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Abstract. In the lattice theory the tensor product $A \otimes B$ is naturally defined on $(0, \vee)-$semilattices. In general, when restricted to lattices this construction will not yield a lattice. However, if the tensor product $A \otimes B$ is *capped*, then $A \otimes B$ is a lattice. It is stated as an open problem in [4] whether the converse is true.

In the present paper we prove that it is not so, that is, there are bounded lattices A and B such that $A \otimes B$ is not capped, but is a lattice. Furthermore, A has length three and is generated by a nine-element set of atoms, while B is the dual lattice of A.

1. Introduction

The tensor product of join-semilattices with zero was introduced in J. Anderson and N. Kimura [1] and G.A. Fraser [2]. The latter paper, in a category-theoretical style approach, defines it as a free object of a special kind. Although this approach can, in principle, be carried over to lattices, this does not look promising. On the other hand, the original definition of tensor product of semilattices can be restricted to lattices, yielding interesting results and problems. This was performed in a number of papers of G.Grätzer and F.Wehrung [6, 4, 5].

In many natural cases the tensor product of lattices appears to be a lattice, for example when both lattices involved are finite. To generalize this situation, G.Grätzer and F.Wehrung introduced the notion of *capped* tensor product. When the tensor product is capped, it is a lattice. Whether the converse is true is a long-standing conjecture, which we solve in this paper.

The paper [7] gives a good insight into the present state of affairs in the field.

This paper is organized as follows. In Section 2 we give basic definitions from lattice theory. In Section 3 we adopt classical results concerning tensor products of lattices from [6] and [5]. This includes the characterization of bi-ideals as homomorphisms of a special kind, antitone step functions and adjustment sequences. Theorem 1 establishes the equivalence between the representations of bi-ideals by homomorphisms and the adjustment sequences.

In Sections 4 and 5 we start to deploy the original technique aiming at a narrow class of *normal* lattices, which we use later to prove the conjecture. As a prerequisite, in Section 4 we introduce *marked lattices* and define the family of *zig-zag* operators acting on them. After that, in Section 5 we define normal lattices as special kinds of marked lattices, and establish some important facts about them. We prove that normal lattices are finitely generated and that all their nontrivial intervals are finite. The essential property of this section is established in Lemma 7, which, informally, states that normal lattices are almost locally finite, that is, every finite subset S of a normal lattice L generates a finite sublattice in a slightly modified lattice L^x.

In Section 6 we introduce the infinite normal lattice L_∞ and claim that the tensor product $L_\infty \otimes L_\infty^{\mathrm{op}}$ satisfies the desired properties, namely that it is not capped but is a lattice. Apart from being infinite and normal, the lattice L_∞ possesses another property captured in Lemma 9, namely that an infinite sublattice of L_∞ can be mapped into L_∞ only the trivial way. As a digression, albeit we do not prove that, this property naturally holds for the most of the infinite normal lattices, so in this sense the lattice L_∞ is in no way exceptional.

Note that the technique developed throughout Sections 4, 5 and 6 is independent of those developed in Section 3.

Finally, in Section 7 we bring together the results of Sections 3, 4, 5 and 6 to establish the key properties of $L_\infty \otimes L_\infty^{\mathrm{op}}$. In Theorems 2 and 3 we prove that there are noncapped elements in $L_\infty \otimes L_\infty^{\mathrm{op}}$ and that all these elements are in a way similar to each other. These results easily yield Theorem 4, which positively solves the conjecture.

2. Preliminary definitions and formulation of the problem

We start with some basic definitions and notations. A subset S of a poset T is called *hereditary* iff $y \leq x$ and $x \in S$ implies $y \in S$, for all $x, y \in T$.

We denote by S^{op} the dual poset of a poset S, that is, the same set with order given by

$$x \leq_{S^{\mathrm{op}}} y \Leftrightarrow x \geq_S y.$$

In an arbitrary poset the *height* of an element x, denoted $\mathbf{h}(x)$, is defined as the supremum of all cardinalities of a chain from a non-minimal element to x. The *length* of a poset is the supremum of the heights of all its elements. We are following a standard convention of calling elements of height one *atoms*, additionally we call elements of height two *lines*.

For a lattice K and $x \in K$ let $(x]$ denote the *principal ideal generated by* x, that is, $(x] = \{y \mid y \leq x\}$. For lattices K, L and an element $(x, y) \in K \times L$ we use the notation $(x, y]$ instead of $\big((x, y)\big]$ to denote the principal ideal.

For a lattice L and a set $A \subseteq L$ we denote by $\langle A \rangle_L$ the sublattice of L generated by A in L, and by $A^\vee$ and $A^\wedge$ the subsets

$$A^\vee = \{\bigvee B \mid B \text{ is a finite subset of } A\},$$
$$A^\wedge = \{\bigwedge B \mid B \text{ is a finite subset of } A\}.$$

For a $(0,1)$-lattice L, $(\vee, 0)$-semilattice K and $(\wedge, 1)$-semilattice P we denote by L^-, K^0 and P^1 the posets $L - \{0, 1\}$, $K - \{0\}$ and $P - \{1\}$ correspondingly.

An element x in a complete lattice A is *compact* iff $x \leq \bigvee S$ for some S implies $x \leq \bigvee T$ for some finite $T \subseteq S$. The set of all compact elements of a complete lattice A is denoted by $\mathbf{C}(K)$. For every complete lattice A the set $\mathbf{C}(A)$ is a $(0, \vee)$-semilattice. A complete lattice A is called *algebraic* iff every element is the join of compact elements.

For a $(0, \vee)$-semilattice K we denote by $\operatorname{Id} K$ the complete algebraic lattice of ideals of K ordered by set inclusion. The semilattice K is a subsemilattice of $\operatorname{Id} K$ with the canonical embedding given by $a \mapsto (a]$.

The tensor product and the capped tensor product are defined in [6]. As these definitions are central to the paper, let us recall them.

Let K and L be lattices with zero. The *lateral join* is the partial function $\vee_{\text{lat}}\colon (K \times L) \times (K \times L) \to K \times L$ defined by

$$(x_1, y_1) \vee_{\text{lat}} (x_2, y_2) = (x_1 \vee x_2,\ y_1 \vee y_2)$$

provided either $x_1 = x_2$ or $y_1 = y_2$.

A subset I of $K \times L$ is a *bi-ideal* iff it is hereditary, it contains the set

$$\perp_{K,L} = (\{0_K\} \times L) \cup (K \times \{0_L\}),$$

and it is closed under lateral joins.

The *extended tensor product* of K and L, denoted by $K \mathbin{\overline{\otimes}} L$, is the set of all bi-ideals in $K \times L$ ordered by set inclusion. It can easily be seen that $K \mathbin{\overline{\otimes}} L$ is a complete algebraic lattice. Define the tensor product of K and L, denoted $K \otimes L$, as the set of compact elements of $K \mathbin{\overline{\otimes}} L$ with order inherited from $K \mathbin{\overline{\otimes}} L$. It can be easily seen that $K \otimes L$ is a $(0, \vee)$-subsemilattice of $K \mathbin{\overline{\otimes}} L$. In general, however, $K \otimes L$ is not a lattice (see Corollary 8.2 in [5]).

A simple example of a compact bi-ideal in $K \times P$ is a set $a \otimes b$, $a \in K$, $b \in L$, called a *pure tensor*:

$$a \otimes b = \{(x, y) \in K \times L \mid x \leq a,\ y \leq b\} \cup \perp_{K,L}.$$

Call a compact bi-ideal $I \subseteq K \times L$ a *capped bi-ideal* if it is a finite union of pure tensors, that is

$$I = \bigcup (a_i \otimes b_i \mid i = 0, \ldots, k-1).$$

Note that the $\bigcup$ symbol above refers to set-theoretical union. In general, every compact bi-ideal can be represented as

$$I = \bigvee (a_i \otimes b_i \mid i = 0, \ldots, k-1),$$

where the join is taken in $K \mathbin{\overline{\otimes}} L$. Nevertheless, the set-theoretic representation as a finite union does not necessarily hold.

We call the tensor product $K \otimes L$ *capped* if all its elements are capped bi-ideals. If the tensor product is capped, then clearly it is a lattice. It is asked in [6, Problem 3] whether the converse holds.

Conjecture 1. *There are bounded lattices K and L such that $K \otimes L$ is a lattice, yet the tensor product $K \otimes L$ is not capped.*

The subsequent sections are dedicated to proving that this conjecture is correct.

3. Preparatory technique

We start with a representation by homomorphisms described in Section 2.3 of [5].

Let A and B be $(\vee, 0)$-semilattices. We consider the set of all semilattice homomorphisms from the semilattice $\left(A^0; \vee\right)$ to the semilattice $(\operatorname{Id} B; \cap)$,

$$A \mathbin{\overrightarrow{\otimes}} B = \operatorname{Hom}\left(\left(A^0; \vee\right), (\operatorname{Id} B; \cap)\right),$$

ordered componentwise. The arrow indicates the way homomorphisms go. Note that the elements of $A \mathbin{\overrightarrow{\otimes}} B$ are antitone functions.

With any element φ of $A \mathbin{\overrightarrow{\otimes}} B$ we associate the subset $\varepsilon(\varphi)$ of $A \times B$:

$$\varepsilon(\varphi) = \{(x, y) \in A \times B \mid y \in \varphi(x)\} \cup \perp_{A,B}.$$

Proposition 1. *The map ε is an isomorphism between $A \overrightarrow{\otimes} B$ and $A \overline{\otimes} B$. The inverse map, ε^{-1}, sends $H \in A \overline{\otimes} B$ to $\varepsilon^{-1}(H)\colon A^0 \to \operatorname{Id} B$, defined by*

$$\varepsilon^{-1}(H)(a) = \{x \in B \mid (a, x) \in H\}\,. \tag{1}$$

Proof. See Proposition 2.4 of [5]. ■

As there is a natural isomorphism between $A \overline{\otimes} B$ and $B \overline{\otimes} A$, we can define an isomorphism η from $B \overrightarrow{\otimes} A$ to $A \overline{\otimes} B$ in a similar way as we defined ε, that is

$$\eta(\psi) = \{(x, y) \in A \times B \mid x \in \psi(y)\} \cup \perp_{A,B},$$

for any $\psi \in B \overrightarrow{\otimes} A$.

Then for every homomorphism $\varphi \in A \overrightarrow{\otimes} B$ we define its *conjugate* $\varphi^* \in B \overrightarrow{\otimes} A$ as $\varphi^* = \eta^{-1} \circ \varepsilon(\varphi)$. Thus, the conjugation defines an isomorphism between $A \overrightarrow{\otimes} B$ and $B \overrightarrow{\otimes} A$. By abuse of terminology, the converse mapping $\varepsilon^{-1} \circ \eta$ is also called conjugation. We subsume the notation defined above by stating that the diagram in Figure 1 below commutes.

$$\begin{tikzcd}
& A \overline{\otimes} B \arrow[dl, shift right, "\varepsilon^{-1}"'] \arrow[dr, shift left, "\eta^{-1}"] & \\
A \overrightarrow{\otimes} B \arrow[ur, shift right, "\varepsilon"'] \arrow[rr, shift left, "\cdot^* = \eta^{-1} \circ \varepsilon"] & & B \overrightarrow{\otimes} A \arrow[ul, shift left, "\eta"] \arrow[ll, shift left, "\cdot^* = \varepsilon^{-1} \circ \eta"]
\end{tikzcd}$$

Figure 1

Proposition 2. *Let φ be a homomorphism from $(A; \vee)$ to $(\operatorname{Id} B; \cap)$, that is, $\varphi \in A \overrightarrow{\otimes} B$. Then $y \in \varphi(x)$ iff $x \in \varphi^*(y)$, for all $x \in A$, $y \in B$.*

Proof. Indeed, $y \in \varphi(x) \Leftrightarrow (x, y) \in \varepsilon(\varphi) \Leftrightarrow x \in \varphi(y)$. ■

As we mentioned earlier, all elements of $A \otimes B$ are finite joins of pure tensors. The following proposition gives the representation of finite joins in a purely arithmetic way. For every positive integer m, denote by $\mathrm{F}(m)$ the free lattice on m generators $x_0, \dots, x_{m-1}$. One can evaluate any element p of $\mathrm{F}(m)$ at any m-tuple of elements of any lattice, thus justifying the notation $p(a_0, \dots, a_{m-1})$. By p^{op} we denote the dual of p.

Proposition 3. *Let A and B be lattices with zero, let n be a positive integer, let $a_0, \dots, a_{n-1} \in A$, and let $b_0, \dots, b_{n-1} \in B$. Then*

$$\bigvee (a_i \otimes b_i \mid i < n) = \bigcup (p(a_0, \dots, a_{n-1}) \otimes p^{\mathrm{op}}(b_0, \dots, b_{n-1}) \mid p \in \mathrm{F}(n))$$

Proof. See Lemma 2.2, (iii) of [5]. ■

To argue about whether the tensor product is a lattice we need the following easy result.

Proposition 4. *$A \otimes B$ is a lattice if and only if it is closed under finite intersection.*

Proof. See Proposition 2.4 of [6]. ■

Proposition 5. *Let A and B be lattices with zero and let A be locally finite. Then $A \otimes B$ is a lattice iff $A \otimes B$ is a capped tensor product.*

Proof. See Theorem 3 of [5]. ■

The following technique described in Section 6 of [5] enables us to obtain a very useful representation of compact bi-ideals.

Let A and B be lattices with zero. We put $(a]_\bullet = (a] \cap A^0$, for all $a \in A$. We denote by $\operatorname{Int} A$ the boolean lattice of the powerset of A^0 generated by all sets of the form $(a]_\bullet$, for $a \in A^0$. Furthermore, we denote by $\operatorname{Int}_* A$ the ideal of $\operatorname{Int} A$ consisting of all $X \in \operatorname{Int} A$ with $X \subseteq (a]_\bullet$, for some $a \in A$.

The map $\xi \colon A^0 \to B$ is a *step function* if the range of ξ is finite and the inverse image $\xi^{-1}\{b\}$ belongs to $\operatorname{Int}_* A$, for all $b \in B^0$.

Proposition 6. *For a positive integer n let $a_0, \dots, a_{n-1} \in A$ and $b_0, \dots, b_{n-1} \in B$. Then the map $\xi \colon A^0 \to B$ defined by*

$$\xi(x) = \bigvee (b_i \mid i < n,\ x \le a_i)\,, \tag{2}$$

for all $x \in A^0$, is an antitone step function. Furthermore,

$$(x, \xi(x)) \in \bigvee (a_i \otimes b_i \mid i < n)\,,$$

for all $x \in A^0$.

Proof. See Lemma 6.4 of [5]. ■

If an antitone function $\xi \colon A^0 \to B$ is obtained by application of formula (2), then we say that an antitone function $\psi \colon B^0 \to A$ is the *dual* of ξ if it is obtained by application of (2) to elements b_i and a_i, $i < n$, that is

$$\psi(x) = \bigvee (a_i \mid i < n,\ x \le b_i)\,.$$

A *join-cover* of the element $a \in A$ is any finite subset $S \subseteq L$ such that $a \le \bigvee S$. A join-cover S of a is nontrivial if $a \not\le s$, for all $s \in S$. Let $\mathrm{C}(a)$ denote the set of all finite join-covers and $\mathrm{C}^*(a)$ the set of all finite nontrivial join-covers of a.

Let $\xi \colon A^0 \to B$ be a map with finite range. Then *one-step adjustment* of ξ is $\xi^{(1)} \colon A^0 \to B$ defined by

$$\xi^{(1)}(x) = \bigvee \Big(\bigwedge \xi[S] \mid S \in \mathrm{C}(x) \Big)\,, \tag{3}$$

for all $x \in A^0$. Note that since the range of ξ is finite, the right hand side of the equation (3) is well-defined.

Proposition 7. *Let $\xi \colon A^0 \to B$ be a map with finite range. Then $\xi^{(1)}$ is antitone and has finite range, and the inequality $\xi(x) \le \xi^{(1)}(x)$ holds for all $x \in A^0$. Furthermore, $\xi^{(1)} = \xi$ iff ξ is a semilattice homomorphism from $\big(A^0; \vee\big)$ to $(B; \wedge)$.*

Proof. See Remark 6.6 and Lemma 6.8 of [5]. ■

Similar techniques are used in connection with bounded homomorphisms, see Chapter 2 of [3].

Proposition 8. *If ξ is an antitone map with finite range, then the one-step adjustment* (3) *takes the form:*

$$\xi^{(1)}(x) = \xi(x) \vee \bigvee \Big(\bigwedge \xi[S] \mid S \in \mathrm{C}^*(x) \Big)\,. \tag{4}$$

Proof. See Remark 6.6 of [5]. ■

Proposition 9. *Let $\xi\colon A^0 \to B$ be a step function, then the one-step adjustment $\xi^{(1)}$ of ξ is an antitone step function.*

Proof. See Proposition 6.10 of [5]. ■

Proposition 9 justifies the following definition. Let $\xi\colon A^0 \to B$ be a step function. The *adjustment sequence* of ξ is the sequence $\left(\xi^{(n)} \mid n < \omega\right)$ defined inductively by $\xi^{(0)} = \xi$, and $\xi^{(n+1)} = (\xi^{(n)})^{(1)}$, for all $n < \omega$.

Proposition 10. *Let $\xi\colon A^0 \to B$ be a step function. Then the adjustment sequence of ξ is increasing, that is, $\xi^{(n)} \leq \xi^{(n+1)}$, for all n. Furthermore, if $n > 0$ then $\xi^{(n)}$ is an antitone step function.*

Proof. See Corollary 6.12 of [5]. ■

Consider the complete lattice $\operatorname{Id} B$ and view the mappings $\xi^{(n)}$, $n < \omega$ as mappings from A^0 to $\operatorname{Id} B$ using the canonical embedding of B into $\operatorname{Id} B$. Define the *ω-adjustment* of ξ as the mapping $\xi^{(\omega)}\colon A^0 \to \operatorname{Id} B$ defined by

$$\xi^{(\omega)}(x) = \bigcup(\xi^{(n)}(x) \mid n < \omega), \tag{5}$$

and the *extended adjustment sequence* of ξ is the sequence $\left(\xi^{(\alpha)} \mid \alpha \leq \omega\right)$.

Note that the range of $\xi^{(\omega)}$ need not be finite, and thus $\xi^{(\omega)}$ need not be a step function. However, $\xi^{(\omega)}$ is antitone. Also, just as the adjustment sequence, the extended adjustment sequence is increasing.

Proposition 11. *The one-step adjustment and the ω-adjustment operations are monotonous, that is, if ξ and ψ are maps with finite range such that $\xi \leq \psi$, then $\xi^{(1)} \leq \psi^{(1)}$ and $\xi^{(\omega)} \leq \psi^{(\omega)}$.*

Proof. Follows immediately from formulas (3) and (5). ■

The following theorem establishes connection between the bi-ideals represented as homomorphisms and the extended adjustment sequences.

Theorem 1. *Let $n < \omega$, $a_0, \dots, a_{n-1} \in A$ and $b_0, \dots, b_{n-1} \in B$. Let the map $\xi\colon A^0 \to B$ be the map defined by (2). Let $I \in A \otimes B$ be the compact bi-ideal*

$$I = \bigvee(a_i \otimes b_i \mid i < n),$$

and let $\varphi_I \in A \overrightarrow{\otimes} B$ be the representation of I defined by (1), that is, $\varphi_I = \varepsilon^{-1}(I)$. Then

(i) *$\xi^{(\omega)}$ is a homomorphism from $\left(A^0, \vee\right)$ to $(\operatorname{Id} B, \cap)$, that is, $\xi^{(\omega)} \in A \overrightarrow{\otimes} B$,*
(ii) *$\varphi_I = \xi^{(\omega)}$.*

Proof. (i): As $\xi^{(\omega)}$ is antitone, then $\xi^{(\omega)}(x \vee y) \subseteq \xi^{(\omega)}(x) \cap \xi^{(\omega)}(y)$. On the other hand since $\{x, y\} \in \mathrm{C}(x \vee y)$, then by (3) we infer $\xi^{(\omega)}(x \vee y) \supseteq \xi^{(n+1)}(x \vee y) \supseteq \xi^{(n)}(x) \cap \xi^{(n)}(y)$, for all $n < \omega$. But from (5) we get

$$\begin{aligned}
\xi^{(\omega)}(x) \cap \xi^{(\omega)}(y) &= \bigcup(\xi^{(n)}(x) \mid n < \omega) \cap \bigcup(\xi^{(n)}(y) \mid n < \omega) \\
&= \bigcup(\xi^{(n)}(x) \cap \xi^{(n)}(y) \mid n < \omega) \\
&\subseteq \xi^{(\omega)}(x \vee y),
\end{aligned}$$

and the result follows.

(ii): The homomorphism φ_I is the smallest homomorphism such that for all $i < n$ holds $b_i \in \varphi_I(a_i)$. Thus, $\varphi_I \leq \xi^{(\omega)}$. On the other hand from Proposition 6 follows $\xi^{(0)} \leq \varphi_I$. The homomorphism φ_I does not have finite range, but as $\operatorname{Id} B$ is a complete lattice then (3) is well-defined for φ_I. As φ_I is a homomorphism from $(A^0, \vee)$ to $(\operatorname{Id} B, \cap)$ then by Proposition 7 the equality $\varphi_I^{(1)} = \varphi_I$ holds. Thus, $\varphi_I^{(\alpha)} = \varphi_I$, for all $\alpha < \omega{+}1$. But then by Proposition 11 we obtain $\xi^{(\omega)} \leq \varphi_I^{(\omega)} = \varphi_I$, which completes the proof. ∎

Remark 1. Under the conditions of Theorem 1, if the lattice B is of finite length then it coincides with $\operatorname{Id} B$. Thus, we can regard $\xi^{(\omega)}$ and φ_I as homomorphisms from $(A^0, \vee)$ to $(B, \wedge)$. This applies to all elements of $A \overrightarrow{\otimes} B$, that is, we can regard $A \overrightarrow{\otimes} B$ as

$$A \overrightarrow{\otimes} B = \operatorname{Hom}\left(\left(A^0; \vee\right), (B; \wedge)\right).$$

Corollary 1. *Under the conditions of Theorem* 1*, if* $\phi\colon B^0 \to A$ *is a dual of* $\xi\colon A^0 \to B$*, then*

(i) $\phi^{(\omega)}$ *is a homomorphism from* $(B^0, \vee)$ *to* $(\operatorname{Id} A, \cap)$*, that is,* $\phi^{(\omega)} \in B \overrightarrow{\otimes} A$*,*

(ii) $\varphi_I^* = \phi^{(\omega)}$.

For an element a of A and an ideal $\mathfrak{I}$ of B, let us define a bi-ideal $a \overset{\bullet}{\otimes} \mathfrak{I}$ as

$$a \overset{\bullet}{\otimes} \mathfrak{I} = \{(x,y) \in A \times B \mid x \leq a,\ y \in \mathfrak{I}\} \cup \perp_{A,B}.$$

Notice that in general $a \overset{\bullet}{\otimes} \mathfrak{I}$ is not a compact bi-ideal, that is, it belongs to $A \overline{\otimes} B$, but not to $A \otimes B$.

For every bi-ideal I the following representation trivially holds

$$I = \bigcup \left(x \overset{\bullet}{\otimes} \varphi_I(x) \mid x \in A\right),$$

where $\varphi_I = \varepsilon^{-1}(I)$. However, this representation can be strengthened, as it is shown in the following lemma.

Lemma 1. *Under the conditions of Theorem* 1*, the bi-ideal* I *can be represented as*

$$I = \bigcup \left(x \overset{\bullet}{\otimes} \varphi_I(x) \mid x \in \langle a_i \mid i < n\rangle,\ x \neq 0_A\right).$$

Proof. Let us denote by J the union from the right hand side of the hypothesis, that is,

$$J = \bigcup \left(x \overset{\bullet}{\otimes} \varphi_I(x) \mid x \in \langle a_i \mid i < n\rangle,\ x \neq 0_A\right).$$

As $I = \bigcup \left(x \overset{\bullet}{\otimes} \varphi_I(x) \mid\ x \neq 0_A\right)$, the inclusion $I \supseteq J$ trivially holds.

Now, we are going to prove that J is a bi-ideal. As J contains all pairs (a_i, b_i) for all $i < n$, and I is the least bi-ideal containing all such pairs, this would yield $I \subseteq J$, proving the lemma.

It is obvious that J is hereditary and contains the set $\perp_{A,B}$, so we only need to prove that it is closed under lateral joins.

Let (c, d_1) and (c, d_2) be elements from J, $c \neq 0_A$, $d_j \neq 0_B$. Then there are two elements x_1 and x_2 from $\langle a_i \mid i < n\rangle$ such that $c \leq x_j$ and $d_j \in \varphi_I(x_j)$, for $j = 1, 2$. Then $c < x_1 \wedge x_2$, and as φ_I is antitone, then $d_j \in \varphi_I(x_j) \subseteq \varphi_I(x_1 \wedge x_2)$ and so $d_1 \vee d_2 \subseteq \varphi_I(x_1 \wedge x_2)$. As the element $x_1 \wedge x_2$ is from $\langle a_i \mid i < n\rangle$, then $(c, d_1 \vee d_2)$ lies in J.

Now, let (c_1, d) and (c_2, d) lie in J, $c_j \neq 0_A$, $d \neq 0_B$. Then again, there are elements x_1 and x_2 from $\langle a_i \mid i < n\rangle$ such that $c_j \leq x_j$ and $d \in \varphi_I(x_j)$, for $j = 1, 2$.

In this case $c_1 \vee c_2 \leq x_1 \vee x_2$ and $d \in \varphi_I(x_1) \cap \varphi_I(x_2) = \varphi_I(x_1 \vee x_2)$, and so $(c_1 \vee c_2, d)$ lies in J. Thus, J is closed under lateral joins and so it is a bi-ideal, which finishes the proof. ∎

Now we can apply Remark 1 to obtain

Corollary 2. *Under the conditions of Theorem* 1*, if the lattice B is of finite length, then the bi-ideal I can be represented as*

$$\begin{aligned} I &= \bigcup \left(x \otimes \varphi_I(x) \mid x \in A^0 \right) \\ &= \bigcup \left(x \otimes \varphi_I(x) \mid x \in \langle a_i \mid i < n \rangle ,\ x \neq 0_A \right). \end{aligned}$$

4. Marked lattices

A *marked lattice* is a structure $(L, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$, where

- L is a lattice of length at most three.
- The elements $a_1, \ldots, a_9$, called the *marked elements* of L, are pairwise distinct atoms of L.
- $a_i \vee a_j = 1_L$, for any distinct $i, j \in \{1, \ldots, 9\}$.

Marked lattices will often be denoted in the form $L[a_1, \ldots, a_9]$, or simply L if the marked elements are understood from the context.

Now, we define operators **CW** and **CCW** which transform a marked lattice by adding new elements into it. It should be observed that each of the additional "lines" of the construction is defined by two distinct marked "atoms" x and y, and then the line is denoted xy or, indifferently, yx.

If $L[a, b, c, d, e, f, g, h, i]$ is a marked lattice, then we define the marked lattice $K[a', b', c', d', e', f', g', h', i'] = \mathbf{CW}(L)$ in the following way. The elements of K are all elements of L plus:

- candidate lines $ab, bc, cd, de, ef, fg, gh, hi$ and ic,
- candidate lines $ce, df, eg, fh, gi, ha, ib, ac$ and dh,
- candidate atoms $ab \cdot ce, bc \cdot df, cd \cdot eg, de \cdot fh, ef \cdot gi, fg \cdot ha, gh \cdot ib, hi \cdot ac$ and $ic \cdot dh$.

And the order on K is defined as:

- restricted to L, order of K coincides with the original order of L,
- all new elements lie above the zero and below the unit of L,
- $a, b \leq ab; \ldots ; d, h \leq dh$,
- $ab \cdot ce \leq ab, ce; \ldots ; ic \cdot dh \leq ic, dh$,
- otherwise elements are incomparable.

Finally, we define marked elements by $a' = ab \cdot ce; \ldots ; i' = ic \cdot dh$.

Similarly, we define the marked lattice $P[a', b', c', d', e', f', g', h', i'] = \mathbf{CCW}(L)$ in the following way. The elements of P are all elements of L plus:

- candidate lines $ad, be, cf, dg, eh, fi, ga, hb$ and ia,
- candidate lines $ei, fa, gb, hc, id, ae, bf, cg$ and bd,
- candidate atoms $ad \cdot ei, be \cdot fa, cf \cdot gb, dg \cdot hc, eh \cdot id, fi \cdot ae, ga \cdot bf, hb \cdot cg$ and $ia \cdot bd$.

The order on P is defined as:

- restricted to L, order of P coincides with the original order of L,
- all new elements lie above the zero and below the unit of L,
- $a, d \leq ad;\ \ldots;\ b, d \leq bd$,

- $ad \cdot ei \leq ad, ei$; $\ldots$; $ia \cdot bd \leq ia, bd$,
- otherwise elements are incomparable.

And the marked elements are $a' = ad \cdot ei$, $\ldots$, $i' = ia \cdot bd$.

The constructions **CW** and **CCW** are illustrated in Figures 2 and 3 below.

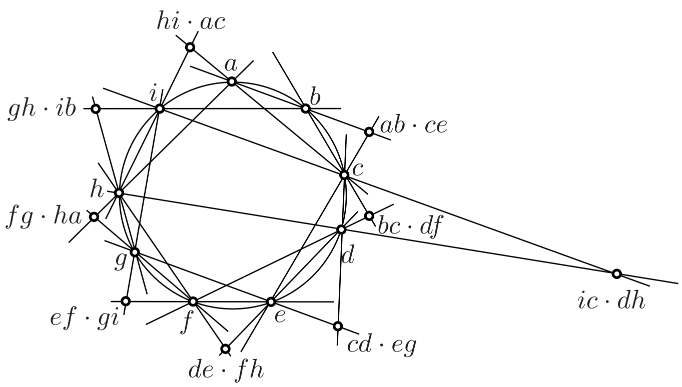

FIGURE 2. Illustrating the construction **CW**

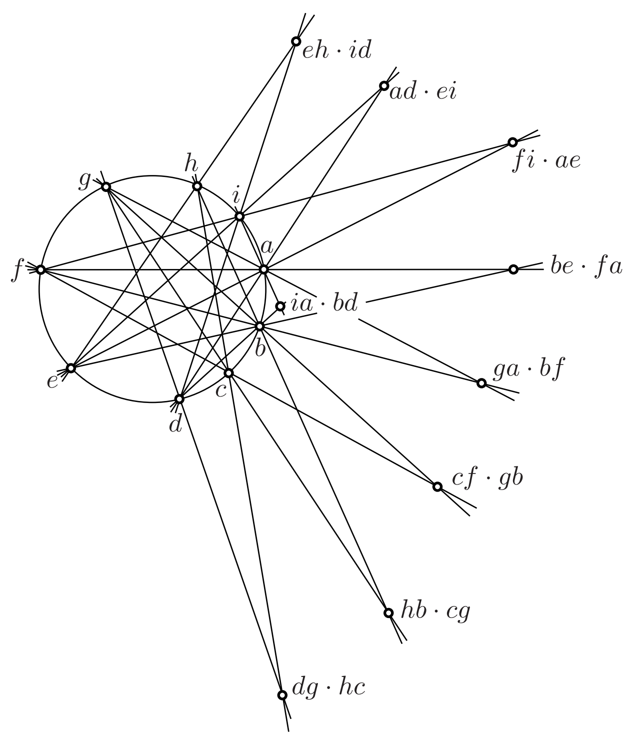

FIGURE 3. Illustrating the construction **CCW**

Lemma 2. *If L is a marked lattice, $K = \mathbf{CW}(L)$, $P = \mathbf{CCW}(L)$, then*

(i) *In K and P candidate atoms are atoms and candidate lines are lines.*
(ii) *Atoms and lines of L remain correspondingly atoms and lines in both K and P (except the "degenerate" line 1).*
(iii) *K and P are lattices.*
(iv) *K and P are marked lattices.*
(v) *L is a meet-subsemilattice of both K and P.*

Proof. (i): The only element that lies below candidate atoms is the zero element of L, so they are atoms. Now if we take candidate line xy, then there are four elements below it: the zero of L, two marked elements of L, namely x and y, and one candidate atom. Thus, xy is a line.

(ii): Trivial, as there are no new elements added below any atom or any line of L.

(iii): As K and P are bounded posets of finite length then to prove that they are lattices it is sufficient to prove that they are meet-semilattices. All meets between atoms trivially exist, as well as meets between a line and an atom. Now, if we take two lines then there are three possibilities. If we take two lines from L then their meet is preserved from L. If we take two candidate lines then, by the construction of the operators **CW** and **CCW**, they meet either at a candidate atom or at a marked element, or else they have no atom in common, in which case they meet at the zero.

Finally, if we take a candidate line l and a line p from L then the only elements that can lie below both of them, except for the zero, are marked elements of L. But as any two distinct marked elements join at the unit, there is at most one marked element of L below p, and so the meet exists.

(iv): Both in K and P the height of the unit element is three. Also any pair of distinct candidate atoms have no common line above them, so they join at the unit. Thus, it is a marked lattice.

(v): Trivial, as there are no new elements added below any atom or any line of L. ∎

Notice that under the conditions of Lemma 2, L is a join-subsemilattice of neither K nor P.

We call an operator on marked lattices a *finite zig-zag operator* if it can be represented as a finite composition of operators **CW** and **CCW**, that is,

$$\mathbf{Z} = \prod_{i=1}^{k} \mathbf{Z}_i \;=\; \mathbf{Z}_k \circ \mathbf{Z}_{k-1} \circ \cdots \circ \mathbf{Z}_1,$$

where $k \in \mathbb{N}$ and each $\mathbf{Z}_i$ is either **CW** or **CCW**. The number k is called the *degree* of Z, denoted $\deg(\mathbf{Z})$. The operators **CW** and **CCW** are themselves finite zig-zag operators of degree one, and we call them *primitive* zig-zag operators. We denote by $\mathbf{Z}_{r\ldots l}$ the operator

$$\mathbf{Z}_{r\ldots l} = \prod_{i=r}^{l} \mathbf{Z}_i \;=\; \mathbf{Z}_l \circ \mathbf{Z}_{l-1} \circ \cdots \circ \mathbf{Z}_r,$$

where $1 \le r \le l \le \deg \mathbf{Z}$.

We follow the convention of treating the product of zero operators as the identity operator Id, which we thus consider a zig-zag operator of degree 0. With this convention we allow the notation

$$\mathbf{Z}_{r\ldots r-1} = \prod_{i=r}^{r-1} \mathbf{Z}_i \ = \ \mathrm{Id}\,.$$

We define an *infinite zig-zag operator* $\mathbf{Z}$ associated to the sequence of primitive zig-zag operators $(\mathbf{Z}_k)_{k=1}^{\infty}$, denoted $\mathbf{Z} = \prod_{i=1}^{\infty} \mathbf{Z}_i$, as an operator which assigns to a marked lattice L the direct limit of the sequence of bounded meet-semilattices $\big(\mathbf{Z}_{1\ldots k}(L)\big)_{k=1}^{\infty}$, where $\mathbf{Z}_{1\ldots k}(L) = \prod_{i=1}^{k} \mathbf{Z}_i(L)$, that is, $\mathbf{Z}(L)$ is a poset

$$\mathbf{Z}(L) = \bigcup_{k=1}^{\infty} \prod_{i=1}^{k} \mathbf{Z}_i(L),$$

and the partial order on $\mathbf{Z}(L)$ is given by

$$x \leq_{\mathbf{Z}(L)} y \Leftrightarrow \exists k\colon x \leq_{\mathbf{Z}_{1\ldots k}(L)} y.$$

The next lemma proves that, thus defined, $\mathbf{Z}(L)$ is an infinite bounded lattice of length three.

Lemma 3. *Let $\mathbf{Z}$ be infinite zig-zag operator and let L be a marked lattice. Then $\mathbf{Z}(L)$ is an infinite lattice of length three, and $\mathbf{Z}_{1\ldots k}(L)$ is a meet-subsemilattice of $\mathbf{Z}(L)$, for every $k < \omega$.*

Proof. It follows from Lemma 2, (v) that all meets from $\mathbf{Z}(L)$ are preserved in $\mathbf{Z}_{1\ldots k}(L)$, for every positive k. In particular, that the order on $\mathbf{Z}_{1\ldots k}(L)$ is the restriction of the order of $\mathbf{Z}(L)$. Then indeed $\mathbf{Z}(L)$ is a meet-semilattice and $\mathbf{Z}_{1\ldots k}(L)$ is its meet-subsemilattice.

Now, observe that the length of $\mathbf{Z}(L)$ is the supremum of lengths of $\mathbf{Z}_{1\ldots k}(L)$ for all k, and so it equals three. Thus, $\mathbf{Z}(L)$ is of finite length from which it follows that it is a lattice. ■

For infinite zig-zag operators we utilize the notation $\mathbf{Z}_{k\ldots l}$ in a similar way as we do in the finite case, however, we let $l = \infty$. Note that in case l is finite, $\mathbf{Z}_{k\ldots l}$ is a finite zig-zag operator, while $\mathbf{Z}_{k\ldots\infty}$ is an infinite zig-zag operator. We write that $\mathbf{Z}$ is a *zig-zag operator* in case it is not essential for us whether $\mathbf{Z}$ is finite or infinite. We say that the degree of an infinite zig-zag operator is ∞.

The following observations of a combinatorial nature about the primitive zig-zag operators will be useful for us in the subsequent sections.

Proposition 12. *Let L be a marked lattice and let $\mathbf{W}$ be a primitive zig-zag operator. Then for every line $l \in \mathbf{W}(L) - L$ there is a unique line $l^* \in \mathbf{W}(L) - L$, called the* conjugate *of l, such that l and l^* meet at an atom in $\mathbf{W}(L) - L$.*

For example, for the operator $\mathbf{CW}$ the conjugate of the line ab is ce, and the conjugate of ic is dh. Notice also that the equality $l = l^{**}$ holds for any line l in $\mathbf{W}(L) - L$.

Proposition 13. *Let $L[a,b,c,d,e,f,g,h,i]$ be a marked lattice. For a marked element $x \in L$ define $\ell_{\mathbf{CW}}x$ (correspondingly $\ell_{\mathbf{CCW}}x$) as the number of candidate lines of $\mathbf{CW}(L)$ (correspondingly $\mathbf{CCW}(L)$) containing x. Also define $\theta_{\mathbf{CW}}(x)$ (correspondingly $\theta_{\mathbf{CCW}}(x)$) as the subset of those marked elements of L that join with x at a candidate line of $\mathbf{CW}$ (correspondingly $\mathbf{CCW}$). Then the values of the functions $\ell_{\mathbf{CW}}$, $\ell_{\mathbf{CCW}}$, $\theta_{\mathbf{CW}}$ and $\theta_{\mathbf{CCW}}$ are given in Table 1 below.*

x	$\ell_{\mathbf{CW}}x$	$\ell_{\mathbf{CCW}}x$
a	3	5
b	3	5
c	5	3
d	4	4
e	4	4
f	4	4
g	4	4
h	5	3
i	4	4

x	$\theta_{\mathbf{CW}}(x)$	$\theta_{\mathbf{CCW}}(x)$
a	b, c, h	d, e, f, g, i
b	a, c, i	d, e, f, g, h
c	a, b, d, e, i	f, g, h
d	c, e, f, h	a, b, g, i
e	c, d, f, g	a, b, h, i
f	d, e, g, h	a, b, c, i
g	e, f, h, i	a, b, c, d
h	a, d, f, g, i	b, c, e
i	b, c, g, h	a, d, e, f

TABLE 1. The functions $\ell_{\mathbf{CW}}$, $\ell_{\mathbf{CCW}}$, $\theta_{\mathbf{CW}}$ and $\theta_{\mathbf{CCW}}$.

In particular, $\ell_{\mathbf{CW}}x = |\theta_{\mathbf{CW}}(x)|$ *and* $\ell_{\mathbf{CCW}}x = |\theta_{\mathbf{CCW}}(x)|$*, for any marked element* x *of* L*.*

Let $L[s_0, \dots, s_8]$ be a marked lattice, let $\mathbf{W}$ be a primitive zig-zag operator and let t be a marked element of $L'[t_0, \dots, t_8] = \mathbf{W}(L)$. Then we define the *base* of t, denoted $\mathrm{Bs}_{\mathbf{W}}(t)$, as the following set of marked elements of L:

$$\mathrm{Bs}_{\mathbf{W}}(t) = \{s_i \mid 0 \le i \le 8,\ t \in \mathrm{Cn}(s_i)\}.$$

Observe that $|\mathrm{Bs}_{\mathbf{CW}}(t_i)| = |\mathrm{Bs}_{\mathbf{CCW}}(t_i)| = 4$, for all $i \in \{0, \dots, 8\}$.

Lemma 4. *Let* $L[s_0, \dots, s_8]$ *be a marked lattice, let* $\mathbf{W}$ *be a primitive zig-zag operator and let* $L'[t_0, \dots, t_8] = \mathbf{W}(L)$*. Then the following inequality holds*

$$|\mathrm{Bs}_{\mathbf{W}}(t_i) \cup \mathrm{Bs}_{\mathbf{W}}(t_j) \cup \mathrm{Bs}_{\mathbf{W}}(t_k)| \ge 6,$$

for any distinct $i, j, k \in \{0, \dots, 7\}$*.*

Proof. In the following proof all indexes are taken modulo 9. First of all, let us observe that for $i \in \{0, \dots, 7\}$ (but not for $i = 8$) the base of t_i can be represented as

$$\mathrm{Bs}_{\mathbf{CW}}(t_i) = \{s_i, s_{i+1}, s_{i+2}, s_{i+4}\},$$
$$\mathrm{Bs}_{\mathbf{CCW}}(t_i) = \{s_i, s_{i+3}, s_{i+4}, s_{i+8}\}.$$

In the case $\mathbf{W} = \mathbf{CW}$ from this representation we can infer

$$|\mathrm{Bs}_{\mathbf{CW}}(t_i) \cap \mathrm{Bs}_{\mathbf{CW}}(t_j)| \le 2$$

and thus

$$|\mathrm{Bs}_{\mathbf{CW}}(t_i) \cup \mathrm{Bs}_{\mathbf{CW}}(t_j)| \ge 6,$$

for any distinct $i, j \in \{0, \dots, 7\}$.

In the case $\mathbf{W} = \mathbf{CCW}$ the latter inequality does not hold, as, for example,

$$\mathrm{Bs}_{\mathbf{CCW}}(t_0) \cup \mathrm{Bs}_{\mathbf{CCW}}(t_4) = \{s_0, s_3, s_4, s_7, s_8\}.$$

On the other hand we can see that the inequality

$$|\mathrm{Bs}_{\mathbf{CW}}(t_i) \cup \mathrm{Bs}_{\mathbf{CW}}(t_j)| \le 5$$

holds for distinct $i, j \in \{0, \ldots, 7\}$ only if $|i - j| = 4$. As from any three distinct elements $i, j, k \in \{0, \ldots, 7\}$ one can always choose a pair that does not satisfy this condition, we infer

$$|\mathrm{Bs}_{\mathbf{CCW}}(t_i) \cup \mathrm{Bs}_{\mathbf{CCW}}(t_j) \cup \mathrm{Bs}_{\mathbf{CCW}}(t_k)| \geq 6,$$

for any distinct $i, j, k \in [0, 7]$, thus finishing the proof. ■

5. Normal lattices

We denote by $L_0[a_1, \ldots, a_9]$, or simply L_0, the marked lattice represented in Figure 4.

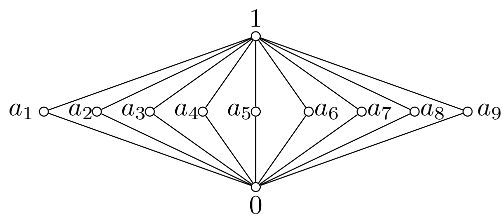

Figure 4. The lattice L_0

We call a lattice L *normal* if there is a zig-zag operator $\mathbf{Z}^L$ such that $L = \mathbf{Z}^L(L_0)$. We understand by the *degree* of a normal lattice the minimal degree of the zig-zag operator $\mathbf{Z}$, such that $L = \mathbf{Z}(L_0)$. We treat a normal lattice of finite degree as marked lattice, that is, we allow a zig-zag operator to be applied to it. Notice also that a normal lattice is of finite degree iff it is finite. Further on, we will work exclusively with normal lattices.

We need the following technical result

Lemma 5. *Let L be a lattice of length at most three and let $x \in L^-$, then the poset obtained from L by removing x, denoted L^x, is a lattice.*

Proof. Let $y, z \in L^x$. If $y \vee_L z = w \neq x$ then $y \vee_{L^x} z = w$. On the other hand, if $y \vee_L z = x$ then $y \vee_{L^x} z = 1$. Thus, every pair of elements in L^x has a join, and the meets are handled dually. ■

Note that as all normal lattices are of length at most three, then Lemma 5 can be applied to them.

Let L be a normal lattice, $x \in L^-$. Define the *rank* of x, denoted $\mathbf{r}(x)$, as

$$\mathbf{r}(x) = \begin{cases} 0, & x \in L_0; \\ 2n - 1, & x \in \mathbf{Z}^L_{1\ldots n}(L_0) - \mathbf{Z}^L_{1\ldots n-1}(L_0),\ x \text{ is a line;} \\ 2n, & x \in \mathbf{Z}^L_{1\ldots n}(L_0) - \mathbf{Z}^L_{1\ldots n-1}(L_0),\ x \text{ is an atom.} \end{cases}$$

That way, all atoms in a normal lattice have even rank, all lines (except the "degenerate" line 1 in L_0) have odd rank, and, informally speaking, an element of rank $k + 1$ is generated by elements of rank k.

Let us define a binary relation $\lessdot$ on a normal lattice L by $w \lessdot y$ if w and y are comparable elements of L^- and $\mathbf{r}(w) = \mathbf{r}(y) - 1$. We say that a subset C of L is a *cone* if $w \lessdot y$ and $w \in C$ implies $y \in C$. For a subset S of L we denote by $\mathrm{Cn}(S)$ the least cone of L containing S.

Occasionally, we write $\mathrm{Cn}(x_1, \dots, x_k)$ instead of $\mathrm{Cn}(\{x_1, \dots, x_k\})$ to denote the cone of $\{x_1, \dots, x_k\}$. For an element x of L we denote by $\mathrm{Cn}^-(x)$ the set $\mathrm{Cn}(x) - \{x\}$.

Proposition 14. *For any normal lattice L the following statements hold:*

(i) *Any distinct comparable elements $x, y \in L^-$ satisfy $\mathbf{r}(x) - \mathbf{r}(y) \in \{-1, 1\}$, that is, either $x \lessdot y$ or $y \lessdot x$.*
(ii) *Every element $x \in L^-$ of rank $k \geq 1$ has exactly two elements u and v such that $u, v \lessdot x$.*
(iii) *Every line of L of rank $k \geq 1$ contains exactly three atoms: two of rank $k-1$ and one of rank $k+1$.*
(iv) *Every atom of L is contained in at most seven lines.*

Note that in case x is an atom the two elements u and v of (ii) are conjugate lines.

Corollary 3. *If L is a normal lattice then all its intervals, except for the trivial interval $[0, 1]$, are finite.*

Let us state several trivial observations about cones.

Proposition 15. *For any normal lattice L the following statements hold:*

(i) *All elements of $\mathrm{Cn}(x)^-$ have rank strictly greater than the rank of x.*
(ii) *The zero of L lies in $\mathrm{Cn}(S)$ only if it lies in S, and similarly for the unit. In particular, $\mathrm{Cn}(x) \subseteq L^-$, for any $x \in L^-$.*
(iii) *For all x, y in L, if $y \in \mathrm{Cn}(x)$ then $\mathrm{Cn}(y) \subseteq \mathrm{Cn}(x)$.*

Lemma 6. *Let K be a normal lattice, then $K - \mathrm{Cn}(x)$ is a sublattice of K^x, for all $x \in K^-$.*

Proof. Suppose, to the contrary, that in K^x there is $y \in \mathrm{Cn}(x)$ such that $y = u \vee v$ or $y = u \wedge v$, for some $u, v \in K - \mathrm{Cn}(x)$. As the corresponding join or meet is proper, then $u, v \in K^-$, and also as $\mathrm{Cn}(x) \subseteq K^-$ then $y \in K^-$. Observe that u and v are comparable to y, thus, by Proposition 14, (i) either $u \lessdot y$ or $y \lessdot u$, and similarly for v. But if $y \lessdot u$, then $u \in \mathrm{Cn}(y)$, which is impossible, and so we get $u, v \lessdot y$.

Now, by Proposition 14, (ii) u and v are the only elements of K such that $u, v \lessdot y$. As $u, v \notin \mathrm{Cn}(x)$, then we may say that $z \not\lessdot y$, for any $z \in \mathrm{Cn}(x)$. But, as $y \neq x$, this implies $y \notin \mathrm{Cn}(x)$, a contradiction. ■

Now we need to establish a crucial fact about infinite normal lattices.

Lemma 7. *Let L be a finite normal lattice and let $\mathbf{W}$ be an infinite zig-zag operator, $K = \mathbf{W}(L)$. Then $\langle L \rangle_{K^x}$ is finite, for any x in $K - L$.*

Proof. As the rank of x is greater than the rank of all elements of L then by Proposition 15 we get $L \subseteq K - \mathrm{Cn}(x)$. Observe that by Lemma 6 the set $K - \mathrm{Cn}(x)$ is a sublattice of K^x and so $\langle L \rangle_{K^x} \subseteq K - \mathrm{Cn}(x)$. Thus, to prove the finiteness of $\langle L \rangle_{K^x}$ it is sufficient to prove the following statement: There exists a positive integer k such that all marked elements of $\mathbf{W}_{1 \dots k}(L)$ belong to $\mathrm{Cn}(x)$.

By Proposition 14, (iii) every line l has exactly one atom y of rank $\mathbf{r}(l) + 1$ in $\mathrm{Cn}(l)$. For example, if $\mathbf{W}_1 = \mathbf{CW}$ and the line is ab, then the relevant atom is $ab \cdot ce$. As $\mathrm{Cn}(y) \subseteq \mathrm{Cn}(l)$, then, without loss of generality we may assume that x is an atom.

Next, by Proposition 13, for both $\mathbf{CW}$ and $\mathbf{CCW}$ every marked element participates in the generation of at least three new lines, which in turn participate in producing three new marked elements, u_1, u_2 and u_3. For example, for the atom a and the operator $\mathbf{CW}$ the corresponding lines would be ab, ha and ac, and the corresponding atoms would be $a_1 = ab \cdot ce$, $f_1 = fg \cdot ha$ and $h_1 = hi \cdot ac$. Thus, $\mathrm{Cn}(x) \supseteq \mathrm{Cn}(u_1, u_2, u_3)$.

Again by Proposition 13, only two out of the three elements u_1, u_2, u_3 can produce three lines, at least one produce four or more. Say, if we consider the marked elements a, f and h, and the operator is $\mathbf{CCW}$, then there are four lines and, correspondingly, four atoms generated from f, namely lines fa, cf, fi and bf, and atoms $b_1 = be \cdot fa$, $c_1 = cf \cdot gb$, $f_1 = fi \cdot ae$ and $g_1 = ga \cdot bf$. In any case, there are at least four distinct marked elements $v_1, \dots, v_4$ of $\mathbf{W}_{1\dots2}(L)$ such that $\mathrm{Cn}(x) \supseteq \mathrm{Cn}(v_1, \dots, v_4)$.

Now, let $L_2[s_0, \dots, s_8] = \mathbf{W}_{1\dots2}(L)$ and $L_3[t_0, \dots, t_8] = \mathbf{W}_{1\dots3}(L)$; this way the elements $v_1, \dots, v_4$ lie in $\{s_0, \dots, s_8\}$. For any $i \in \{0, \dots, 8\}$, the element t_i lies outside of $\mathrm{Cn}(v_1, \dots, v_4)$ iff no v_j lies in $\mathrm{Bs}_{\mathbf{W}_3}(t_i)$. As $|\{s_0, \dots, s_8\} - \{v_1, \dots, v_4\}| = 5$, then by Lemma 4 it follows that at most two elements from $t_0, \dots, t_7$ and possibly the element t_8 are outside of $\mathrm{Cn}(v_1, \dots, v_4)$. So there are at most three elements outside of $\mathrm{Cn}(v_1, \dots, v_4)$, and thus at least six elements $w_1, \dots, w_6$ in $\mathrm{Cn}(v_1, \dots, v_4)$.

Regarding the example, if we take the marked elements b, c, f, g and the operator $\mathbf{CW}$, then every marked element of $\mathbf{CW}(L)$ lies in $\mathrm{Cn}(b, c, f, g)$. For the operator $\mathbf{CCW}$, however, there are two marked elements, $a_1 = ad \cdot ei$ and $e_1 = ef \cdot gi$ that lie outside of $\mathrm{Cn}(b, c, f, g)$.

Finally, as $|\mathrm{Bs}_{\mathbf{W}_4}(x)| = 4$, for any marked element $x \in \mathbf{W}_{1\dots4}(L)$, and as there are at most three marked elements from $\mathbf{W}_{1\dots3}(L)$ outside of $\mathrm{Cn}(v_1, \dots, v_4)$, then all marked elements of $\mathbf{W}_{1\dots4}(L)$ lie in $\mathrm{Cn}(v_1, \dots, v_4)$, and, consequently, in $\mathrm{Cn}(x)$. ∎

Lemma 8. *Let L be an infinite normal lattice, $L = \mathbf{Z}(L_0)$, let $L^n[a_1^n, \dots, a_9^n] = \mathbf{Z}_{1\dots n}(L_0)$, and let ϕ be a one-to-one lattice homomorphism from $\langle a_1^n, \dots, a_9^n\rangle_L$ to L. Then the following two statements hold:*

(i) *The heights of elements of L remain unaltered by the homomorphism ϕ.*
(ii) *The set $\{\phi(a_1^n), \dots, \phi(a_9^n)\}$ coincides with the set of marked elements of $\mathbf{Z}_{1\dots m}(L_0)$, for some $m \geq 0$. That is, $\{\phi(a_1^n), \dots, \phi(a_9^n)\} = \{a_1^m, \dots, a_9^m\}$, where $L^m[a_1^m, \dots, a_9^m] = \mathbf{Z}_{1\dots m}(L_0)$.*

It should be pointed out that $\{0, 1\} \cup \{a_1^n, \dots, a_9^n\}$ is not a sublattice of L. Moreover, the set $L - \langle a_1^n, \dots, a_9^n\rangle_L$ is finite, so, informally, $\{a_1^n, \dots, a_9^n\}$ generates almost entire L.

Proof. (i): As the lattice $\langle a_1^n, \dots, a_9^n\rangle_L$ is infinite, and as all intervals of L, except for $[0, 1]$, are finite, then $\phi(0) = 0$ and $\phi(1) = 1$. Now, suppose that $\phi(x) = l$, for some atom x and line l. But then there is a line $p \in \langle a_1^n, \dots, a_9^n\rangle_L$, $x < p$, and so $l < \phi(p) < 1$, which is impossible. Thus, $\phi(x)$ is an atom whenever x is an atom, and, by the similar argument, $\phi(l)$ is a line whenever l is a line.

(ii): We denote the set of marked elements of $\mathbf{Z}_{1\dots k}(L_0)$ by P_k, for any $k \geq 0$. Observe that the elements of P_k are exactly the elements of L of rank $2k$.

For a set X of atoms of L let us regard X as an undirected graph graph with respect to relation $R = \{(x, y) \mid x \vee y \text{ is a line}\}$. Then the set P_k and, consequently, the set $\phi(P_k)$ are connected graphs. Let us denote the set of indexes $I_k = \{i \mid P_i \cap \phi(P_k) \neq \varnothing\}$. Note that if two atoms $x \in P_i$ and $y \in P_j$ join at a line, then either $i = j \pm 1$ or $i = j$. This and the fact that P^k is connected imply that I^k is an interval in N, which we denote by $[m_0^k, m_1^k]$.

Notice that the statement $\{\phi(a_1^n), \dots, \phi(a_9^n)\} = \{a_1^m, \dots, a_9^m\}$ can be restated as $m_0^n = m_1^n = m$, for some m. We claim that the proof of this fact can further be reduced to proving the following statement:

(†) If $m_0^k < m_1^k$ for some k then $m_0^{k+1} < m_1^{k+1}$ and $m_0^{k+1} \le m_0^k$.

Indeed, let us assume that (†) holds and, arguing by contradiction, suppose that $m_0^n < m_1^n$. Then iteratively applying (†) we get $m_0^k < m_1^k$, for all $k \ge n$, and the sequence $\{m_0^k\}_{k=n}^{\infty}$ is decreasing and nonnegative. Thus, there is $N \ge n$ such that all m_0^k are equal to $M = m_0^N$, for all $k \ge N$. By the definition of I_k we obtain that P_M has a nonempty intersection with $\phi(P_k)$, for all $k \ge N$. As ϕ is one-to-one, the sets P_k are disjoint for all k, and P_M is finite, we get a contradiction.

Now, let us prove that (†) is correct. To simplify the notation, in the proof below we write m_0 instead of m_0^k.

Take $x, y \in P_k$ such that $\phi(x) \in P_{m_0}$, $\phi(y) \in P_{m_0+1}$, and $l = x \vee y$ is a line. As x and y have rank $2k$ then by Proposition 14 the rank of l is $2k+1$ and there is an atom u of rank $2k+2$, that is, $u \in P_{k+1}$, such that $u \le l$. Applying Proposition 14 once again, we infer that u is the meet of two conjugate lines l and l^* of rank $2k+1$. Furthermore, l^* is the join of two atoms z and w of rank $2k$, that is, $z, w \in P^k$, $z \vee w = l^*$. It is clear that the elements x, y, z, w, u, l and l^* are pairwise distinct.

Let us consider the rank of the images of elements x, y, z, w, u, l and l^* under the homomorphism ϕ. As $\mathbf{r}(\phi(x)) = 2m_0$ and $\mathbf{r}(\phi(y)) = 2m_0+2$, then $\mathbf{r}(\phi(l)) = 2m_0+1$ and $\mathbf{r}(\phi(u)) = 2m_0$. For the rank of the line $\phi(l^*)$ we need to consider two cases: either $\mathbf{r}(\phi(l^*)) = 2m_0 - 1$, or $\mathbf{r}(\phi(l^*)) = 2m_0 + 1$. However, in the first case we get $\mathbf{r}(\phi(z)) = \mathbf{r}(\phi(w)) = 2m_0 - 2$, which is impossible as z and w lie in P_k. Thus, $\mathbf{r}(\phi(l^*)) = 2m_0 + 1$ and out of the atoms $\phi(z)$ and $\phi(w)$ one (say z) has rank $2m_0$, and another (say w) has rank $2m_0 + 2$.

The above argument is illustrated in Figure 5 below.

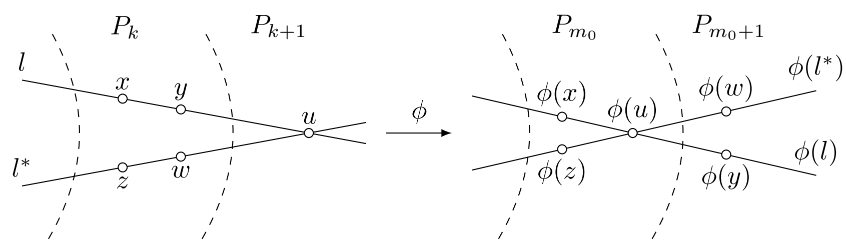

FIGURE 5

All in all, as $u \in P_{k+1}$ and $\phi(u) \in P_{m_0}$, then $m_0 \in [m_0^{k+1}, m_1^{k+1}]$, yielding $m_0^{k+1} \le m_0^k$. Also notice that as the sets P_{m_0} and $\phi(P_{k+1})$ are equal in size, and as in P_{m_0} there are elements $\phi(x)$ and $\phi(z)$, which do not belong to $\phi(P_{k+1})$, then some element of $\phi(P_{k+1})$ lies outside of P_{m_0}, thus $m_0^{k+1} < m_1^{k+1}$. ■

6. THE COUNTEREXAMPLE

Let

$$L_\infty = \left(\prod_{k=1}^{\infty} \mathbf{C}_k \right) L_0$$

where

$$\mathbf{C}_k = \mathbf{CCW} \circ \prod_{i=1}^{k} \mathbf{CW}\,.$$

We denote the infinite zig-zag operator $\prod_{k=1}^{\infty} \mathbf{C}_k$ by $\mathbf{Z}$, that is, $L_\infty = \mathbf{Z}(L_0)$. Notice that $\mathbf{Z}_{1\ldots n} \neq \prod_{k=1}^{n} \mathbf{C}_k$, as the operators $\mathbf{C}_k$ are not primitive. Furthermore, the operator $\prod_{k=1}^{n} \mathbf{C}_k$ can be expressed as

$$\prod_{k=1}^{n} \mathbf{C}_k = \mathbf{Z}_{1\ldots\frac{n(n+3)}{2}}\,. \tag{6}$$

We also denote the marked lattice $\mathbf{Z}_{1\ldots n}(L_0)$ by L_n, the set of marked elements of L_n by P_n, and the set of lines from which P_n is generated by Q_n. This way, P_n is the set of all elements of rank $2n$, and Q_n of rank $2n-1$.

Now we are going to prove that the only automorphism of L_∞ is the trivial identity automorphism, in fact even a stronger property.

Lemma 9. *Let $L_n[a_1^n, \ldots, a_9^n] = \mathbf{Z}_{1\ldots n}(L_0)$. Then the only one-to-one homomorphism from $\langle a_1^n, \ldots, a_9^n \rangle_{L_\infty}$ to L_∞ is a trivial embedding $x \mapsto x$.*

Proof. Let ϕ be a one-to-one homomorphism from $\langle a_1^n, \ldots, a_9^n \rangle_{L_\infty}$ to L_∞, notice that $\{a_1^n, \ldots, a_9^n\} = P_n$. By Lemma 8 the homomorphism ϕ preserves heights of the elements of $\langle a_1^n, \ldots, a_9^n \rangle_{L_\infty}$, and $\phi[P_n] = P_m$ for some m. Observe that the sets Q_{k+1} and P_{k+1} can be expressed as

$$\begin{aligned} Q_{k+1} &= \{x \in P_k^{\vee} \mid x \text{ is a line}\}, \\ P_{k+1} &= \{x \in P_k^{\vee\wedge} - P_k \mid x \text{ is an atom}\}, \end{aligned}$$

for any $k \geq 0$. From this representation it follows that $\phi\left[P_{n+d}\right] = P_{m+d}$ and $\phi\left[Q_{n+d+1}\right] = Q_{m+d+1}$, that is, ϕ establishes a bijection between Q_{n+d+1} and Q_{m+d+1}, and between P_{n+d} and P_{m+d}, for any $d \geq 0$.

We claim that for all $d \geq 0$ and for all $x \in P_{n+d}$ the following equality holds

$$\phi\left[\theta_{\mathbf{Z}_{n+d+1}}(x)\right] = \theta_{\mathbf{Z}_{m+d+1}}(\phi(x)).$$

Indeed,

$$\begin{aligned} y \in \phi\left[\theta_{\mathbf{Z}_{n+d+1}}(x)\right] &\Leftrightarrow \phi^{-1}(y) \in \theta_{\mathbf{Z}_{n+d+1}}(x) \Leftrightarrow \phi^{-1}(y) \vee x \in Q_{n+d+1} \\ &\Leftrightarrow y \vee \phi(x) \in Q_{m+d+1} \Leftrightarrow y \in \theta_{\mathbf{Z}_{m+d+1}}(\phi(x)). \end{aligned}$$

In particular, we get $\theta_{\mathbf{Z}_{n+d+1}}(x) = \theta_{\mathbf{Z}_{m+d+1}}(\phi(x))$. Notice that as $\mathbf{Z}_{n+d+1}$ and $\mathbf{Z}_{m+d+1}$ are primitive zig-zag operators, then the notation $\theta_{\mathbf{Z}_{n+d+1}}$ and $\theta_{\mathbf{Z}_{m+d+1}}$ is justified.

Let us consider the case when $m \neq n$. We claim that in this case we can choose $d \geq 0$ such that $\mathbf{Z}_{n+d+1} = \mathbf{CW}$ and $\mathbf{Z}_{m+d+1} = \mathbf{CCW}$. Indeed, take $N \geq |m-n|$ and denote $m_0 = \frac{N(N+1)}{2}$. Then by (6) we get $\mathbf{Z}_{m_0} = \mathbf{CCW}$, and $\mathbf{Z}_{n_0} = \mathbf{CW}$ for all n_0 such that $n_0 \neq m_0$ and $|n_0 - m_0| \leq N$. Additionally, let N be big enough so that $m < m_0$, and put $d = m_0 - m - 1$. Then $\mathbf{Z}_{m+d+1} = \mathbf{Z}_{m_0} = \mathbf{CCW}$. Furthermore, $|n+d+1-m_0| = |n-m| \leq N$, thus, $\mathbf{Z}_{n+d+1} = \mathbf{CW}$.

Denote the marked elements of P_{n+d} by $a, b, c, d, e, f, g, h, i$ and of P_{m+d} by $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, \overline{g}, \overline{h}, \overline{i}$. As $\ell_{\mathbf{CW}}(a) = \ell_{\mathbf{CW}}(b) = 3$, and $\ell_{\mathbf{CCW}}(\overline{c}) = \ell_{\mathbf{CCW}}(\overline{e}) = 3$, we get $\phi(ab) = \overline{ch}$. Trivially, ϕ maps conjugate lines to conjugate lines, so $\phi(ce) = \phi(ab^*) = \overline{ch}^* = \overline{dg}$. But this is impossible because $\ell_{\mathbf{CW}}(c) = 5$ and $\ell_{\mathbf{CCW}}(\overline{d}) = \ell_{\mathbf{CCW}}(\overline{g}) = 4$.

We finally need to consider the case when $\phi[P_n] = P_n$. In the remaining part of the theorem we denote the marked elements of P_n by $a, b, c, d, e, f, g, h, i$.

If $\mathbf{Z}_{n+1} = \mathbf{CW}$ then by the above argument we get $\phi(ab) = ab$, and $\phi(ab^*) = \phi(ce) = ce$. As $\phi(c) = \phi(h) = 5$, we get $\phi(c) = c, \phi(e) = e$ and $\phi(h) = h$. Now we use $\theta_{\mathbf{CW}}$ to determine the values of ϕ on the remaining marked elements. From $\theta_{\mathbf{CW}}(\phi(d)) = \{c, e, h, \phi(f)\}$ we get $\phi(d) = d$ and $\phi(f) = f$. From $\theta_{\mathbf{CW}}(\phi(g)) = \{e, f, h, \phi(i)\}$ we get $\phi(g) = g$ and $\phi(i) = i$. And from $\theta_{\mathbf{CW}}(\phi(b)) = \{c, i, \phi(a)\}$ we get $\phi(b) = b$ and $\phi(a) = a$. Thus, ϕ is an identity.

If $\mathbf{Z}_{n+1} = \mathbf{CCW}$ then by considering $\ell_{\mathbf{CCW}}$ we get $\phi(ch) = ch$, and $\phi(ch^*) = \phi(dg) = dg$. Thus, either $\phi(h) = h$, or $\phi(h) = c$. But in the latter case we get $\theta_{\mathbf{CCW}}(\phi(h)) = \{\phi(b), \phi(c), \phi(e)\} = \{f, g, h\}$, which is impossible because $\ell_{\mathbf{CCW}}(b) = 5$ and so $\phi(b) \in \{a, b\}$. Thus, $\phi(h) = h$ and consequently $\phi(c) = c$. Also from $\theta_{\mathbf{CCW}}(\phi(h)) = \{\phi(b), c, \phi(e)\} = \{b, c, e\}$ we get $\phi(b) = b$, $\phi(e) = e$ and as $\ell_{\mathbf{CCW}}(a) = 5$ then $\phi(a) = a$. From $\theta_{\mathbf{CCW}}(\phi(g)) = \{a, b, c, \phi(d)\}$ we get $\phi(g) = g$ and $\phi(d) = d$. And from $\theta_{\mathbf{CCW}}(\phi(i)) = \{a, d, e, \phi(f)\}$ we get $\phi(i) = i$ and $\phi(f) = f$. Again, ϕ is an identity. ■

7. The proof

We are going to prove that the tensor product $L_\infty \otimes L_\infty^{\mathrm{op}}$ is a lattice and that it contains a noncapped compact bi-ideal, confirming Conjecture 1.

Throughout this section in order not to get lost between the order of L_∞ and its dual we will stick to the order of L_∞. This way, the elements of $L_\infty \overrightarrow{\otimes} L_\infty^{\mathrm{op}}$, which by Remark 1 lie in $\operatorname{Hom}\left(\left(L_\infty^0; \vee\right), \left(L_\infty^{\mathrm{op}}; \wedge_{L_\infty^{\mathrm{op}}}\right)\right)$, become join-homomorphisms from L_∞^0 to L_∞, and correspondingly the elements of $L_\infty^{\mathrm{op}} \overrightarrow{\otimes} L_\infty$ become meet-homomorphisms from L_∞^1 to L_∞.

Rewritten in these terms, the pure tensor takes form

$$a \otimes b = \{(x, y) \in L_\infty \times L_\infty \mid x \leq a,\ y \geq b\} \cup \bot$$

where

$$\bot = (\{0\} \times L_\infty) \cup (L_\infty \times \{1\}),$$

for all $a, b \in L_\infty$. Formula (2) takes form

$$\xi(x) = \bigwedge (b_i \mid i < N,\ x \leq a_i), \tag{2'}$$

for all $x \in L_\infty^0$, where $a_i, b_i \in L_\infty$, for $i < N$. Thus defined, $\xi(x)$ is an isotone step function. Similarly, formulas (3) and (4) for one-step adjustment take form

$$\xi^{(1)}(x) = \bigwedge \left(\bigvee \xi[S] \mid S \in \mathrm{C}(x) \right), \tag{3'}$$

$$\xi^{(1)}(x) = \xi(x) \wedge \bigwedge \left(\bigvee \xi[S] \mid S \in \mathrm{C}^*(x) \right). \tag{4'}$$

We start with the following technical lemma

Lemma 10. *Let K be a lattice of length at most three. Then*

(i) *For any $x \in K^-$ and any $y \in K$, $y \neq x$, the following equality holds*

$$\mathrm{C}(y) - \mathrm{C}(x) = \mathrm{C}_{K^x}(y) - \mathrm{C}(x).$$

(ii) *Let x be an atom and let w be a line in K, $x < w$. For a subset $S \subseteq K$ let us define the subset $\overline{S} \subseteq K^x$ by*

$$\overline{S} = \begin{cases} S, & x \notin S; \\ S - \{x\} + \{w\} & x \in S, \end{cases}$$

then for all $y \in K^x$ the following equality holds

$$\{\overline{S} \mid S \in \mathrm{C}(y)\} = \{S \mid S \in \mathrm{C}_{K^x}(y)\}.$$

We recall that $\mathrm{C}(a)$ and $\mathrm{C}_{K^x}(a)$ are the sets of finite join-covers of a in K and in K^x correspondingly.

Proof. (i): If a subset $A \subseteq K$ is finite and $x \notin A$ then either $\bigvee_K A = \bigvee_{K_x} A$ or $\bigvee_K A = x$. Thus, for any finite subset $S \subseteq K$ such that $S \notin \mathrm{C}(x)$ we get $\bigvee_K S = \bigvee_{K_x} S$, which immediately proves the claim.

(ii): Take $y \in K^x$ and $S \in \mathrm{C}(y)$. Then $\bigvee \overline{S} \geq \bigvee S \geq y$ and $x_0 \notin \overline{S}$, thus, $\overline{S} \in \mathrm{C}_{K^x}(y)$ and $\{\overline{S} \mid S \in \mathrm{C}(y)\} \subseteq \{S \mid S \in \mathrm{C}_{K^x}(y)\}$. To prove the opposite inclusion we take $S \in \mathrm{C}_{K^x}(y)$. Then either $\bigvee_{K^x} S = \bigvee S$ or $\bigvee S = x$, but the latter is impossible as x is join-irreducible. Thus, $\bigvee_{K^x} S = \bigvee S$ and $S \in \mathrm{C}(y)$. As $S = \overline{S}$ we get $\{\overline{S} \mid S \in \mathrm{C}(y)\} \supseteq \{S \mid S \in \mathrm{C}_{K^x}(y)\}$, and consequently $\{\overline{S} \mid S \in \mathrm{C}(y)\} = \{S \mid S \in \mathrm{C}_{K^x}(y)\}$. ∎

The following two theorems establish a crucial property of $L_\infty \times L_\infty^{\mathrm{op}}$, namely, that in $L_\infty \times L_\infty^{\mathrm{op}}$ there is at least one noncapped compact bi-ideal, and that all noncapped compact bi-ideals are "almost" identical.

Theorem 2. *The set $E = \{(x, y) \mid x \leq y\}$ is a noncapped compact bi-ideal in $L_\infty \times L_\infty^{\mathrm{op}}$.*

Proof. For every element $x \in L_\infty^-$ the pair (x, x) lies in E. Additionally, $x \otimes x$ is the only pure tensor which is a subset of E and which contains (x, x) as en element. This means that the smallest representation of E as a union of pure tensors is

$$E = \bigcup (x \otimes x \mid x \in L_\infty^-),$$

which is obviously infinite, thus proving that E is noncapped.

On the other hand, we use the fact that the elements $a_1, \ldots, a_9$ generate entire L_∞, thus by Proposition 3 the set E can be represented as

$$\begin{aligned} E &= \bigcup (x \otimes x \mid x \in L_\infty) \\ &= \bigcup (p(a_1, \ldots, a_9) \otimes p^{\mathrm{op}}(a_1, \ldots, a_9) \mid p \in \mathrm{F}(n)) \\ &= \bigvee (a_i \otimes a_i \mid i = 1, \ldots, 9)\,. \end{aligned}$$

So, E is a join of the finite number of pure tensors, and thus it is compact. ∎

Lemma 11. *Let I be a noncapped compact bi-ideal in $L_\infty \times L_\infty^{\mathrm{op}}$,*

$$I = \bigvee (a_i \otimes b_i \mid i < N)\,.$$

Let $\varphi_I = \varepsilon^{-1}(X)$, $\varphi_E = \varepsilon^{-1}(E)$. Then φ_I differs from φ_E only in a finite number of elements.

Proof. Let ξ be the isotone step function from L_∞^0 to L_∞^{op} obtained from the elements a_i and b_i, $i < N$ by formula $(2')$, that is,

$$\xi(x) = \bigwedge (b_i \mid i < N,\ x \leq a_i)\,,$$

and let $\rho\colon L^1_\infty \to L_\infty$ be its dual. Without loss of generality take $a_i > 0$ and $b_i < 1$, for all $i < N$. By Theorem 1 and Corollary 1, $\varphi_I = \xi^{(\omega)}$ and $\varphi^*_I = \rho^{(\omega)}$.

We recall that $\xi^{(\omega)}$ is a join-homomorphism from L^0_∞ to L_∞ and $\rho^{(\omega)}$ is a meet-homomorphism from L^1_∞ to L_∞. Notice that both $\xi^{(\omega)}$ and $\rho^{(\omega)}$ are isotone mappings and the following statement holds

$$(\ddagger) \qquad \begin{aligned}(x, y) \in I &\Leftrightarrow x = 0 \text{ or } \xi^{(\omega)}(x) \le y \\ &\Leftrightarrow y = 1 \text{ or } x \le \rho^{(\omega)}(y),\end{aligned}$$

for all $x, y \in L_\infty$.

Fix k_0 such that $a_i, b_i \in L_{k_0}$, for all $i < N$. Now, step by step, we prove that there is some r_0 such that $\xi^{(\omega)}$ preserves heights of all elements in L^0_∞, except for the atoms and lines of L_{r_0-1}; that is, $\xi^{(\omega)}$ maps points, lines and the unit of L^0_∞ to points, lines and the unit of L_∞ correspondingly, or, alternatively, to lines, points and the zero of L^{op}_∞ correspondingly.

(i) $\xi^{(\omega)}(1) = 1$.

If $\xi^{(\omega)}(1) = a < 1$ then by ($\ddagger$) we get $1 = \rho^{(\omega)}(a)$. Then

$$\rho^{(\omega)}(y) = \rho^{(\omega)}(y) \wedge \rho^{(\omega)}(a) = \rho^{(\omega)}(y \wedge a),$$

for all $y \in L^1_\infty$. Thus, $\rho^{(\omega)}(y) \otimes y \subseteq \rho^{(\omega)}(y \wedge a) \otimes (y \wedge a)$, for all $y \in L^1_\infty$, and by Corollary 2 we get

$$\begin{aligned} I &= \bigcup \left(\rho^{(\omega)}(y) \otimes y \mid y \in L^1_\infty \right) \\ &= \bigcup \left(\rho^{(\omega)}(y \wedge a) \otimes (y \wedge a) \mid y \in L^1_\infty \right) \\ &= \bigcup \left(\rho^{(\omega)}(y) \otimes y \mid y \in [0, a] \right). \end{aligned}$$

As the interval $[0, a]$ is finite, I is a finite union of pure tensors and thus it is capped, a contradiction.

(ii) $\rho^{(\omega)}(0) = 0$.

Just as in (i), if $0 < b = \rho^{(\omega)}(0)$ then $\xi^{(\omega)}(b) = 0$. Consequently, $\xi^{(\omega)}(x) = \xi^{(\omega)}(b \vee x)$ for all $x \in L^0_\infty$, and the rest of the proof follows.

(iii) $\xi^{(\omega)}(x) > 0$, for all $x > 0$.

If not, then $\rho^{(\omega)}(0) \ge x > 0$, a contradiction.

(iv) $\rho^{(\omega)}(x) < 1$, for all $x < 1$.

If not, then $\xi^{(\omega)}(1) \le x < 1$, a contradiction.

(v) $\xi^{(\omega)}(x) < 1$, for all $x \in L_\infty - L_{k_0}$.

Suppose otherwise, so there is $x_0 \in L_\infty - L_{k_0}$ such that $\xi^{(\omega)}(x_0) = 1$. Then $\xi^{(n)}(x_0) = 1$, for all $n < \omega$. Notice that $x < 1$ and $x \ne a_i$, for all $i < N$. Let $\xi_{x_0}\colon (L^{x_0}_\infty)^0 \to L_\infty$ be the isotone step function defined by ($2'$), just as the function ξ, but with the domain $(L^{x_0}_\infty)^0$ instead of L^0_∞, that is

$$\xi_{x_0}(y) = \bigwedge \left(b_i \mid i < m,\ y \le_{L^{x_0}_\infty} a_i \right).$$

We claim that the following statement is true: For all $n > 0$ and for all $y \in (L^{x_0}_\infty)^0$ the equality $\xi^{(n)}_{x_0}(y) = \xi^{(n)}(y)$ holds. The proof is by induction on n.

The base of induction is given by $\xi^{(0)}_{x_0}(y) = \xi_{x_0}(y) = \xi^{(0)}(y)$, for all $y \in (L^{x_0}_\infty)^0$, which is trivial. On the other hand, by (3′) we get

$$\xi^{(n+1)}(y) = \bigwedge\Big(\bigvee \xi^{(n)}[S] \mid S \in \mathrm{C}(y)\Big),$$

for all $n > 0$ and all $y \in (L^{x_0}_\infty)^0$. If $S \in \mathrm{C}(x_0)$ then $\bigvee \xi^{(n)}[S] = 1$, and thus the above statement can be rewritten as

$$\xi^{(n+1)}(y) = \bigwedge\Big(\bigvee \xi^{(n)}[S] \mid S \in \mathrm{C}(y) - \mathrm{C}(x_0)\Big).$$

By Lemma 10, $\mathrm{C}(y) - \mathrm{C}(x) = \mathrm{C}_{L^x_\infty}(y) - \mathrm{C}(x)$ and so

$$\begin{aligned}\xi^{(n+1)}(y) &= \bigwedge\Big(\bigvee \xi^{(n)}[S] \mid S \in \mathrm{C}_{L^x_\infty}(y) - \mathrm{C}(x_0)\Big)\\ &= \bigwedge\Big(\bigvee \xi^{(n)}_{x_0}[S] \mid S \in \mathrm{C}_{L^x_\infty}(y) - \mathrm{C}(x_0)\Big),\end{aligned}$$

and by the same argument as above

$$\xi^{(n+1)}(y) = \bigwedge\Big(\bigvee \xi^{(n)}_{x_0}[S] \mid S \in \mathrm{C}_{L^x_\infty}(y)\Big) = \xi^{(n+1)}_{x_0}(y).$$

Thus, $\xi^{(n)}_{x_0}(y) = \xi^{(n)}(y)$ and consequently $\xi^{(\omega)}_{x_0}(y) = \xi^{(\omega)}(y)$, for all $n > 0$ and all $y \in (L^{x_0}_\infty)^0$.

Notice that $\xi^{(\omega)}_{x_0}$ is an element of $L_\infty \overrightarrow{\otimes} (L^{x_0}_\infty)^{\mathrm{op}}$ and $\xi^{(\omega)}_{x_0} = \varepsilon(J)$ where J is the compact bi-ideal in $L_\infty \otimes (L^{x_0}_\infty)^{\mathrm{op}}$,

$$J = \bigvee (a_i \otimes b_i \mid i < N).$$

Let $\psi_J \colon L^1_\infty \to L_\infty$ be the meet-homomorphism defined by $\psi_J = \eta(J) = (\xi^{(\omega)}_{x_0})^*$. Using the representation from Corollary 2 we get

$$\begin{aligned}J &= \bigcup \big(\psi_J(y) \otimes y \mid y \in \langle b_i \mid i < N\rangle_{L^{x_0}_\infty}\big)\\ &= \bigcup \big(\psi_J(y) \otimes y \mid y \in \langle L_{k_0}\rangle_{L^{x_0}_\infty}\big).\end{aligned}$$

By Lemma 7, the set $\langle L_{k_0}\rangle_{L^x_\infty}$ is finite, thus J is capped. But from $\xi^{(\omega)}_{x_0}(y) = \xi^{(\omega)}(y)$ we get

$$I = J \cup (x_0 \otimes \varphi_I(x_0)),$$

thus I is also capped, a contradiction.

(vi) $\rho^{(\omega)}(x) > 0$ for all $x \in L_\infty - L_{k_0}$.

Just as in (v), we take $x_0 \in L_\infty - L_{k_0}$ such that $\rho^{(\omega)}(x_0) = 0$, and by the same argument prove that $\rho^{(\omega)} = \rho^{(\omega)}_{x_0}$. As the bi-ideal J defined by $\rho^{(\omega)}_{x_0}$ is capped, then so is I, a contradiction.

(vii) For every atom x from L_∞ such that $x \notin L_{k_0}$, $\xi^{(\omega)}(x)$ is an atom in L_∞.

Arguing by contradiction, take an atom $x_0 \in L_\infty$, $x_0 \notin L_{k_0}$ such that $u = \xi^{(\omega)}(x_0)$ is not an atom. By the above argument $0 < \xi^{(\omega)}(x_0) < 1$, which means that u is a line. Similarly as in (v), we introduce the isotone step function $\xi_{x_0} \colon (L^{x_0}_\infty)^0 \to L_\infty$ and argue by induction that the equality $\xi^{(n)}_{x_0}(y) = \xi^{(n)}(y)$ holds for all $n > 0$ and all $y \in (L^{x_0}_\infty)^0$. The base of induction, $n = 0$, is obvious.

Notice that the sequence $\left(\xi^{(k)}(x_0) \mid k < \omega\right)$ is decreasing in L_∞, and all its elements lie in $[\xi^{(\omega)}(x_0), 1] = \{u, 1\}$. Thus, there exists a unique m such that $\xi^{(i)}(x_0) = 1$ for $i \leq m$ and $\xi^{(j)}(x_0) = u$ for $j \geq m+1$. Then by $(4')$ we get

$$u = \xi^{(m+1)}(x_0) = \bigwedge \left(\bigvee \xi^{(m)}[S] \mid S \in \mathrm{C}^*(x_0) \right).$$

Thus, there exists a join-cover $S_0 \in \mathrm{C}^*(x_0)$ such that $u = \bigvee \xi^{(m)}[S_0]$. As S_0 is a nontrivial join-cover of x_0, and as x_0 is an atom, and, consequently, is join-irreducible, it follows that $\bigvee S_0 = l > x_0$.

As $\xi^{(k)}$ is isotone, we get $\xi^{(k)}(x_0) \leq \xi^{(k)}(l)$, for all $k \geq 0$. On the other hand,

$$\xi^{(k)}(l) \leq \xi^{(m+1)}(l) \leq \bigvee \xi^{(m)}[S_0] = u = \xi^{(k)}(x_0),$$

for all $k \geq m+1$. And $\xi^{(k)}(l) \leq 1 = \xi^{(k)}(x_0)$, for all $k \leq m$. Thus, $\xi^{(k)}(x_0) = \xi^{(k)}(l)$, for all $k \geq 0$. Notice also, that as $\xi^{(m+1)}(l) = u < 1$, then by (i) we infer that $x_0 < l < 1$, thus l is a line.

For a subset $S \subseteq L_\infty$ let us define the subset $\overline{S} \subseteq L_\infty^{x_0}$ by

$$\overline{S} = \begin{cases} S, & x_0 \notin S; \\ S - \{x_0\} + \{l\} & x_0 \in S. \end{cases}$$

Notice that by the above argument we get $\bigvee \xi^{(k)}[S] = \bigvee \xi^{(k)}[\overline{S}]$, for all $k \geq 0$.

By Lemma 10, $\{\overline{S} \mid S \in \mathrm{C}(y)\} = \{S \mid S \in \mathrm{C}_{L_\infty^{x_0}}(y)\}$, for all $y \in L_\infty^{x_0}$. Now, let us prove the induction step

$$\begin{aligned} \xi^{(n+1)}(y) &= \bigwedge \left(\bigvee \xi^{(n)}[S] \mid S \in \mathrm{C}(y) \right) \\ &= \bigwedge \left(\bigvee \xi^{(n)}[\overline{S}] \mid S \in \mathrm{C}(y) \right) \\ &= \bigwedge \left(\bigvee \xi_{x_0}^{(n)}[S] \mid S \in \mathrm{C}_{L_\infty^{x_0}}(y) \right) \\ &= \xi_{x_0}^{(n+1)}(y), \end{aligned}$$

for all $y \in L_\infty^{x_0}$. Then by the same argument as in (v) we infer that I is capped together with the compact bi-ideal J in $L_\infty \otimes (L_\infty^{x_0})^{\mathrm{op}}$, defined by

$$J = \bigvee \left(a_i \otimes b_i \mid i < N \right),$$

a contradiction.

(viii) For every line l of L_∞ such that $l \notin L_{k_0}$, $\rho^{(\omega)}(l)$ is a line in L_∞.

Same argument as in (vii).

(ix) For every line l of L_∞ such that $l \notin L_{k_0+1}$, $\xi^{(\omega)}(l)$ is a line in L_∞.

Arguing by contradiction, let us take a line $l \in L_\infty$ such that $l \notin L_{k_0+1}$ and $a = \xi^{(\omega)}(l)$ is an atom in L_∞. Now, take an atom $y \leq l$. As $l \notin L_{k_0+1}$ and y is comparable with l then $y \notin L_{k_0}$, and so by (vii) the element $\xi^{(\omega)}(y)$ is an atom in L_∞. But $\xi^{(\omega)}(y) \leq \xi^{(\omega)}(l) = a$, and so $\xi^{(\omega)}(y) = a$.

Next, take a line $p \neq l$ such that $p \geq y$ and let $b = \xi^{(\omega)}(p)$. From the above argument, b is either a line or an atom. As $p \geq y$ then $\xi^{(\omega)}(p) = b \geq \xi^{(\omega)}(y) = a$, and consequently $\rho^{(\omega)}(b) \geq \rho^{(\omega)}(a)$. But by Proposition 2 we get $\rho^{(\omega)}(a) \geq l$ and $\rho^{(\omega)}(b) \geq p$. Thus $\rho^{(\omega)}(b) \geq l \vee p = 1$, a contradiction.

(x) For every atom x from L_∞ such that $x \notin L_{k_0+1}$, $\rho^{(\omega)}(x)$ is an atom in L_∞.

Same argument as in (ix).

If we take $r_0 = k_0+2$ then $\xi^{(\omega)}$ preserves heights of all elements in $L_\infty - L_{r_0-1} = L_\infty - L_{k_0+1}$. As all atoms and lines of $\langle P_{r_0}\rangle$ lie in $L_\infty - L_{r_0-1}$, then $\xi^{(\omega)}$ preserves heights of all elements in $\langle P_{r_0}\rangle$.

We claim that the mappings $\xi^{(\omega)}$ and $\rho^{(\omega)}$ are one-to-one on $\langle P_{r_0}\rangle$. Indeed, take two distinct elements $a_1, a_2 \in \langle P_{r_0}\rangle$ and suppose that $\xi^{(\omega)}(a_1) = \xi^{(\omega)}(a_2) = b$. Then either all three elements a_1, a_2 and b are atoms, or all three are lines. However, as $\xi^{(\omega)}$ is a join-homomorphism then $\xi^{(\omega)}(a_1 \vee a_2) = b$. But then the elements a_1 and $a_1 \vee a_2$ have same heights, a contradiction. For $\rho^{(\omega)}$ the argument is similar.

Take

$$r_1 = \max\left\{k \mid \exists x \in L_\infty - L_{k-1} \colon \xi^{(\omega)}(x) \in L_{r_0}\right\},$$
$$r_2 = \max\left\{k \mid \exists x \in L_\infty - L_{k-1} \colon \rho^{(\omega)}(x) \in L_{r_0}\right\}$$

and

$$\overline{r} = \max(r_0, r_1, r_2).$$

Notice that as L_{r_0} is finite and the mappings $\xi^{(\omega)}$ and $\rho^{(\omega)}$ are one-to-one, then the maximums in the above definitions exist, thus r_1 and r_2 are defined correctly. Then $\xi^{(\omega)}$ and $\rho^{(\omega)}$ preserve heights on $L_\infty - L_{\overline{r}}$, $\xi^{(\omega)}(x) \in L_\infty - L_{r_0}$ and $\xi^{(\omega)}(x) \in L_\infty - L_{r_0}$, for all $x \in L_\infty - L_{\overline{r}}$, from which it follows that the mappings $\rho^{(\omega)} \circ \xi^{(\omega)}$ and $\xi^{(\omega)} \circ \rho^{(\omega)}$ also preserve heights on $L_\infty - L_{\overline{r}}$.

Now we argue that $\xi^{(\omega)}$ is a homomorphism on $\langle P_{\overline{r}}\rangle$. Indeed, we already know that it is a join-homomorphism. From (‡) it follows that the inequalities

$$x \leq \rho^{(\omega)} \circ \xi^{(\omega)}(x)$$

and

$$x \geq \xi^{(\omega)} \circ \rho^{(\omega)}(x)$$

hold, for all $x \in L_\infty^-$, whenever $\xi^{(\omega)}(x), \rho^{(\omega)}(x) \in L_\infty^-$. Then, as $\xi^{(\omega)} \circ \rho^{(\omega)}$ and $\xi^{(\omega)} \circ \rho^{(\omega)}$ preserve heights on $L_\infty - L_{\overline{r}}$, it follows that $\rho^{(\omega)} \circ \xi^{(\omega)}(x) = x$ and $\xi^{(\omega)} \circ \rho^{(\omega)}(y) = y$, for all $x, y \in L_\infty - L_{\overline{r}}$. Thus

$$\begin{aligned}\xi^{(\omega)}(a \wedge b) &= \xi^{(\omega)}\left(\rho^{(\omega)}\xi^{(\omega)}(a) \wedge \rho^{(\omega)}\xi^{(\omega)}(b)\right) = \xi^{(\omega)}\rho^{(\omega)}\left(\xi^{(\omega)}(a) \wedge \xi^{(\omega)}(b)\right)\\ &= \xi^{(\omega)}(a) \wedge \xi^{(\omega)}(b).\end{aligned}$$

Finally, we see that $\xi^{(\omega)}$ is a one-to-one homomorphism from $\langle P_{\overline{r}}\rangle$ to L_∞, thus, by Lemma 9 the homomorphism $\xi^{(\omega)}$ is an identity mapping, which means that $\xi^{(\omega)}(x)$ coincides with $\varphi_E(x)$ on $\langle P_{\overline{r}}\rangle$. As $L_\infty - \langle P_{\overline{r}}\rangle$ is finite, this proves the lemma. ■

Theorem 3. *For every pair of compact bi-ideals $I, J \in L_\infty \otimes L_\infty^{\mathrm{op}}$, the bi-ideal $I \cap J$ is compact.*

Proof. If both I and J are capped then it is straightforward.

If I is capped and J is not, then one can write $I = \bigcup(x_i \otimes y_i \mid i = 1, \ldots, n)$. Then $I \cap J = \bigcup_{i=1}^n (x_i \otimes y_i \cap J)$. As all nontrivial intervals in L_∞ and L_∞^{op} are finite, then every $(x_i \otimes y_i \cap J)$ is a compact bi-ideal, and thus so is their finite union, so $I \cap J$ is compact.

Finally, if both I and J are not capped, then by Lemma 11 there is a finite k such that $\varphi_{I\cap J} = \varphi_I = \varphi_J = \varphi_E$ on $L_\infty - L_k$, and so $I \cap J$ is also compact. ■

Finally, by putting together Theorems 2 and 3 we obtain immediately the following solution of Conjecture 1.

Theorem 4. *The tensor product $L_\infty \otimes L_\infty^{\mathrm{op}}$ is noncapped and is a lattice.*

References

[1] J. Anderson and N. Kimura, *The tensor product of semilattices*, Semigroup Forum **16** (1968), 83-88.

[2] G.A.Fraser, *The semilattice tensor product of distributive semilattices*, Trans. Amer. Math. Soc. **217** (1976), 183-194.

[3] R.Freese, J.Ježek and J.B.Nation, *Free lattices*. Math. Surveys and Monographs **42**, American Mathematical Society, Providence, Rhode Island, 1995.

[4] G.Grätzer, F.Wehrung, *A survey of tensor products and related constructions in two lectures*, Algebra Universalis **45** (2001), 117–134.

[5] G.Grätzer, F.Wehrung, *Tensor products and transferability of semilattices*, Canad. J. Math. **51** (1999), 792–815.

[6] G.Grätzer, F.Wehrung, *Tensor products of semilattices with zero, revisited*, J. Pure Appl. Algebra **147** (2000), 273–301.

[7] F.Wehrung, *Solutions to five problems on tensor products of lattices and related matters*, Algebra Universalis **47** (2002), 479–493.

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