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# Domain decomposition for Full-Wave simulation in a tokamak plasma

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## Abstract

The aim of this work is to develop a numerical method for the full-wave simulation of electromagnetic wave propagation in a plasma. The propagation and the absorption of lower hybrid (LH) electromagnetic waves is a powerful method to generate current in tokamaks by Landau wave particle resonance. Full-wave calculations of the LH wave propagation is a challenging issue because of the short wave length with respect to the machine size. We propose a Fourier finite element method for solving the Maxwell equations based on a mixed augmented variational formulation. In order to develop a parallel version of the simulation and consider non homogenous plasma response, a nonoverlapping domain decomposition approach is presented.

## Introduction

Let the domain  $\Omega$  be a torus (tokamak plasma volume) with strong external time-invariant magnetic field  $\mathbf{B}_{ext}$ . We study a second order partial differential equation for the time-harmonic electric field  $\mathbf{E}$  arising from Maxwell equations:

$$\mathbf{curl} \mathbf{curl} \mathbf{E} - \frac{\omega^2}{c^2} \underline{\mathbf{K}} \mathbf{E} = \mathbf{f} \quad \text{in } \Omega, \quad (1)$$

$$\text{div}(\underline{\mathbf{K}} \mathbf{E}) = g \quad \text{in } \Omega \quad (2)$$

where  $\omega > 0$  is the excited wave frequency and  $c$  denotes the speed of light in free space. The plasma response is described by the matrix  $\underline{\mathbf{K}}$ , in Stix frame (third coordinate parallel to  $\mathbf{B}_{ext}$ ). It includes a cold plasma approximation of the relative dielectric permittivity tensor and Landau damping:

$$\underline{\mathbf{K}}(\mathbf{x}) = \begin{pmatrix} S(\mathbf{x}) & -iD(\mathbf{x}) & 0 \\ iD(\mathbf{x}) & S(\mathbf{x}) & 0 \\ 0 & 0 & P_L(\mathbf{x}) \end{pmatrix}$$

Expressions of the entries  $S$ ,  $D$  and  $P_L$  involve plasma frequencies, cyclotron frequencies of each species (ion and electron) and also the collision frequency. In general, the matrix  $\underline{\mathbf{K}}$  is complex-valued and non-hermitian. Let  $\Gamma$  be the boundary of the domain  $\Omega$

and  $\Gamma_A \subset \Gamma$  be an antenna on the tokamak, then several boundary conditions are possible:

$$\begin{aligned} \text{Neumann:} \quad & \mathbf{curl} \mathbf{E} \times \mathbf{n} = i\omega\mu_0 \mathbf{j}_s \quad \text{on } \Gamma_A \\ \text{Dirichlet:} \quad & \mathbf{E} \times \mathbf{n} = \mathbf{E}_A \times \mathbf{n} \quad \text{on } \Gamma_A. \end{aligned}$$

On the other part of the boundary  $\Gamma_C = \Gamma \setminus \Gamma_A$ , we assume a perfectly conducting condition:

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_C.$$

## 1 Finite element method

### 1.1 Variational formulation and well-posedness

Taking the divergence condition (2) as constraint, we use a mixed augmented variational formulation (MAVF) [3], which gives rise to a  $\mathbf{H}^1$  conforming variational space,  $\mathbf{X}_N(\underline{\mathbf{K}}, \Omega) := \mathbf{H}_0(\mathbf{curl}, \Omega) \cap \mathbf{H}(\text{div } \underline{\mathbf{K}}, \Omega)$ . We obtain the following variational formulation of the Dirichlet problem :

Find  $(\mathbf{E}, p) \in \mathbf{X}_N(\underline{\mathbf{K}}, \Omega) \times L^2(\Omega)$  such that

$$\begin{aligned} a_s(\mathbf{E}, \mathbf{F}) + \overline{b(\mathbf{F}, p)} &= L_s(\mathbf{F}) \quad \forall \mathbf{F} \in \mathbf{X}_N(\underline{\mathbf{K}}, \Omega) \\ b(\mathbf{E}, q) &= l(q) \quad \forall q \in L^2(\Omega). \end{aligned}$$

where

$$\begin{aligned} a_s(\mathbf{E}, \mathbf{F}) &:= (\mathbf{curl} \mathbf{E} \mid \mathbf{curl} \mathbf{F}) - \frac{\omega^2}{c^2} (\underline{\mathbf{K}} \mathbf{E} \mid \mathbf{F}) \\ &\quad + s(\text{div } \underline{\mathbf{K}} \mathbf{E} \mid \text{div } \underline{\mathbf{K}} \mathbf{F}) \\ L_s(\mathbf{F}) &:= (\mathbf{f} \mid \mathbf{F}) + s(g \mid \text{div } \underline{\mathbf{K}} \mathbf{F}) \\ b(\mathbf{E}, q) &:= (\text{div } \underline{\mathbf{K}} \mathbf{E} \mid q) \\ l(q) &:= (g \mid q), \end{aligned}$$

with parameter  $s \in \mathbb{C}$ . Here,  $(\cdot \mid \cdot)$  denotes the standard  $L^2$  inner product in  $\Omega$ .

The well-posedness of the considered formulation follows from the Babuska-Brezzi theorem. Thanks to spectral properties of the dielectric tensor, the sesquilinear form  $a_s$  is coercive if  $\Re(s) > 0$  and  $\Im(s) \leq 0$ .

### 1.2 Dimension reduction and discretization

The 3D problem can be reduced to a series of 2D one by using cylindrical coordinates  $(R, Z, \phi)$  and by

expanding all functions  $f(R, Z, \phi)$  as Fourier series in the angular coordinate  $\phi$

$$f(R, Z, \phi) = \frac{1}{\sqrt{2\pi}} \sum_{\nu \in \mathbb{Z}} f_\nu(R, Z) e^{i\nu\phi}$$

where the coefficients  $f_\nu(R, Z)$  are defined on a cross section of  $\Omega$  [4]. Then the sesquilinear forms of the variational formulation can be written as sum of modal forms

$$a_s(\mathbf{u}, \mathbf{v}) = \sum_{\nu \in \mathbb{Z}} a_{s,\nu}(\mathbf{u}_\nu, \mathbf{v}_\nu), \quad b(\mathbf{v}, p) = \sum_{\nu \in \mathbb{Z}} b_\nu(\mathbf{v}_\nu, p_\nu)$$

The modal variational formulation is then discretized using a Taylor-Hood  $P_2$ -iso- $P_1$  finite element.

### 1.3 Numerical results

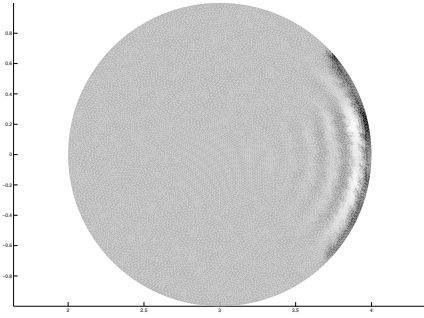


Figure 1: Real part of a component of the electric field for  $\omega = \omega_{LH} = 1.3 \times 10^{10}$  rad/s

## 2 Domain decomposition

Consider a nonoverlapping decomposition  $\overline{\Omega} = \bigcup_k \overline{\Omega}_k$ . In the domain decomposition method considered here, we solve the original problem in each subdomain  $\Omega_i$ ; the equivalence with the one-domain formulation is obtained by continuity conditions

$$[\mathbf{E} \times \mathbf{n}]_{\Sigma_{ij}} = 0 \quad \text{and} \quad [\mathbf{K} \mathbf{E} \cdot \mathbf{n}]_{\Sigma_{ij}} = 0 \quad (3)$$

which ensure the  $\mathbf{X}(\mathbf{K}, \Omega)$  regularity of the electric field and

$$[\mathbf{curl} \mathbf{E} \times \mathbf{n}]_{\Sigma_{ij}} = 0, \quad (4)$$

which implies that the one-domain formulation holds in the sense of distributions. We have denoted, as usual,  $[f]_{\Sigma_{ij}}$  the jump of a quantity  $f$  across the interface  $\Sigma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$ . The conditions (3) are dualized

by introducing the associated Lagrange multipliers  $\boldsymbol{\lambda}_{ij} \in \mathbf{H}^{1/2}(\Sigma_{ij})$ , while (4) is treated as a natural condition. The existence and uniqueness of the solution  $(\mathbf{E}_i, p_i, \boldsymbol{\lambda}_{ij})$  to the multidomain formulation was proved and :

$$\mathbf{E}_i = \mathbf{E}|_{\Omega_i} \quad \text{and} \quad p_i = p|_{\Omega_i}$$

where  $(\mathbf{E}, p)$  is the solution to the one-domain formulation.

The full linear system involving all subdomains (the *outer system*) is a generalized saddle-point problem:

$$\begin{pmatrix} \mathbf{Q} & \mathbf{G}^H \\ \mathbf{G} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ 0 \end{pmatrix} \quad (5)$$

where  $\mathbf{Q}$  is a block sparse non-hermitian matrix. Each block corresponds to a problem in one subdomain. The sparse matrix  $\mathbf{G}$  expresses the interactions between subdomains. The outer system (5) is solved using a preconditioned GMRES algorithm. The inner problem on each subdomain is also a generalized saddle-point problem, and is solved using a direct method.

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