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To cite this version:
Frédéric Chapoton, Jiang Zeng. A curious polynomial interpolation of Carlitz-Riordan’s $q$-ballot numbers. Contributions to Discrete Mathematics, 2014, 10.11575/cdm.v10i1.62222 . hal-00909816v2

HAL Id: hal-00909816
https://hal.science/hal-00909816v2
Submitted on 14 Dec 2013

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A CURIOUS POLYNOMIAL INTERPOLATION OF CARLITZ-RIORDAN'S $q$-BALLOT NUMBERS

FRÉDÉRIC CHAPOTON AND JIANG ZENG

Abstract. We study a polynomial sequence $C_n(x|q)$ defined as a solution of a $q$-difference equation. This sequence, evaluated at $q$-integers, interpolates Carlitz-Riordan’s $q$-ballot numbers. In the basis given by some kind of $q$-binomial coefficients, the coefficients are again some $q$-ballot numbers. We obtain in a combinatorial way another curious recurrence relation for these polynomials.

1. Introduction

This paper was motivated by a previous work of the first author on flows on rooted trees [Cha13b], where the well-known Catalan numbers and the closely related ballot numbers played an important role. In fact, one can easily introduce one more parameter $q$ in this work, and then Catalan numbers and ballots numbers get replaced by their $q$-analogues introduced a long time ago by Carlitz-Riordan [CR64], see also [Car72, FH85].

These $q$-Catalan numbers have been recently considered by many people, see for example [Cig05, BF07, Hag08, BP12], including some work by Reineke [Rein05] on moduli space of quiver representations.

Inspired by an analogy with another work of the first author on rooted trees [Cha13a], it is natural to try to interpolate the $q$-ballot numbers. In the present article, we prove that this is possible and study the interpolating polynomials.

Our main object of study is a sequence of polynomials in $x$ with coefficients in $\mathbb{Q}(q)$, defined by the $q$-difference equation:

$$\Delta_q C_{n+1}(x|q) = qC_n(q\,x + q + 1|q),$$

where $\Delta_q f(x) = (f(1 + qx) - f(x))/(1 + (q - 1)x)$ is the Hahn operator.

After reading a previous version of this paper, Johann Cigler has kindly brought the two related references [Cig97, Cig98] to our attention, where a sequence of more general polynomials $G_n(x, r)$ was introduced through a $q$-difference operator for $q$-integer $x$ and positive integer $r$. Comparing these two sequences one has

$$G_n(qx + 1, 2) = C_{n+1}(x|q).$$
In the next section we recall classical material on Carlitz-Riordan’s $q$-analogue for Catalan and ballot numbers and define our polynomials. In the third section, we evaluate our polynomials at $q$-integers in terms of $q$-ballot numbers and prove a product formula when $q = 1$. In the fourth section, we find their expansion in a basis made of a kind of $q$-binomial coefficients and obtain another recurrence for these polynomials. This recurrence is not usual even in the special $x = q = 1$ case and we have only a combinatorial proof in the general case. We conclude the paper with some open problems.

Nota Bene: Figures are best viewed in color.

2. Carlitz-Riordan’s $q$-ballot numbers

Recall that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ may be defined as solutions to

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad (n \geq 0), \quad C_0 = 1.$$\hspace{1cm} (2.1)

The first values are

$$\begin{array}{cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 C_n & 1 & 1 & 2 & 5 & 14 & 42 & 132 & 429 & 1430
\end{array}$$

It is well known that $C_n$ is the number of lattice paths from $(0,0)$ to $(n,n)$ with steps $(1,0)$ and $(0,1)$, which do not pass above the line $y = x$. As a natural generalization, one considers the set $P(n,k)$ of lattice paths from $(0,0)$ to $(n+1,k)$ with steps $(1,0)$ and $(0,1)$, such that the last step is $(1,0)$ and they never rise above the line $y = x$. Let $f(n,k)$ be the cardinality of $P(n,k)$. The first values of $f(n,k)$ are given in Table 1. These numbers are called ballot numbers and have a long history in the literature of combinatorial theory. Moreover, one (see [Com74]) has the explicit formula

$$f(n,k) = \frac{n-k+1}{n+1}\binom{n+k}{k}, \quad (n \geq k \geq 0).$$\hspace{1cm} (2.2)

Carlitz and Riordan [CR64] introduced the following $q$-analogue of these numbers

$$f(n,k|q) = \sum_{\gamma \in P(n,k)} q^{A(\gamma)},$$\hspace{1cm} (2.3)

where $A(\gamma)$ is the area under the path (and above the $x$-axis). The first values of $f(n,k|q)$ are given in Table 2. Furthermore, Carlitz [Car72] uses a variety of elegant techniques to derive several basic properties of the $f(n,k|q)$, among which the following is the basic recurrence relation

$$f(n,k|q) = qf(n,k-1|q) + q^k f(n-1,k|q), \quad (n,k \geq 0),$$\hspace{1cm} (2.4)
A CURIOUS INTERPOLATION OF CARLITZ-RIORDAN’S $q$-BALLOT NUMBERS

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Table 1. The first values of ballot numbers $f(n,k)$

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<tr>
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<td>$q^2 + q^3 + 2q^4 + 2q^5 + 2q^6 + q^7$</td>
<td>$q^3 Y + q^4 Y$</td>
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Table 2. The first values of $q$-ballot numbers $f(n,k|q)$ with $Y = q^6 + 3q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1$

where $f(n,k|q) = 0$ if $n < k$ and $f(0,0|q) = 1$.

It is also easy to see that the polynomial $f(n,k|q)$ is of degree $kn - k(k - 1)/2$ and satisfies the equation $f(n,n|q) = qf(n,n - 1|q)$. If one defines the $q$-Catalan numbers by

$$
C_{n+1}(q) = \sum_{k=0}^{n} f(n,k|q) = q^{-n-1} f(n+1,n+1|q) \quad (n \geq 0),
$$

then, one obtains the following analogue of (2.1) for the Catalan numbers

$$
C_{n+1}(q) = \sum_{i=0}^{n} C_i(q) C_{n-i}(q) q^{(i+1)(n-i)},
$$

where $C_0(q) = 1$. Setting $\widetilde{C}_n(q) = q^{\binom{n}{2}} C_n(q^{-1})$, one has a simpler $q$-analog of (2.1)

$$
\widetilde{C}_{n+1}(q) = \sum_{i=0}^{n} q^i \widetilde{C}_i(q) \widetilde{C}_{n-i}(q).
$$
The first values are $C_1(q) = 1$, $C_2(q) = 1 + q$, $C_3(q) = 1 + q + 2q^2 + q^3$ and

$$C_4(q) = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + q^6.$$ 

No explicit formula is known for Carlitz-Riordan’s $q$-Catalan numbers. However, Andrews [And75] proved the following recurrence formula

$$(2.8) \quad C_n(q) = \frac{q^n}{[n + 1]_q} \left[ \begin{array}{c} 2n \vspace{3pt} \\
\end{array} \right]_q + q \sum_{j=0}^{n-1} (1 - q^{n-j})q^{(n+1-j)j} \left[ \begin{array}{c} 2j + 1 \vspace{3pt} \\
\end{array} \right]_q C_{n-1-j}(q),$$

where $[x]_q = \frac{q^x - 1}{q - 1}$.

Recall that the $q$-shifted factorial $(x; q)_n$ is defined by

$$(x; q)_n = (1 - x)(1 - xq) \cdots (1 - xq^{n-1}) \quad (n \geq 1) \quad \text{and} \quad (x; q)_0 = 1.$$ 

The two kinds of $q$-binomial coefficients are defined by

$$\left[ \begin{array}{c} n \\
\end{array} \right]_q := \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}, \quad \left[ \begin{array}{c} x \\
\end{array} \right]_k := \frac{x(x - 1) \cdots (x - [k-1]_q)}{[k]_q !},$$

with $(x)_0 = 1$. Note that

$$\left[ \begin{array}{c} n \\
\end{array} \right]_k_q = q^{\binom{n}{k}} \left[ \begin{array}{c} n \\
\end{array} \right]_k, \quad \left[ \begin{array}{c} -n \\
\end{array} \right]_k_q = (-1)^k q^{-kn} \left[ \begin{array}{c} k + n - 1 \\
\end{array} \right]_k_q.$$

The $q$-derivative operator $D_q$ and Hahn operator $\Delta_q$ are defined by

$$(2.9) \quad D_q f(x) = \frac{f(qx) - f(x)}{(q - 1)x} \quad \text{and} \quad \Delta_q f(x) = \frac{f(1 + qx) - f(x)}{1 + (q - 1)x}.$$

**Definition 1.** The sequence of polynomials $\{C_n(x|q)\}_{n \geq 1}$ is defined by the $q$-difference equation (1.1) or equivalently

$$(2.10) \quad \frac{C_{n+1}(x|q) - C_{n+1}(q^{-1}x - q^{-1}|q)}{1 + (q - 1)x} = C_n(qx + 1|q) \quad (n \geq 1),$$

with the initial condition $C_1(x|q) = 1$ and $C_n(-\frac{1}{q}|q) = 0$ for $n \geq 2$. 
For example, we have

\[
C_2(x|q) = 1 + q \binom{x}{1}_q,
\]

\[
C_3(x|q) = (1 + q) + (q + q^2 + q^3) \binom{x}{1}_q + q^4 \binom{x}{2}_q,
\]

\[
C_4(x|q) = (q^3 + q^2 + 2q + 1) + (q^5 + q^4 + 2q^3 + 2q^2 + q) \binom{x}{1}_q
\]

\[
+ (q^9 + q^8 + q^7 + q^6 + q^5)q^{-1} \binom{x}{2}_q + q^9 \binom{x}{3}_q.
\]

It is clear that \( C_n(x|q) \) is a polynomial in \( \mathbb{Q}(q)[x] \) of degree \( n - 1 \) for \( n \geq 1 \).

3. Some preliminary results

We first show that the evaluation of the polynomials \( C_n(x|q) \) at \( q \)-integers is always a polynomial in \( \mathbb{N}[q] \). Note that formulae (3.1) and (3.4) were implicitly given in [Cig97].

**Proposition 2.** When \( x = [k]_q \) we have

\[
C_{n+1}([k]_q|q) = q^{kn} \binom{n(n+1)}{2} f(k + n, n|q^{-1}) \quad (n, k \geq 0). \tag{3.1}
\]

**Proof.** When \( x = [k]_q \) Eq. (2.10) becomes

\[
C_{n+1}([k]_q|q) = q^{k}C_n([k + 1]_q|q) + C_{n+1}([k - 1]_q|q). \tag{3.2}
\]

It is easy to see that (3.1) is equivalent to (2.4). \( \square \)

**Corollary 3.** We have

\[
C_{n+1}(0|q) = C_n(1|q) \quad \text{and} \quad C_{n+1}(1|q) = \tilde{C}_{n+1}(q). \tag{3.3}
\]

**Proof.** Letting \( x = 0 \) in (2.10) we get \( C_{n+1}(0|q) = C_n(1|q) \). Letting \( k = 1 \) in (3.1) we have

\[
C_{n+1}(1|q) = q^{n+\binom{n+1}{2}} f(1 + n, n|q^{-1})
\]

\[
= q^{n+\binom{n+1}{2}} f(n + 1, n + 1|q^{-1})
\]

\[
= q^{\binom{n+1}{2}} C_{n+1}(q^{-1}),
\]

which is equal to \( \tilde{C}_{n+1}(q) \) by definition. \( \square \)
The shifted factorial is defined by
\[(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x + 1) \cdots (x + n - 1), \quad n = 1, 2, 3, \ldots,\]
and \((x)_{-n} = 1 / (x - n)_n\).

**Proposition 4.** When \(q = 1\) we have the explicit formula
\[
C_{n+1}(x|1) = (x + 1)(x + n + 2)_{n-1} / n! = x + 1 + x + n \binom{x + 2n}{n} \quad (n \geq 0).
\]

**Proof.** When \(q = 1\) the equation (2.10) reduces to
\[
C_{n+1}(x|1) = C_{n+1}(x-1|1) + C_n(x+1|1).
\]
Since \(C_{n+1}(x|1)\) is a polynomial in \(x\) of degree \(n\), it suffices to prove that the right-hand side of (3.4) satisfy (3.5) for \(x\) being positive integers \(k\). By Proposition 2 and (2.2) it suffices to check the following identity
\[
k + 1 + x + n \binom{x + 2n}{n} = k + 1 + x + 1 + x + n \binom{x + 2n}{n} + k + 2 + x + 1 + x + n \binom{x + 2n}{n}.
\]
This is straightforward. \(\square\)

To motivate our result in the next section we first prove two \(q\)-versions of a folklore result on the polynomials which take integral values on integers (see [St86, p. 38]). Introduce the polynomials \(p_k(x)\) by
\[
p_0(x) = 1 \quad \text{and} \quad p_k(x) = (-1)^k q^{-\binom{k}{2}} (x-1)(x-q) \cdots (x-q^{k-1}) / (q;q)_k, \quad k \geq 1.
\]
So \(p_k(q^n) = \left[\begin{array}{c} n \\ k \end{array}\right]_q\) for \(n \in \mathbb{N}\).

**Proposition 5.** The following statements hold true.

(i) The polynomial \(f(x)\) of degree \(k\) assumes values in \(\mathbb{Z}[q]\) at \(x = 1, q, \ldots, q^k\) if and only if
\[
f(x) = c_0 + c_1 p_1(x) + \cdots + c_k p_k(x),
\]
where \(c_j = q^{\binom{j}{2}} (1-q)^2 \Delta_j^q f(1)\) are polynomials in \(\mathbb{Z}[q]\) for \(0 \leq j \leq k\).

(ii) The polynomial \(\tilde{f}(x)\) of degree \(k\) assumes values in \(\mathbb{Z}[q]\) at \(x = 0, [1]_q, \ldots, [k]_q\) if and only if
\[
\tilde{f}(x) = \sum_{j=0}^{k} \tilde{c}_j q^{-\binom{j}{2}} (x)_j, \quad \tilde{c}_j = q^{\binom{j}{2}} \Delta_j^q \tilde{f}(0)\]
are polynomials in \(\mathbb{Z}[q]\) for \(0 \leq j \leq k\).
Proof. Clearly we can expand any polynomial $f(x)$ of degree $k$ in the basis $\{p_j(x)\}_{0 \leq j \leq k}$ as in (3.7). Besides, it is easy to see that

$$\Delta_q \left( \frac{x}{k-1} \right)_q = \left( \frac{x}{k-1} \right)_q,$$

we obtain the expansion (3.8).

**Remark.** (1) We can also derive (3.8) from (3.7) as follows. Let $y = \frac{x-1}{q-1}$. For any polynomial $f(x)$ define $\tilde{f}(y) = f(1 + (q-1)y)$. Since $q^n = 1 + (q-1)[n]_q$, it is clear that $f(q^n) \in \mathbb{Z}[q] \iff \tilde{f}([n]_q) \in \mathbb{Z}[q]$. Writing $1 + qx - [j]_q = q(x - [j-1]_q)$ we see that

$$\tilde{f}_j(x) = q^{-\binom{j}{2}} \binom{x}{j}_q.$$

The expansion (3.8) follows from (3.7) immediately.

(2) When $\tilde{f}(x) = x^n$, it is known (see, for example, [Ze06]) that

$$\tilde{c}_k = \Delta_q 0^n = [k]_q! S_q(n, k),$$

where $[k]_q! = [1]_q \cdots [k]_q$ and $S_q(n, k)$ are classical $q$-Stirling numbers of the second kind defined by

$$S_q(n, k) = S_q(n - 1, k - 1) + [k]_q S_q(n - 1, k) \quad \text{for } n \geq k \geq 1,$$

with $S_q(n, 0) = S_q(0, k) = 0$ except $S_q(0, 0) = 1$.

(3) The two formulas (3.7) and (3.8) are special cases of the Newton interpolation formula, namely, for any polynomial $f$ of degree less than or equal to $n$ one has

$$f(x) = \sum_{k=0}^{n} \left( \sum_{j=0}^{k} \frac{f(b_j)}{\prod_{r=0, r \neq j}^{k} (b_j - b_r)} \right) (x - b_0) \cdots (x - b_{k-1}),$$

(3.10)
where $b_0, b_1, \ldots, b_{n-1}$ are distinct complex numbers. Some recent applications of (3.10) in the computation of moments of Askey-Wilson polynomials are given in [GITZ].

4. Main results

In the light of Propositions 2 and 5 it is natural to consider the expansion of $C_{n+1}(x|q)$ and $C_{n+1}(qx + 1|q)$ in the basis $(\gamma)_q^j$ ($j \geq 0$). It turns out that the coefficients in such expansions are Carlitz-Riordan’s $q$-ballot numbers. Note that formula (4.2) was implicitly given in [Cig97].

Theorem 6. For $n \geq 0$ we have

\begin{align}
(4.1) \quad C_{n+1}(x|q) &= \sum_{j=0}^{n} f(n+j, n-j|q^{-1})q^{j(n+1)}(x)^{j}_q, \\
(4.2) \quad C_n(qx + 1|q) &= \sum_{j=0}^{n-1} f(n+j, n-1-j|q^{-1})q^{j(n+1)}(x)^{j}_q.
\end{align}

Proof. It is sufficient to prove the theorem for $x = [k]_q$ with $k = 0, 1, \ldots, n$. By Proposition 2 the two equations (4.1) and (4.2) are equivalent to

\begin{align}
(4.3) \quad f(k+n, n|q^{-1}) &= \sum_{j=0}^{k} f(n+j, n-j|q^{-1})q^{j-k-n-j}k_{j}^{k}, \\
(4.4) \quad f(k+n, n-1|q^{-1}) &= \sum_{j=0}^{k} f(n+j, n-j-1|q^{-1})q^{j-k-n-2j+k}k_{j}^{k},
\end{align}

for $n \geq k \geq j$. Replacing $q$ by $1/q$ and using $[k]_{q^{-1}} = q^{j(k-j)}[k]_{j}^{j}$ we get

\begin{align}
(4.5) \quad f(n+k, n|q) &= \sum_{j=0}^{k} f(n+j, n-j|q)q^{(n-j)(k-j)+j}k_{j}^{k}, \\
(4.6) \quad f(k+n, n-1|q) &= \sum_{j=0}^{k} f(n+j, n-j-1|q)q^{(n-j-1)(k-j)+j}k_{j}^{k}.
\end{align}

We only prove (4.5). By definition, the left-hand side $f(n+k, n|q)$ is the enumerative polynomial of lattice paths from $(0, 0)$ to $(n + k + 1, n)$ with $(1, 0)$ as the last step. Each such path $\gamma$ must cross the line $y = -x + 2n$. Suppose it crosses this line at the point $(n+j, n-j)$, $0 \leq j \leq k$. Then the path corresponds to a unique pair $(\gamma_1, \gamma_2)$, where $\gamma_1$ is a path from $(0, 0)$ to $(n+j, n-j)$ and $\gamma_2$ is a path from $(n+j, n-j)$
Figure 1. The decomposition $\gamma \mapsto (\gamma_1, \gamma_2)$ to $(n+k, n)$. It is clear that the area under the path $\gamma$ is equal to $S_1 + S_2 + S_3 + j$, where

- $S_1$ is the area under the path $\gamma_1'$, which is obtained from $\gamma_1$ plus the last step $(n+j, n-j) \rightarrow (n+j+1, n-j)$;
- $S_2$ is the area under the path $\gamma_2$ and above the line $y = n-j$;
- $S_3$ is the area of the rectangle delimited by the four lines $y = 0, y = n-j, x = n+j+1$ and $x = n+k+1$, i.e., $(n-j)(k-j)$.

This decomposition is depicted in Figure 1. Clearly, summing over all such lattice paths gives the summand on the right-hand side of (4.5). This completes the proof. □

**Remark.** When $q = 1$, by (2.2), the above theorem implies that

\[
\frac{x+1}{x+n+1} \binom{x+2n}{n} = \sum_{j=0}^{n} \frac{2j+1}{n+j+1} \binom{2n}{n-j} \binom{x}{j},
\]

\[
\frac{x+2}{x+n+1} \binom{x+2n-1}{n-1} = \sum_{j=0}^{n-1} \frac{2j+2}{n+j+1} \binom{2n-1}{n-j-1} \binom{x}{j}.
\]

Note that the two sequences

\[
\{f(n+j, n-j|1)\} \text{ and } \{f(n+j, n-j-1|1)\} \quad (0 \leq j \leq n)
\]

correspond, respectively, to the $(2n-1)$-th and $2n$-th anti-diagonal coefficients of the triangle $\{f(n,k)\}_{0 \leq k \leq n}$, see Table 1.
Theorem 7. The polynomials $C_n(x|q)$ satisfy $C_1(x|q) = 1$ and
\begin{equation}
[n]_q C_{n+1}(x|q) = ([2n - 1]_q + xq^{2n-1})C_n(x|q)
+ \sum_{j=0}^{n-2} [n - j - 1]_q \tilde{C}_{j}(q)C_{n-j}(x|q)q^{2j+1}.
\end{equation}

Proof. Since $C_{n+1}(x|q)$ is a polynomial in $x$ of degree $n$, it suffices to prove (4.9) for $x = [k]_q$, where $k$ is any positive integer, namely,
\begin{equation}
[n]_q C_{n+1}([k]_q|q) = [2n + k - 1]_q C_n([k]_q|q)
+ \sum_{j=0}^{n-2} [n - j - 1]_q \tilde{C}_{j}(q)C_{n-j}([k]_q|q)q^{2j+1}.
\end{equation}

Let $m \geq n$ and
\begin{equation}
\tilde{f}(m, n|q) = q^{(m-n)n+\left(\frac{n+1}{2}\right)}f(m, n|q^{-1}).
\end{equation}
In view of the definition (2.3) it is clear that
\begin{equation}
\tilde{f}(m, n|q) = \sum_{\gamma \in \mathcal{P}(m,n)} q^{A'(\gamma)},
\end{equation}
where $A'(\gamma)$ denotes the area above the path $\gamma$ and under the line $y = x$ and $y = n$.

Since
\[ \tilde{C}_j(q) = q^{\left(\frac{j}{2}\right)}C_j(q^{-1}) = q^{\left(\frac{j+1}{2}\right)}f(j, j|q^{-1}) = \tilde{f}(j, j|q), \]
using Proposition 2 and (4.11) with $m = k + n$, we can rewrite (4.10) as
\begin{equation}
[n]_q \tilde{f}(m, n|q) = [n + m - 1]_q \tilde{f}(m - 1, n - 1|q)
+ \sum_{j=0}^{n-2} q^j [n - j - 1]_q \tilde{f}(j, j|q)q^{j+1}[n - j - 1]_q \tilde{f}(m - j - 1, n - j - 1|q).
\end{equation}
A pointed lattice path is a pair $(\alpha, \gamma)$ such that $\alpha \in \{(1, 1), \ldots, (n, n)\}$ and $\gamma \in \mathcal{P}(m, n)$. If $\alpha = (i, i)$ we call $i$ the height of $\alpha$ and write $h(\alpha) = i$. Let $\mathcal{P}^*(m, n)$ be the set of all such pointed lattice paths. It is clear that the left-hand side of (4.12) has the following interpretation
\begin{equation}
[n]_q \tilde{f}(m, n|q) = \sum_{(\alpha, \gamma) \in \mathcal{P}^*(m,n)} q^{h(\alpha) - 1 + A'(\gamma)}.
\end{equation}

Now, we compute the above enumerative polynomial of $\mathcal{P}^*(m, n)$ in another way in order to obtain the right-hand side of (4.12). We distinguish two cases.
Let $P^*_1(m, n, j)$ be the set of all pointed lattice paths $(\alpha, \gamma)$ in $P^*(m, n)$ such that $h(\alpha) \in \{j + 2, \ldots, n\}$, where $j$ is the smallest integer such that $(j + 1, j) \to (j + 1, j + 1)$ is a step of $\gamma$, i.e., the first step of $\gamma$ touching the line $y = x$. If $(\alpha, \gamma) \in P^*_1(m, n, j)$, then we have the correspondence $(\alpha, \gamma) \to (\gamma_1, (\alpha', \gamma_2))$, where $\gamma_1$ is a lattice path from $(0, 0)$ to $(j + 1, j + 1)$ which touches the line $y = x$ only at the two extremities, and $(\alpha', \gamma_2)$ is a pointed lattice path from $(0, 0)$ to $(m - j, n - j - 1)$ with $h(\alpha') = h(\alpha) - j - 1$. This decomposition is depicted in Figure 2.

Thus the corresponding enumerative polynomial of such paths for the fixed $j$ is

$$
\sum_{(\alpha, \gamma) \in P^*_1(m, n, j)} q^{h(\alpha) - 1 + A(\gamma)} = q^j \tilde{f}(j, j|q) \cdot q^{j+1}[n - j - 1]_q \tilde{f}(m - j - 1, n - j - 1|q).
$$

Summing over all $j$ ($0 \leq j \leq n - 2$) we obtain the second term on the right-hand side of (4.12).

Let $P^*_2(m, n)$ be the set of all pointed lattice paths $(\alpha, \gamma)$ in $P^*(m, n)$ such that $h(\alpha) \in \{1, \ldots, n\}$ and $h(\alpha) \leq j + 1$ where $j$ (if any) is the smallest integer such that $(j + 1, j) \to (j + 1, j + 1)$ is a step of $\gamma$, i.e., the first step of $\gamma$ touching the line $y = x$. If $(\alpha, \gamma) \in P^*_2(m, n)$, where $\gamma = (p_0, \ldots, p_{m+n+1})$ with $p_0 = (0, 0)$ and $p_{m+n+1} = (m + n + 1, n)$, we can associate a pair $(i, \gamma')$ where $\gamma' \in P(m - 1, n - 1)$ is obtained from $\gamma$ by deleting the vertical step $(x, h(\alpha) - 1) \to (x, h(\alpha))$ and the first horizontal step $(0, 0) \to (1, 0)$, i.e.,

$$
\gamma' = (p'_1, \ldots, p'_i, p'_{i+2}, \ldots, p'_{n+m+1})
$$

where $i = x + h(\alpha) - 1$, $p'_k = p_k - (1, 0)$ if $k = 1, \ldots, i$ and $p'_k = p_k - (0, 1)$ if $k = i + 2, \ldots, m + n + 1$. It is easy to see that the mapping $(\alpha, \gamma) \to (i, \gamma')$ is a bijection, which is depicted in Figure 3.
\[ (\alpha, \gamma) \mapsto (i, \gamma') \text{ with } m = 15, n = 8, \alpha = (6, 6) \text{ and } i = 15 \]

Since \( 1 \leq x \leq m \) and \( 0 \leq h(\alpha) - 1 \leq n - 1 \) we have \( i \in \{1, \ldots, m + n - 1\} \). As \( A'(\gamma) = x - 1 + A'(\gamma') \) we have

\[ h(\alpha) - 1 + A'(\gamma) = i - 1 + A'(\gamma'). \]

It follows that

\[ \sum_{(\alpha, \gamma) \in P_{2}(m,n)} q^{h(\alpha)+A'(\gamma)} = \sum_{i=1}^{m+n-1} q^{i-1} \sum_{\gamma'} q^{A'(\gamma')} = [n + m - 1]_{q} \tilde{f}(m - 1, n - 1|q). \]

Summing up the two cases we obtain the right-hand side of (4.12). \( \square \)

When \( q = 1 \) we have an alternative proof of Theorem 7.

\textit{Another proof of the }q = 1\textit{ case.} When \( q = 1 \) Eq. (4.9) reduces to

\[ nC_{n+1}(x|1) = (2n - 1 + x)C_{n}(x|1) + \sum_{j=0}^{n-2} (n - j - 1)C_{j}C_{n-j}(x|1) \quad (n \geq 2). \]  

This yields immediately \( C_{1}(x|1) = 1, \ C_{2}(x|1) = x + 1, \ C_{3}(x|1) = (x + 1)(x + 4)/2, \) in accordance with the formula (3.4). For \( n \geq 3, \) letting \( k = n - j - 3, \ N = n - 3 \) and \( z = x + 3, \) by (3.4), the recurrence (4.14) is equivalent to the following identity

\[ \frac{(z + N + 2)_{N}}{N!} = \sum_{k=0}^{N} 4^{N-k} \frac{(3/2)^{N-k}}{(3)_{N-k}} \frac{(z + k)_{k}}{k!} \quad (N \geq 0). \]
Notice that we can rewrite the right-hand side as
\[
\frac{(3/2)_N}{(3)_N} A^N \sum_{k=0}^{N} \frac{(-2 - N)_k ((z + 1)/2)_k (z/2)_k}{k! (-1/2 - N)_k (z)_k}
\]
\[
= \frac{(3/2)_N}{(3)_N} A^N \left( 3F_2 \left( \begin{array}{c} -2 - N, \ (z + 1)/2, \ z/2 \\ -1/2 - N, \ z \end{array} ; 1 \right) - \frac{(-2 - N)_{N+1} ((z + 1)/2)_{N+1} (z/2)_{N+1}}{(-1/2 - N)_{N+1} (z)_{N+1} (N + 1)!} - \frac{(-2 - N)_{N+2} ((z + 1)/2)_{N+2} (z/2)_{N+2}}{(-1/2 - N)_{N+2} (z)_{N+2} (N + 2)!} \right).
\]

Invoking Pfaff-Saalschütz formula [AAR99, Theorem 2.2.6] we obtain
\[
3F_2 \left( \begin{array}{c} -2 - N, \ (z + 1)/2, \ z/2 \\ -1/2 - N, \ z \end{array} ; 1 \right) = \frac{(z/2)_{N+2} ((z - 1)/2)_{N+2}}{(z)_{N+2} (-1/2)_{N+2}}.
\]

Substituting this in the previous expression yields \( \frac{(z+N+2)N}{N!} \) after simplification. \( \square \)

When \( x = 1 \) Eq. (4.14) reduced to the following identity for Catalan numbers:

\[
(4.15) \quad nC_{n+1} = 2nC_n + \sum_{j=0}^{n-2} (n - j - 1)C_j C_{n-j}.
\]

5. Concluding remarks

We conclude this paper with a few open problems. By Theorem 3 it is clear that
\( C_{n+1}(x|q) \) is a polynomial in \( x \) of degree \( n \) with leading coefficient \( q^n / [1]_q [2]_q \cdots [n]_q \).

**Conjecture 8.** One can write \( C_n(x|q) \) as an irreducible fraction
\[
\frac{P_n(x|q)}{[1]_q \cdots [n-1]_q},
\]
where \( P_n \) has only positive coefficients.

This has been checked up to \( n = 27 \). The similar conjecture is true when \( q = 1 \) by Proposition 4.

Finally, the Newton polytope of the numerator of \( C_n(x|q) \) seems to have a nice shape. This is illustrated in Figure 4, where the horizontal axis is associated with powers of \( q \) and the vertical axis with powers of \( x \). The slopes of the upper part seems to be given in general by the odd integers 1, 3, . . . , \( 2n - 3 \).
Figure 4. Newton polytope of the numerator of $C_n(x|q)$ for $n = 6$

References


[Cha13a] F. Chapoton, Sur une série en arbres à deux paramètres, Séminaire Lotharingien de Combinatoire vol. 70 (2013)


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