$ can be thought as conjugacy classes of pairs of elements of $\text{Out}_k(\mathfrak{g})$.

Remark 4.4. It is well known that the abstract group $\text{Out}_k(\mathfrak{g})$ is the symmetric group S_n with $n = 1, 2, 3$. Furthermore, if the pair of elements of S_n assigned to e_1 and e_2 generate a subgroup in S_n acting transitively on $\{1, \dots, n\}$ then the conjugacy class of this pair corresponds to a dessin d'enfant of degree n and conversely (see below for details and references). As we shall see the classification of generalized Onsager algebras, as Lie algebras over R' , is entirely given in terms of dessins.

5. DESSINS D'ENFANTS

In this section we briefly recall the definition of dessins d'enfants and their properties. For more details and applications we refer to the surveys [LZ], [Sch], [W].

A dessin d'enfant is a bipartite connected graph Γ which is embedded into an orientable closed (connected) topological surface X such that it fills the surface, i.e. $X \setminus \Gamma$ is a union of open cells. Two dessins d'enfants (X_1, Γ_1) and (X_2, Γ_2) are called equivalent if there exists a homeomorphism $f : X_1 \rightarrow X_2$ such that $f(\Gamma_1) = \Gamma_2$.

Such graphs appear in a nice way to describe all possible coverings $\beta : X \rightarrow \mathbb{P}^1(\mathbb{C})$ from a closed (connected) Riemann surface X to the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ which are ramified at most over the points $0, 1$ and ∞ . Such pair (X, β) is called a Belyi morphism. The remarkable fact, known as Belyi's theorem, is that a projective smooth connected curve X over \mathbb{C} is defined over $\overline{\mathbb{Q}}$ if and only if there exists a finite map $X \rightarrow \mathbb{P}^1$ which is unramified outside $0, 1$ and ∞ .

Given a Belyi pair (X, β) we associate a bipartite graph Γ on X as follows. We may identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$. Let $I = [0, 1]$ be the closed segment on the real line \mathbb{R} . Then its preimage $\beta^{-1}(I)$ is a connected graph on X . Its vertices are the

preimages of 0 and 1. We may colour all preimages of 0 with one colour (say white) and all preimages of 1 with another colour (say black). One checks that $\beta^{-1}(I)$ fills X and hence it is a dessin d'enfants.

It is a striking fact that the above correspondence is one-to-one.

Theorem 5.1. *A Belyi pair (X, β) of degree d is uniquely determined up to equivalence by*

- *a dessin d'enfants up to equivalence;*
- *a monodromy map $\alpha : F_2 \rightarrow S_d$, i.e. a transitive action of the free group F_2 on two generators on a d -elements set $\{1, 2, \dots, d\}$, up to conjugation;*

Proof. See [HS]. □

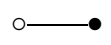
6. CLASSIFICATION OF GENERALIZED ONSAGER ALGEBRAS OVER R'

Theorem 6.1. *The classification of affine 3-pointed Lie algebras over $R' = k[t]_{t(t-1)}$ is as follows.*

- (i): *If \mathfrak{g} is of type $A_1, B_\ell (\ell \geq 2), C_\ell (\ell \geq 3), G_2, F_4, E_7$ or E_8 then all 3-pointed affine algebras are trivial i.e. isomorphic to $\mathfrak{g} \otimes_k R'$.*
- (ii): *If \mathfrak{g} is of type $A_\ell (\ell > 1), D_\ell (\ell > 4)$ or E_6 there are four isomorphism classes of 3-pointed affine algebras. The trivial algebra, and three “quadratic” algebras.*
- (iii): *If \mathfrak{g} is of type D_4 there are eleven 3-pointed affine algebras. The trivial algebra, three “quadratic” algebras, and seven “trialitarian” algebras of which four are “cyclic cubic” and three are “non-cyclic cubic”-algebras.*

Proof. In view of Corollary 3.2 and Remarks 4.3 and 4.4, the classification is given by computing conjugacy classes of pairs of elements of the symmetric groups S_n where $n = 1, 2, 3$ in cases (i), (ii) and (iii) respectively and the corresponding dessins.

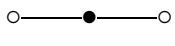
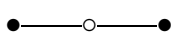

(i) It is obvious that there is only one conjugacy class whose dessin is just the interval I , corresponding to the trivial cover of the Riemann sphere.

S_1 pair representative	Dessin	n_0	n_1	n_∞	g
(1, 1)		1	1	1	0

Here (and everywhere below) n_0, n_1 and n_∞ are the number of vertices over 0, 1 and ∞ respectively under the corresponding cover $\beta : X \rightarrow \mathbb{P}^1$ and g is the genus of X ¹.





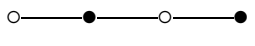

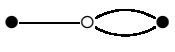
(ii) Let $r = (12)$ be the generator of S_2 . There are four conjugacy classes of pairs of elements of S_2 . The trivial class (1, 1) and three transitive classes with representatives and dessins given by the following table.

¹Note that if $\beta : X \rightarrow \mathbb{P}^1$ has degree n then the genus g of X is equal to $(n - n_0 - n_1 - n_\infty + 2)/2$.

S_2 pair representative	Dessin	n_0	n_1	n_∞	g
$(1, r)$		2	1	1	0
$(r, 1)$		1	2	1	0
(r, r)		1	1	2	0

Each class in fact consists of a single element and the corresponding covers are given by the étale extension S/R' where $S = R'(\sqrt{t-1})$, $R'(\sqrt{t})$ and $R'(\sqrt{t(t-1)})$ respectively.

(iii) Let $r = (12)$, $s = (23)$ and $c = sr = (123)$. There are eleven conjugacy classes of pairs of elements of S_3 . The trivial class $(1, 1)$, three “quadratic” classes $(1, r)$, $(r, 1)$, (r, r) (whose corresponding algebras are obtained as in (ii) above), and seven transitive classes represented by

S_3 pair representative	Dessin	n_0	n_1	n_∞	g	Trialitarian Type
$(1, c)$		3	1	1	0	cyclic
$(c, 1)$		1	3	1	0	cyclic
(c, c^2)		1	1	3	0	cyclic
(c, c)		1	1	1	1	cyclic
(r, s)		2	2	1	0	non-cyclic
(r, c)		2	1	2	0	non-cyclic
(c, r)		1	2	2	0	non-cyclic

For explicit descriptions of the corresponding covers (X, β) we refer to [Z]. □

7. CLASSIFICATION OF GENERALIZED ONSAGER ALGEBRAS OVER k

Just as in the case of Kac-Moody Lie algebras, in infinite dimensional Lie Theory one is interested in viewing generalized Onsager algebras as algebras over k and

not over R' . Clearly, if two Onsager algebras are isomorphic over R' then they are isomorphic as k -algebras as well. However the converse is not true. Before proceeding to their classification over k we first state and prove a useful criterion for studying the R vs k -isomorphism question for arbitrary twisted forms of $\mathfrak{g} \otimes_k R$.

Consider an *arbitrary* (commutative, associative and unital) k -algebra R and the set of all R -isomorphism classes of forms of the R -Lie algebra $\mathfrak{g} \otimes_k R$. These are classified by the pointed set $H^1(R, \mathbf{Aut}(\mathfrak{g}))$. We have a natural action of the abstract group $\Gamma = \text{Aut}_k(R)$ on the set of isomorphism classes of $\mathbf{Aut}(\mathfrak{g})$ -torsors. Namely, every element $\gamma \in \Gamma$ gives rise to the scheme automorphism $\gamma_* : \text{Spec}(R) \rightarrow \text{Spec}(R)$. If $\mathcal{T} \rightarrow \text{Spec}(R)$ is an $\mathbf{Aut}(\mathfrak{g})$ -torsor the base change γ_* produces a scheme ${}^\gamma\mathcal{T} = \mathcal{T} \times_{\gamma_*} \text{Spec}(R)$ which is obviously an $\mathbf{Aut}(\mathfrak{g})$ -torsor. Thus γ yields a natural bijection

$$\gamma_* : H^1(R, \mathbf{Aut}(\mathfrak{g})) \rightarrow H^1(R, \mathbf{Aut}(\mathfrak{g})).$$

Similarly, Γ acts on the set of isomorphism classes of twisted R -forms of the simply connected group \mathbf{G}^{sc} attached to \mathfrak{g} and their Lie algebras.

On the level of Lie algebras over R the action of $\gamma \in \Gamma$ can be seen as follows. Let \mathcal{L} be an R -form of \mathfrak{g}_R . Then we define ${}^\gamma\mathcal{L}$ to be \mathcal{L} (as a set) with the same Lie bracket structure, but the R -module structure is given by composing the automorphism $\gamma : R \rightarrow R$ with the standard action of R on \mathcal{L} . Thus for all $x, y \in \mathcal{L} = {}^\gamma\mathcal{L}$ we have

$$[x, y]_{{}^\gamma\mathcal{L}} = [x, y]_{\mathcal{L}}$$

and for all $r \in R$ we have

$$r \cdot {}^\gamma\mathcal{L} x = \gamma(r) \cdot_{\mathcal{L}} x.$$

In particular, it follows that the identity mapping $f_\gamma : {}^\gamma\mathcal{L} \rightarrow \mathcal{L}$ is an isomorphism of k -Lie algebras.

Proposition 7.1. *Let \mathcal{L} and \mathcal{L}' be R -forms of \mathfrak{g}_R . Then \mathcal{L} and \mathcal{L}' are isomorphic as k -Lie algebras if and only if there exists $\gamma \in \Gamma = \text{Aut}_k(R)$ such that $\mathcal{L}' \simeq {}^\gamma\mathcal{L}$ over R .*

Proof. This follows essentially from the fact that the centroids $C(\mathcal{L})$ and $C(\mathcal{L}')$ of \mathcal{L} and \mathcal{L}' are naturally isomorphic to R , as we now explain. For $r \in R$ consider the homothety $\chi_r \in \text{End}_k(\mathcal{L})$ given by $\chi_r : x \mapsto rx$. Clearly $\chi_r \in C(\mathcal{L})$. This gives a natural morphism of associative associative and commutative k -algebras $R \rightarrow C(\mathcal{L})$ which is known to be an isomorphism (see lemma 3.4 of [P2] and lemma 4.6 of [GP]).

We now turn to the proof of the Proposition. Assume that $f : \mathcal{L} \rightarrow \mathcal{L}'$ is a k -algebra isomorphism. It is immediate to verify that the map $C(f) : C(\mathcal{L}') \rightarrow \text{End}_k(\mathcal{L})$ defined by $\chi \mapsto f^{-1} \circ \chi \circ f$ is in fact a k -algebra isomorphism of $C(\mathcal{L}')$ onto $C(\mathcal{L})$. Under our identification $C(\mathcal{L}') \simeq R \simeq C(\mathcal{L})$ we thus have $C(f) := \gamma \in \text{Aut}_k(R)$.

The reader can easily verify from the definition that for the isomorphism $f_\gamma : {}^\gamma\mathcal{L} \rightarrow \mathcal{L}$ above one has $C(f_\gamma) = \gamma^{-1}$. Thus, if we consider the composition

$$f' : {}^\gamma\mathcal{L} \xrightarrow{f_\gamma} \mathcal{L} \xrightarrow{f} \mathcal{L}'$$

then $C(f') = 1$. But this means that f' is R -linear.

The converse is clear since we have already observed that $f_\gamma : \gamma\mathcal{L} \rightarrow \mathcal{L}$ is an isomorphism of k -Lie algebras. \square

Remark 7.2. The above Proposition has nothing to do with Lie algebras. We could replace \mathfrak{g} by any finite dimensional central simple algebra over k .

We now come back to the ring R' . For such a ring we have $\text{Aut}_k(R') \simeq S_3$. Indeed, an arbitrary k -automorphism $\gamma : R' \rightarrow R'$ induces an automorphism $\text{Spec}(R') \rightarrow \text{Spec}(R')$ and hence an automorphism of the function field $k(t)$. On the other hand, by [Hart, Chapter 1, Exercise 6.6] one has $\text{Aut}_k k(t) \simeq \text{Aut}(\mathbb{P}_k^1)$ and thus we obtain a mapping from $\text{Aut}_k(R')$ into the permutation group S_3 of the points $\{0, 1, \infty\}$ on \mathbb{P}_k^1 . Also, it is known [Hart, Chapter IV, Exercise 2.2] that any permutation of $\{0, 1, \infty\}$ is induced by a unique automorphism of \mathbb{P}_k^1 . This induces an automorphism of $\mathbb{P}_k^1 \setminus \{0, 1, \infty\}$, hence of $\text{Spec}(R')$.

We now pass to the classification of k -isomorphism classes of generalized Onsager Lie algebras.

Theorem 7.3. *The classification of generalized Onsager Lie algebras over k is as follows.*

- (i): *If \mathfrak{g} is of type $A_1, B_\ell \ell \geq 2, C_\ell \ell \geq 3, G_2, F_4, E_7$ or E_8 then all Onsager algebras are trivial i.e. isomorphic to $\mathfrak{g} \otimes_k R'$.*
- (ii): *If \mathfrak{g} is of type $A_\ell \ell > 1, D_\ell \ell > 4$ or E_6 there are two isomorphism classes of Onsager algebras. The trivial algebra, and one quadratic algebra arising from the three quadratic algebras over R' . More precisely, the three non-isomorphic quadratic algebras over R' are isomorphic as k -algebras.*
- (iii): *If \mathfrak{g} is of type D_4 there are five Onsager algebras.*
 - (1) *The trivial algebra.*
 - (2) *One quadratic algebra arising from the three quadratic algebras over R' .*
 - (3) *One cyclic cubic algebra arising from the three cyclic cubic algebras over R' corresponding to dessins of genus 0.*
 - (4) *One cyclic cubic algebra corresponding to the cyclic cubic R' -algebra whose dessin is of genus 1.*
 - (5) *One non-cyclic cubic algebra arising from the three non-cyclic cubic algebras over R' .*

Proof. (i) is obvious because there exists a unique R' -form of $\mathfrak{g} \otimes_k R'$. Indeed, in this case we have $\text{Out}(\mathfrak{g}) = 1$.

(ii) and (iii): The problem is equivalent to classifying the set of R' -forms of $\mathfrak{g} \otimes_k R'$ up to the action of $\text{Aut}_k(R') = S_3$. Let \mathcal{L} be an R' -form of $\mathfrak{g} \otimes_k R'$ and let $\gamma \in S_3$. The form \mathcal{L} corresponds to some class $[\xi] \in H^1(R', \mathbf{Out}(\mathfrak{g}))$ and hence to a homomorphism $F_2 \rightarrow S_d, d = 2, 3$, which we may assume to be transitive. By Theorem 5.1 this leads us to a unique Belyi pair $\beta : X \rightarrow \mathbb{P}^1$.

The form $\gamma\mathcal{L}$ corresponds to the class of the torsor $\gamma\xi$ and hence to another Belyi pair $\gamma\beta : \gamma X \rightarrow \mathbb{P}^1$. This new Belyi pair $(\gamma X, \gamma\beta)$ can be explicitly described as

follows: as we noticed before the permutation γ of the set $\{0, 1, \infty\}$ gives rise to a unique automorphism $\gamma_* : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and then we can take ${}^\gamma X$ to be equal to $X \times_{\gamma_*} \mathbb{P}^1$ and ${}^\gamma \beta$ to be a natural projection ${}^\gamma X = X \times_{\gamma_*} \mathbb{P}^1 \rightarrow \mathbb{P}^1$. As a consequence of this construction we conclude that the numbers n_0, n_1, n_∞ for the pair $({}^\gamma X, {}^\gamma \beta)$ are obtained from those of for the Belyi pair (X, β) by permuting them with the use of γ . Looking at the tables 2 and 3 we obtain that in case (ii) three of the twisted forms of $\mathfrak{g}_{R'}$ presented in table 2 are in the same S_3 -orbit and that in case (iii) there are three S_3 -orbits as claimed in the theorem. \square

Remark 7.4. (i) An étale extension of R' of degree 2 is isomorphic, up to $\text{Aut}_k(R')$ -base change, to either $R' \times R'$ or $R'[\sqrt{t}]$.

(ii) An étale extension of R' of degree 3 is isomorphic, up to $\text{Aut}_k(R')$ -base change, to one (and only one) of the algebras of the following list:

- (a) $R' \times R' \times R'$;
- (b) $R'[\sqrt{t}] \times R'$ (the quadratic case);
- (c) $R'[\sqrt[3]{t}]$ (the cyclic cubic case of genus 0);
- (d) $R'[\sqrt[3]{t(t-1)}]$ (the cyclic cubic case of genus 1);
- (e) $R'[X]/(X^3 + 3X^2 - 4t)$ (the non-cyclic cubic case of genus 0).

8. COHOMOLOGICAL OBSTRUCTION TO CONJUGACY

Let \mathfrak{G} be a reductive group scheme of constant type over a base k -scheme \mathbf{X} and let \mathfrak{S} be a split subtorus of \mathfrak{G} . We define its Weyl group scheme by

$$\mathbf{W}(\mathfrak{S}) = \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})/\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})$$

which is a constant finite group scheme [SGA3, XI.5.9]. The fppf sheaf $\mathbf{Y} = \mathfrak{G}/\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$ is representable (*ibid*, 5.3.bis) and for each \mathbf{X}'/\mathbf{X} elements of the set $\mathbf{Y}(\mathbf{X}')$ are in one-to-one correspondence with \mathbf{X}' -subtori \mathfrak{S}' of $\mathfrak{G}_{\mathbf{X}'}$ which are fppf-locally conjugate to $\mathfrak{S}_{\mathbf{X}'}$. Since $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$ is smooth, locally conjugation with respect to étale topology is equivalent to that of in fppf topology.

The orbits of $\mathfrak{G}(\mathbf{X})$ on $\mathbf{Y}(\mathbf{X})$ can be described by means of the exact sequence

$$\mathbf{Y}(\mathbf{X}) \xrightarrow{\phi} H^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})) \longrightarrow H^1(\mathbf{X}, \mathfrak{G})$$

arising from an exact sequence

$$1 \longrightarrow \mathbf{N}_{\mathfrak{G}}(\mathfrak{S}) \longrightarrow \mathfrak{G} \longrightarrow \mathbf{Y} \longrightarrow 1.$$

More precisely, we have a natural bijection [Gi, III.3.2.4]

$$\mathfrak{G}(\mathbf{X}) \setminus \mathbf{Y}(\mathbf{X}) \xrightarrow{\sim} \ker\left(H^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})) \rightarrow H^1(\mathbf{X}, \mathfrak{G})\right).$$

To summarize,

$$\ker\left(H^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})) \rightarrow H^1(\mathbf{X}, \mathfrak{G})\right)$$

classifies the $\mathfrak{G}(\mathbf{X})$ -conjugacy classes of subtori \mathfrak{G}' of \mathfrak{G} which are locally-étale \mathfrak{G} -conjugate to \mathfrak{G} . For each such \mathfrak{G}' , we denote by $\gamma(\mathfrak{G}') \in H^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{G}))$ its $\mathfrak{G}(\mathbf{X})$ -conjugacy class. In terms of torsors, $\gamma(\mathfrak{G}')$ is the class of the strict transporter $\text{Transpstr}_{\mathfrak{G}}(\mathfrak{G}, \mathfrak{G}')$ which is an $\mathbf{N}_{\mathfrak{G}}(\mathfrak{G})$ -torsor [Gi]. We recall here its definition: for each $\mathbf{X}' \rightarrow \mathbf{X}$

$$\text{Transpstr}_{\mathfrak{G}}(\mathfrak{G}, \mathfrak{G}')(\mathbf{X}') = \left\{ g \in \mathfrak{G}(\mathbf{X}'') \mid g \mathfrak{G}(\mathbf{X}'') g^{-1} = \mathfrak{G}'(\mathbf{X}'') \quad \forall \mathbf{X}'' \rightarrow \mathbf{X}' \right\}.$$

The converse map is given as follows ([Gi, lemme 2.1]). Consider an $\mathbf{N}_{\mathfrak{G}}(\mathfrak{G})$ -torsor \mathbf{E} equipped with a trivialization

$$u : \mathfrak{G} \xrightarrow{\sim} \mathbf{E} \wedge^{\mathbf{N}_{\mathfrak{G}}(\mathfrak{G})} \mathfrak{G}.$$

Then $u^{-1}(\mathbf{E} \wedge^{\mathbf{N}_{\mathfrak{G}}(\mathfrak{G})} \mathfrak{G})$ is a subtorus of \mathfrak{G} .

Furthermore, each such \mathfrak{G}' is Zariski-locally \mathfrak{G} -conjugated to \mathfrak{G} if and only if

$$\gamma(\mathfrak{G}') \in H_{Zar}^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{G})) \subset H^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{G})).$$

It follows that the set

$$\ker\left(H_{Zar}^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{G})) \rightarrow H_{Zar}^1(\mathbf{X}, \mathfrak{G})\right)$$

classifies the $\mathfrak{G}(\mathbf{X})$ -conjugacy classes of tori which are locally conjugate to \mathfrak{G} for the Zariski topology.

Now we consider the following exact sequence of \mathbf{X} -group-schemes

$$1 \rightarrow \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G}) \rightarrow \mathbf{N}_{\mathfrak{G}}(\mathfrak{G}) \rightarrow \mathbf{W}(\mathfrak{G}) \rightarrow \mathbf{1}.$$

As before, there is a natural action of $\mathbf{W}(\mathfrak{G})(\mathbf{X})$ on $H^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G}))$ and it induces a bijection [Gi, III.3.3.1]

$$H^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})) / \mathbf{W}(\mathfrak{G})(\mathbf{X}) \xrightarrow{\sim} \ker\left(H^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{G})) \rightarrow H^1(\mathbf{X}, \mathbf{W}(\mathfrak{G}))\right).$$

We consider then the subset $H_{Zar}^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})) / \mathbf{W}(\mathfrak{G})(\mathbf{X})$ of

$$H_{Zar}^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{G})) \subset H^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{G})).$$

Proposition 8.1. (1) *The pointed set*

$$\ker\left(H^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})) / \mathbf{W}(\mathfrak{G})(\mathbf{X}) \rightarrow H^1(\mathbf{X}, \mathfrak{G})\right)$$

classifies split subtori of \mathfrak{G} which are locally conjugate to \mathfrak{G} for the étale topology.

(2) *We have*

$$\ker\left(H_{Zar}^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})) \rightarrow H_{Zar}^1(\mathbf{X}, \mathfrak{G})\right) \xrightarrow{\sim} \ker\left(H^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})) \rightarrow H^1(\mathbf{X}, \mathfrak{G})\right).$$

(3) *The pointed set*

$$\ker\left(H_{Zar}^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})) / \mathbf{W}(\mathfrak{G})(\mathbf{X}) \rightarrow H_{Zar}^1(\mathbf{X}, \mathfrak{G})\right)$$

classifies $\mathfrak{G}(\mathbf{X})$ -conjugacy classes of split subtori of \mathfrak{G} which are locally conjugate to \mathfrak{S} for the Zariski topology or equivalently locally for the étale topology.

The proof uses the fact that $\mathbf{W}(\mathfrak{S})$ is an open subgroup of the constant \mathbf{X} -group $\mathbf{Aut}(\mathfrak{S})$.

Lemma 8.2. (1) *The map $H^1(\mathbf{X}, \mathbf{W}(\mathfrak{S})) \rightarrow H^1(\mathbf{X}, \mathbf{Aut}(\mathfrak{S}))$ has trivial kernel.*

(2) *The compositum*

$$(\mathfrak{G}/\mathbf{N}_{\mathfrak{G}}(\mathfrak{S}))(\mathbf{X}) \longrightarrow H^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})) \longrightarrow H^1(\mathbf{X}, \mathbf{W}(\mathfrak{S})) \longrightarrow H^1(\mathbf{X}, \mathbf{Aut}(\mathfrak{S}))$$

maps a subtorus \mathfrak{S}' of \mathfrak{G} which is étale-locally conjugate to \mathfrak{S} to the class of the $\mathbf{Aut}(\mathfrak{S})$ -torsor $\text{Isom}_{gr}(\mathfrak{S}, \mathfrak{S}')$.

Proof. (1) Since \mathfrak{S} is split, $\mathbf{Aut}(\mathfrak{S})$ is the constant \mathbf{X} -group associated to $\mathbf{GL}_r(\mathbb{Z})$. Hence $\mathbf{W}(\mathfrak{S})$ is the constant \mathbf{X} -group associated to a finite subgroup Γ of $\mathbf{GL}_r(\mathbb{Z})$. We consider an exact sequence of étale \mathbf{X} -sheaves

$$1 \rightarrow \Gamma_{\mathbf{X}} \rightarrow \mathbf{GL}_r(\mathbb{Z})_{\mathbf{X}} \rightarrow (\mathbf{GL}_r(\mathbb{Z})/\Gamma)_{\mathbf{X}} \rightarrow 1.$$

It gives rise to a long exact sequence of pointed sets

$$1 \rightarrow \Gamma^{\pi_0(\mathbf{X})} \rightarrow \mathbf{GL}_r(\mathbb{Z})^{\pi_0(\mathbf{X})} \rightarrow (\mathbf{GL}_r(\mathbb{Z})/\Gamma)^{\pi_0(\mathbf{X})} \rightarrow H^1(\mathbf{X}, \Gamma_{\mathbf{X}}) \xrightarrow{\lambda} H^1(\mathbf{X}, \mathbf{GL}_r(\mathbb{Z})_{\mathbf{X}}).$$

It follows that λ has trivial kernel, because $\pi_0(\mathbf{X})$ acts trivially on Γ , $\mathbf{GL}_r(\mathbb{Z})$ and $\mathbf{GL}_r(\mathbb{Z})/\Gamma$.

(2) The $\mathfrak{G}(\mathbf{X})$ -conjugacy class of \mathfrak{S}' is nothing but the class of the strict transporter $\text{Transpstr}_{\mathfrak{G}}(\mathfrak{S}, \mathfrak{S}')$ which is an $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S})$ -torsor. The change of groups $\mathbf{N}_{\mathfrak{G}}(\mathfrak{S}) \rightarrow \mathbf{W}(\mathfrak{S}) \rightarrow \mathbf{Aut}(\mathfrak{S})$ applied to that transporter yields indeed the $\mathbf{Aut}(\mathfrak{S})$ -torsor $\text{Isom}_{gr}(\mathfrak{S}, \mathfrak{S}')$. \square

We now proceed with the proof of the Proposition.

Proof. (1) Consider the diagram of exact sequences of pointed sets

$$\begin{array}{ccccc} (\mathfrak{G}/\mathbf{N}_{\mathfrak{G}}(\mathfrak{S}))(\mathbf{X}) & & & & 1 \\ & & \varphi \downarrow & & \downarrow \\ H^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})) & \xrightarrow{i_*} & H^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{S})) & \longrightarrow & H^1(\mathbf{X}, \mathbf{W}(\mathfrak{S})) \\ & & & & \downarrow \\ & & & & H^1(\mathbf{X}, \mathbf{Aut}(\mathfrak{S})). \end{array}$$

Let \mathfrak{S}' be a subtorus of \mathfrak{G} which is étale-locally conjugate to \mathfrak{S} . According to Lemma 8.2.(2), \mathfrak{S}' is split if and only if the image of $\gamma(\mathfrak{S}')$ in $H^1(\mathbf{X}, \mathbf{Aut}(\mathfrak{S}))$ vanishes. Since $H^1(\mathbf{X}, \mathbf{W}(\mathfrak{S})) \rightarrow H^1(\mathbf{X}, \mathbf{Aut}(\mathfrak{S}))$ has trivial kernel by the first part of that Lemma, it follows that \mathfrak{S}' is split if and only if the image of $\gamma(\mathfrak{S}')$ in $H^1(\mathbf{X}, \mathbf{W}(\mathfrak{S}))$ vanishes, that is if and only if $\gamma(\mathfrak{S}')$ admits a reduction to $\mathbf{Z}_{\mathfrak{G}}(\mathfrak{S})$.

(2) The statement is local so we may assume that $\mathbf{X} = \mathrm{Spec}(R)$ with R a local ring. We need to show that the kernel $\ker\left(H^1(R, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})) \rightarrow H^1(R, \mathfrak{G})\right)$ vanishes. But this is [SGA3, XXVI.5.9].

(3) By (2), the pointed set

$$\ker\left(H_{Zar}^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G}))/\mathbf{W}(\mathfrak{G})(\mathbf{X}) \rightarrow H_{Zar}^1(\mathbf{X}, \mathfrak{G})\right)$$

maps bijectively to $\ker\left(H^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G}))/\mathbf{W}(\mathfrak{G})(\mathbf{X}) \rightarrow H^1(\mathbf{X}, \mathfrak{G})\right)$ so it classifies $\mathfrak{G}(\mathbf{X})$ -conjugacy classes of split subtori of \mathfrak{G} which are étale-locally conjugated to \mathfrak{G} by (2). But for such a subtorus \mathfrak{G}' , its conjugacy class $\gamma(\mathfrak{G}')$ belongs then to $H_{Zar}^1(\mathbf{X}, \mathbf{N}_{\mathfrak{G}}(\mathfrak{G}))$ so that \mathfrak{G}' is locally conjugated to \mathfrak{G} for the Zariski topology. \square

The following strengthens Proposition 10.1 of [CGP].

Proposition 8.3. *Assume that \mathbf{X} is a connected scheme and \mathfrak{G} is a split subtorus of \mathfrak{G}/\mathbf{X} such that \mathfrak{G} is generically maximal split in \mathfrak{G} .*

(1) *The torus \mathfrak{G} is a maximal split subtorus of \mathfrak{G} and $\mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})$ is a Levi subgroup of a parabolic subgroup of \mathfrak{G} which is generically minimal.*

(2) *The generically maximal split \mathbf{X} -split subtori of \mathfrak{G} are the subtori locally conjugated to \mathfrak{G} for the Zariski topology. Their $\mathfrak{G}(\mathbf{X})$ -conjugacy class are classified by the pointed set $\ker\left(H_{Zar}^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G}))/\mathbf{W}(\mathfrak{G})(\mathbf{X}) \rightarrow H_{Zar}^1(\mathbf{X}, \mathfrak{G})\right)$.*

Proof. (1) The first assertion is obvious. It is known that $\mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})$ is a Levi subgroup of a parabolic subgroup \mathbf{P} of \mathfrak{G} [SGA3, XXVI.6.2]. Since \mathfrak{G} remains maximal split at the generic point of \mathbf{X} , it follows that $\mathbf{P} \times_{\mathbf{X}} \kappa(\mathbf{X})$ is a minimal parabolic subgroup of $\mathfrak{G}_{\kappa(\mathbf{X})}$ [SGA3, XXVI.6].

(2) If \mathfrak{G}' is a split torus of \mathfrak{G} which is generically maximal split then it is Zariski-locally maximal split. In particular, \mathfrak{G}' is Zariski-locally \mathfrak{G} -conjugate to \mathfrak{G} by Demazure's conjugacy theorem [SGA3, XXVI.6.16]. Conversely, if \mathfrak{G}' is \mathfrak{G} -conjugate to \mathfrak{G} in Zariski topology, it is generically maximal split because so is \mathfrak{G} . The second assertion follows from Proposition 8.1(3). \square

We apply the above general considerations to the particular case of $\mathbf{X} = \mathrm{Spec}(R')$, a semisimple adjoint group scheme \mathfrak{G} over R' and a split torus \mathfrak{G} of \mathfrak{G} which is generically split. By Proposition 3.1 we have $H^1(\mathbf{X}, \mathfrak{G}) = 1$. Thus $H_{Zar}^1(\mathbf{X}, \mathfrak{G}) = 1$ and therefore

$$\ker\left(H_{Zar}^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})) \rightarrow H_{Zar}^1(\mathbf{X}, \mathfrak{G})\right) = H_{Zar}^1(\mathbf{X}, \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})).$$

Since by Corollary 3.2 our groups are always quasisplit, the centralizer $\tilde{\mathfrak{G}} = \mathbf{Z}_{\mathfrak{G}}(\mathfrak{G})$ is always a maximal torus of \mathfrak{G} . Since \mathfrak{G} is of adjoint type this maximal torus is a Weil restriction $R_{S'/R'}(\mathfrak{T})$ of a split S' -torus \mathfrak{T} where S' is a Galois extension of R' that splits \mathfrak{G} .

By Shapiro's lemma $H^1(R', \tilde{\mathfrak{G}})$ vanishes whenever S' is isomorphic to the coordinate ring of the affine line minus a finite number of points. This is the case whenever our dessin is of genus 0, namely in all cases except for case (c, c) in type D_4 . In this exceptional case the situation is quite the opposite. The scheme S' is a non-empty open affine subscheme of an elliptic curve, hence $H^1(S', \mathfrak{T})$ is infinite. Now Shapiro's lemma yields that $H_{Zar}^1(\mathbf{X}, \tilde{\mathfrak{G}})$ is infinite. It follows that there is an infinite number of conjugacy classes of maximal split tori inside the corresponding twisted adjoint R' -group \mathfrak{G} .

9. CONJUGACY QUESTIONS FOR GENERALIZED ONSAGER ALGEBRAS

Let \mathcal{L} be a twisted form of $\mathfrak{g} \otimes_k R'$. The role of split Cartan subalgebras for the infinite dimensional k -Lie algebra \mathcal{L} is played by maximal abelian k -diagonalizable subalgebras. Recall that a subalgebra \mathbf{k} of \mathcal{L} is k -diagonalizable if there exists a k -basis v_λ , $\lambda \in \Lambda$, of \mathcal{L} consisting of eigenvectors for the adjoint action of \mathbf{k} , i.e. for all $\lambda \in \Lambda$ and $x \in \mathbf{k}$, there exists $\lambda(x) \in k$ such that $[x, v_\lambda] = \lambda(x)v_\lambda$. A k -diagonalizable subalgebra of \mathcal{L} is necessarily abelian. The MADs are those k -diagonalizable subalgebras which are maximal with respect to inclusion.

Using the correspondence between maximal split R' -tori of \mathfrak{G} and MADs of $\mathcal{L} = \text{Lie}(\mathfrak{G})$ given in [CGP, Theorem 7.1] and an obvious fact that a maximal split torus in \mathfrak{G} is generically maximal split (because by Corollary 3.2 there are no R' -anisotropic semisimple group schemes we obtain the following.

Theorem 9.1. *If \mathcal{L} is not of dessin type (c, c) there exists a single conjugacy class of MADs under the adjoint action of $\mathfrak{G}(R)$ on \mathcal{L} . If \mathcal{L} is of type (c, c) the number of conjugacy classes of MADs is infinite. \square*

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