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Abstract. ω -semigroups are to languages of infinite strings what semigroups are to languages of finite strings: they allow for obtaining an algebraic characterization of regular languages of infinite words. Defined as two-sorted algebras, ω -semigroups distinguish (images of) finite words from (images of) infinite ones. Recently, it has been shown that there are some benefits in embedding finite words (strings) and infinite words (streams) into finite or infinite birooted words (tiled streams).

In this paper, we show that such an embedding is robust enough to be extended to the algebraic level by embedding the two-sorted ω -semigroups into (some notion of) one-sorted Ehresmann ordered ω -monoids. As a byproduct, we obtain an algebraic characterization of regular languages of finite and infinite birooted words that generalizes and unifies the algebraic characterizations of regular languages of finite and infinite words.

1 Introduction

Infinite words naturally arise in the field of verification, as specifications of behaviors of reactive systems. Derived from Büchi's result on infinite string automata [2], efficient tools have been developed for model-checking. A robust algebraic theory of infinite words [25, 23] (see also [24] for an overview) gives a deep insight of the underlying mathematical framework.

Infinite words also appear in software engineering. Programming languages with lazy evaluation mechanisms such as Haskell [9] or synchronous languages such as Lustre or Esterel [3] allow for defining infinite streams of values by means of lazy evaluation mechanisms or co-inductive constructs. In that field, it is common practice (see e.g. [8]) to mix finite word types (strings) and infinite word types (streams), covering a part of the ω -semigroup structure [24]. This is especially clear in view of the distinction between finite string concatenation and mixed product of (finite) strings with (infinite) streams.

When modeling time-sensitive applications [15], it has recently been advocated that there are benefits in embedding finite strings as well as infinite streams into (some notion of) tiled streams, here called *birooted words*. Such an embedding yields a decomposition of every operation on strings and streams into a

succession of *synchronizations*: how the objects are positioned in time with respect to each other and *fusions*: how they are combined and/or merged with each other. In the most general case, strings and streams are embedded into possibly infinite tiled streams.

In this paper, we aim at proving that the underlying ideas and methods used in [15] are robust enough to be lifted to the abstract setting of semigroup theory. More precisely, the embedding proposed in [15], formally stated and proved here (Theorem 3 and Theorem 4), is defined over concrete strings and streams, i.e. over elements of the free ω -semigroup [23]. In this paper, we generalize this embedding by lifting it to *arbitrary* ω -semigroups (Theorem 12). This leads to an original definition of ordered ω -monoid (Definition 6). Then, by applying adequate algebraic tools [11, 10, 14], we obtain an algebraic characterization of regular languages of finite and infinite birooted words (Theorem 19).

Since regular languages of finite and infinite tiled streams can be used as programming language subtypes, this result constitutes a new step in developing the idea that, in the long term, inverse semigroups [19] and, beyond, tiling semigroups [16, 17] and semigroups with local units [6, 7] – from which most of the proposed constructions derive – may be tuned towards robust engineering practice.

This result also shows that concepts and tools arising from the ongoing development of a language theory and algebraic tools for inverse semigroups can be used as a unifying theme throughout the classical language theory. Indeed, it has already been shown [12] that the theories of finite word languages and of finite tree languages can be unified into the theory of birooted tree languages. Based on our former study of languages of one-dimensional overlapping tiles [13, 11, 14], the present paper unifies the language theories of finite words and of infinite words into a similar framework.

For the sake of simplicity, we restrict our presentation to one sided birooted words with an input root at the beginning of the underlying domain. Clearly, all proposed concepts and constructions can be extended to two-sided birooted words. Thus our result also applies to bi-infinite words with the resulting ω -monoids equipped with both left and right infinite products.

2 From finite or infinite words to birooted words

In this section we define the monoid of positive finite or infinite birooted words. From a mathematical point of view, that monoid is a submonoid of the filter completion of the monoid of McAlister [20]. The language theory of the McAlister monoid has been developed in [13, 11, 14, 4].

2.1 Finite and infinite words

We review here the definition of finite and infinite words in order to fix the notation. Strictly speaking, we define here *right* finite and infinite words. Extending our proposal to bi-infinite birooted words would require to define *left* finite and

infinite words as well, which, at least for infinite words, increases the underlying algebraic signature.

Let A be a finite alphabet. A (right) word on the alphabet A is a partial function $u : \omega \rightarrow A$ with $\text{dom}(u)$ of the form $[0, l[= \{z < \omega : z < l\}$ for some $l \leq \omega$. The length $|u|$ of the word u is defined as $|u| = l$. The word u is finite when $|u| < \omega$ and infinite when $|u| = \omega$. The unique word of length 0 (empty domain) is denoted by 1 and called the *empty word*. The set of finite (resp. finite and non-empty) words is denoted by A^* (resp. A^+). The set of infinite words is denoted by A^ω . The set $A^* + A^\omega$ of finite and infinite words is denoted by A^∞ .

The *finite product* of two finite words $u \in A^*$ and $v \in A^*$ is defined as $u \cdot v$ with $\text{dom}(u \cdot v) = [0, |u| + |v|[$ and $u \cdot v(k) = u(k)$ when $0 \leq k < |u|$ and $u \cdot v(k) = v(k - |u|)$ when $|u| \leq k < |u| + |v|$. The finite product is *associative* with neutral element 1. The set A^* (resp. A^+) equipped with the finite product is the free monoid (resp. semigroup) generated by A .

The *mixed product* of a finite word $u \in A^*$ and an infinite word $v \in A^\omega$ is defined similarly as the word $u * v$ with $\text{dom}(u * v) = [0, \omega[$ and $u * v(k) = u(k)$ when $0 \leq k < |u|$ and $u * v(k) = v(k - |u|)$ otherwise. The finite and mixed products satisfy the *mixed associativity* law: $(u \cdot v) * w = u * (v * w)$ for all $u, v \in A^*$ and each $w \in A^\omega$.

Remark. Associativity in A^* and mixed associativity allow for simplifying notations as follows. For every $u \in A^*$ and every $v \in A^\infty$, we may denote by $uv \in A^\infty$ either the product $u \cdot v \in A^*$ when $v \in A^*$ or the mixed product $u * v \in A^\omega$ when $v \in A^\omega$.

The *infinite product* π of an infinite sequence $(u_i)_{i \geq 0}$ of non-empty finite words is the infinite word defined by $(\pi(u_i)_{i \geq 0})(k) = u_{i_k} \left(k - \sum_{0 \leq i < i_k} |u_i| \right)$ for every integer $k \geq 0$, where i_k is the unique positive integer such that $\sum_{0 \leq i < i_k} |u_i| \leq k < \sum_{0 \leq i \leq i_k} |u_i|$. Finite and infinite product are compatible in the following sense: the *infinite associativity law* holds, i.e. for every infinite sequence of non-empty words $(u_i)_{i \geq 0} \in (A^+)^\omega$, for every strictly monotonic sequence $(i_j)_{j \geq 0} \in \omega^\omega$ with $i_0 = 0$, given the infinite sequence of words $(v_j)_{j \geq 0}$ defined by $v_j = \prod_{i_j \leq i < i_{j+1}} u_i$ for every $j \geq 0$, then $\pi(u_i)_{i \geq 0} = \pi(v_j)_{j \geq 0}$.

The pair (A^+, A^ω) equipped with the finite product, the mixed product and the infinite product is known [23] as the free ω -semigroup generated by the alphabet A .

The set A^∞ is also ordered by the *prefix order* \leq_p , i.e. for every u and $v \in A^\infty$, we have $u \leq_p v$ when either $u = v$, or u is finite and there is $w \in A^\infty$ such that either v is finite and $u \cdot w = v$ or v is infinite and $u * w = v$. Extended with a maximum element denoted by 0, the set $A^\infty + 0$ ordered by the prefix order \leq_p is a complete lattice. The *prefix join* $u \vee_p v$ of two words u and $v \in A^\infty$ is then the least word $w \in A^\infty$, if it exists, such that we have both $u \leq_p w$ and $v \leq_p w$, or 0 otherwise.

Finally, for all $u \in A^*$ and $v \in A^\infty$, the *right residual* $u^{-1}(v)$ of v by u , is defined as the word $w \in A^\infty$, if it exists, such that $v = u \cdot w$ when $v \in A^*$ or

$v = u * w$ when $v \in A^\infty$. We take $u^{-1}(v) = 0$ otherwise. By definition, $u^{-1}(v) \neq 0$ if and only if $u \leq_p v$.

2.2 Birooted words

A *positive birooted (right) word* u is a pair $u = (u_1, u_2)$ where $u_1 \in A^*$ and $u_2 \in A^\infty$. The word $u_1 u_2 \in A^\infty$ is called the domain of the birooted word u and the word u_1 the root path of the birooted word. The birooted word u is finite when u_2 is finite. The set of *finite or infinite positive birooted words* on the alphabet A is denoted by $T^\infty(A)$. The set of *finite positive birooted words* on the alphabet A is denoted by $T^+(A)$.

Remark. Compared to our previous work [13, 11] on finite birooted words (called overlapping tiles), in a tile (u_1, u_2) , we only have empty domain on the left of the root path: the birooted words defined here are *one-sided*. A two-sided positive birooted word would be a triple (u_1, u_2, u_3) with a finite or infinite left word $u_1 \in {}^\infty A$, a finite root path $u_2 \in A^*$, and a finite or infinite right word $u_3 \in A^\infty$. All definitions and results presented here are easily extended to the two-sided case.

The product $u \cdot v$ of two birooted words $u = (u_1, u_2)$ and $v = (v_1, v_2)$ is defined as the birooted word $w = (w_1, w_2)$ with $w_1 = u_1 v_1$ and $w_2 = v_1^{-1}(u_2) \vee_p v_2$ when $v_1^{-1}(u_2) \vee_p v_2 \neq 0$. Otherwise, we take $u \cdot v = 0$ for some new birooted word 0 , with $0 \cdot u = u \cdot 0 = 0$ for every $u \in T^\infty(A) + 0$.

The set $T^\infty(A)$ extended with a zero and equipped with the above product is a monoid with unit $1 = (1, 1)$. It is denoted by $T_0^\infty(A)$. The subset $T_0^+(A) = T^+(A) + 0$ of finite birooted words is a submonoid of $T_0^\infty(A)$. It is also known as the (one-sided version of the) submonoid of positive tiles of the inverse monoid $T_0(A)$ of McAlister (see [20] and [13]). One can also check that it is isomorphic to the polycyclic monoid of Nivat and Perrot [22].

The set of idempotents of $T_0^\infty(A)$ is denoted by $U(T_0^\infty(A))$. It is a commutative submonoid of $T^\infty(A)$. Indeed, one can easily check that an idempotent is either 0 or a birooted word of the form $u = (1, u_2)$.

The monoid $T_0^\infty(A)$ is equipped with the *left projection* $u \mapsto u^L$ and the *right projection* $u \mapsto u^R$ defined by $0^L = 0 = 0^R$ and, for every $u = (u_1, u_2)$, by $u^L = (1, u_2)$ and $u^R = (1, u_1 * u_2)$. One can check that:

Lemma 1. *The mappings $u \mapsto u^L$ and $u \mapsto u^R$ are indeed projections from $T_0^\infty(A)$ onto $U(T_0^\infty(A))$, i.e. for every $u \in T_0^\infty(A)$, we have $u^L \in U(T_0^\infty(A))$ and $u^R \in U(T_0^\infty(A))$ and, for every $e \in U(T_0^\infty(A))$, we have $e^L = e = e^R$. Moreover, for every $u \in T_0^\infty$, we have $u^R \cdot u = u = u \cdot u^L$.*

The *natural order relation* is defined by $u \leq v$ when $u = u^R v u^L$ for every u and $v \in T_0^\infty(A)$. One can easily check that:

Lemma 2. *The relation \leq over $T_0^\infty(A)$ is an order relation. It is stable under product, 0 is the least element of $T_0^\infty(A)$ and, for all birooted words $u = (u_1, u_2)$ and $v = (v_1, v_2)$ we have $u \leq v$ if and only if $u_1 = v_1$ and $u_2 \geq_p v_2$.*

We also have $U(T_0^\infty(A)) = \{u \in T_0^\infty(A) : u \leq 1\}$ and, for every $u \in T_0^\infty$, the idempotent u^L (resp. u^R) is the least element $e \leq 1$ such that $u \cdot e = u$ (resp. $e \cdot u = u$), i.e. the left projection of u is the least right local unit of u and the right projection of u is the least left local unit of u .

Remark. The monoid $T_0^\infty(A)$ of (right) birooted words is isomorphic to the *filter completion* of the set $T_0^+(A)$ of (one-sided) finite birooted words ordered by the natural order. Indeed, an (order) filter is a *non-empty* subset¹ X of $T^+(A)$ that is *upward closed*, i.e. for every u and $v \in T^+(A)$, if $u \leq v$ and $u \in X$ then $v \in X$, and *downward directed*, i.e. for every u and $v \in X$ there exists $w \in X$ such that $w \leq u$ and $w \leq v$. Then, one can show that the mapping $\varphi : T^\infty(A) \rightarrow \mathcal{P}(T^+(A))$ that maps every bi-rooted word $u \in T^\infty(A)$ to the set of finite birooted words $\varphi(u) = \{v \in T^+(A) : u \leq v\}$ is a bijection from $T^\infty(A)$ to the set of filters of $T^+(A)$.

2.3 Embedding words into birooted words

We show how finite and infinite words on the alphabet A can be embedded in the monoid $T_0^\infty(A)$ of positive finite or infinite birooted words.

The finite word mapping $\theta_f : A^* \rightarrow T_0^\infty(A)$ is defined by $\theta_f(v) = (v, 1)$ for every finite word $v \in A^*$. The infinite word mapping $\theta_\omega : A^\omega \rightarrow T_0^\infty(A)$ is defined by $\theta_\omega(w) = (1, w)$ for every infinite word $w \in A^\omega$. Then we have:

Theorem 3. *The mapping $\theta_f : A^* \rightarrow T_0^\infty(A)$ is a one-to-one monoid morphism, i.e. $\theta_f(1) = 1$ and $\theta_f(v \cdot v') = \theta_f(v) \cdot \theta_f(v')$ for every v and $v' \in A^*$.*

The mapping $\theta_\omega : A^\omega \rightarrow T_0^\infty(A)$ is one-to-one. Moreover, for every $v \in A^$ and $w \in A^\omega$, we have $\theta_\omega(v * w) = (\theta_f(v) \cdot \theta_\omega(w))^R$.*

In other words, the mappings θ_f and θ_ω define an embedding of finite and infinite words into the monoid of birooted words, preserving the finite product and inducing a mixed product on birooted words, defined by $u * v = (u \cdot v)^R$ for all birooted words u and $v \in T_0^\infty(A)$. Then we have $\theta_\omega(v * w) = \theta_f(v) * \theta_\omega(w)$ for all $v \in A^*$ and $w \in A^\omega$.

Provided $T_0^\infty(A)$ is equipped with the adequate infinite product, these two mappings actually define an ω -semigroup embedding. More precisely, we first observe that $U(T_0^\infty(A))$ ordered by the natural order is complete lattice with product as meet. For every infinite sequence of birooted words $(u_i)_{i \in \omega}$, let then $\pi(u_i)_{i \in \omega}$ be the infinite product defined by

$$\pi(u_i)_{i \in \omega} = \bigwedge_{n \in \omega} (u_0 \cdot u_1 \cdot \dots \cdot u_{n-1})^R$$

Then we have:

¹ Another definition, often encountered in the literature, amounts to taking proper subsets instead.

Theorem 4. *The pair $(T_0^+(A), U(T_0^\infty(A)))$ equipped with the finite product, the mixed product and the infinite product is a well-defined ω -semigroup and the pair of mappings $(\theta_f, \theta_\omega) : (A^+, A^\omega) \rightarrow (T_0^+(A), U(T_0^\infty(A)))$ is an ω -semigroup embedding, i.e. a one-to-one mapping that preserves the finite, mixed and infinite products.*

In the particular case when the infinite sequence $(u_i)_{i \in \omega}$ is *constant*, i.e. there is some $v \in T_0^\infty(A)$ such that $u_i = v$ for every $i \in \omega$, then we write v^ω for $\pi(u_i)_{i \in \omega}$. One can easily check that for every idempotent $u \leq 1$ we have $u^\omega = u$.

Remark. Still writing A for the images by θ_f of the one-letter words of A , then, within the monoid $T_0^\infty(A)$, and extending the product, induced star and ω -product operations to sets of non-zero birooted words, the sets $A^* = \{(v, 1) \in T_0^\infty(A) : v \in A^*\}$ and $A^\omega = \{(1, w) \in T_0^\infty(A) : w \in A^\omega\}$ are in a one-to-one correspondence with, respectively, finite words and infinite words.

3 Embedding ω -semigroups into ordered ω -monoids

We show here that the embedding for the free ω -semigroup (A^+, A^ω) into the monoid of finite and infinite birooted words $T^\infty(A)$ can be generalized to an embedding of any ω -semigroup S into (some notion of) ordered ω -monoid $M(S)$.

3.1 Ordered ω -monoids

Let M be a monoid partially ordered by a relation \leq . We assume that the order relation \leq is stable under product, i.e. if $x \leq y$ then $xz \leq yz$ and $zx \leq zy$ for every x, y and $z \in M$. The set $U(M)$ of *subunits* of the partially ordered monoid M is defined by $U(M) = \{y \in M : y \leq 1\}$.

The following definition first appears in [11]. It was later turned into the present form [14, 12] and then, following [18], refined with the congruence property [10, 5].

Definition 5 (Adequately ordered and Ehresmann ordered monoid).

A (stable) partially ordered monoid M is an *adequately ordered monoid* when:

- (A1) the subunits are idempotent: for every $x \in M$, if $x \leq 1$ then $xx = x$;
- (A2) the left and right projections are well-defined: for every $x \in M$, both $x^L = \min\{y \in U(M) : xy = x\}$ and $x^R = \min\{y \in U(M) : yx = x\}$ exist in $U(M)$,

It is an Ehresmann ordered monoid (or E-ordered monoid) when moreover

- (A3) it satisfies the congruence property: for every $x, y, z \in M$, if $x^L = y^L$ then $(xz)^L = (yz)^L$ and if $x^R = y^R$ then $(zx)^R = (zy)^R$,

Remark. In an adequately ordered monoid, the subunits commute: indeed, for all $x, y \leq 1$ we have $xy \leq y$ and $xy \leq x$ by stability, thus $xy = xyxy \leq yx$ and by symmetry $yx \leq xy$. It follows that $U(M)$, ordered by the monoid order, is a meet semilattice with the product as the meet operator; thus for any subunit x we have $x = x^L = x^R$. In other words, in an adequately ordered monoid, the left and right projection mappings are indeed projections of M onto $U(M)$.

Examples. Every monoid trivially ordered is an adequately ordered monoid. Every inverse monoid ordered by the natural order [19] is also an adequately ordered monoid with $x^R = x \cdot x^{-1}$ and $x^L = x^{-1} \cdot x$ for every x .

The notion of adequately ordered monoid is extended here with an infinite (right) product as follows.

Definition 6 (E-ordered ω -monoid). An *E-ordered ω -monoid* if an E-ordered monoid M equipped with an infinite product operator $\pi : M^\omega \rightarrow U(M)$, satisfying the following properties:

- (I1) *subunit preservation:* for every $(x_i)_{i \in \omega}$ such that for every $i \in \omega$ we have $x_i = x$ for some $x \in U(M)$, then $\pi(x_i)_{i \in \omega} = x$,
- (I2) *monotonicity:* for all infinite sequences $(x_i)_{i \in \omega} \in M^\omega$ and $(y_i)_{i \in \omega}$, if $x_i \leq y_i$ for every $i \in \omega$ then $\pi(x_i)_{i \in \omega} \leq \pi(y_i)_{i \in \omega}$,
- (I3) *mixed associativity:* for every infinite sequence $(x_i)_{i \in \omega} \in M^\omega$, for every $x \in M$, defining $x'_i = x$ when $i = 0$ and $x'_i = x_{i-1}$ when $i > 0$, we have $(x(\pi(x_i)_{i \in \omega}))^R = \pi(x'_i)_{i \in \omega}$
- (I4) *infinite associativity:* for every infinite sequence $(x_i)_{i \in \omega} \in M^\omega$, for every strictly increasing sequence $(k_i)_{i \in \omega}$ of positive integers with $k_0 = 0$, if $y_i = x_{k_i} x_{k_i+1} \cdots x_{k_{i+1}-1}$ is defined for every $i \in \omega$ then $\pi(y_i)_{i \in \omega} = \pi(x_i)_{i \in \omega}$.

3.2 The ω -monoid completion

We define here the completion of any ω -semigroup S into an adequately ordered ω -monoid $M(S)$ and prove that S is embedded, as an ω -monoid, into $M(S)$.

Let $S = (S_f, S_\omega)$ be an ω -semigroup with finite product $\cdot : S_f \times S_f \rightarrow S_f$, mixed product $* : S_f \times S_\omega \rightarrow S_\omega$ and infinite product $\pi : (S_f)^\omega \rightarrow S_\omega$. By definition (see [24]) the finite, mixed and infinite product are related by mixed and infinite associativity laws. Let S_f^1 be the semigroup S_f extended with a unit and let $\mathcal{P}^*(S_\omega)$ be the set of non-empty subsets of S_ω .

Definition 7 (Monoid completion). The *monoid completion* $M(S)$ of the ω -semigroup S is defined as $M(S) = S_f^1 \times \mathcal{P}^*(S_\omega) + 0$, equipped with the product \cdot defined, for all non-zero (x, X) and $(y, Y) \in M(S)$, by $(x, X) \cdot (y, Y) = (xy, y^{-1}(X) \cap Y)$ when $y^{-1}(X) \cap Y \neq \emptyset$ and 0 otherwise, with $y^{-1}(X) = \{z \in S_\omega : y * z \in X\}$; for all $m \in M(S)$ we take $0.m = m.0 = 0$. $M(S)$ is ordered by the relation \leq defined as follows: 0 is the smallest element, and for all non-zero (x, X) and $(y, Y) \in M(S)$, $(x, X) \leq (y, Y)$ iff $x = y$ and $X \subseteq Y$.

Lemma 8 (Soundness). $M(S)$ is a partially ordered monoid with unit $1 = (1, S_\omega)$.

Proof. By the associativity of the mixed product $*$, for all $X \in \mathcal{P}^*(S_\omega)$ and $y, z \in S_f^1$, $z^{-1}(y^{-1}(X)) = (yz)^{-1}(X)$: hence the associativity of the product in $M(S)$. The neutrality of $(1, S_\omega)$ and the stability of the order relation are trivial. \square

Lemma 9 (Adequacy). The set $M(S)$ is an adequately ordered monoid.

Proof. By definition, a subunit of $M(S)$ is either 0 or an element of the form $(1, X)$ for some $X \subseteq S_\omega$. It follows that (A1) holds: subunits are indeed idempotent elements.

Let us prove (A2). We easily check that $0^L = 0^R = 0$. Let $x \in M(S)$ be some non-zero element of $M(S)$, i.e. of the form $x = (s, X)$. We prove that we have $x^R = (1, s * X)$ and $x^L = (1, X)$.

For the right projection, let $z = (1, s * X)$. We have $z \cdot x = (x, X \cap s^{-1}(s * X))$ but $X \subseteq s^{-1}(s * X)$ hence $z \cdot x = x$. Now, if $z' \cdot x = x$ for some $z' = (1, Y)$ then we have $X = X \cap s^{-1}(Y)$ hence $X \subseteq s^{-1}(Y)$ hence $s * X \subseteq Y$ that is, following the order definition, $z \leq z'$.

For the left projection, let $z = (1, X)$. We have $x \cdot z = (s, X \cap 1^{-1}(X)) = (s, X) = z$. If $x \cdot z' = x$ for some $z' = (1, Y)$ we have $x \cdot z' = (s, Y \cap 1^{-1}(X)) = (s, X)$ hence $X \subseteq Y$ that is $z \leq z'$. \square

Lemma 10. The monoid $M(S)$ is an E-ordered monoid.

Proof. We prove that $M(S)$ satisfies (A3). Let $x, y, z \in S$. If any of x, y or z is zero then the claim is satisfied. Assume $x = (x_1, X)$, $y = (y_1, Y)$ and $z = (z_1, Z)$.

For the left projection, assume that $x^L = y^L$. This means that $X = Y$. We have $xz = (x_1 z_1, z_1^{-1}(X) \cap Z)$ hence $(xz)^L = (1, z_1^{-1}(X) \cap Z)$. A similar computation shows that $(yz)^L = (1, z_1^{-1}(Y) \cap Z)$ hence the claim since $X = Y$.

For the right projection, assume that $x^R = y^R$. This means that $x_1 * X = y_1 * Y$. We have $zx = (z_1 x_1, x_1^{-1}(Z) \cap X)$ hence $(zx)^R = (1, z_1 * x_1 * (x_1^{-1}(Z) \cap X))$. A similar computation shows that $(zy)^R = (1, z_1 y_1 (y_1^{-1}(Z) \cap Y))$. Now let $z' \in z_1 * x_1 * (x_1^{-1}(Z) \cap X)$. By definition, there is $x' \in X$ such that $z' = z_1 * x_1 * x'$ and $x_1 * x' \in Z$. Since $x_1 * X = y_1 * Y$, there exists $y' \in Y$ such that $x_1 * x' = y_1 * y' \in Z$ hence $y' \in y_1^{-1}(Z) \cap Y$ and thus $z' = z_1 * x_1 * x' = z_1 * y_1 * y' \in z_1 * y_1 * (y_1^{-1}(Z) \cap Y)$. This proves that $(zy)^R \leq (zx)^R$. The reverse inequality is proved by applying a symmetrical argument. \square

Remark. The notion of E-ordered monoid, extending Ehresmann semigroups [18], appears in [5]. It is examined quite in detail in [10]. There, a general construction, based on left and right ideals, is provided to embed every monoid S into a non-trivial adequately ordered monoid $\mathcal{Q}(S)$: the quasi-inverse expansion of S that turned out to be E-ordered as well. One can observe that the construction of $M(S)$ is quite similar with right ideals replaced by non-empty subsets of S_ω .

We define the infinite product π almost by iteration although, as well known in ω -language theory, the infinite product itself cannot be defined as a limit since many regular languages are not closed in prefix topology.

Among subunits of $M(S)$, the meet \wedge in the order corresponds to the product. Indeed, a subunit in $M(S)$ is either zero (behaving like $(1, \emptyset)$) or of the form $(1, X)$ for some non-empty $X \subseteq S_\omega$ with $(1, X) \cdot (1, Y) = (1, X \cap Y)$. The monoid of subunits $U(S)$ is thus isomorphic to the power set $\mathcal{P}(S_\omega)$ with intersection as product. In the following definition, we sometimes use the meet operator \wedge in place of the product.

Definition 11 (Infinite product). Let $(x_i)_{i \in \omega} \in (M(S))^\omega$. Let $\pi^0() = 1$ and for every $n \in \omega$, let $\pi^{n+1}(x_i)_{i \leq n} = (x_0 \cdot \pi^n(x_{i+1})_{i < n})^R$. Then, the infinite product $\pi(x_i)_{i \in \omega}$ is defined by

$$\pi(x_i)_{i \in \omega} = (1, X_\omega) \wedge \bigwedge_{n \in \omega} \pi^n(x_i)_{i \leq n}$$

with X_ω defined by $X_\omega = S_\omega$ when $J = \{i \in \omega : x_i \leq 1\}$ is finite and X_ω defined by $X_\omega = \{x_\omega\}$ when J is infinite, with $x_\omega = \pi(s_{j_i})_{i < \omega}$ where $(j_i)_{i \in \omega}$ is the increasing enumeration of the elements of J and, for every $j \in J$, the (non-zero) element x_j is of the form $x_j = (s_j, X_j)$.

Theorem 12. *The partially monoid $M(S)$ equipped with the above infinite product is an adequately ordered ω -monoid.*

Proof. Subunit preservation (I1) immediately follows from the definition. Monotonicity (I2) is also easily checked. Indeed, if $(x_i)_{i \leq \omega} \leq (y_i)_{i \in \omega}$ then, by definition of the order relation, this means that for every $i \in \omega$ if $x_i = (s_i, X_i)$ for some $s_i \in S_f$ and $X_i \subseteq S_\omega$ then we have $y_i = (s_i, Y_i)$ for some $Y_i \subseteq S_\omega$ with $X_i \subseteq Y_i$. It is then routine to check that we indeed have $\pi(x_i)_{i \leq \omega} \leq \pi(y_i)_{i \in \omega}$.

Mixed associativity (I3) follows from Lemma 10 and the mixed associativity in S .

It remains to check that the product and the infinite product also satisfy the infinite associativity law (I4). Let $(x_i)_{i \in \omega} \in (S_f \times \{S_\omega\})^\omega$, let $(j_i)_{i \in \omega}$ a strictly increasing sequence of integers with $j_0 = 0$, and, for every $i \in \omega$, let $y_i = x_{j_i} \cdot x_{j_{i+1}} \cdot \dots \cdot x_{j_{i+1}-1}$. We have to prove that $\pi(x_i)_{i \in \omega} = \pi(y_i)_{i \in \omega}$. By definition, for every $i \in \omega$, we have

$$\pi^{j_{i+1}}(x_j)_{j < j_{i+1}} \leq \pi^i(y_j)_{j < i} \leq \pi^{j_i}(x_j)_{j < j_i}$$

hence $\bigwedge_{i \in \omega} \pi^i(x_j)_{j < i} = \bigwedge_{i \in \omega} \pi^i(y_j)_{j < i}$. In the case when either product is zero then we are done. Otherwise, assuming that, for every $i < \omega$ we have $x_i = (s_i, X_i)$ and $y_i = (t_i, Y_i)$ with $t_i = s_{j_i} \cdot s_{j_{i+1}} \cdot \dots \cdot s_{j_{i+1}-1}$ we have to check that $X_\omega = Y_\omega$ as defined above. But this immediately follows from the definition and the fact S is an ω -semigroup. Indeed, either we have $X_\omega = Y_\omega = S_\omega$, or we have $X_\omega = \{x_\omega\}$ and $Y_\omega = \{y_\omega\}$ with $x_\omega = y_\omega$ that follows from the infinite associativity property in the ω -semigroup S . \square

Last, we aim at showing that the ω -monoid $S = (S_f, S_\omega)$ can be embedded as an ω -monoid into $M(S)$. This means defining over $M(S)$ an ω -monoid structure. This is done by taking $M(S)$ for the “finitary” part and $U(M(S))$ for the “infinitary” part. The finite and infinite product are the one already defined, and the *mixed product* of two elements $x \in M(S)$ and $y \in U(M(S))$ is defined by $x * y = (x \cdot y)^R$.

Lemma 13. *The pair of sets $(M(S), U(M(S)))$ equipped with the product \cdot , the mixed product $*$ and the infinite product π is an ω -semigroup.*

Proof. Lemma 8 ensures that $M(S)$ equipped with the product \cdot is a semigroup. The mixed associativity law follows from Lemma 10. The infinite associativity law follows from Theorem 12. \square

Theorem 14. *The mapping $\theta = (\theta_f, \theta_\omega) : (S_f, S_\omega) \rightarrow (M(S), U(M(S)))$, defined by $\theta_f(x) = (x, S_\omega)$ for every $x \in S_f$ and by $\theta_\omega(y) = (1, \{y\})$ for every $y \in S_\omega$, is a one-to-one ω -monoid morphism.*

Proof. θ_f is a monoid morphism: indeed, for each $x \in S_f$, we have $x^{-1}(S_\omega) = \{y \in S_\omega : x * y \in S_\omega\} = S_\omega$, hence $\theta_f(x) \cdot \theta_f(y) = (x \cdot y, S_\omega) = \theta_f(x \cdot y)$ for all $x, y \in S_f$.

We show that it preserves the mixed product. Let $x \in S_f$ and $y \in S_\omega$. We have $\theta(x * y) = (1, \{x * y\})$ and $\theta(x) * \theta(y) = ((x, S_\omega) \cdot (1, \{y\}))^R = ((x, \{y\}))^R = (1, \{x * y\})$ hence we indeed have equality. Observe that this property is also a particular case of the congruence property proved in Lemma 10.

It remains to prove that θ also preserves infinite products. Let $(x_i)_{i < \omega} \in (S_f)^\omega$. By definition of θ_ω , we have $\theta_\omega(\pi(x_i)_{i \in \omega}) = (1, \{\pi(x_i)_{i \in \omega}\})$. By definition of the infinite product, since $\theta_f(x_i) \not\leq 1$ for every $i \in \omega$, we have $\pi(\theta_f(x_i))_{i \in \omega} = \theta_\omega(\pi(x_i)_{i \in \omega}) \wedge \bigwedge_{i \in \omega} \pi^i(\theta(x_j))_{j < i}$. Now, by definition of θ_f , for every $i \in \omega$, we have $\theta_f(x_i) = (x_i, S_\omega)_{i \in \omega}$ hence, by definition of π^i , $\pi^{i+1}(\theta(x_j))_{j \leq i} = (1, \bigcap_{j \leq i} x_0 * x_1 * \dots * x_j * S_\omega)$. Now, by mixed associativity in S , we have we have $\pi(x_i)_{i \in \omega} \in x_0 * x_1 * \dots * x_i * S_\omega$ for every $i \in \omega$, hence $\theta_\omega(\pi(x_i)_{i \in \omega}) \leq \pi^i(\theta(x_j))_{j < i}$ henceforth $\pi(\theta_f(x_i))_{i \in \omega} = \theta_\omega(\pi(x_i)_{i \in \omega})$. \square

4 Application to language theory

We provide here a language theory for birooted words. Since every non-zero birooted word may be seen as a pair composed of a finite and a finite or infinite word, the logical definability in Monadic Second-Order (MSO) logic of languages of birooted words boils down to the classical finite and infinite word language theory. More interestingly, we also provide a notion of finite-state birooted word automaton that *essentially* captures definability in MSO logic.

4.1 Languages of birooted words and their automata

From now on, a language of birooted words is a set $X \subseteq T^\infty(A)$ of *non-zero* birooted words.

The class of such languages is equipped with the boolean operators union (also called sum), intersection and complement, plus operators derived from the structure of $T_0^\infty(A)$: for all X and $Y \subseteq T^\infty(A)$, the product $X \cdot Y$, the star X^* , as well as X^+ and the omega X^ω , are defined by extending the corresponding operators on $T_0^\infty(A)$ in a point-wise manner, *always omitting* the zero possibly resulting from these products. Additionally, we define the left and right projections X^L and X^R of the language X as the sets $X^L = \{x^L \in T^\infty(A) : x \in X\}$ and $X^R = \{x^R \in T^\infty(A) : x \in X\}$.

Theorem 15. *The class of languages of (positive) birooted words definable in MSO is closed under all the operators defined above. Moreover, it is finitely generated from the finite languages of (positive) birooted words, sum, product, star, omega and left and right projections.*

Proof. Proving the closure properties essentially amounts to extend to infinite birooted words the proof arguments provided in [13] (Robustness Theorem). The second part is easily proved by applying ω -language theory (see [24]) and arguments analogous to the one provided in [13] (Simplicity Theorem). \square

Combining the notions of Muller ω -word automata [21] and of tile automata [14], we define birooted words ω -automata as follows.

Definition 16. A (finite) *birooted word ω -automaton* is a tuple $\mathcal{A} = \langle Q, \delta, K, W \rangle$ with a (finite) *set of states* Q , a *transition function* $\delta : A \rightarrow \mathcal{P}(Q \times Q)$, a *finitary acceptance condition* $K \subseteq Q \times Q$ and an *infinitary acceptance condition* $W \subseteq \mathcal{P}(Q)$. For technical reasons, we always assume that $\emptyset \in W$.

A *run* of the automaton \mathcal{A} on a birooted word $u = (u_1, u_2)$ is a labeling mapping $\rho : [0, |u_1 * u_2| + 1[\rightarrow Q$ that satisfies the *local consistency* property : $(\rho(k), \rho(k+1)) \in \delta((u_1 * u_2)[k])$ for every $0 \leq k \leq |u_1| + |u_2|$.

The run ρ of the automaton \mathcal{A} on the birooted word $u = (u_1, u_2)$ is *locally accepting* when the pair of states $(\rho(0), \rho(|u_1|))$ that marks the input and output roots belongs to K .

The run ρ is *globally accepting* when the set of states that occur infinitely often belongs to W , i.e. $\{q \in Q : |\varphi^{-1}(q)| = \infty\} \in W$. Observe that when (u_1, u_2) is finite then the global acceptance constraint is always satisfied since we assume that $\emptyset \in W$.

The language $L(\mathcal{A})$ is then defined as the set of birooted words $(u_1, u_2) \in T^\infty(A)$ for which there exists a locally and globally accepting run of \mathcal{A} on (u_1, u_2) .

Theorem 17. *A language $L \subseteq T^\infty(A)$ of non-zero birooted words is recognized by a finite-state birooted word ω -automaton if and only if L is definable in MSO and upward closed.*

Proof. Assume that L is recognizable by a finite-state automaton. Since the existence of an accepting run on a birooted word amounts to checking the existence of some labeling of the vertices of (the graph representation of) the birooted word, this property is classically definable in MSO. Then, by definition of birooted word automata, the language L is upward closed w.r.t. in the birooted word order.

Conversely, assume that L is definable in MSO. By applying classical automata theoretic techniques, there exists an finite-state *word* automaton \mathcal{A}' recognizing the language of finite and infinite words $L' = \{u_1 \# u_2 \in (A + \#)^\infty : (u_1, u_2) \in L\}$. It is not difficult to derive from the automaton \mathcal{A}' a birooted word automaton \mathcal{A} recognizing, under upward closure hypothesis, the language L . \square

Corollary 18. *The class of languages recognized by a finite-state birooted word ω -automaton is closed under the sum, intersection, product, star and omega operations. Moreover, it is finitely generated from finite languages of (positive) birooted words, sum, product, star and omega, and the upward closure of left and right projections.*

4.2 Premorphism and quasi-recognizable languages

A mapping $\varphi : T_0^\infty(A) \rightarrow T$ from birooted words to an adequately ordered monoid T is a *premorphism* when $\varphi(1) = 1$ and $\varphi(xy) \leq \varphi(x)\varphi(y)$ and if $x \leq y$ then $\varphi(x) \leq \varphi(y)$ for every x and $y \in T_0^\infty(A)$.

The premorphism φ is *adequate* when, moreover, $\varphi(x^L) = (\varphi(x))^L$ and $\varphi(x^R) = (\varphi(x))^R$ for every $x \in S$, and if $x^L y^R \neq 0$ and $x^L \vee y^R = 1$ then $\varphi(xy) = \varphi(x)\varphi(y)$ for every x and $y \in S$. In the latter case we say that the product xy is disjoint.

The adequate premorphism φ is *ω -adequate* when T is an adequately ordered ω -monoid and, for every infinite sequence $(u_i)_{0 \leq i} \in (T_\infty^0(A))^\omega$, we have $\varphi(\pi(u_i)_{0 \leq i}) \leq \pi(\varphi(u_i))_{0 \leq i}$ and, moreover, if the product $\pi(u_i)_{0 \leq i}$ is disjoint then we also have $\varphi(\pi(u_i)_{0 \leq i}) = \pi(\varphi(u_i))_{0 \leq i}$ – that is, in the case for every $0 < k$, given $x = u_0 \cdot u_1 \cdots u_{k-1}$ and $y = \pi(u_{i+k})_{0 \leq i}$ we have $x^L y^R \neq 0$ and $x^L \vee y^R = 1$.

A language $L \subseteq T^\infty(A)$ is *quasi-recognizable* when there exists a finite adequately ordered ω -monoid S and an ω -adequate premorphism $\varphi : T^\infty(A) \rightarrow S$ such that $L = \varphi^{-1}(\varphi(T))$.

The next theorem extends a similar result proved in [14] for languages of finite birooted words.

Theorem 19. *Let $L \subseteq T_0^\infty(A)$ be a language of non-zero finite or infinite birooted words. The language L is quasi-recognizable if and only if it is a finite boolean combination of upward-closed (for the natural order) languages definable in monadic second order logic (MSO).*

Proof. Assume that there is an ω -adequate premorphism $\varphi : T_0^\infty(A) \rightarrow S$ that recognizes L . Without loss of generality, possibly by adding a new zero to S , we may assume that $\varphi^{-1}(0) = 0$. Then, for every non-zero $s \in S$, $\varphi^{-1}(s)$ is definable in MSO.

Indeed, the mapping that maps every finite word $u \in A^*$ to $\varphi((u, 1))$ is a monoid morphism since, for every $a \in A$, every $u \in A^*$, the product $(a, 1) \cdot (u, 1)$ is disjoint. Then, given $(u_1, u_2) \in T^\infty(A)$, since $(u_1, u_2) = (u_1, 1) \cdot (1, u_2)$ with a disjoint product, we have $\varphi((u_1, u_2)) = \varphi((u_1, 1)) \cdot \varphi((1, u_2))$. In the case u_2 is also finite, we are done since $\varphi((1, u_2)) = \varphi((u_2, 1)^R) = \varphi((u_2, 1))^R$. In the case when u_2 is infinite, we can apply Ramsey's theorem and infinite associativity, following the technique used for infinite words [24], to show that $(1, u_2) = \pi((v_i, 1)_{0 \leq i})$ with $\varphi((v_i, 1)) = \varphi((v_1, 1))$ henceforth $\varphi((1, u_2)) = (\varphi((v_0, 1)) \cdot (\varphi((v_1, 1)))^\omega)^R$.

Then, by applying classical arguments of algebraic language theory, the fact that L is a finite boolean combination of upward closed languages is a consequence of the monotonicity of φ and the finiteness of S since, for every $s \in \varphi(L)$, we have $\varphi^{-1}(s) = \varphi^{-1}(s^\uparrow) \cap \varphi^{-1}(s^\downarrow)$ with s^\uparrow (resp. s^\downarrow) is the upward closure (downward closure) of the singleton $\{s\}$.

Conversely, assume that L is a finite boolean combination of upward closed languages $\{L_i\}_{0 \leq i \leq n}$ definable in MSO. For every $0 \leq i \leq n$, by applying Theorem 17, the language L_i is recognized by a finite-state automaton \mathcal{A}_i . Now, by applying Lemma 20 below, for every $i \in I$, there exists an ω -adequate premorphisms $\varphi_i : T_0^\infty \rightarrow S_i$ with finite S_i such that $L_i = \varphi_i^{-1}(\varphi_i(L_i))$ hence we conclude by applying classical algebraic techniques, proving that the product premorphism $\varphi : T_0^\infty(A) \rightarrow S = S_0 \times S_1 \times \cdots \times S_n$ defined for every $x \in T_0^\infty(A)$ by $\varphi(x) = (\varphi_0(x), \varphi_1(x), \cdots, \varphi_n(x))$ recognizes L . Proving that the product monoid S is a (finite) adequately ordered ω -monoid raises no particular difficulties. \square

Lemma 20. *Let \mathcal{A} be a finite-state birooted word ω -automaton. There exists a (finite) adequately ordered ω -monoid $S_{\mathcal{A}}$ and an ω -adequate premorphism $\varphi_{\mathcal{A}} : T_0^\infty(A) \rightarrow S_{\mathcal{A}}$ such that $L(\mathcal{A}) = \varphi_{\mathcal{A}}^{-1}(\varphi_{\mathcal{A}}(L(\mathcal{A})))$.*

Proof. Let $\mathcal{A} = \langle Q, \delta, K, W \rangle$ be the automaton. Let $T_Q \subseteq Q \times \mathcal{P}(Q) \times Q \times \mathcal{P}(Q)$ be the set of automaton traces, that is, quadruples of the form $(p, P, q, R) \in Q \times \mathcal{P}(Q) \times Q \times \mathcal{P}(Q)$ with $p \in P$ when $p \neq q$, with, intendedly, a input root state p , a set of states P occurring between the input and the output root, an output root state q and a set of states R occurring infinitely often.

The set T_Q is equipped with the partial product \cdot defined, for every (p, P, q, R) and $(p', P', q', R') \in T_Q$, when $q = p'$ and either $R = \emptyset$ or $R' = \emptyset$ or $R = R'$ by

$$(p, P, q, R) \cdot (p', P', q', R') = (p, P \cup P', q', R \cup R')$$

One can check that, although partial, it is nevertheless associative in the sense that for every traces t, t', t'' , we have the product $(t \cdot t') \cdot t''$ is defined if and only if the product $t \cdot (t' \cdot t'')$ is defined and, in that case, they are equal.

The T_Q is then equipped with an infinite product π defined, for every sequence $(p_i, P_i, q_i, R_i)_{0 \leq i}$ of traces of T_Q , when $q_i = p_{i+1}$ and $R_i = \emptyset$ for every $0 \leq i$, by

$$\pi(p_i, P_i, q_i, \emptyset)_{0 \leq i} = (p_0, \{p_0\}, p_0, R)$$

with $R = \{q \in Q : |\{i \in \omega : q \in P_i\}| = \infty\}$.

Then we define $S_{\mathcal{A}}$ as the power set of T_Q , with product and infinite product defined as the point-wise extensions of the product and infinite product of T_Q . That is, for every X and $Y \subseteq T_Q$, we define $X \cdot Y$ as the set of all defined products $x \cdot y$ in T_Q with $x \in X$ and $y \in Y$, and, for every sequence $(X_i)_{0 \leq i}$ of subsets of traces, we define $\pi(X_i)_{0 \leq i}$ as the set of all defined products of the form $\pi(x_i)_{0 \leq i}$ with $x_i \in X_i$ for every $0 \leq i$.

It is routine to check that the monoid $S_{\mathcal{A}}$ obtained, ordered by inclusion, is an adequately ordered ω -monoid (following the technicalities used in [14, 12] for finite birooted words or trees).

We define $\varphi_{\mathcal{A}} : T_0^\infty(A) \rightarrow S_{\mathcal{A}}$ by $\varphi(0) = \emptyset$ and, for all birooted words (u_1, u_2) , $\varphi((u_1, u_2))$ is the set of traces of the form (p, P, q, R) such that there exists a run ρ of the automaton \mathcal{A} on (u_1, u_2) with $\rho(0) = p$, $P = \rho([0, |u_1| - 1])$, $q = \rho(|u_1|)$ and $R = \{r \in |Q| : |\rho^{-1}(q)| = \infty\}$. Again, it is routine to check that $\varphi_{\mathcal{A}}$ is an ω -adequate premorphism.

We finally check that, by construction, a birooted word $x \in T_0^\infty(A)$ belongs to L if and only if there exist $(p, P, q, R) \in \varphi_{\mathcal{A}}(x)$ such that $(p, q) \in K$ and $R \in W$. \square

5 Conclusion

We have shown that the embedding of strings and streams into tiled streams, as done for programming purposes [15], can be lifted to arbitrary ω -semigroups, thus unifying the finite and infinite word language theories into the emerging language theory of birooted words.

Following [12], we hope to generalize the present approach to trees, possibly leading to further developments in the very subtle and difficult emerging algebraic theory of languages of infinite trees [1]. However, there is no evidence yet that such a generalization could lead to a successful algebraic characterization of infinite tree languages.

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