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# LOCAL LIMITS OF CONDITIONED GALTON-WATSON TREES II: THE CONDENSATION CASE

ROMAIN ABRAHAM AND JEAN-FRANÇOIS DELMAS

ABSTRACT. We provide a complete picture of the local convergence of critical or subcritical Galton-Watson tree conditioned on having a large number of individuals with out-degree in a given set. The generic case, where the limit is a random tree with an infinite spine has been treated in a previous paper. We focus here on the non-generic case, where the limit is a random tree with a node with infinite out-degree. This case corresponds to the so-called condensation phenomenon.

## 1. INTRODUCTION

Conditioning critical or sub-critical Galton-Watson (GW) trees comes from the seminal work of Kesten, [10]. Let  $p = (p(n), n \in \mathbb{N})$  be an offspring distribution such that:

$$(1) \quad p(0) > 0, \quad p(0) + p(1) < 1.$$

Let  $\mu(p) = \sum_{n=0}^{+\infty} np(n)$  be its mean. If  $\mu(p) < 1$  (resp.  $\mu(p) = 1$ ,  $\mu(p) > 1$ ), we say that the offspring distribution and the associated GW tree are sub-critical (resp. critical, super-critical). In the critical and sub-critical cases, the tree is a.s. finite, but Kesten considered in [10] the limit of a sub-critical or critical tree conditioned to have height greater than  $n$ . When  $n$  goes to infinity, this conditioned tree converges in distribution to the so-called size-biased GW tree. This random tree has an infinite spine on which are grafted a random number of independent GW trees with the same offspring distribution  $p$ . This limit tree can be seen as the GW tree conditioned on non-extinction.

Since then, other conditionings have been considered for critical GW trees: large total progeny see Kennedy [9] and Geiger and Kaufmann [5], large number of leaves see Curien and Kortchemski [3]. In [1], we generalized those previous results by conditioning the GW tree to have a large number of individuals whose number of offspring belongs to a set  $\mathcal{A} \subset \mathbb{N}$ . Let

$$(2) \quad p(\mathcal{A}) = \sum_{k \in \mathcal{A}} p(k).$$

If  $p(\mathcal{A}) > 0$ , then the limiting tree is again the same size-biased tree as for Kesten [10].

However, the results are different in the subcritical case. We first define for an offspring distribution  $p$  that satisfies (1) and a set  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$  a modified offspring distribution  $p_{\mathcal{A},\theta}$  by:

$$(3) \quad \forall k \geq 0, \quad p_{\mathcal{A},\theta}(k) = \begin{cases} c_{\mathcal{A}}(\theta)\theta^k p(k) & \text{if } k \in \mathcal{A}, \\ \theta^{k-1}p(k) & \text{if } k \in \mathcal{A}^c, \end{cases}$$

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where the normalizing constant  $c_{\mathcal{A}}(\theta)$  is given by:

$$(4) \quad c_{\mathcal{A}}(\theta) = \frac{\theta - \mathbb{E} [\theta^X \mathbf{1}_{\{X \in \mathcal{A}^c\}}]}{\theta \mathbb{E} [\theta^X \mathbf{1}_{\{X \in \mathcal{A}\}}]},$$

where  $X$  is a random variable distributed according to  $p$ . Let  $I_{\mathcal{A}}$  be the set of positive  $\theta$  for which  $p_{\mathcal{A},\theta}$  is a probability distribution. If  $p$  is sub-critical, according to Lemma 5.2, either there exists (a unique)  $\theta_{\mathcal{A}}^c \in I_{\mathcal{A}}$  such that  $p_{\mathcal{A},\theta_{\mathcal{A}}^c}$  is critical or  $\theta_{\mathcal{A}}^* := \max I_{\mathcal{A}} \in I_{\mathcal{A}}$  and  $p_{\mathcal{A},\theta_{\mathcal{A}}^*}$  is sub-critical. We shall say, see Definition 5.3, that  $p$  is **generic** for the set  $\mathcal{A}$  in the former case and that  $p$  is **non-generic** for the set  $\mathcal{A}$  in the latter case. See Lemma 5.4 and Remark 5.5 on the non-generic property.

For a tree  $\mathbf{t}$ , let  $\mathcal{L}_{\mathcal{A}}(\mathbf{t})$  be the set of nodes of  $\mathbf{t}$  whose number of offspring belongs to  $\mathcal{A}$  and  $L_{\mathcal{A}}(\mathbf{t})$  be its cardinal (see definition in Section 6). It is proven in [1] that, for every  $\theta \in I_{\mathcal{A}}$ , if  $\tau$  is a GW tree with offspring distribution  $p$  and  $\tau_{\mathcal{A},\theta}$  is a GW tree with offspring distribution  $p_{\mathcal{A},\theta}$ , then the conditional distributions of  $\tau$  given  $\{L_{\mathcal{A}}(\tau) = n\}$  and that of  $\tau_{\mathcal{A},\theta}$  given  $\{L_{\mathcal{A}}(\tau_{\mathcal{A},\theta}) = n\}$  are the same. Therefore, if  $p$  is generic for the set  $\mathcal{A}$ , that is there exists a  $\theta_{\mathcal{A}}^c \in I_{\mathcal{A}}$  such that  $p_{\theta_{\mathcal{A}}^c, \mathcal{A}}$  is critical, then the GW tree  $\tau$  conditioned on  $L_{\mathcal{A}}(\tau)$  being large converges to the size-biased tree associated with  $p_{\mathcal{A},\theta_{\mathcal{A}}^c}$ .

When the sub-critical offspring distribution is non-generic for  $\mathbb{N}$ , a condensation phenomenon has been observed when conditioning with respect to the total population size, see Jonnsson and Stefansson [7] and Janson [6]: the limiting tree is no more the size-biased tree but a tree that contains a single node with infinitely many offspring. The goal of this paper is to give a short proof of this result and to show that such a condensation also appears when  $p$  is non-generic for  $\mathcal{A}$  and conditioning by  $L_{\mathcal{A}}(\tau)$  being large. This and [1] give a complete description of the limit in distribution of a critical or subcritical GW tree  $\tau$  conditioned on  $\{L_{\mathcal{A}}(\tau) = n\}$  as  $n$  goes to infinity.

We summarize this complete description as follows. Let  $p$  be an offspring distribution that satisfies (1) which is critical or sub-critical (that is  $\mu(p) \leq 1$ ). Let  $\tau^*(p)$  denote the random tree which is defined by:

- i) There are two types of nodes: *normal* and *special*.
- ii) The root is special.
- iii) Normal nodes have offspring distribution  $p$ .
- iv) Special nodes have offspring distribution the biased distribution  $\tilde{p}$  on  $\mathbb{N} \cup \{+\infty\}$  defined by:

$$\tilde{p}(k) = \begin{cases} k p(k) & \text{if } k \in \mathbb{N}, \\ 1 - \mu & \text{if } k = +\infty. \end{cases}$$

- v) The offsprings of all the nodes are independent of each others.
- vi) All the children of a normal node are normal.
- vii) When a special node gets a finite number of children, one of them is selected uniformly at random and is special while the others are normal.
- viii) When a special node gets an infinite number of children, all of them are normal.

Notice that:

- If  $p$  is critical, then a.s.  $\tau^*(p)$  has one infinite spine and all its nodes have finite degrees. This is the size-biased tree considered in [10].
- If  $\mu(p) < 1$  then a.s.  $\tau^*(p)$  has exactly one node of infinite degree and no infinite spine. This tree has been considered in [7, 6].

**Definition 1.1.** Let  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$ . We define  $p_{\mathcal{A}}^*$  as:

- *critical case* ( $\mu(p) = 1$ ):

$$p_{\mathcal{A}}^* = p.$$

- *subcritical and generic for  $\mathcal{A}$*  ( $\mu(p) < 1$  and there exists (a unique)  $\theta_{\mathcal{A}}^c \in I_{\mathcal{A}}$  such that  $\mu(p_{\mathcal{A}, \theta_{\mathcal{A}}^c}) = 1$ ):

$$p_{\mathcal{A}}^* = p_{\mathcal{A}, \theta_{\mathcal{A}}^c}.$$

- *subcritical and non-generic for  $\mathcal{A}$*  ( $\mu(p) < 1$  and  $\mu(p_{\mathcal{A}, \theta_{\mathcal{A}}^*}) < 1$ ):

$$(5) \quad p_{\mathcal{A}}^* = p_{\mathcal{A}, \theta_{\mathcal{A}}^*}, \quad \text{with } \theta_{\mathcal{A}}^* = \max I_{\mathcal{A}}.$$

We state our main result (the convergence of random discrete trees is precisely defined in Section 2 and GW trees are presented in Section 3).

**Theorem 1.2.** Let  $\tau$  be a GW tree with offspring distribution  $p$  which satisfies (1) and  $\mu(p) \leq 1$ . Let  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$ . We have the following convergence in distribution:

$$(6) \quad \text{dist}(\tau \mid L_{\mathcal{A}}(\tau) = n) \xrightarrow{n \rightarrow +\infty} \text{dist}(\tau^*(p_{\mathcal{A}}^*)),$$

where the limit is understood along the infinite subsequence  $\{n \in \mathbb{N}^*; \mathbb{P}(L_{\mathcal{A}}(\tau) = n) > 0\}$ , as well as:

$$(7) \quad \text{dist}(\tau \mid L_{\mathcal{A}}(\tau) \geq n) \xrightarrow{n \rightarrow +\infty} \text{dist}(\tau^*(p_{\mathcal{A}}^*)).$$

The theorem has already been proven in the critical case and the subcritical generic case in [1]. We concentrate here on the case of the subcritical non-generic case. The non-generic case for  $\mathcal{A} = \mathbb{N}$ ,  $0 \in \mathcal{A}$ ,  $0 \notin \mathcal{A}$  are respectively proven in Sections 4, 6 and 7. Let us add that a subcritical offspring distribution  $p$  is either generic for all  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$  or non-generic at least for  $\{0\}$  and eventually for other sets and generic for other sets  $\mathcal{A}$  such that  $p(\mathcal{A}) > 0$ , see Lemma 5.4. It is not possible for a subcritical offspring distribution  $p$  to be non-generic for all  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$ , see Remark 5.5. By considering the last example of Remark 5.5, we exhibit a distribution  $p$  which is non-generic for  $\{0\}$  but generic for  $\mathbb{N}$ . Thus the associated GW tree conditioned on having  $n$  vertices converges in distribution (as  $n$  goes to infinity) to a tree with an infinite spine whereas the same tree conditioned on having  $n$  leaves converges in distribution to a tree with an infinite node.

In Section 2, we recall the setting of the discrete trees (which is close to [1], but has to include discrete trees with infinite nodes). We also give in Lemma 2.2, in the same spirit of Lemma 2.1 in [1], a convergence determining class which is the key result to prove the convergence in the non-generic case. Section 3 is devoted to some remarks on GW trees. We study in detail the distribution  $p_{\mathcal{A}, \theta}$  defined by (3) in Section 5. The proof of Theorem 1.2 is given in the following three sections. More precisely, the case  $\mathcal{A} = \mathbb{N}$  is presented in Section 4. This provides an elementary and self-contained proof of the results from [7, 6]. The case  $0 \in \mathcal{A}$  can be handled in the same spirit, see Section 6, using that the set  $\mathcal{L}_{\mathcal{A}}(\tau)$  can be encoded into a GW tree  $\tau^{\mathcal{A}}$ , see [11] or [14]. Notice that if  $0 \notin \mathcal{A}$ , then  $\mathcal{L}_{\mathcal{A}}(\tau)$ , when non empty, can also be encoded into a GW tree  $\tau^{\mathcal{A}}$ , see [14]. However, we didn't use this result, but rather use in Section 7 a more technical version of the previous proofs to treat the case  $0 \notin \mathcal{A}$ . We prove in the appendix, Section 8, consequences of the strong ratio limit property we used in the previous sections.

## 2. THE SET OF DISCRETE TREES

We recall Neveu's formalism [13] for ordered rooted trees. We set

$$\mathcal{U} = \bigcup_{n \geq 0} (\mathbb{N}^*)^n$$

the set of finite sequences of positive integers with the convention  $(\mathbb{N}^*)^0 = \{\emptyset\}$ . For  $n \geq 0$  and  $u = (u_1, \dots, u_n) \in \mathcal{U}$ , we set  $|u| = n$  the length of  $u$  and:

$$|u|_\infty = \max(|u|, (u_i, 1 \leq i \leq |u|))$$

with the convention  $|\emptyset| = |\emptyset|_\infty = 0$ . We will call  $|u|_\infty$  the norm of  $u$  although it is not a norm since  $\mathcal{U}$  is not even a vector space. If  $u$  and  $v$  are two sequences of  $\mathcal{U}$ , we denote by  $uv$  the concatenation of the two sequences, with the convention that  $uv = u$  if  $v = \emptyset$  and  $uv = v$  if  $u = \emptyset$ . The set of ancestors of  $u$  is the set:

$$(8) \quad A_u = \{v \in \mathcal{U}; \text{there exists } w \in \mathcal{U}, w \neq \emptyset, \text{ such that } u = vw\}.$$

The most recent common ancestor of a subset  $\mathbf{s}$  of  $\mathcal{U}$ , denoted by  $\text{MRCA}(\mathbf{s})$ , is the unique element  $v$  of  $\bigcap_{u \in \mathbf{s}} A_u$  with maximal length  $|v|$ . For  $u, v \in \mathcal{U}$ , we denote by  $u < v$  the lexicographic order on  $\mathcal{U}$  i.e.  $u < v$  if  $u \in A_v$  or, if we set  $w = \text{MRCA}(\{u, v\})$ , then  $u = wiu'$  and  $v = wjv'$  for some  $i, j \in \mathbb{N}^*$  with  $i < j$ .

A tree  $\mathbf{t}$  is a subset of  $\mathcal{U}$  that satisfies:

- $\emptyset \in \mathbf{t}$ ,
- If  $u \in \mathbf{t}$ , then  $A_u \subset \mathbf{t}$ .
- For every  $u \in \mathbf{t}$ , there exists  $k_u(\mathbf{t}) \in \mathbb{N} \cup \{+\infty\}$  such that, for every positive integer  $i$ ,  $ui \in \mathbf{t}$  iff  $1 \leq i \leq k_u(\mathbf{t})$ .

The integer  $k_u(\mathbf{t})$  represents the number of offsprings of the vertex  $u \in \mathbf{t}$ . (Notice that  $k_u(\mathbf{t})$  has to be finite in [1], whereas  $k_u(\mathbf{t})$  might take the value  $+\infty$  here.) The vertex  $u \in \mathbf{t}$  is called a leaf if  $k_u(\mathbf{t}) = 0$  and it is said infinite if  $k_u(\mathbf{t}) = +\infty$ . By convention, we shall set  $k_u(\mathbf{t}) = -1$  if  $u \notin \mathbf{t}$ . The vertex  $\emptyset$  is called the root of  $\mathbf{t}$ . We set:

$$|\mathbf{t}| = \text{Card}(\mathbf{t}).$$

Let  $\mathbf{t}$  be a tree. The set of its leaves is  $\mathcal{L}_0(\mathbf{t}) = \{u \in \mathbf{t}; k_u(\mathbf{t}) = 0\}$ . Its height and its "norm" are resp. defined by

$$H(\mathbf{t}) = \sup\{|u|, u \in \mathbf{t}\} \quad \text{and} \quad H_\infty(\mathbf{t}) = \sup\{|u|_\infty, u \in \mathbf{t}\} = \max(H(\mathbf{t}), \sup\{k_u(\mathbf{t}), u \in \mathbf{t}\});$$

they can be infinite. For  $u \in \mathbf{t}$ , we define the sub-tree  $\mathcal{S}_u(\mathbf{t})$  of  $\mathbf{t}$  "above"  $u$  as:

$$\mathcal{S}_u(\mathbf{t}) = \{v \in \mathcal{U}, uv \in \mathbf{t}\}.$$

For  $u \in \mathbf{t} \setminus \mathcal{L}_0(\mathbf{t})$ , we also define the forest  $\mathcal{F}_u(\mathbf{t})$  "above"  $u$  as the following sequence of trees:

$$\mathcal{F}_u(\mathbf{t}) = (\mathcal{S}_{ui}(\mathbf{t}); i \in \mathbb{N}^*, i \leq k_u(\mathbf{t})).$$

For  $u \in \mathbf{t} \setminus \{\emptyset\}$ , we also define the sub-tree  $\mathcal{S}^u(\mathbf{t})$  of  $\mathbf{t}$  "below"  $u$  as:

$$\mathcal{S}^u(\mathbf{t}) = \{v \in \mathbf{t}; u \notin A_v\}.$$

Notice that  $u \in \mathcal{S}^u(\mathbf{t})$ .

For  $v = (v_k, k \in \mathbb{N}^*) \in (\mathbb{N}^*)^{\mathbb{N}}$ , we set  $\bar{v}_n = (v_1, \dots, v_n)$  for  $n \in \mathbb{N}$ , with the convention that  $\bar{v}_0 = \emptyset$  and  $\bar{\mathbf{v}} = \{\bar{v}_n, n \in \mathbb{N}\}$  defines a tree consisting of an infinite spine or branch. We denote by  $\mathbb{T}_\infty$  the set of trees. We denote by  $\mathbb{T}_0$  the subset of finite trees,

$$\mathbb{T}_0 = \{\mathbf{t} \in \mathbb{T}_\infty; |\mathbf{t}| < +\infty\},$$

by  $\mathbb{T}_\infty^{(h)}$  the subset of trees with norm less than  $h$ ,

$$\mathbb{T}_\infty^{(h)} = \{\mathbf{t} \in \mathbb{T}_\infty; H_\infty(\mathbf{t}) \leq h\},$$

by  $\mathbb{T}_0^*$  the subset of trees with no infinite branch,

$$\mathbb{T}_0^* = \{\mathbf{t} \in \mathbb{T}_\infty; \forall v \in (\mathbb{N}^*)^{\mathbb{N}}, \bar{\mathbf{v}} \not\subset \mathbf{t}\},$$

and by  $\mathbb{T}_2$  the subset of trees with no infinite branch and with exactly one infinite vertex,

$$\mathbb{T}_2 = \{\mathbf{t} \in \mathbb{T}_\infty; \text{Card}\{u \in \mathbf{t}; k_u(\mathbf{t}) = +\infty\} = 1\} \cap \mathbb{T}_0^*.$$

Notice that  $\mathbb{T}_0$  is countable and  $\mathbb{T}_2$  is uncountable.

For  $h \in \mathbb{N}$ , the restriction function  $r_{h,\infty}$  from  $\mathbb{T}_\infty$  to  $\mathbb{T}_\infty$  is defined by:

$$r_{h,\infty}(\mathbf{t}) = \{u \in \mathbf{t}, |u|_\infty \leq h\}.$$

We endow the set  $\mathbb{T}_\infty$  with the ultra-metric distance

$$d_\infty(\mathbf{t}, \mathbf{t}') = 2^{-\max\{h \in \mathbb{N}, r_{h,\infty}(\mathbf{t}) = r_{h,\infty}(\mathbf{t}')\}}.$$

A sequence  $(\mathbf{t}_n, n \in \mathbb{N})$  of trees converges to a tree  $\mathbf{t}$  with respect to the distance  $d_\infty$  if and only if, for every  $h \in \mathbb{N}$ ,

$$r_{h,\infty}(\mathbf{t}_n) = r_{h,\infty}(\mathbf{t}) \quad \text{for } n \text{ large enough,}$$

that is for all  $u \in \mathcal{U}$ ,  $\lim_{n \rightarrow +\infty} k_u(\mathbf{t}_n) = k_u(\mathbf{t}) \in \mathbb{N} \cup \{-1, +\infty\}$ . The Borel  $\sigma$ -field associated with the distance  $d_\infty$  is the smallest  $\sigma$ -field containing the singletons for which the restriction functions  $(r_{h,\infty}, h \in \mathbb{N})$  are measurable. With this distance, the restriction functions are contractant. Since  $\mathbb{T}_0$  is dense in  $\mathbb{T}_\infty$  and  $(\mathbb{T}_\infty, d_\infty)$  is complete and compact, we get that  $(\mathbb{T}_\infty, d_\infty)$  is a compact Polish metric space.

*Remark 2.1.* In [1], we considered

$$\mathbb{T} = \{\mathbf{t} \in \mathbb{T}_\infty; k_u(\mathbf{t}) < +\infty \forall u \in \mathbf{t}\}$$

the subset of trees with no infinite vertex. On  $\mathbb{T}$ , we defined the distance:

$$d(\mathbf{t}, \mathbf{t}') = 2^{-\max\{h \in \mathbb{N}, r_h(\mathbf{t}) = r_h(\mathbf{t}')\}},$$

with  $r_h(\mathbf{t}) = \{u \in \mathbf{t}, |u| \leq h\}$ . Notice that  $(\mathbb{T}, d)$  is Polish but not compact and that  $\mathbb{T}$  is not closed in  $(\mathbb{T}_\infty, d_\infty)$ . If a sequence  $(\mathbf{t}_n, n \in \mathbb{N}^*)$  converges in  $(\mathbb{T}, d)$  then it converges in  $(\mathbb{T}_\infty, d_\infty)$ . And if a sequence  $(\mathbf{t}_n, n \in \mathbb{N}^*)$  of elements of  $\mathbb{T}$  converges in  $(\mathbb{T}_\infty, d_\infty)$  to a limit in  $\mathbb{T}$  then it converges to the same limit in  $(\mathbb{T}, d)$ .

Consider the closed ball  $B_\infty(\mathbf{t}, 2^{-h}) = \{\mathbf{t}' \in \mathbb{T}_\infty; d_\infty(\mathbf{t}, \mathbf{t}') \leq 2^{-h}\}$  for some  $\mathbf{t} \in \mathbb{T}_\infty$  and  $h \in \mathbb{N}$  and notice that:

$$B_\infty(\mathbf{t}, 2^{-h}) = r_{h,\infty}^{-1}(\{r_{h,\infty}(\mathbf{t})\}).$$

Since the distance is ultra-metric, the closed balls are open and the open balls are closed, and the intersection of two balls is either empty or one of them. We deduce that the family  $((r_{h,\infty}^{-1}(\{\mathbf{t}\}), \mathbf{t} \in \mathbb{T}_\infty^{(h)}), h \in \mathbb{N})$  is a  $\pi$ -system, and Theorem 2.3 in [2] implies that this family is convergence determining for the convergence in distribution. Let  $(T_n, n \in \mathbb{N}^*)$  and  $T$  be  $\mathbb{T}_\infty$ -valued random variables. We denote by  $\text{dist}(T)$  the distribution of the random variable  $T$  (which is uniquely determined by the sequence of distributions of  $r_{h,\infty}(T)$  for every  $h \geq 0$ ), and we denote:

$$\text{dist}(T_n) \xrightarrow[n \rightarrow +\infty]{} \text{dist}(T)$$

for the convergence in distribution of the sequence  $(T_n, n \in \mathbb{N}^*)$  to  $T$ . Notice that this convergence in distribution is equivalent to the finite dimensional convergence in distribution of  $(k_u(T_n), u \in \mathcal{U})$  to  $(k_u(T), u \in \mathcal{U})$  as  $n$  goes to infinity.

We deduce from the portmanteau theorem that the sequence  $(T_n, n \in \mathbb{N}^*)$  converges in distribution to  $T$  if and only if for all  $h \in \mathbb{N}$ ,  $\mathbf{t} \in \mathbb{T}_\infty^{(h)}$ :

$$\lim_{n \rightarrow +\infty} \mathbb{P}(r_{h,\infty}(T_n) = \mathbf{t}) = \mathbb{P}(r_{h,\infty}(T) = \mathbf{t}).$$

As we shall only consider  $\mathbb{T}_0$ -valued random variables that converge in distribution to a  $\mathbb{T}_2$ -valued random variable, we give an other characterization of convergence in distribution that holds for this restriction. To present this result, we introduce some notations. If  $v = (v_1, \dots, v_n) \in \mathcal{U}$ , with  $n > 0$ , and  $k \in \mathbb{N}$ , we define the shift of  $v$  by  $k$  as  $\theta(v, k) = (v_1 + k, v_2, \dots, v_n)$ . If  $\mathbf{t} \in \mathbb{T}_0$ ,  $\mathbf{s} \in \mathbb{T}_\infty$  and  $x \in \mathbf{t}$  we denote by:

$$\mathbf{t} \otimes (\mathbf{s}, x) = \mathbf{t} \cup \{x\theta(v, k_x(\mathbf{t})), v \in \mathbf{s} \setminus \{\emptyset\}\}$$

the tree obtained by grafting the tree  $\mathbf{s}$  at  $x$  on “the right” of the tree  $\mathbf{t}$ , with the convention that  $\mathbf{t} \otimes (\mathbf{s}, x) = \mathbf{t}$  if  $\mathbf{s} = \{\emptyset\}$  is the tree reduced to its root. Notice that if  $x$  is a leaf of  $\mathbf{t}$  and  $\mathbf{s} \in \mathbb{T}$ , then this definition coincides with the one given in [1].

For every  $\mathbf{t} \in \mathbb{T}_0$  and every  $x \in \mathbf{t}$ , we consider the set of trees obtained by grafting a tree at  $x$  on “the right” of  $\mathbf{t}$ :

$$\mathbb{T}(\mathbf{t}, x) = \{\mathbf{t} \otimes (\mathbf{s}, x), \mathbf{s} \in \mathbb{T}_\infty\}$$

as well as for  $k \in \mathbb{N}$ :

$$\mathbb{T}(\mathbf{t}, x, k) = \{\mathbf{s} \in \mathbb{T}(\mathbf{t}, x); k_x(\mathbf{s}) = k\} \quad \text{and} \quad \mathbb{T}_+(\mathbf{t}, x, k) = \{\mathbf{s} \in \mathbb{T}(\mathbf{t}, x); k_x(\mathbf{s}) \geq k\}$$

the subsets of  $\mathbb{T}(\mathbf{t}, x)$  such that the number of offspring of  $x$  are resp.  $k$  and  $k$  or more. It is easy to see that  $\mathbb{T}_+(\mathbf{t}, x, k)$  is closed. It is also open, as for all  $\mathbf{s} \in \mathbb{T}_+(\mathbf{t}, x, k)$  we have that  $B_\infty(\mathbf{s}, 2^{-\max(k, H_\infty(\mathbf{t})) - 1}) \subset \mathbb{T}_+(\mathbf{t}, x, k)$ .

Moreover, notice that the set  $\mathbb{T}_2$  is a Borel subset of the set  $\mathbb{T}$ . The next lemma gives another criterion for the convergence in distribution in  $\mathbb{T}_0 \cup \mathbb{T}_2$ . Its proof is very similar to the proof of Lemma 2.1 in [1].

**Lemma 2.2.** *Let  $(T_n, n \in \mathbb{N}^*)$  and  $T$  be  $\mathbb{T}_\infty$ -valued random variables which belong a.s. to  $\mathbb{T}_0 \cup \mathbb{T}_2$ . The sequence  $(T_n, n \in \mathbb{N}^*)$  converges in distribution to  $T$  if and only if for every  $\mathbf{t} \in \mathbb{T}_0$ ,  $x \in \mathbf{t}$  and  $k \in \mathbb{N}$ , we have:*

$$(9) \quad \lim_{n \rightarrow +\infty} \mathbb{P}(T_n \in \mathbb{T}_+(\mathbf{t}, x, k)) = \mathbb{P}(T \in \mathbb{T}_+(\mathbf{t}, x, k)) \quad \text{and} \quad \lim_{n \rightarrow +\infty} \mathbb{P}(T_n = \mathbf{t}) = \mathbb{P}(T = \mathbf{t}).$$

*Remark 2.3.* Let

$$\mathbb{T}_1 = \{\mathbf{t} \in \mathbb{T}; \exists! v \in (\mathbb{N}^*)^\infty \text{ s.t. } \bar{\mathbf{v}} \subset \mathbf{t}\},$$

be the subset of trees with only one infinite spine (or branch). We give in [1] a characterization of the convergence in  $\mathbb{T}_0 \cup \mathbb{T}_1$  as follows. Let  $(T_n, n \in \mathbb{N}^*)$  and  $T$  be  $\mathbb{T}$ -valued random variables which belong a.s. to  $\mathbb{T}_0 \cup \mathbb{T}_1$ . The sequence  $(T_n, n \in \mathbb{N}^*)$  converges in distribution to  $T$  if and only if (9) holds for every  $\mathbf{t} \in \mathbb{T}_0$ ,  $x \in \mathcal{L}_0(\mathbf{t})$  and  $k = 0$ . In a sense, the convergence in  $\mathbb{T}_0 \cup \mathbb{T}_1$  is thus easier to check.

*Proof.* The subclass  $\mathcal{F} = \{\mathbb{T}_+(\mathbf{t}, x, k) \cap (\mathbb{T}_0 \cup \mathbb{T}_2), \mathbf{t} \in \mathbb{T}_0, x \in \mathbf{t}, k \in \mathbb{N}\} \cup \{\{\mathbf{t}\}, \mathbf{t} \in \mathbb{T}_0\}$  of Borel sets on  $\mathbb{T}_0 \cup \mathbb{T}_2$  forms a  $\pi$ -system since we have

$$\mathbb{T}_+(\mathbf{t}_1, x_1, k_1) \cap \mathbb{T}_+(\mathbf{t}_2, x_2, k_2) = \begin{cases} \mathbb{T}_+(\mathbf{t}_1, x_1, k_1) & \text{if } \mathbf{t}_1 \in \mathbb{T}(\mathbf{t}_2, x_2) \text{ and } x_2 \in A_{x_1}, \\ \mathbb{T}_+(\mathbf{t}_1, x_1, k_1 \vee k_2) & \text{if } \mathbf{t}_1 \in \mathbb{T}(\mathbf{t}_2, x_2) \text{ and } x_1 = x_2, \\ \{\mathbf{t}_1\} & \text{if } \mathbf{t}_1 \in \mathbb{T}(\mathbf{t}_2, x_2) \text{ and } x_2 \notin A_{x_1} \cup \{x_1\}, \\ \emptyset & \text{in the other (non-symmetric) cases.} \end{cases}$$

For every  $h \in \mathbb{N}$  and every  $\mathbf{t} \in \mathbb{T}_\infty^{(h)}$ , we have that  $\mathbf{t}'$  belongs to  $r_{h,\infty}^{-1}(\{\mathbf{t}\}) \cap \mathbb{T}_2$  if and only if  $\mathbf{t}'$  belongs to some  $\mathbb{T}_+(\mathbf{s}, x, k) \cap \mathbb{T}_2$  with  $x \in \mathbf{t}$  such that  $|x|_\infty = h$  and  $\mathbf{s}$  belongs to  $r_{h,\infty}^{-1}(\{\mathbf{t}\}) \cap \mathbb{T}_0$  with  $x \in \mathbf{s}$ . Since  $\mathbb{T}_0$  is countable, we deduce that  $\mathcal{F}$  generates the Borel  $\sigma$ -field on  $\mathbb{T}_0 \cup \mathbb{T}_2$ . In particular  $\mathcal{F}$  is a separating class in  $\mathbb{T}_0 \cup \mathbb{T}_2$ . Since  $A \in \mathcal{F}$  is closed and open as well, according to Theorem 2.3 of [2], to prove that the family  $\mathcal{F}$  is a convergence determining class, it is enough to check that, for all  $\mathbf{t} \in \mathbb{T}_0 \cup \mathbb{T}_2$  and  $h \in \mathbb{N}$ , there exists  $A \in \mathcal{F}$  such that:

$$(10) \quad \mathbf{t} \in A \subset B_\infty(\mathbf{t}, 2^{-h}).$$

If  $\mathbf{t} \in \mathbb{T}_0$ , this is clear as  $\{\mathbf{t}\} = B_\infty(\mathbf{t}, 2^{-h})$  for all  $h > H_\infty(\mathbf{t})$ . If  $\mathbf{t} \in \mathbb{T}_2$ , for all  $\mathbf{s} \in \mathbb{T}_0$  and  $x \in \mathbf{s}$  such that  $\mathbf{t} \in \mathbb{T}_+(\mathbf{s}, x, k)$ , with  $k = k_x(\mathbf{s})$ , we have  $\mathbf{t} \in \mathbb{T}_+(\mathbf{s}, x, k) \subset B_\infty(\mathbf{t}, 2^{-|x|_\infty})$ . Since we can find such a  $\mathbf{s}$  and  $x$  such that  $|x|_\infty$  is arbitrary large, we deduce that (10) is satisfied. This proves that the family  $\mathcal{F}$  is a convergence determining class in  $\mathbb{T}_0 \cup \mathbb{T}_2$ . Since, for  $\mathbf{t} \in \mathbb{T}_0$ ,  $x \in \mathbf{t}$  and  $k \in \mathbb{N}$ , the sets  $\mathbb{T}_+(\mathbf{t}, x, k)$  and  $\{\mathbf{t}\}$  are open and closed, we deduce from the portmanteau theorem that if  $(T_n, n \in \mathbb{N}^*)$  converges in distribution to  $T$ , then (9) holds for every  $\mathbf{t} \in \mathbb{T}_0$ ,  $x \in \mathbf{t}$  and  $k \in \mathbb{N}$ .  $\square$

### 3. GW TREES

**3.1. Definition.** Let  $p = (p(n), n \in \mathbb{N})$  be a probability distribution on the set of the non-negative integers. We assume that  $p$  satisfies (1). Let  $g(z) = \sum_{k \in \mathbb{N}} p(k) z^k$  be the generating function of  $p$ . We denote by  $\rho(p)$  its convergence radius and we will write  $\rho$  for  $\rho(p)$  when it is clear from the context. We say that  $p$  is aperiodic if  $\{k; p(k) > 0\} \subset d\mathbb{N}$  implies  $d = 1$ .

A  $\mathbb{T}$ -valued random variable  $\tau$  is a Galton-Watson (GW) tree with offspring distribution  $p$  if the distribution of  $k_\emptyset(\tau)$  is  $p$  and for  $n \in \mathbb{N}^*$ , conditionally on  $\{k_\emptyset(\tau) = n\}$ , the sub-trees  $(\mathcal{S}_1(\tau), \mathcal{S}_2(\tau), \dots, \mathcal{S}_n(\tau))$  are independent and distributed as the original tree  $\tau$ . Equivalently, for every  $h \in \mathbb{N}^*$  and  $\mathbf{t} \in \mathbb{T}_\infty^{(h)}$ , we have:

$$\mathbb{P}(r_{h,\infty}(\tau) = \mathbf{t}) = \prod_{u \in r_{h-1,\infty}(\mathbf{t})} p(k_u(\mathbf{t})).$$

In particular, the restriction of the distribution of  $\tau$  on the set  $\mathbb{T}_0$  is given by:

$$(11) \quad \forall \mathbf{t} \in \mathbb{T}_0, \quad \mathbb{P}(\tau = \mathbf{t}) = \prod_{u \in \mathbf{t}} p(k_u(\mathbf{t})).$$

The GW tree is called critical (resp. sub-critical, super-critical) if  $\mu(p) = 1$  (resp.  $\mu(p) < 1$ ,  $\mu(p) > 1$ ). In the critical and sub-critical case, we have that a.s.  $\tau$  belongs to  $\mathbb{T}_0$ .

Let  $\mathbb{P}_k$  be the distribution of the forest  $\tau^{(k)} = (\tau_1, \dots, \tau_k)$  of i.i.d. GW trees with offspring distribution  $p$ . We set:

$$|\tau^{(k)}| = \sum_{j=1}^k |\tau_j|.$$



When there is no confusion, we shall write  $\tau$  for  $\tau^{(k)}$ .

**3.2. Condensation tree.** We say that the offspring distribution  $p$  is non-generic if  $g$  has convergence radius 1 and  $\mu(p) = g'(1) < 1$ . The corresponding GW tree is also called non-generic.

Assume that  $p$  satisfies (1) with  $\mu(p) < 1$ . Recall the definition of the tree  $\tau^*(p)$  in the introduction. Remark that, as  $\mu(p) < 1$ , the tree  $\tau^*(p)$  belongs a.s. to  $\mathbb{T}_2$  if  $p$  is non-generic.

For  $\mathbf{t} \in \mathbb{T}_0$ ,  $x \in \mathbf{t}$ , we set:

$$D(\mathbf{t}, x) = \frac{\mathbb{P}(\tau = \mathcal{S}^x(\mathbf{t}))}{p(0)} \mathbb{P}_{k_x(\mathbf{t})}(\tau = \mathcal{F}_x(\mathbf{t})).$$

For  $z \in \mathbb{R}$ , we set  $z_+ = \max(z, 0)$ . Let  $X$  be a random variable with distribution  $p$ . The following lemma is elementary.

**Lemma 3.1.** *Assume that  $p$  satisfies (1) and  $\mu(p) < 1$ . The distribution of  $\tau^*(p)$  is also characterized by: a.s.  $\tau^*(p) \in \mathbb{T}_2$  and for  $\mathbf{t} \in \mathbb{T}_0$ ,  $x \in \mathbf{t}$ ,  $k \in \mathbb{N}$ ,*

$$(12) \quad \mathbb{P}(\tau^*(p) \in \mathbb{T}_+(\mathbf{t}, x, k)) = D(\mathbf{t}, x) \left(1 - \mu(p) + \mathbb{E}[(X - k_x(\mathbf{t}))_+ \mathbf{1}_{\{X \geq k\}}]\right).$$

In particular, we have that if  $x \in \mathcal{L}_0(\mathbf{t})$ :

$$\mathbb{P}(\tau^*(p) \in \mathbb{T}(\mathbf{t}, x), k_x(\tau^*(p)) = +\infty) = (1 - \mu(p)) \frac{\mathbb{P}(\tau = \mathbf{t})}{p(0)}$$

and

$$(13) \quad \mathbb{P}(\tau^*(p) \in \mathbb{T}(\mathbf{t}, x)) = \frac{\mathbb{P}(\tau = \mathbf{t})}{p(0)}.$$

*Remark 3.2.* Let  $\tau^S(p)$  denote the limit (in distribution) of a critical or sub-critical GW tree  $\tau$  conditionally on  $\{H(\tau) = n\}$  or  $\{H(\tau) \geq n\}$  as  $n$  goes to infinity. The distribution of  $\tau^S(p)$  is characterized by the properties i) to vii) with  $\tilde{p}$  in iv) replaced by the size-biased distribution  $p^\circ$ :

$$p^\circ(k) = \frac{k p(k)}{\mu} \text{ for } k \in \mathbb{N}.$$

Remark that, when  $p$  is critical, the definitions of  $\tau^*(p)$  and  $\tau^S(p)$  coincide. We have that a.s.  $\tau^S(p)$  belongs to  $\mathbb{T}_1$ . Following [1], we notice that the distribution of  $\tau^S(p)$  is characterized by: a.s.  $\tau^S(p) \in \mathbb{T}_1$  and for all  $\mathbf{t} \in \mathbb{T}_0$ ,  $x \in \mathcal{L}_0(\mathbf{t})$ ,

$$(14) \quad \mathbb{P}(\tau^S(p) \in \mathbb{T}(\mathbf{t}, x)) = \frac{\mathbb{P}(\tau = \mathbf{t})}{\mu(p)^{|x|} p(0)}.$$

#### 4. CONDITIONING ON THE TOTAL POPULATION SIZE ( $\mathcal{A} = \mathbb{N}$ )

We prove Theorem 1.2 for  $\mathcal{A} = \mathbb{N}$  and  $p$  non-generic for  $\mathbb{N}$ . The results of this section appear already in [6] see also [7]. It is a special case of Theorem 1.2 with  $\mathcal{A} = \mathbb{N}$ . We provide here an elementary proof relying on the strong ratio limit property of random walks on the integers.

4.1. **The case  $\rho(p) = 1$ .** We first consider the case  $\rho(p) = 1$  and  $\mu(p) < 1$ .

**Theorem 4.1.** *Assume that  $p$  satisfies (1) and is non-generic for  $\mathbb{N}$ . We have that:*

$$(15) \quad \text{dist}(\tau \mid |\tau| = n) \xrightarrow{n \rightarrow +\infty} \text{dist}(\tau^*(p)),$$

where the limit is understood along the infinite subsequence  $\{n \in \mathbb{N}^*; \mathbb{P}(|\tau| = n) > 0\}$ , and:

$$(16) \quad \text{dist}(\tau \mid |\tau| \geq n) \xrightarrow{n \rightarrow +\infty} \text{dist}(\tau^*(p)).$$

*Proof.* For simplicity, we shall assume that  $p$  is aperiodic, that is  $\mathbb{P}(|\tau| = n) > 0$  for all  $n$  large enough. The adaptation to the periodic case is left to the reader.

Recall  $\rho(p) = 1$ . Let  $k \in \mathbb{N}$ ,  $\mathbf{t} \in \mathbb{T}_0$ ,  $x \in \mathbf{t}$ ,  $\ell = k_x(\mathbf{t})$  and  $m = |\mathbf{t}|$ . We have:

$$\mathbb{P}(\tau \in \mathbb{T}_+(\mathbf{t}, x, k), |\tau| = n) = D(\mathbf{t}, x) \sum_{j \geq \max(\ell+1, k)} p(j) \mathbb{P}_{j-\ell}(|\tau| = n - m).$$

Let  $(X_n, n \in \mathbb{N}^*)$  be a sequence of independent random variables taking values in  $\mathbb{N}$  with distribution  $p$  and set  $S_n = \sum_{k=1}^n X_k$ . Let us recall Dwass formula (see [4]): for every  $k \in \mathbb{N}^*$  and every  $n \geq k$ , we have

$$(17) \quad \mathbb{P}_k(|\tau| = n) = \frac{k}{n} \mathbb{P}(S_n = n - k).$$

Let  $\tau_n$  be distributed as  $\tau$  conditionally on  $\{|\tau| = n\}$ . Using Dwass formula (17), we have

$$\begin{aligned} \mathbb{P}(\tau_n \in \mathbb{T}_+(\mathbf{t}, x, k)) &= \frac{\mathbb{P}(\tau \in \mathbb{T}_+(\mathbf{t}, x, k), |\tau| = n)}{\mathbb{P}(|\tau| = n)} \\ &= D(\mathbf{t}, x) \sum_{j \geq \max(\ell+1, k)} p(j) \frac{\mathbb{P}_{j-\ell}(|\tau| = n - m)}{\mathbb{P}(|\tau| = n)} \\ &= D(\mathbf{t}, x) \sum_{j \geq \max(\ell+1, k)} p(j) n \frac{j - \ell}{n - m} \frac{\mathbb{P}(S_{n-m} = n - m - j + \ell)}{\mathbb{P}(S_n = n - 1)}. \end{aligned}$$

We then set

$$(18) \quad \delta_n^0(k, \ell) = \frac{1}{\mathbb{P}(S_n = n)} \sum_{j \geq k} p(j) \mathbb{P}(S_n = n + \ell - j)$$

and

$$(19) \quad \delta_n^1(k, \ell) = \frac{1}{\mathbb{P}(S_n = n)} \sum_{j \geq k} j p(j) \mathbb{P}(S_n = n + \ell - j).$$

We get:

$$\begin{aligned} \mathbb{P}(\tau_n \in \mathbb{T}_+(\mathbf{t}, x, k)) &= D(\mathbf{t}, x) \frac{n}{n - m} \frac{\mathbb{P}(S_{n-m} = n - m)}{\mathbb{P}(S_n = n - 1)} \\ &\quad (\delta_{n-m}^1(\max(\ell + 1, k), \ell) - \ell \delta_{n-m}^0(\max(\ell + 1, k), \ell)). \end{aligned}$$

Then use the strong ratio limit property (44) as well as its consequences (45) and (46), to get that:

$$(20) \quad \lim_{n \rightarrow +\infty} \mathbb{P}(\tau_n \in \mathbb{T}_+(\mathbf{t}, x, k)) = D(\mathbf{t}, x) \left( 1 - \mu(p) + \sum_{j \geq \max(\ell+1, k)} (j - \ell) p(j) \right).$$

Thanks to (12), we get:

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\tau_n \in \mathbb{T}_+(\mathbf{t}, x, k)) = \mathbb{P}(\tau^*(p) \in \mathbb{T}_+(\mathbf{t}, x, k)).$$

Then use Lemma 2.2 to get (15). Since  $\text{dist}(\tau \mid |\tau| \geq n)$  is a mixture of  $\text{dist}(\tau \mid |\tau| = k)$  for  $k \geq n$ , we deduce that (16) holds.  $\square$

*Remark 4.2.* The proof of (20) also holds if  $\mu = 1$ . In this case we get in particular that for all  $\mathbf{t} \in \mathbb{T}_0$  and  $x \in \mathcal{L}_0(\mathbf{t})$ :

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\tau_n \in \mathbb{T}(\mathbf{t}, x)) = \frac{\mathbb{P}(\tau = \mathbf{t})}{p(0)}.$$

Then the application  $\mathbb{T}(\mathbf{t}, x) \mapsto \mathbb{P}(\tau = \mathbf{t})/p(0)$  can be extended into a probability distribution on  $\mathbb{T}_1$  which is given by the distribution of  $\tau^*(p)$  (also equal to the distribution of  $\tau^S$  defined in Remark 3.2). Then use Remark 2.3 to get that  $\text{dist}(\tau \mid |\tau| = n)$  converges to  $\text{dist}(\tau^*(p))$ .

**4.2. The case  $\rho(p) > 1$ .** We consider the case  $\rho(p) > 1$ . The offspring distribution  $p_{\mathbb{N}, \theta}$  of (3) has generating function:

$$g_\theta(z) = \frac{g(\theta z)}{g(\theta)}.$$

Recall  $I_{\mathbb{N}}$  is the set of positive  $\theta$  for which  $p_{\mathbb{N}, \theta}$  is a well defined probability distribution. Furthermore, according to [9] (see also Proposition 5.5 in [1] for a more general setting), if  $\tau_{\mathbb{N}, \theta}$  denotes a GW tree with offspring distribution  $p_{\mathbb{N}, \theta}$ , then the distribution of  $\tau_{\mathbb{N}, \theta}$  conditionally on  $|\tau_{\mathbb{N}, \theta}|$  does not depend on  $\theta \in I_{\mathbb{N}}$ . It is easy to check that  $\mu(p_{\mathbb{N}, \theta})$  is increasing in  $\theta$ . Following [6], we shall say that  $p$  is non-generic for  $\mathbb{N}$  if  $\lim_{\theta \uparrow \rho(p)} \mu(p_{\mathbb{N}, \theta}) < 1$ . In that case, we have  $I_{\mathbb{N}} = (0, \rho(p)]$  and  $p_{\mathbb{N}}^*$  defined by (5) is  $p_{\mathbb{N}}^* = p_{\mathbb{N}, \rho(p)}$ .

**Corollary 4.3.** *Assume that  $p$  satisfies (1) and is non-generic for  $\mathbb{N}$ . We have that:*

$$\text{dist}(\tau \mid |\tau| = n) \xrightarrow[n \rightarrow +\infty]{} \text{dist}(\tau^*(p_{\mathbb{N}}^*)),$$

where the limit is understood along the infinite subsequence  $\{n \in \mathbb{N}^*; \mathbb{P}(|\tau| = n) > 0\}$ , and:

$$\text{dist}(\tau \mid |\tau| \geq n) \xrightarrow[n \rightarrow +\infty]{} \text{dist}(\tau^*(p_{\mathbb{N}}^*)).$$

*Proof.* The first convergence is a direct consequence of (15) and the fact that  $\tau$  conditionally on  $\{|\tau| = n\}$  is distributed as  $\tau_{\mathbb{N}, \rho(p)}$  conditionally on  $\{|\tau_{\mathbb{N}, \rho(p)}| = n\}$ . The proof of the second convergence is similar to the proof of (16).  $\square$

This result with Proposition 4.6 and Corollary 5.9 in [1] ends the proof of Theorem 1.2 for the case  $\mathcal{A} = \mathbb{N}$  and gives a complete description of the asymptotic distribution of critical and sub-critical GW trees conditioned to have a large total population size.

## 5. GENERIC AND NON-GENERIC DISTRIBUTIONS

Let  $p$  be a distribution on  $\mathbb{N}$  satisfying (1) and let  $X$  be a random variable with distribution  $p$ . Recall  $\rho(p)$  denotes the convergence radius of the generating function  $g$  of  $p$ . Let  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$ . We consider the modified distribution  $p_{\mathcal{A}, \theta}$  on  $\mathbb{N}$  given by (3) and let  $I_{\mathcal{A}}$  be the set of positive  $\theta$  for which  $p_{\mathcal{A}, \theta}$  is a probability distribution. We have  $\theta \in I_{\mathcal{A}}$  if and only if  $\theta > 0$  and:

$$(21) \quad \mathbb{E}[\theta^X \mathbf{1}_{\{X \in \mathcal{A}\}}] < +\infty \quad \text{and} \quad \mathbb{E}[\theta^X \mathbf{1}_{\{X \in \mathcal{A}^c\}}] \leq \theta.$$

In particular,  $I_{\mathcal{A}}$  is an interval of  $(0, +\infty)$  which contains 1. We have  $\inf I_{\mathcal{A}} = 0$  if  $0 \in \mathcal{A}$  and  $1 > \inf I_{\mathcal{A}} \geq p(0)$  if  $0 \notin \mathcal{A}$ . Let:

$$(22) \quad \theta_{\mathcal{A}}^* = \sup I_{\mathcal{A}} \in [1, \rho(p)].$$

We deduce from the definition of  $p_{\mathcal{A},\theta}$  the following rule of composition, for  $\theta \in \mathcal{A}$  and  $\theta q \in \mathcal{A}$ :

$$(23) \quad p_{\mathcal{A},\theta q} = (p_{\mathcal{A},\theta})_{\mathcal{A},q}.$$

The generating function,  $g_{\mathcal{A},\theta}$ , of  $p_{\mathcal{A},\theta}$  is given by:

$$g_{\mathcal{A},\theta}(z) = \mathbb{E} \left[ (z\theta)^X \left( \frac{1}{\theta} \mathbf{1}_{\mathcal{A}^c}(X) + c_{\mathcal{A}}(\theta) \mathbf{1}_{\mathcal{A}}(X) \right) \right].$$

And we have:

$$(24) \quad \mu(p_{\mathcal{A},\theta}) = \mathbb{E} [X\theta^{X-1} \mathbf{1}_{\{X \in \mathcal{A}^c\}}] + c_{\mathcal{A}}(\theta) \mathbb{E} [X\theta^X \mathbf{1}_{\{X \in \mathcal{A}\}}].$$

Let:

$$(25) \quad \theta_{\mathcal{A}}^c = \inf \{ \theta \in I_{\mathcal{A}}; \mu(p_{\mathcal{A},\theta}) = 1 \},$$

with the convention that  $\inf \emptyset = +\infty$ . Notice that the function  $\theta \mapsto \mu(p_{\mathcal{A},\theta})$  is continuous over  $I_{\mathcal{A}}$ .

**Lemma 5.1.** *Let  $p$  be a distribution on  $\mathbb{N}$  satisfying (1) and  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$ . The function  $\theta \mapsto \mu(p_{\mathcal{A},\theta})$  is increasing over  $(0, \theta_{\mathcal{A}}^c + \varepsilon) \cap I_{\mathcal{A}}$  for some strictly positive  $\varepsilon$  depending on  $p$ . If  $0 \in \mathcal{A}$ , then the function  $\theta \mapsto \mu(p_{\mathcal{A},\theta})$  is increasing over  $I_{\mathcal{A}}$ .*

*Proof.* Notice it is enough to consider  $\theta < \theta_{\mathcal{A}}^*$ . Since  $p$  satisfies (1), it is easy to check that  $p_{\mathcal{A},\theta}$  satisfies (1) for all  $\theta \in I_{\mathcal{A}}$  such that  $\theta < \theta_{\mathcal{A}}^*$ . Thanks to the composition rule, it is enough to prove that  $\theta \mapsto \mu_{\mathcal{A},\theta}$  is increasing at  $\theta = 1$  if  $\mu(p) \leq 1 + \varepsilon$  for some  $\varepsilon > 0$ , with  $p$  satisfying (1) and  $\rho(p) > 1$ .

Let  $\theta \in I_{\mathcal{A}}$ . We have:

$$\mu_{\mathcal{A},\theta} - \mathbb{E}[X] = \frac{h_{\mathcal{A}}(\theta)}{\theta \mathbb{E}[\theta^X \mathbf{1}_{\mathcal{A}}(X)]},$$

with

$$h_{\mathcal{A}}(\theta) = \mathbb{E} [X\theta^X \mathbf{1}_{\mathcal{A}^c}(X)] \mathbb{E} [\theta^X \mathbf{1}_{\mathcal{A}}(X)] + \theta \mathbb{E} [X\theta^X \mathbf{1}_{\mathcal{A}}(X)] \\ - \mathbb{E} [\theta^X \mathbf{1}_{\mathcal{A}^c}(X)] \mathbb{E} [X\theta^X \mathbf{1}_{\mathcal{A}}(X)] - \theta \mathbb{E}[X] \mathbb{E} [\theta^X \mathbf{1}_{\mathcal{A}}(X)].$$

Of course we have  $h_{\mathcal{A}}(1) = 0$ . The function  $h_{\mathcal{A}}$  is of class  $\mathcal{C}^\infty$  on  $[0, \rho(p))$ . We obtain:

$$h'_{\mathcal{A}}(1) = \mathbb{E} [(X-1)(Xp(\mathcal{A}) - \mathbb{E}[X\mathbf{1}_{\mathcal{A}}(X)])] = p(\mathcal{A})\mathbb{E}[X(X-1)] + (1 - \mathbb{E}[X])\mathbb{E}[X\mathbf{1}_{\mathcal{A}}(X)].$$

In particular, we deduce from this last expression that  $h'_{\mathcal{A}}(1) > 0$  if  $\mathbb{E}[X] \leq 1$ . However, since  $p(\mathcal{A})\mathbb{E}[X(X-1)] > 0$  as  $p$  satisfies (1), we deduce that  $h'_{\mathcal{A}}(1) > 0$  as soon as  $\mathbb{E}[X] < 1 + \varepsilon$  for some small positive  $\varepsilon$ . This ends the proof of the first part of the lemma.

Let us assume that  $0 \in \mathcal{A}$ . Thanks to the first part, if  $\mathbb{E}[X] = \mu(p) > 1$ , elementary computations yield that  $h'_{\mathcal{A}}(1)/\mathbb{P}(\mathcal{A})$  is minimal, that is  $\mathbb{E}[X\mathbf{1}_{\mathcal{A}}(X)]/\mathbb{P}(\mathcal{A})$  is maximal, (for all subsets  $\mathcal{A}$  of  $\mathbb{N}$  containing 0) for  $\mathcal{A}$  of the form  $\mathcal{A}_n = \{0\} \cup \{k; k \geq n\}$ . It is then easy to check that the function  $n \mapsto h'_{\mathcal{A}_n}(1)$  is first non decreasing and then non increasing. Since  $h'_{\mathcal{A}_0}(1)$  and  $h'_{\mathcal{A}_\infty}(1)$  are positive, we get that  $h'_{\mathcal{A}_n}(1)$  is positive for all  $n \in \mathbb{N}$  and thus  $h'_{\mathcal{A}}(1)$  is positive. This ends the proof of the second part of the lemma.  $\square$

Let us consider the equation:

$$(26) \quad \mu(p_{\mathcal{A},\theta}) = 1.$$

**Lemma 5.2.** *Let  $p$  be a distribution on  $\mathbb{N}$  satisfying (1) and  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$ . Equation (26) has at most one solution. If there is no solution to Equation (26), then we have  $\mu(p) < 1$ ,  $\theta_{\mathcal{A}}^*$  belongs to  $I_{\mathcal{A}}$  and  $\mu(p_{\mathcal{A},\theta_{\mathcal{A}}^*}) < 1$ .*

The (unique) solution of (26), if it exists, is denoted  $\theta_{\mathcal{A}}^c$ . Notice that  $p_{\mathcal{A},\theta_{\mathcal{A}}^c}$  is critical.

*Proof.* Lemma 5.1 directly implies that Equation (26) has at most one solution.

If  $0 \in \mathcal{A}$ , then we have  $\inf_{I_{\mathcal{A}}} \mu(p_{\mathcal{A},\theta}) = p(1)\mathbf{1}_{\mathcal{A}^c}(1) < 1$ . If  $0 \notin \mathcal{A}$ , then set  $q = \min I_{\mathcal{A}} \in (0, 1)$ . Notice that  $c_{\mathcal{A}}(q) = 0$  and  $\mathbb{E}[q^X \mathbf{1}_{\{X \in \mathcal{A}^c\}}] = q$ . Use that the function  $\theta \mapsto \mathbb{E}[\theta^X \mathbf{1}_{\{X \in \mathcal{A}^c\}}]$  is convex and less than the identity map on  $(q, 1]$  to deduce that  $\mathbb{E}[Xq^{X-1} \mathbf{1}_{\{X \in \mathcal{A}^c\}}]$  is strictly less than 1. Then use (24) to deduce that:

$$\lim_{\theta \downarrow q} \mu(p_{\mathcal{A},\theta}) = \mathbb{E}[Xq^{X-1} \mathbf{1}_{\{X \in \mathcal{A}^c\}}] < 1.$$

In conclusion, we deduce that  $\inf_{I_{\mathcal{A}}} \mu(p_{\mathcal{A},\theta}) < 1$ . Hence, if  $\mu(p) \geq 1$  then Equation (26) has at least one solution.

From what precedes, if there is no solution to Equation (26), this implies that  $\mu(p) < 1$  and thus:

$$(27) \quad \mu(p_{\mathcal{A},\theta}) < 1 \quad \text{for all } \theta \in I_{\mathcal{A}}.$$

We only need to consider the case  $\theta_{\mathcal{A}}^* > 1$ . Since  $\theta_{\mathcal{A}}^* \leq \rho(p)$ , we have  $\rho(p) > 1$ . Since  $\mu(p) < 1$ , the interval  $J = \{\theta; g(\theta) < \theta\}$  is non-empty and  $\inf J = 1$ . On  $J \cap I_{\mathcal{A}}$ , we deduce from (4) that  $\theta c_{\mathcal{A}}(\theta) > 1$  and then from (24) that  $\mu(p_{\mathcal{A},\theta}) > g'(\theta)$  and thus  $g'(\theta) < 1$ . Notice this implies that  $I_{\mathcal{A}} \cap (1, +\infty)$  is a subset of  $\bar{J}$  the closure of  $J$ . The properties on  $g$  imply that  $\bar{J} = \{\theta; g(\theta) \leq \theta\}$ . This clearly implies that (21) holds for  $\theta_{\mathcal{A}}^*$  that is  $\theta_{\mathcal{A}}^* \in I_{\mathcal{A}}$ . Then conclude using (27).  $\square$

**Definition 5.3.** *Let  $p$  be a distribution on  $\mathbb{N}$  satisfying (1) and  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$ . If Equation (26) has a (unique) solution, then  $p$  is called generic for  $\mathcal{A}$ . If Equation (26) has no solution, then  $p$  is called non-generic for  $\mathcal{A}$ .*

In the next lemma, we write  $\rho$  for  $\rho(p)$ .

**Lemma 5.4.** *Let  $p$  be a distribution on  $\mathbb{N}$  satisfying (1) such that  $\mu(p) < 1$ .*

- If  $\rho = +\infty$  or  $\rho < +\infty$  and  $g'(\rho) \geq 1$ , then  $p$  is generic for any  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$ .
- If  $\rho = 1$  and  $g'(1) < 1$ , then  $p$  is non-generic for all  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$ .
- If  $1 < \rho < +\infty$  and  $g'(\rho) < 1$  (and thus  $g(\rho) < \rho$ ), then  $p$  is non-generic for  $\{0\}$  and  $p$  is generic for  $\{k\}$  for all  $k$  large enough and such that  $p(k) > 0$ . Furthermore  $p$  is non-generic for  $\mathcal{A} \subset \mathbb{N}$  (with  $p(\mathcal{A}) > 0$ ) if and only if:

$$\mathbb{E}[Y|Y \in \mathcal{A}] < \frac{\rho - \rho g'(\rho)}{\rho - g(\rho)},$$

with  $Y$  distributed as  $p_{\mathbb{N},\rho}$ , that is  $\mathbb{E}[f(Y)] = \mathbb{E}[f(X)\rho^X]/g(\rho)$  for every non-negative measurable function  $f$ . We also have  $\theta_{\mathcal{A}}^* = \rho$ .

*Remark 5.5.* We give some consequences and remarks related to the previous Lemma.

- (1) If  $p$  is generic for  $\{0\}$  then it is generic for all  $\mathcal{A} \subset \mathbb{N}$  with  $p(\mathcal{A}) > 0$ .
- (2) If  $\mathcal{A}$  and  $\mathcal{B}$  are disjoint subsets of  $\mathbb{N}$  such that  $p(\mathcal{A}) > 0$  and  $p(\mathcal{B}) > 0$ , then if  $p$  is non-generic for  $\mathcal{A}$  and for  $\mathcal{B}$  then it is non-generic for  $\mathcal{A} \cup \mathcal{B}$ .
- (3) If  $\mathcal{A}$  and  $\mathcal{B}$  are disjoint subsets of  $\mathbb{N}$  such that  $p(\mathcal{A}) > 0$  and  $p(\mathcal{B}) > 0$ , then if  $p$  is generic for  $\mathcal{A}$  and for  $\mathcal{B}$  then it is generic for  $\mathcal{A} \cup \mathcal{B}$ .

- (4) Assume  $\rho(p) > 1$  and  $\mathcal{A} \subset \mathcal{B}$  with  $p(\mathcal{B}) > p(\mathcal{A}) > 0$ .
- Then  $p$  non-generic for  $\mathcal{A}$  does not imply in general that  $p$  is non-generic for  $\mathcal{B}$ . (See case (6) below with  $\mathcal{A} = \{0\}$  and  $\mathcal{B} = \mathbb{N}$ .)
  - Then  $p$  non-generic for  $\mathcal{B}$  does not imply in general that  $p$  is non-generic for  $\mathcal{A}$ . (Let  $p$  satisfying (1) be such that  $\rho(p) > 1$  and  $p$  non-generic for  $\mathcal{B} = \mathbb{N}$ . Then, according to Lemma 5.4, there exists  $k$  large enough such that  $p(k) > 0$  and  $p$  is generic for  $\mathcal{A} = \{k\}$ .)
- (5) According to the second part of the proof of Lemma 5.1, we get that there exists  $n_0 \in \mathbb{N}^*$  such that:

$$\sup_{\mathcal{A} \ni 0} \mathbb{E}[Y|Y \in \mathcal{A}] = \mathbb{E}[Y|Y \in \mathcal{A}_{n_0}],$$

with  $\mathcal{A}_n = \{0\} \cup \{k; k \geq n\}$ . In particular, if  $p$  is non-generic for  $\mathcal{A}_{n_0}$  then it is non-generic for all  $\mathcal{A}$  containing 0.

- (6) Let  $G$  be a generating function with radius of convergence  $\rho_G = 1$ . Let  $c \in (0, 1)$ . Let  $p$  be the distribution with generating function:

$$g(z) = \frac{G(cz)}{G(c)}.$$

The radius of convergence of  $g$  is thus  $\rho = 1/c$  and we have:

$$g_{\mathbb{N},\rho}(z) = G(z) \quad \text{and} \quad g_{\{0\},\rho}(z) = \frac{cG(z)}{G(c)} + 1 - \frac{c}{G(c)}.$$

Therefore, we have:

$$g'_{\mathbb{N},\rho}(1) = G'(1) \quad \text{and} \quad g'_{\{0\},\rho}(1) = \frac{cG'(1)}{G(c)}.$$

If  $G'(1) = 1$ , then we have  $G(c) > c$ . This implies  $g'_{\{0\},\rho}(1) < g'_{\mathbb{N},\rho}(1) = 1$ . Thus  $p$  is generic for  $\mathbb{N}$  but non generic for  $\{0\}$ .

*Proof.* For  $\mathcal{A} \subset \mathbb{N}$  such that  $p(\mathcal{A}) > 0$  and  $\theta \in I_{\mathcal{A}}$ , notice that:

$$(28) \quad \mu(p_{\mathcal{A},\theta}) - 1 = G_{\mathcal{A}}(\theta) \frac{\theta - g(\theta)}{\theta} - (1 - g'(\theta)) \quad \text{with} \quad G_{\mathcal{A}}(\theta) = \frac{\mathbb{E}[X\theta^X \mathbf{1}_{\mathcal{A}}(X)]}{\mathbb{E}[\theta^X \mathbf{1}_{\mathcal{A}}(X)]}.$$

If  $\rho = +\infty$  or  $\rho < +\infty$  and  $g'(\rho) \geq 1$ , then there exists  $q > 1$  finite such that  $g'(q) = 1$  which implies that  $q$  satisfies (21). We also have  $g(q) < q$ . This implies, thanks to (28), that  $\mu(p_{\mathcal{A},q}) > 1$ . Therefore,  $p$  is generic for  $\mathcal{A}$ .

If  $\rho < +\infty$  and  $g'(\rho) < 1$ , then we have  $g(\rho) < \rho$  and  $\rho$  satisfies (21). This implies that  $\theta_{\mathcal{A}}^* = \rho \in I_{\mathcal{A}}$ . According to Lemma 5.2,  $p$  is non-generic for  $\mathcal{A}$  if and only if  $\mu(p_{\mathcal{A},\rho}) < 1$  that is, using (28):

$$G_{\mathcal{A}}(\rho) < \frac{\rho - \rho g'(\rho)}{\rho - g(\rho)}.$$

We have  $G_{\{0\}}(\rho) = 0$  and thus  $p$  is non-generic for  $\{0\}$ . For  $k$  such that  $p(k) > 0$ , we have  $G_{\{k\}}(\rho) = k/\rho$  and thus  $p$  is generic for  $k$  large enough such that  $p(k) > 0$ . To conclude, notice that  $\rho G_{\mathcal{A}}(\rho) = \mathbb{E}[Y|Y \in \mathcal{A}]$ .  $\square$

6. VERTICES WITH A GIVEN NUMBER OF CHILDREN I: CASE  $0 \in \mathcal{A}$ 

Assume  $0 \in \mathcal{A} \subset \mathbb{N}$  and  $\mathcal{A} \neq \mathbb{N}$ . Assume that  $p$  satisfies (1),  $\mu(p) < 1$ . We prove Theorem 1.2 for  $p$  non-generic for  $\mathcal{A}$ .

In what follows, we denote by  $X$  a random variable distributed according to  $p$ . We consider only  $\mathbb{P}(X \in \mathcal{A}) < 1$ , as the case  $\mathbb{P}(X \in \mathcal{A}) = 1$  corresponds to  $\mathcal{A} = \mathbb{N}$  of Section 4. For  $\mathbf{t} \in \mathbb{T}_0$ , we set  $\mathcal{L}_{\mathcal{A}}(\mathbf{t}) = \{u \in \mathbf{t}, k_u(\mathbf{t}) \in \mathcal{A}\}$  the set of nodes whose number of children belongs to  $\mathcal{A}$  and define  $L_{\mathcal{A}}(\mathbf{t}) = \text{Card}(\mathcal{L}_{\mathcal{A}}(\mathbf{t}))$ .

For a tree  $\mathbf{t} \in \mathbb{T}_0$ , following [11, 14], we can map the set  $\mathcal{L}_{\mathcal{A}}(\mathbf{t})$  onto a tree  $\mathbf{t}^{\mathcal{A}}$ . We first define a map  $\phi$  from  $\mathcal{L}_{\mathcal{A}}(\mathbf{t})$  on  $\mathcal{U}$  and a sequence  $(\mathbf{t}_k)_{1 \leq k \leq n}$  of trees (where  $n = L_{\mathcal{A}}(\mathbf{t})$ ) as follows. Recall that we denote by  $<$  the lexicographic order on  $\mathcal{U}$ . Let  $u^1 < \dots < u^n$  be the ordered elements of  $\mathcal{L}_{\mathcal{A}}(\mathbf{t})$ .

- $\phi(u^1) = \emptyset$ ,  $\mathbf{t}_1 = \{\emptyset\}$ .
- For  $1 < k \leq n$ , set  $w^k = \text{MRCOA}(\{u^{k-1}, u^k\})$  the most recent common ancestor of  $u^{k-1}$  and  $u^k$  and recall that  $S_{w^k}(\mathbf{t})$  denotes the tree above  $w^k$ . We set  $\mathbf{s} = \{w^k u, u \in S_{w^k}\}$  the subtree above  $w^k$  and  $v = \min(\mathcal{L}_{\mathcal{A}}(\mathbf{s}))$ . Then, we set

$$\phi(u^k) = \phi(v)(k_{\phi(v)}(\mathbf{t}_{k-1}) + 1)$$

the concatenation of the node  $\phi(v)$  with the integer  $k_{\phi(v)}(\mathbf{t}_{k-1}) + 1$ , and

$$\mathbf{t}_k = \mathbf{t}_{k-1} \cup \{\phi(u^k)\}.$$

In other words,  $\phi(u^k)$  is a child of  $\phi(v)$  in  $\mathbf{t}_k$  and we add it “on the right” of the other children (if any) of  $\phi(v)$  in the previous tree  $\mathbf{t}_{k-1}$  to get  $\mathbf{t}_k$ .

It is clear by construction that  $\mathbf{t}_k$  is a tree for every  $k \leq n$ . We set  $\mathbf{t}^{\mathcal{A}} = \mathbf{t}_n$ . Then  $\phi$  is a one-to-one map from  $\mathcal{L}_{\mathcal{A}}(\mathbf{t})$  onto  $\mathbf{t}^{\mathcal{A}}$ . The construction of the tree  $\mathbf{t}^{\mathcal{A}}$  is illustrated on Figure 1. Notice that  $L_{\mathcal{A}}(\mathbf{t})$  is just the total progeny of  $\mathbf{t}^{\mathcal{A}}$ .

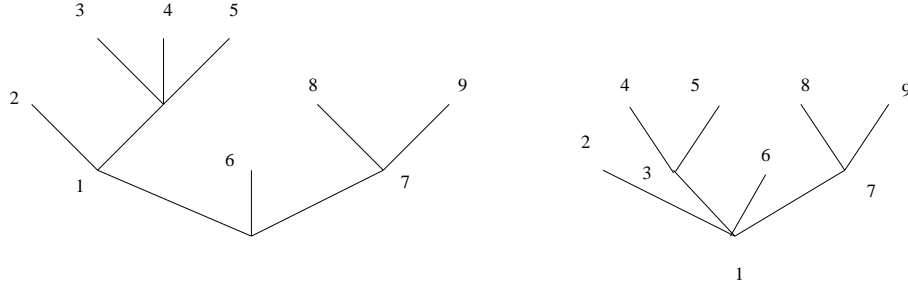


FIGURE 1. left: a tree  $\mathbf{t}$ , right: the tree  $\mathbf{t}^{\mathcal{A}}$  for  $\mathcal{A} = \{0, 2\}$

If  $\tau$  is a GW tree with offspring distribution  $p$ , the tree  $\tau^{\mathcal{A}}$  associated with  $\mathcal{L}_{\mathcal{A}}(\tau)$  is then, according to [14] Theorem 6 (for the particular case  $0 \in \mathcal{A}$ ), a GW tree whose offspring distribution  $p^{\mathcal{A}}$  is defined as follows. Let  $N$ ,  $Y''$  and  $(Y'_k, k \in \mathbb{N})$  be independent random variables such that  $N$  is geometric with parameter  $p(\mathcal{A})$ ,  $Y''$  is distributed as  $X$  conditionally on  $\{X \in \mathcal{A}\}$  and  $(Y'_k, k \in \mathbb{N})$  are independent random variables distributed as  $X - 1$  conditionally on  $\{X \notin \mathcal{A}\}$ . We set:

$$(29) \quad X_{\mathcal{A}} = \sum_{k=1}^{N-1} Y'_k + Y'',$$

with the convention that  $\sum_{\emptyset} = 0$ . Then  $p^{\mathcal{A}}$  is the distribution of  $X_{\mathcal{A}}$ . Let  $g^{\mathcal{A}}$  denote its generating function:

$$(30) \quad g^{\mathcal{A}}(z) = \frac{z \mathbb{E} [z^X \mathbf{1}_{\{X \in \mathcal{A}\}}]}{z - \mathbb{E} [z^X \mathbf{1}_{\{X \notin \mathcal{A}\}}]}.$$

An elementary computation gives:

$$(31) \quad \mu(p^{\mathcal{A}}) = 1 - \frac{1 - \mu(p)}{p(\mathcal{A})} \quad \text{and} \quad g^{\mathcal{A}}(\theta) = \frac{1}{c_{\mathcal{A}}(\theta)}.$$

We recover that if  $\tau$  is critical ( $\mu(p) = 1$ ) then  $\tau^{\mathcal{A}}$  is critical as  $\mu(p^{\mathcal{A}}) = 1$ , see also [14] Lemma 6. Notice in particular that for all  $k \in \mathcal{A}$ :

$$(32) \quad p^{\mathcal{A}}(k) = \mathbb{P}(X_{\mathcal{A}} = k) \geq \mathbb{P}(N = 1, Y'' = k) = p(k),$$

and for  $k \in \mathcal{A}^c$ :

$$(33) \quad p^{\mathcal{A}}(k-1) = \mathbb{P}(X_{\mathcal{A}} = k-1) \geq \mathbb{P}(N = 2, Y'_1 = k-1) = p(\mathcal{A})p(k).$$

**Lemma 6.1.** *Assume that  $p$  satisfies (1),  $\mu(p) < 1$ . Then  $p^{\mathcal{A}}$  satisfies (1),  $\mu(p^{\mathcal{A}}) < 1$  and  $\rho(p^{\mathcal{A}}) = \rho(p)$  if  $\rho(p) = 1$  or if  $\rho(p) > 1$  and  $g'(\rho(p)) < 1$ .*

*Proof.* Since (32) implies  $p^{\mathcal{A}}(0) \geq p(0)$  and that  $\mu(p) < 1$  with (31) implies  $\mu(p^{\mathcal{A}}) < 1$ , we deduce that  $p^{\mathcal{A}}$  satisfies (1).

Let  $\rho_{\mathcal{A}}$  be the convergence radius of the serie given by  $\mathbb{E} [z^X \mathbf{1}_{\{X \in \mathcal{A}\}}]$  and  $\rho_{\mathcal{A}^c}$  be the convergence radius of the series given by  $\mathbb{E} [z^X \mathbf{1}_{\{X \in \mathcal{A}^c\}}]$ . We get that  $\min(\rho_{\mathcal{A}}, \rho_{\mathcal{A}^c}) = \rho(p)$ . We deduce that the convergence radius of  $g^{\mathcal{A}}$  is  $\rho(p)$  if  $\rho(p) = 1$  or if  $\rho(p) > 1$  and  $g'(\rho(p)) < 1$ .  $\square$

**6.1. The case  $\rho(p) = 1$ .** We state now the main result of this section.

**Theorem 6.2.** *Assume that  $p$  satisfies (1),  $\mu(p) < 1$  and  $\rho(p) = 1$ . We have that:*

$$(34) \quad \text{dist}(\tau \mid L_{\mathcal{A}}(\tau) = n) \xrightarrow{n \rightarrow +\infty} \text{dist}(\tau^*(p)),$$

where the limit is understood along the infinite subsequence  $\{n \in \mathbb{N}^*; \mathbb{P}(L_{\mathcal{A}}(\tau) = n) > 0\}$ , as well as

$$(35) \quad \text{dist}(\tau \mid L_{\mathcal{A}}(\tau) \geq n) \xrightarrow{n \rightarrow +\infty} \text{dist}(\tau^*(p)).$$

*Proof.* For simplicity, we shall assume that  $p^{\mathcal{A}}$  is aperiodic. The adaptation to the periodic case is left to the reader. We define for  $j \in \mathbb{N}$  and  $n \geq 2$ :

$$(36) \quad n_j = n - \mathbf{1}_{\mathcal{A}}(j).$$

Let  $k \in \mathbb{N}$ ,  $\mathbf{t} \in \mathbb{T}_0$ ,  $x \in \mathbf{t}$ ,  $\ell = k_x(\mathbf{t})$  and  $m = |\mathbf{t}^{\mathcal{A}}| - \mathbf{1}_{\{x \in \mathcal{L}_{\mathcal{A}}(\mathbf{t})\}}$ . We have:

$$\mathbb{P}(\tau \in \mathbb{T}_+(\mathbf{t}, x, k), L_{\mathcal{A}}(\tau) = n) = D(\mathbf{t}, x) \sum_{j \geq \max(\ell+1, k)} p(j) \mathbb{P}_{j-\ell}(|\tau^{\mathcal{A}}| = n_j - m).$$

Let  $(X_n, n \in \mathbb{N}^*)$  be independent random variables taking values in  $\mathbb{N}$  with distribution  $p^{\mathcal{A}}$  and set  $S_n = \sum_{k=1}^n X_k$ . According to Dwass formula (17), we have:

$$\mathbb{P}_{j-\ell}(|\tau^{\mathcal{A}}| = n_j - m) = \frac{j-\ell}{n_j-m} \mathbb{P}(S_{n_j-m} = n_j - m - j + \ell).$$



Let  $\tau_n$  be distributed as  $\tau$  conditionally on  $\{L_{\mathcal{A}}(\tau) = n\}$ . Then we have, using (47) and (48):

$$\begin{aligned} \mathbb{P}(\tau_n \in \mathbb{T}_+(\mathbf{t}, x, k)) &= D(\mathbf{t}, x) \sum_{j \geq \max(\ell+1, k)} p(j)n \frac{j-\ell}{n_j - m} \\ &\quad \frac{\mathbb{P}(S_{n_j - m} = n_j - m - j + \ell)}{\mathbb{P}(S_n = n - 1)} \\ &= D(\mathbf{t}, x) \frac{n}{n - m} \frac{\mathbb{P}(S_{n-m} = n - m)}{\mathbb{P}(S_n = n - 1)} \\ &\quad \left( \delta_{n-m}^{1, \mathcal{A}}(\max(\ell + 1, k), \ell) - \ell \delta_{n-m}^{0, \mathcal{A}}(\max(\ell + 1, k), \ell) \right). \end{aligned}$$

Then use the generalizations of the strong ratio limit properties (44), (50) and (51) to get that:

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\tau_n \in \mathbb{T}_+(\mathbf{t}, x, k)) = D(\mathbf{t}, x) \left( 1 - \mu(p) + \sum_{j \geq \max(\ell, k)} (j - \ell)p(j) \right).$$

Thanks to (12), we get:

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\tau_n \in \mathbb{T}_+(\mathbf{t}, x, k)) = \mathbb{P}(\tau^*(p) \in \mathbb{T}_+(\mathbf{t}, x, k)).$$

Then use Lemma 2.2 to get (34). Since  $\text{dist}(\tau \mid L_{\mathcal{A}}(\tau) \geq n)$  is a mixture of  $\text{dist}(\tau \mid L_{\mathcal{A}}(\tau) = k)$  for  $k \geq n$ , we deduce that (35) holds.  $\square$

**6.2. The case  $\rho(p) > 1$ .** We consider the case  $p$  non-generic for  $\mathcal{A}$  with  $\rho(p) > 1$ . In particular, we have  $g'(\rho) < 1$  and  $g(\rho) < \rho$  thanks to Lemma 5.4. Recall the offspring distribution  $p_{\mathcal{A}, \theta}$  defined by (3). Notice that the normalizing constant  $c_{\mathcal{A}}(\theta)$  is given by:

$$(37) \quad c_{\mathcal{A}}(\theta) = \frac{\theta - \mathbb{E}[\theta^X \mathbf{1}_{\{X \in \mathcal{A}^c\}}]}{\theta \mathbb{E}[\theta^X \mathbf{1}_{\{X \in \mathcal{A}\}}]} = \frac{1}{g^{\mathcal{A}}(\theta)}.$$

Notice that  $p_{\mathcal{A}, 1} = p$ . Since  $\rho(p)$  is also the convergence radius of  $g^{\mathcal{A}}$ , see Lemma 6.1, we deduce that  $p_{\mathcal{A}, \theta}$  is well defined for  $\theta \in [0, \rho(p)]$  and  $\theta_{\mathcal{A}}^* = \rho(p)$ . Let  $g_{\mathcal{A}, \theta}$  be the generating function of  $p_{\mathcal{A}, \theta}$ .

According to [9] if  $\mathcal{A} = \{0\}$  and Proposition 5.5 in [1] for the general setting, if  $\tau_{\mathcal{A}, \theta}$  denotes a GW tree with offspring distribution  $p_{\mathcal{A}, \theta}$ , then the distribution of  $\tau_{\mathcal{A}, \theta}$  conditionally on  $L_{\mathcal{A}}(\tau_{\mathcal{A}, \theta})$  does not depend on  $\theta \in [0, \rho(p)]$ .

*Remark 6.3.* It is easy to check that:

$$(38) \quad (g_{\mathcal{A}, \theta})^{\mathcal{A}}(z) = \frac{g^{\mathcal{A}}(\theta z)}{g^{\mathcal{A}}(\theta)} = (g^{\mathcal{A}})_{\mathbb{N}, \theta}(z).$$

The distribution of  $\tau_{\mathcal{A}, \theta}$  is the distribution of  $\tau$  “shifted” by  $\theta$  such that the conditional distribution given the number of vertices having a number of children in  $\mathcal{A}$  is the same. Then, according to (38), the tree  $(\tau_{\mathcal{A}, \theta})^{\mathcal{A}}$  of vertices having a number of children in  $\mathcal{A}$  associated with  $\tau_{\mathcal{A}, \theta}$  is distributed as the distribution of  $\tau^{\mathcal{A}}$  “shifted” by  $\theta$  such that the conditional distribution given the total number of vertices is the same.

The proof of the following corollary is similar to the one of Corollary 4.3.

**Corollary 6.4.** *Assume that  $p$  satisfies (1) and is non-generic for  $\mathcal{A}$ . Let  $p_{\mathcal{A}}^* = p_{\mathcal{A}, \rho(p)}$ . We have that:*

$$\text{dist}(\tau \mid L_{\mathcal{A}}(\tau) = n) \xrightarrow{n \rightarrow +\infty} \text{dist}(\tau^*(p_{\mathcal{A}}^*)),$$

where the limit is understood along the infinite subsequence  $\{n \in \mathbb{N}^*; \mathbb{P}(L_{\mathcal{A}}(\tau) = n) > 0\}$ , as well as

$$\text{dist}(\tau \mid L_{\mathcal{A}}(\tau) \geq n) \xrightarrow[n \rightarrow +\infty]{} \text{dist}(\tau^*(p_{\mathcal{A}}^*)).$$

This result with Proposition 4.6 and Corollary 5.7 in [1] ends the proof of Theorem 1.2 for the case  $0 \in \mathcal{A}$ , and gives a complete description of the asymptotic distribution of critical and sub-critical GW trees conditioned to have a large number vertices with given number of children.

## 7. VERTICES WITH A GIVEN NUMBER OF CHILDREN II: CASE $0 \notin \mathcal{A}$

Let  $\mathcal{A} \subset \mathbb{N}$ . We assume in this section that  $0 \notin \mathcal{A}$  and  $p(\mathcal{A}) > 0$ . We prove Theorem 1.2 for  $p$  non-generic for  $\mathcal{A}$ . Notice we follow the spirit of the case  $0 \in \mathcal{A}$ .

**7.1. Setting and notations.** Although the construction of the previous section also holds in that case with a different offspring distribution, we failed to get analogues to formulas (32) and (33). Therefore, we prefer to map  $\mathcal{L}_{\mathcal{A}}(\tau)$  onto a forest  $\mathcal{F}_{\mathcal{A}}(\tau)$  of independent GW trees. Let us describe this map.

Let  $\mathbf{t} \in \mathbb{T}_0$ . We define a map  $\tilde{\phi}$  from  $\mathcal{L}_{\mathcal{A}}(\mathbf{t})$  into the set  $\bigcup_{n \geq 1} \mathbb{T}_0^n$  of forests of finite trees as follows.

First, for  $u \in \mathbf{t}$  we define  $S_u^{\mathcal{A}}(\mathbf{t})$  the subtree rooted at  $u$  with no progeny in  $\mathcal{A}$  by

$$S_u^{\mathcal{A}}(\mathbf{t}) = \{w \in uS_u(\mathbf{t}), A_w \cap A_u^c \cap \mathcal{L}_{\mathcal{A}}(\mathbf{t}) = \emptyset\}.$$

For  $u \in \mathbf{t}$ , we define  $C_u^{\mathcal{A}}(\mathbf{t})$  as the leaves of  $S_u^{\mathcal{A}}(\mathbf{t})$  that belong to  $\mathcal{A}$ .

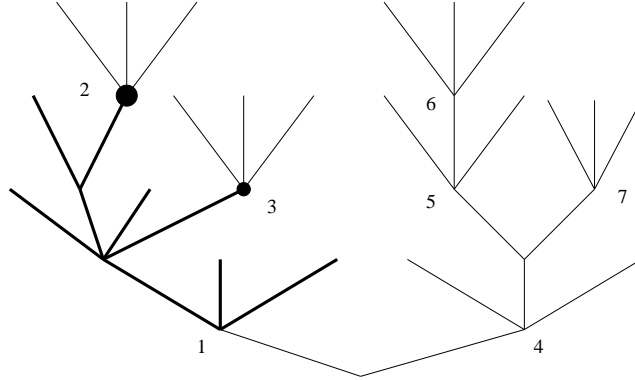


FIGURE 2. The subtree  $S_1^{\mathcal{A}}(\mathbf{t})$  in bold for  $\mathcal{A} = \{3\}$ , and the elements of  $C_1^{\mathcal{A}}(\mathbf{t})$ .

We set

$$\tilde{S}_{\emptyset}^{\mathcal{A}}(\mathbf{t}) = \begin{cases} S_{\emptyset}^{\mathcal{A}}(\mathbf{t}) & \text{if } \emptyset \notin \mathcal{L}_{\mathcal{A}}(\mathbf{t}) \\ \{\emptyset\} & \text{if } \emptyset \in \mathcal{L}_{\mathcal{A}}(\mathbf{t}) \end{cases}$$

and we set  $\tilde{C}_{\emptyset}^{\mathcal{A}}(\mathbf{t})$  the set of leaves of  $\tilde{S}_{\emptyset}^{\mathcal{A}}(\mathbf{t})$  that belong to  $\mathcal{L}_{\mathcal{A}}(\mathbf{t})$ .

Let  $\tilde{N}_{\emptyset}(\mathbf{t}) = \text{Card}(\tilde{C}_{\emptyset}^{\mathcal{A}}(\mathbf{t}))$ . Then the range of  $\tilde{\phi}$  belongs to  $\mathbb{T}_0^{\tilde{N}_{\emptyset}(\mathbf{t})}$ . Moreover if  $u_1 < u_2 < \dots < u_{\tilde{N}_{\emptyset}(\mathbf{t})}$  are the elements of  $\tilde{C}_{\emptyset}^{\mathcal{A}}(\mathbf{t})$  ranked in lexicographic order, we set for every  $1 \leq i \leq \tilde{N}_{\emptyset}(\mathbf{t})$

$$\tilde{\phi}(u_i) = \emptyset^{(i)}$$

where  $\emptyset^{(i)}$  denotes the root of the  $i$ -th tree in  $\mathbb{T}_0^{\tilde{N}_\emptyset(\mathbf{t})}$ .

We then construct  $\tilde{\phi}$  recursively: if  $u \in \mathcal{L}_{\mathcal{A}}(\mathbf{t})$  and  $\tilde{\phi}(u) = v^{(i)}$  (which is an element of the  $i$ -th tree), then we denote by  $u_1 < \dots < u_k$  the elements of  $C_u^{\mathcal{A}}(\mathbf{t})$  ranked in lexicographic order and we set for  $1 \leq j \leq k$

$$\tilde{\phi}(u_j) = v^{j^{(i)}}.$$

Finally, we set  $\mathcal{F}_{\mathcal{A}}(\mathbf{t}) = \tilde{\phi}(\mathbf{t})$ .

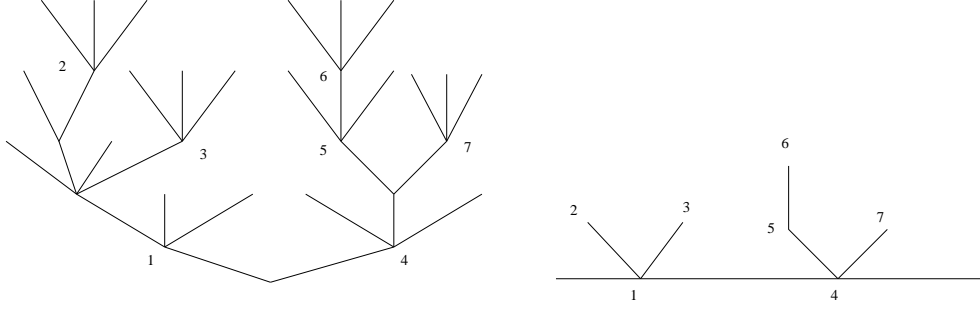


FIGURE 3. A tree  $\mathbf{t}$  and the forest  $\mathcal{F}_{\mathcal{A}}(\mathbf{t})$  for  $\mathcal{A} = \{3\}$ .

Let  $\tau$  be a Galton-Watson tree with offspring distribution  $p$ . Let us describe the distribution of  $\mathcal{F}_{\mathcal{A}}(\tau)$ .

We define the offspring distribution  $\tilde{p}$  by

$$\begin{cases} \tilde{p}(k) = p(k) \mathbf{1}_{\{k \notin \mathcal{A}\}} & \text{for } k \geq 1, \\ \tilde{p}(0) = p(0) + p(\mathcal{A}). \end{cases}$$

Then  $\tilde{S}_\emptyset^{\mathcal{A}}(\tau)$  is distributed as a (subcritical) GW tree with offspring distribution  $\tilde{p}$ . In particular, if we denote by  $L$  the number of leaves of  $\tilde{S}_\emptyset^{\mathcal{A}}(\tau)$ , then we have

$$\mathbb{E}[L] = \frac{p(0) + p(\mathcal{A})}{1 - \mathbb{E}[X \mathbf{1}_{\{X \notin \mathcal{A}\}}]}$$

where  $X$  is a random variable distributed according to  $p$ . Moreover, conditionally given  $L$ , the random variable  $N := N_\emptyset(\tau)$  has a binomial distribution with parameter  $(L, p(\mathcal{A})/(p(0) + p(\mathcal{A})))$ .

Let  $X^{\mathcal{A}}$  be the random variable

$$X^{\mathcal{A}} = \sum_{k=1}^{Z'} N_k$$

where  $Z'$  is distributed as  $X$  conditionally given  $\{X \in \mathcal{A}\}$  and  $(N_k, k \in \mathbb{N})$  is a sequence of independent random variables, independent of  $Z'$ , and distributed as  $N$ . We denote by  $p^{\mathcal{A}}$  the law of  $X^{\mathcal{A}}$ . Then the forest  $\mathcal{F}_{\mathcal{A}}(\tau)$  is distributed as  $N$  independent GW trees with offspring distribution  $p^{\mathcal{A}}$ .

**7.2. Main result.** We recall that  $L_{\mathcal{A}}(\tau)$  is aperiodic since  $0 \notin \mathcal{A}$ , see [1].

**Theorem 7.1.** *Assume that  $p$  satisfies (1) and  $\mu(p) < 1$  and  $\rho(p) = 1$ . We have that:*

$$(39) \quad \text{dist}(\tau \mid L_{\mathcal{A}}(\tau) = n) \xrightarrow{n \rightarrow +\infty} \text{dist}(\tau^*(p)),$$

as well as

$$(40) \quad \text{dist}(\tau \mid L_{\mathcal{A}}(\tau) \geq n) \xrightarrow{n \rightarrow +\infty} \text{dist}(\tau^*(p)).$$

*Proof.* It is enough to prove that for all  $\mathbf{t} \in \mathbb{T}_0$ ,  $x \in \mathbf{t}$  and  $k \in \mathbb{N}$ :

$$(41) \quad \lim_{n \rightarrow +\infty} \mathbb{P}(\tau \in \mathbb{T}_+(\mathbf{t}, x, k), L_{\mathcal{A}}(\tau) = n) = D(\mathbf{t}, x) \mathbb{P}(\tau^*(p) \in \mathbb{T}_+(\mathbf{t}, x, k)).$$

Set  $M_0 = 0$  and  $M_n = \sum_{k=1}^n N_k$  for  $n \in \mathbb{N}^*$ . Let  $m = L_{\mathcal{A}}(\mathbf{t}) - \mathbf{1}_{\mathcal{A}}(k_x(\mathbf{t}))$  and  $\ell = k_x(\mathbf{t})$ . Recall (36). We have

$$\begin{aligned} \mathbb{P}(\tau \in \mathbb{T}_+(\mathbf{t}, x, k), L_{\mathcal{A}}(\tau) = n) &= D(\mathbf{t}, x) \sum_{j \geq \max(\ell+1, k)} p(j) \mathbb{P}_{j-\ell}(L_{\mathcal{A}}(\tau) = n_j - m) \\ &= D(\mathbf{t}, x) \sum_{j \geq \max(\ell+1, k)} p(j) \frac{j-\ell}{n_j - m} \mathbb{E} \left[ N \mathbf{1}_{\{S_{n_j-m} + M_{j-1-\ell} + N = n_j - m\}} \right], \end{aligned}$$

where we used Dwass formula (17) for the last equality where  $S_n = \sum_{k=1}^n X_k$  with  $(X_k, k \in \mathbb{N}^*)$  independent random variables distributed as  $X^{\mathcal{A}}$ , see also (58). Recall (59). In particular, we have:

$$(42) \quad \mathbb{P}(\tau \in \mathbb{T}_+(\mathbf{t}, x, k) \mid L_{\mathcal{A}}(\tau) = n) = D(\mathbf{t}, x) \left( B_{n-m, \ell} - \sum_{j=\ell+1}^{k-1} p(j)(j-\ell) a_{n-m, j} \right),$$

with:

$$a_{n, j} = \frac{n}{n_j} \frac{\mathbb{E} \left[ N \mathbf{1}_{\{S_{n_j} + M_{j-1-\ell} + N = n_j\}} \right]}{\mathbb{E} \left[ N \mathbf{1}_{\{S_n + N = n\}} \right]}.$$

Notice that Lemma 8.6 implies that  $\lim_{n \rightarrow +\infty} a_{n, j} = 1$ . Then use Lemma 8.9 to get:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbb{P}(\tau \in \mathbb{T}_+(\mathbf{t}, x, k) \mid L_{\mathcal{A}}(\tau) = n) &= D(\mathbf{t}, x) (1 - \ell + \mathbb{E}[(X - \ell)_+ \mathbf{1}_{\{X \geq k\}}]) \\ &= \mathbb{P}(\tau^*(p) \in \mathbb{T}_+(\mathbf{t}, x, k)). \end{aligned}$$

This ends the proof.  $\square$

**Corollary 7.2.** *Assume that  $p$  satisfies (1), is non-generic for  $\mathcal{A}$ . Let  $p_{\mathcal{A}}^* = p_{\mathcal{A}, \rho(p)}$ . We have that:*

$$\text{dist}(\tau \mid L_{\mathcal{A}}(\tau) = n) \xrightarrow{n \rightarrow +\infty} \text{dist}(\tau^*(p_{\mathcal{A}}^*)),$$

as well as

$$\text{dist}(\tau \mid L_{\mathcal{A}}(\tau) \geq n) \xrightarrow{n \rightarrow +\infty} \text{dist}(\tau^*(p_{\mathcal{A}}^*)).$$

This result with Proposition 4.6 and Corollary 5.7 in [1] for the generic case ends the proof of Theorem 1.2 for  $0 \notin \mathcal{A}$  and gives a complete description of the asymptotic distribution of critical and sub-critical GW trees conditioned to have a large population.

## 8. APPENDIX

**8.1. Strong ratio limit property.** Let  $(X_n, n \in \mathbb{N})$  be independent random variables taking values in  $\mathbb{N}$  with distribution  $p = (p(k), k \in \mathbb{N})$ . We assume that:

$$(43) \quad \mu(p) \leq 1 \text{ and either } \mu(p) = 1 \text{ or, for all } \theta > 0, \mathbb{E} \left[ e^{\theta X_1} \right] = +\infty.$$

Let  $S_n = \sum_{k=1}^n X_k$ . We assume that  $p$  is aperiodic (that is  $\mathbb{P}(S_n = n) > 0$  for all  $n$  large enough). According to [8] or [12], we have the following strong ratio limit property for all  $m, k \in \mathbb{Z}$ :

$$(44) \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{P}(S_{n-m} = n - k)}{\mathbb{P}(S_n = n)} = 1.$$

We deduce the following corollary. Recall the definition of  $\delta_n^0$  and  $\delta_n^1$  of (18) and (19).

**Corollary 8.1.** *Assume that  $p$  satisfies (43) and is aperiodic. For all  $k \in \mathbb{Z}$  and  $\ell \in \mathbb{N}$ , we have:*

$$(45) \quad \lim_{n \rightarrow +\infty} \delta_n^0(k, \ell) = \sum_{j \geq k} p(j).$$

and

$$(46) \quad \lim_{n \rightarrow +\infty} \delta_n^1(k, \ell) = 1 - \mu(p) + \sum_{j \geq k} jp(j).$$

*Proof.* Since  $\mathbb{P}(S_{n+1} = n + \ell) = \sum_{j \in \mathbb{N}} p(j) \mathbb{P}(S_n = n + \ell - j)$ , we have:

$$\delta_n^0(k, \ell) = \frac{\mathbb{P}(S_{n+1} = n + \ell)}{\mathbb{P}(S_n = n)} - \sum_{j < k} p(j) \frac{\mathbb{P}(S_n = n + \ell - j)}{\mathbb{P}(S_n = n)}.$$

Then use (44) to get (45).

Notice that, by exchangeability:

$$\sum_{j \in \mathbb{N}} jp(j) \mathbb{P}(S_n = n + \ell - j) = \mathbb{E} [X_1 \mathbf{1}_{\{S_{n+1} = n + \ell\}}] = \frac{n + \ell}{n + 1} \mathbb{P}(S_{n+1} = n + \ell).$$

Thus we have:

$$\delta_n^1(k, \ell) = \frac{n + \ell}{n + 1} \frac{\mathbb{P}(S_{n+1} = n + \ell)}{\mathbb{P}(S_n = n)} - \sum_{j < k} jp(j) \frac{\mathbb{P}(S_n = n + \ell - j)}{\mathbb{P}(S_n = n)}.$$

Then use (44) to get:

$$\lim_{n \rightarrow +\infty} \delta_n^1(k, \ell) = 1 - \sum_{j < k} jp(j).$$

Since  $1 - \sum_{j < \ell} jp(j) = 1 - \mu(p) + \sum_{j \geq \ell} jp(j)$ , this gives (46).  $\square$

**8.2. Generalization of the strong ratio limit property I.** Assume that  $p$  satisfies (43) and is aperiodic. Let  $X$  be a random variable taking values in  $\mathbb{N}$  with distribution  $p$ . Recall  $g$  denote the generating function of  $p$ .

Let  $\mathcal{A} \subset \mathbb{N}$  such that  $0 \in \mathcal{A}$ . Let  $p^{\mathcal{A}}$  be the distribution on  $\mathbb{N}$  with generating function  $g^{\mathcal{A}}$  given by (30) and  $X_{\mathcal{A}}$  distributed according to  $p^{\mathcal{A}}$ . Recall  $\mu(p^{\mathcal{A}})$  is given by (31). In particular  $\mu(p) = 1$  (resp.  $\mu(p) \leq 1$ ) implies  $\mu(p^{\mathcal{A}}) = 1$  (resp.  $\mu(p^{\mathcal{A}}) \leq 1$ ). And from the proof of Lemma 6.1, we get that  $\mathbb{E} [e^{\theta X}] = +\infty$  for all  $\theta > 0$  implies that  $\mathbb{E} [e^{\theta X_{\mathcal{A}}}] = +\infty$  for all  $\theta > 0$ .

Let  $(X_n, n \in \mathbb{N})$  be independent random variables, independent of  $X$ , taking values in  $\mathbb{N}$  with distribution  $p^{\mathcal{A}}$ . Let  $S_n = \sum_{k=1}^n X_k$ . We assume that  $p^{\mathcal{A}}$  is aperiodic (that is  $\mathbb{P}(S_n = n) > 0$  for all  $n$  large enough). In particular the strong ratio limit property (44) holds as well as (45) and (46) hold with  $p$  replaced by  $p^{\mathcal{A}}$ .

Recall (36), that is  $n_j = n - \mathbf{1}_{\mathcal{A}}(j)$ , and let:

$$(47) \quad \delta_n^{0,\mathcal{A}}(k, \ell) = \frac{1}{\mathbb{P}(S_n = n)} \sum_{j \geq k} p(j) \frac{n}{n_j} \mathbb{P}(S_{n_j} = n_j + \ell - j)$$

and

$$(48) \quad \delta_n^{1,\mathcal{A}}(k, \ell) = \frac{1}{\mathbb{P}(S_n = n)} \sum_{j \geq k} j p(j) \frac{n}{n_j} \mathbb{P}(S_{n_j} = n_j + \ell - j).$$

We stress that in (18) and (19),  $(S_n, n \in \mathbb{N})$  is a random walk with increments distributed according to  $p$ ; whereas in (47) and (48),  $(S_n, n \in \mathbb{N})$  is a random walk with increments distributed according to  $p^{\mathcal{A}}$ .

**Lemma 8.2.** *Assume that  $p$  satisfies (43) and is aperiodic. For all  $k \in \mathbb{Z}$  and  $\ell \in \mathbb{N}$ , we have:*

$$(49) \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{E} \left[ \frac{n}{n_X} \mathbf{1}_{\{X+S_{n_X}=n_X+\ell\}} \right]}{\mathbb{P}(S_n = n)} = 1,$$

$$(50) \quad \lim_{n \rightarrow +\infty} \delta_n^{0,\mathcal{A}}(k, \ell) = \sum_{j \geq k} p(j)$$

and

$$(51) \quad \lim_{n \rightarrow +\infty} \delta_n^{1,\mathcal{A}}(k, \ell) = 1 - \mu(p) + \sum_{j \geq k} j p(j).$$

*Proof.* We define:

$$a_n(j) = p(j) \frac{\mathbb{P}(S_{n_j} = n_j + \ell - j)}{\mathbb{P}(S_n = n)} \frac{n}{n_j}$$

as well as

$$b_n(j) = p^{\mathcal{A}}(j) \frac{\mathbb{P}(S_{n-1} = n + \ell - j - 1)}{\mathbb{P}(S_n = n)} + \frac{p^{\mathcal{A}}(j-1)}{p(\mathcal{A})} \frac{\mathbb{P}(S_n = n + \ell - j)}{\mathbb{P}(S_n = n)},$$

with the convention that  $p^{\mathcal{A}}(-1) = 0$ .

Thanks to the strong ratio limit property (that is (44) with  $p^{\mathcal{A}}$  instead of  $p$ ), we have  $\lim_{n \rightarrow +\infty} a_n(j) = p(j)$  and  $\lim_{n \rightarrow +\infty} b_n(j) = p^{\mathcal{A}}(j) + p^{\mathcal{A}}(j-1)/p(\mathcal{A})$ . We have:

$$\sum_{j \in \mathbb{N}} b_n(j) = \frac{\mathbb{P}(S_n = n + \ell - 1)}{\mathbb{P}(S_n = n)} + \frac{1}{p(\mathcal{A})} \frac{\mathbb{P}(S_{n+1} = n + \ell + 1)}{\mathbb{P}(S_n = n)}.$$

We deduce from the strong ratio limit property (that is (44) with  $p^{\mathcal{A}}$  instead of  $p$ ) that:

$$\lim_{n \rightarrow +\infty} \sum_{j \in \mathbb{N}} b_n(j) = 1 + \frac{1}{p(\mathcal{A})} = \sum_{j \in \mathbb{N}} \lim_{n \rightarrow +\infty} b_n(j).$$

Then use (32) and (33) to get that  $a_n(j) \leq 2b_n(j)$  for  $n \geq 2$  and the dominated convergence theorem to get that:

$$\lim_{n \rightarrow +\infty} \sum_{j \in \mathbb{N}} a_n(j) = \sum_{j \in \mathbb{N}} \lim_{n \rightarrow +\infty} a_n(j) = 1.$$

Notice that  $\sum_{j \in \mathbb{N}} a_n(j) = \mathbb{E} \left[ \frac{n}{n_X} \mathbf{1}_{\{X+S_{n_X}=n_X+\ell\}} \right] / \mathbb{P}(S_n = n)$  to deduce that (49) holds. Since  $\delta_n^{0,A}(k, \ell) = \sum_{j \geq k} a_n(j)$ , the proof of (50) is then similar to the proof of (45).

Set  $c_n(\ell) = \delta_n^{1,A}(0, \ell)$  that is:

$$c_n(\ell) = \frac{\mathbb{E} \left[ \frac{n}{n_X} X \mathbf{1}_{\{X+S_{n_X}=n_X+\ell\}} \right]}{\mathbb{P}(S_n = n)}.$$

According to Lemma 8.3 below, (44) and (49), we have that  $\lim_{n \rightarrow +\infty} c_n(\ell) = 1$  for all  $\ell \in \mathbb{Z}$ . Then arguing as in the proof of (46), we easily get (51).  $\square$

**Lemma 8.3.** *For all  $\ell \in \mathbb{Z}$ ,  $n \geq 2$ , we have:*

$$(52) \quad \mathbb{E} \left[ \frac{n}{n_X} X \mathbf{1}_{\{X+S_{n_X}=n_X+\ell\}} \right] = \ell \mathbb{E} \left[ \frac{n}{n_X} \mathbf{1}_{\{X+S_{n_X}=n_X+\ell\}} \right] - (\ell - 1) \mathbb{P}(S_n = n + \ell - 1).$$

*Proof.* We first prove (52) for  $\ell \leq 0$ . Let  $k \geq 1$ . By decomposing according to the number of children of the root of the first tree in the forest, we have:

$$\mathbb{P}_k(|\tau^A| = n) = \sum_{j \in \mathbb{N}} p(j) \mathbb{P}_{j+k-1}(|\tau^A| = n_j),$$

with the convention that  $\mathbb{P}_0(\cdot) = 0$ . Then using Dwass formula (17) in each side of this equality, we get:

$$k \mathbb{P}(S_n = n - k) = \mathbb{E} \left[ \frac{n}{n_X} (X + k - 1) \mathbf{1}_{\{X+S_{n_X}=n_X-k+1\}} \right].$$

Take  $\ell = 1 - k$  to get that (52) holds for  $\ell \leq 0$ .

Unfortunately, we didn't get a similar proof for  $\ell \geq 1$  and we prove (52) for  $\ell \geq 1$  by induction. Let  $\ell \geq 0$ . Assume that (52) holds for all  $\ell' \leq \ell$  and all  $n \geq 2$ , and let us prove it holds for  $\ell + 1$  and all  $n \geq 2$ . We have:

$$(53) \quad \mathbb{E} \left[ \frac{n+1}{n_X+1} X \mathbf{1}_{\{X+S_{n_X+1}=n_X+1+\ell\}} \right] = A_1 + \mathbb{E} \left[ \frac{n_X - n}{n_X(n_X+1)} X \mathbf{1}_{\{X+S_{n_X+1}=n_X+1+\ell\}} \right],$$

with

$$A_1 = \mathbb{E} \left[ \frac{n}{n_X} X \mathbf{1}_{\{X+S_{n_X+1}=n_X+1+\ell\}} \right].$$

Using (52), we have:

$$\begin{aligned} A_1 &= \sum_{j \in \mathbb{N}} p^A(j) \mathbb{E} \left[ \frac{n}{n_X} X \mathbf{1}_{\{X+S_{n_X}=n_X+1+\ell-j\}} \right] \\ &= p^A(0) \mathbb{E} \left[ \frac{n}{n_X} X \mathbf{1}_{\{X+S_{n_X}=n_X+1+\ell\}} \right] \\ &\quad + \sum_{j \in \mathbb{N}^*} p^A(j) \left( (\ell + 1 - j) \mathbb{E} \left[ \frac{n}{n_X} \mathbf{1}_{\{X+S_{n_X}=n_X+1+\ell-j\}} \right] - (\ell - j) \mathbb{P}(S_n = n + \ell - j) \right). \end{aligned}$$

So we have:

$$(54) \quad A_1 = p^A(0) A_2 + A_3 - \mathbb{E} \left[ (\ell - X_1) \mathbf{1}_{\{S_{n+1}=n+\ell\}} \right],$$

with

$$(55) \quad A_2 = \mathbb{E} \left[ \frac{n}{n_X} X \mathbf{1}_{\{X+S_{n_X}=n_X+1+\ell\}} \right] - (\ell+1) \mathbb{E} \left[ \frac{n}{n_X} \mathbf{1}_{\{X+S_{n_X}=n_X+\ell+1\}} \right] + \ell \mathbb{P}(S_n = n + \ell)$$

and

$$A_3 = \mathbb{E} \left[ (\ell+1 - X_1) \frac{n}{n_X} \mathbf{1}_{\{X+S_{n_X+1}=n_X+\ell+1\}} \right].$$

We compute the last term of (54). We have:

$$\mathbb{E} [(\ell - X_1) \mathbf{1}_{\{S_{n+1}=n+\ell\}}] = \mathbb{E} \left[ \left( \ell - \frac{S_{n+1}}{n+1} \right) \mathbf{1}_{\{S_{n+1}=n+\ell\}} \right] = \frac{n}{n+1} (\ell-1) \mathbb{P}(S_{n+1} = n + \ell).$$

We compute  $A_3$ :

$$\begin{aligned} A_3 &= \mathbb{E} \left[ \left( \ell+1 - \frac{S_{n_X+1}}{n_X+1} \right) \frac{n}{n_X} \mathbf{1}_{\{X+S_{n_X+1}=n_X+\ell+1\}} \right] \\ &= \mathbb{E} \left[ \left( \ell+1 - \frac{n_X+1+\ell-X}{n_X+1} \right) \frac{n}{n_X} \mathbf{1}_{\{X+S_{n_X+1}=n_X+\ell+1\}} \right] \\ &= \ell \mathbb{E} \left[ \frac{n}{n_X+1} \mathbf{1}_{\{X+S_{n_X+1}=n_X+\ell+1\}} \right] + \mathbb{E} \left[ \frac{n}{n_X(n_X+1)} X \mathbf{1}_{\{X+S_{n_X+1}=n_X+\ell+1\}} \right]. \end{aligned}$$

Plugging the result in (53), we get:

$$\begin{aligned} &\mathbb{E} \left[ \frac{n+1}{n_X+1} X \mathbf{1}_{\{X+S_{n_X+1}=n_X+1+\ell\}} \right] \\ &= p^A(0) A_2 + \ell \mathbb{E} \left[ \frac{n}{n_X+1} \mathbf{1}_{\{X+S_{n_X+1}=n_X+\ell+1\}} \right] \\ &\quad + \mathbb{E} \left[ \frac{1}{n_X+1} X \mathbf{1}_{\{X+S_{n_X+1}=n_X+1+\ell\}} \right] - \frac{n}{n+1} (\ell-1) \mathbb{P}(S_{n+1} = n + \ell). \end{aligned}$$

We obtain, using that  $(n+1)_X = n_X + 1$  and (52) with  $n+1$  instead of  $n$ :

$$\begin{aligned} p^A(0) A_2 &= \frac{n}{n+1} \mathbb{E} \left[ \frac{n+1}{n_X+1} X \mathbf{1}_{\{X+S_{n_X+1}=n_X+1+\ell\}} \right] - \frac{\ell n}{n+1} \mathbb{E} \left[ \frac{n+1}{n_X+1} \mathbf{1}_{\{X+S_{n_X+1}=n_X+\ell+1\}} \right] \\ &\quad + \frac{n}{n+1} (\ell-1) \mathbb{P}(S_{n+1} = n + \ell) \\ &= 0. \end{aligned}$$

Recall (55). The fact that  $A_2 = 0$  gives exactly that (52) holds with  $\ell$  replaced by  $\ell+1$ . This proves the induction and ends the proof of the lemma.  $\square$

**8.3. Generalization of the strong ratio limit property II.** We use notations from Section 7.2. We have the following generalization of the strong ratio limit property.

**Lemma 8.4.** *Assume that  $p^A$  is aperiodic,  $\mu(p^A) < 1$ ,  $\rho(p^A) = 1$  and  $\mathbb{E} [e^{\theta X_A}] = +\infty$  for all  $\theta > 0$ . Then for all  $m, k \in \mathbb{Z}$ , we have:*

$$(56) \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{E} [N \mathbf{1}_{\{S_{n-m}+N=n-k\}}]}{\mathbb{E} [N \mathbf{1}_{\{S_n+N=n\}}]} = 1.$$

Note that if  $p^A$  is periodic, then (56) still holds along the subsequence for which the denominator is positive.



*Proof.* We shall mimic the proof of the strong ratio limit property provided in [12]. Since  $p^A$  is aperiodic, the denominator of (56) is positive for  $n$  large enough and it is enough to prove the result for  $m = 1$  and  $k$  such that  $p^A(k) > 0$ . Denote  $\hat{p}_n^A(k) = \sum_{i=1}^n \mathbf{1}_{\{X_i=k\}}/n$ . We have:

$$\mathbb{E} [N \hat{p}_n^A(k) \mathbf{1}_{\{S_n+N=n\}}] = \mathbb{E} [N \mathbf{1}_{\{X_n=k\}} \mathbf{1}_{\{S_n+N=n\}}] = p^A(k) \mathbb{E} [N \mathbf{1}_{\{S_{n-1}+N=n-k\}}].$$

The proof will be complete as soon as we prove that:

$$J_n = \frac{\mathbb{E} [N \mathbf{1}_{\{|\hat{p}_n^A(k)-p^A(k)|>\varepsilon\}} \mathbf{1}_{\{S_n+N=n\}}]}{\mathbb{E} [N \mathbf{1}_{\{S_n+N=n\}}]}$$

converges to 0 for all  $\varepsilon > 0$ . Notice that:

$$J_n \leq \frac{\mathbb{E} [N \mathbf{1}_{\{|\hat{p}_n^A(k)-p^A(k)|>\varepsilon\}}]}{\mathbb{E} [N \mathbf{1}_{\{S_n+N=n\}}]} = \frac{\mathbb{P}(|\hat{p}_n^A(k) - p^A(k)| > \varepsilon)}{\mathbb{P}(S_n = n)} \frac{\mathbb{E}[N] \mathbb{P}(S_n = n)}{\mathbb{E} [N \mathbf{1}_{\{S_n+N=n\}}]}.$$

According to [12], since  $p^A$  is non-generic with  $\rho(p^A) = 1$ , we have  $\lim_{n \rightarrow +\infty} \mathbb{P}(|\hat{p}_n^A(k) - p^A(k)| > \varepsilon) / \mathbb{P}(S_n = n) = 0$ . By Fatou and using the strong ratio limit property, we have:

$$\limsup_{n \rightarrow +\infty} \frac{\mathbb{E}[N] \mathbb{P}(S_n = n)}{\mathbb{E} [N \mathbf{1}_{\{S_n+N=n\}}]} \leq 1.$$

Since  $\varepsilon > 0$  is arbitrary, we deduce that  $\lim_{n \rightarrow +\infty} J_n = 0$ .  $\square$

*Remark 8.5.* Notice that, from the proof of the lemma, we see that  $N$  could be replaced by any non-negative integrable random variable independent of  $(X_k, k \in \mathbb{N}^*)$ .

Recall that  $M_0 = 0$  and for  $n \in \mathbb{N}^*$ :

$$M_n = \sum_{k=1}^n N_k.$$

We assume that  $(N_k, k \in \mathbb{N}^*)$  and  $(X_k, k \in \mathbb{N}^*)$  are independent. We have the following result.

**Lemma 8.6.** *Assume  $p^A$  is aperiodic, with  $\mu(p^A) < 1$  and  $\rho(p^A) = 1$ . Let  $m \in \mathbb{N}$  and  $k \in \mathbb{Z}$ , we have:*

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E} [N \mathbf{1}_{\{S_n+N+M_m=n-k\}}]}{\mathbb{E} [N \mathbf{1}_{\{S_n+N=n\}}]} = 1.$$

*Proof.* Let

$$c_{n,\ell} = \frac{\mathbb{E} [N \mathbf{1}_{\{S_n+N=n-\ell-k\}}]}{\mathbb{E} [N \mathbf{1}_{\{S_n+N=n\}}]}.$$

Denote by  $q = (q(\ell), \ell \in \mathbb{N})$  the distribution of  $M_k$  and by  $r = (r(\ell), \ell \in \mathbb{N})$  the distribution of  $S_m$ . We have, thanks to Lemma 8.4, that  $\lim_{n \rightarrow +\infty} c_{n,\ell} = 1$  and:

$$\lim_{n \rightarrow +\infty} \sum_{\ell \in \mathbb{N}} r(\ell) c_{n,\ell} = \lim_{n \rightarrow +\infty} \frac{\mathbb{E} [N \mathbf{1}_{\{S_{n+m}+N=n-k\}}]}{\mathbb{E} [N \mathbf{1}_{\{S_n+N=n\}}]} = 1 = \sum_{\ell \in \mathbb{N}} r(\ell) \lim_{n \rightarrow +\infty} c_{n,\ell}.$$

Let  $j_0$  such that  $\mathbb{P}(Z_1 = j_0) > 0$ . Notice that:

$$r(\ell) = \mathbb{P}(S_m = \ell) \geq \mathbb{P}(Z_1 + \dots + Z_m = mj_0, M_m = \ell, N_{m+1} + \dots + N_{mj_0} = 0).$$

We deduce that there exists  $c > 0$  such that  $q(\ell) \leq Cr(\ell)$  for all  $\ell \in \mathbb{N}$ . By dominated convergence, we deduce that  $\lim_{n \rightarrow +\infty} \sum_{\ell \in \mathbb{N}} q(\ell) c_{n,\ell} = \sum_{\ell \in \mathbb{N}} q(\ell) \lim_{n \rightarrow +\infty} c_{n,\ell} = 1$ .  $\square$

Let  $p_N$  be the distribution of  $N$ . We have, using the decomposition of the GW tree with respect to the descendants of  $\emptyset$  in  $\mathcal{A}$  and Dwass formula (17):

$$(57) \quad \mathbb{P}(L_{\mathcal{A}}(\tau) = n) = \sum_{j \in \mathbb{N}} p_N(j) \mathbb{P}_j(|\tau^{\mathcal{A}}| = n) = \frac{1}{n} \mathbb{E} [N \mathbf{1}_{\{S_n + N = n\}}].$$

More generally, we have

$$(58) \quad \mathbb{P}_j(L_{\mathcal{A}}(\tau) = n) = \frac{1}{n} \mathbb{E} [M_j \mathbf{1}_{\{S_n + M_j = n\}}] = \frac{j}{n} \mathbb{E} [N \mathbf{1}_{\{S_n + M_{j-1} + N = n\}}],$$

with  $N$  independent of  $S_n$  and  $M_{j-1}$ .

We set for  $\ell \in \mathbb{Z}$ :

$$(59) \quad B_{n,\ell} = \sum_{j > \ell} p(j)(j - \ell) \frac{n}{n_j} \frac{\mathbb{E} [N \mathbf{1}_{\{S_{n_j} + M_{j-1-\ell} + N = n_j\}}]}{\mathbb{E} [N \mathbf{1}_{\{S_n + N = n\}}]}.$$

The next lemma is the analogue of Lemma 8.3 in our current setting.

**Lemma 8.7.** *For  $\ell \leq 0$ , we have  $\lim_{n \rightarrow +\infty} B_{n,\ell} = 1 - \ell$ .*

*Proof.* Recall that  $\mathbb{E} [N \mathbf{1}_{\{S_n + N = n\}}] = \mathbb{P}(L_{\mathcal{A}}(\tau) = n)$ . Let  $k \geq 0$ . By decomposing  $\tau$  under  $\mathbb{P}_{k+1}$  with respect to the number of children of the first tree in the forest, we get:

$$\begin{aligned} \mathbb{P}_{k+1}(L_{\mathcal{A}}(\tau) = n) &= \sum_{j \in \mathbb{N}} p(j) \mathbb{P}_{k+j}(L_{\mathcal{A}}(\tau) = n_j) \\ &= \sum_{j \in \mathbb{N}} p(j) \frac{k+j}{n_j} \mathbb{E} [N \mathbf{1}_{\{S_{n_j} + M_{k+j-1} + N = n_j\}}] \\ &= B_{-k,n} \frac{1}{n} \mathbb{E} [N \mathbf{1}_{\{S_n + N = n\}}]. \end{aligned}$$

Then use (58) and Lemma 8.6 to deduce that:

$$\lim_{n \rightarrow +\infty} \frac{n \mathbb{P}_{k+1}(L_{\mathcal{A}}(\tau) = n)}{\mathbb{E} [N \mathbf{1}_{\{S_n + N = n\}}]} = k + 1.$$

This gives the lemma.  $\square$

In order to extend Lemma 8.7 in a weaker form for  $\ell > 0$ , we give a preliminary lemma. Set for  $\ell \geq k$ ,  $\ell, k \in \mathbb{Z}$ :

$$C_{n,\ell}(k) = \mathbb{E} \left[ \frac{n}{n_X} N(X - \ell)_+ \mathbf{1}_{\{S_{n_X} + M_{X-k-1} + N = n_X\}} \right].$$

Notice that for  $\ell \in \mathbb{Z}$ :

$$(60) \quad C_{n,\ell}(\ell) = n B_{n,\ell} \mathbb{P}(L_{\mathcal{A}}(\tau) = n).$$

We define  $z_+ = \max(z, 0)$ .

**Lemma 8.8.** *Assume  $p^{\mathcal{A}}$  is aperiodic, non-generic with  $\rho(p^{\mathcal{A}}) = 1$ . We have for  $k \in \mathbb{Z}$  such that  $k \leq \ell$ :*

$$\lim_{n \rightarrow +\infty} \frac{C_{n,\ell}(k)}{C_{n,\ell}(\ell)} = 1.$$

*Proof.* Notice that  $nN(X - \ell)_+/n_X$  is integrable. Mimicking the proof of Lemma 8.4 and using that  $n_X$  takes only two possible values a.s., we get for  $m, k \in \mathbb{Z}$ :

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E} \left[ \frac{n}{n_X} N(X - \ell)_+ \mathbf{1}_{\{S_{n_X - m} + M_{X-1-\ell} + N = n_X - k\}} \right]}{\mathbb{E} \left[ \frac{n}{n_X} N(X - \ell)_+ \mathbf{1}_{\{S_{n_X} + M_{X-1-\ell} + N = n_X\}} \right]} = 1.$$

Then mimicking the proof of Lemma 8.6, we get for  $m \in \mathbb{N}$  and  $k \in \mathbb{Z}$ :

$$\lim_{n \rightarrow +\infty} \frac{\mathbb{E} \left[ \frac{n}{n_X} N(X - \ell)_+ \mathbf{1}_{\{S_{n_X} + M_{X-1-\ell} + m + N = n_X - k\}} \right]}{\mathbb{E} \left[ \frac{n}{n_X} N(X - \ell)_+ \mathbf{1}_{\{S_{n_X} + M_{X-1-\ell} + N = n_X\}} \right]} = 1.$$

Then take  $m = \ell - k \geq 0$  to get the result.  $\square$

**Lemma 8.9.** *Assume  $p^A$  is aperiodic, non-generic with  $\rho(p^A) = 1$ . For  $\ell > 0$ , we have:*

$$\lim_{n \rightarrow +\infty} B_{n,\ell} = 1 - \mu + \mathbb{E}[(X - \ell)_+].$$

*Proof.* Let  $\ell \geq -1$ . We have:

$$(61) \quad C_{n,\ell}(-1) = C_{n,0}(-1) - \sum_{j=0}^{\ell-1} p(j)(j - \ell) \mathbb{E} \left[ \frac{n}{n_j} N \mathbf{1}_{\{S_{n_j} + M_j + N = n_j\}} \right] \\ - \ell \mathbb{E} \left[ \frac{n}{n_X} N \mathbf{1}_{\{S_{n_X} + M_X + N = n_X\}} \right],$$

with the convention that  $\sum_{\emptyset} = 0$ . Recall that  $\lim_{n \rightarrow +\infty} B_{n,-1} = 2$  and  $\lim_{n \rightarrow +\infty} B_{n,0} = 1$ , thanks to Lemma 8.7 and thus (60) implies that:

$$C_{n,-1}(-1) = 2\mathbb{E}[N \mathbf{1}_{\{S_n + N = n\}}] \quad \text{and} \quad C_{n,0}(0) = \mathbb{E}[N \mathbf{1}_{\{S_n + N = n\}}].$$

We deduce from Lemma 8.8 that

$$\lim_{n \rightarrow +\infty} \frac{C_{n,0}(-1)}{\mathbb{E}[N \mathbf{1}_{\{S_n + N = n\}}]} = \lim_{n \rightarrow +\infty} \frac{C_{n,0}(-1)}{C_{n,0}(0)} = 1.$$

We deduce from (61) with  $\ell = -1$  and Lemma 8.6 that:

$$(62) \quad \lim_{n \rightarrow +\infty} \frac{\mathbb{E} \left[ \frac{n}{n_X} N \mathbf{1}_{\{S_{n_X} + M_X + N = n_X\}} \right]}{\mathbb{E}[N \mathbf{1}_{\{S_n + N = n\}}]} = 1.$$

Let  $\ell \geq 1$ . We deduce from (61) with  $\ell \geq 1$ , (60), (57), Lemma 8.6 and (62) that:

$$\lim_{n \rightarrow +\infty} B_{n,\ell} = 1 - \sum_{j=0}^{\ell-1} p(j)(j - \ell) - \ell = 1 - \mu + \mathbb{E}[(X - \ell)_+].$$

$\square$

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