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EUGEN HELLMANN AND BENJAMIN SCHRAEN

Abstract. Let K be a finite extension of $\mathbb{Q}_p$ and let $\bar{\rho}$ be a continuous, absolutely irreducible representation of its absolute Galois group with values in a finite field of characteristic p. We prove that the Galois representations that become crystalline of a fixed regular weight after an abelian extension are Zariski-dense in the generic fiber of the universal deformation ring of $\bar{\rho}$. In fact we deduce this from a similar density result for the space of trianguline representations. This uses an embedding of eigenvarieties for unitary groups into the spaces of trianguline representations as well as the corresponding density claim for eigenvarieties as a global input.

1. introduction

The density of crystalline representations in the generic fiber of a local deformation ring plays an important role in the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and was proven by Colmez [Co] and Kisin [Ki4] for 2-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. This density statement was generalized by Nakamura [Na2] and Chenevier [Ch1] to the case of 2-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ for finite extensions K of $\mathbb{Q}_p$ resp.. to the case of d-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and finally the general case was treated in [Na3].

In this paper we prove a slightly different density result in the generic fiber of a local deformation ring. The above density statements make heavy use of the fact that the Hodge-Tate weights of the crystalline representations may vary arbitrarily. Contrary to this case, we fix the Hodge-Tate weights but vary the level, or, more precisely, we allow finite (abelian) ramification and allow the representation to be potentially crystalline (more precisely crystabeline).

Note that this density statement is of a different nature than the density of crystalline representations. The density of crystalline representations holds true in the rigid generic fiber $(\mathrm{Spf}\, R_{\bar{\rho}})^{\mathrm{rig}}$ of the universal deformation ring $R_{\bar{\rho}}$ of a given residual $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$-representation $\bar{\rho}$. In contrast to this result, the density of potentially crystalline representations of fixed weight only holds true in the "naive" generic fiber $\mathrm{Spec}(R_{\bar{\rho}}[1/p])$, as the set of representations with fixed (generalized) Hodge-Tate weights is Zariski-closed in the rigid generic fiber $(\mathrm{Spf}\, R_{\bar{\rho}})^{\mathrm{rig}}$.

In the special case of 2-dimensional potentially Barsotti-Tate representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ our result gives a positive answer to a question of Colmez [Co].

A proof of this result (for 2-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$), using the p-adic local Langlands correspondence, was announced previously by Emerton and Paskunas. Our approach does not make use of such a correspondence and works in all dimensions and for arbitrary finite extensions of $\mathbb{Q}_p$. We were motivated by the case of 2-dimensional potentially Barsotti-Tate representations, as for these representations an automorphy lifting theorem is known [Ki2]. We hope to apply our density result to patching techniques in the future.

More precisely our results are as follows. Let K be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{G}_K = \mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ denote its absolute Galois group. Fix a continuous absolutely irreducible representation $\bar{\rho} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ with values in a finite extension $\mathbb{F}$ of $\mathbb{F}_p$. As the representation is assumed to be absolutely irreducible the universal deformation ring $R_{\bar{\rho}}$ of $\bar{\rho}$ exists.

Theorem 1.1. *Let $p \nmid 2d$ and let K be a finite extension of $\mathbb{Q}_p$. Let $\bar{r} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ be an aboslutely irreducible continuous representation which has a potentially diagonalizable lift and let $R_{\bar{r}}$ be its universal deformation ring. Assume that $\bar{r} \ncong \bar{r}(1)$. Let $\mathbf{k} = (k_{i,\sigma}) \in \prod_{\sigma : K \hookrightarrow \bar{\mathbb{Q}}_p} \mathbb{Z}^d$ be a regular weight. Then the representations that are crystabeline of labeled Hodge-Tate weight $\mathbf{k}$ are Zariski-dense in $\mathrm{Spec}\, R_{\bar{r}}[1/p]$.*

Similarly to the proof of density of crystalline representations we use a so called *space of trianguline representations*. This space should be seen as a local Galois-theoretic counterpart of an eigenvariety of Iwahori level. Indeed it was shown in [He2] that certain eigenvarieties embed into a space of trianguline representations in the case $K = \mathbb{Q}_p$. This result is generalized to the case of an arbitrary extension K of $\mathbb{Q}_p$ in section 3.2 below. In fact we prove the following density result for eigenvarieties which might be of independent interest.

Let E be an imaginary quadratic extension of a totally real field F such that $[F : \mathbb{Q}]$ is even and let G be a definite unitary group over F which is quasi-split at all finite places. Let Y be an eigenvariety for a certain set of automorphic representations of $G(\mathbb{A}_F)$ as in [Ch3, 3] which comes along with a Galois pseudo-character interpolating the Galois representations attached to the automorphic representations at the classical points of Y. Given an absolutely irreducible residual representation $\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/E) \to \mathrm{GL}_d(\mathbb{F})$ there is an open and closed subspace $Y_{\bar{\rho}} \subset Y$ where the pseudo-character reduces to (the pseudo-character attached to) $\bar{\rho}$ modulo p. This gives rise to a map $Y_{\bar{\rho}} \to (\mathrm{Spf}\, R_{\bar{\rho}})^{\mathrm{rig}}$ to the rigid generic fiber of the universal deformation ring $R_{\bar{\rho}}$ of $\bar{\rho}$.

Theorem 1.2. *Fix an algebraic irreducible representation W of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$. Let $f \in R_{\bar{\rho}}$ such that f vanishes on all classical points $z \in Y_{\bar{\rho}}$ corresponding to irreducible automorphic representations Π with $\Pi_\infty = W$. Then f vanishes in $\Gamma(Y_{\bar{\rho}}, \mathcal{O}_Y)$.*

We prove Theorem 1.1 by extending Theorem 1.2 to the space of trianguline representations $X(\bar{\rho}_{w_0})$, using a map $f : Y_{\bar{\rho}} \to X(\bar{\rho}_{w_0})$ constructed in Theorem 3.5 below. Here $\bar{\rho}_{w_0}$ is the restriction of $\bar{\rho}$ to the decomposition group at some place w_0 of E. The second step in the proof of Theorem 1.1 then is to realize a given residual representation $\bar{\rho} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ as the restriction to the decomposition group at w_0 of a $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$-representation arising from an automorphic representation of $G(\mathbb{A}_F)$. This construction was already carried out in [GK] or [EG] for example.

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2. The space of trianguline representations

Let $\bar{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. Let $K \subset \bar{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ and let K_0 denote the maximal unramified subextension of $\mathbb{Q}_p$ in K. We fix a compatible system $\epsilon_n \in \bar{\mathbb{Q}}_p$ of p^n-th roots of unity. Let $K_n = K(\epsilon_n) \subset \bar{\mathbb{Q}}_p$ and $K_\infty = \bigcup_n K_n$. We will write $\mathcal{G}_L = \mathrm{Gal}(\bar{\mathbb{Q}}_p/L)$ for any subfield $L \subset \bar{\mathbb{Q}}_p$. Finally we write $\Gamma = \Gamma_K = \mathrm{Gal}(K_\infty/K)$. We define the Hodge-Tate weights of a de Rham representation as the opposite of the gaps of the filtration on the covariant de Rham functor, so that the Hodge-Tate weight of the cyclotomic character is $+1$.

We choose a uniformizer $\varpi \in \mathcal{O}_K$ and normalize the reciprocity isomorphism $\mathrm{rec}_K : K^\times \to W_K^{\mathrm{ab}}$ of local class field theory such that ϖ is mapped to a geometric Frobenius automorphism. Here W_K^{ab} is the abelization of the Weil group $W_K \subset \mathcal{G}_K$ and the reciprocity map allows us to identify $\mathcal{O}_K^\times$ with a subgroup of $\mathcal{G}_K^{\mathrm{ab}}$, the maximal abelian quotient of $\mathcal{G}_K$. Further we write $\varepsilon : \mathcal{G}_K \to Z_p^\times$ for the cyclotomic character.

Let X be a rigid analytic space and recall the definition of the sheaf of relative Robba rings $\mathcal{R}_X = \mathcal{R}_{X,K}$ for K. If the base field K is understood we will omit the subscript K from the notation. This is the sheaf of functions that converge on the product of X with some boundary part of the open unit disc over K_0, see [He1, 2.2] or [KPX, Defintion 2.2.3] for example[1]. If $X = \mathrm{Sp}\, L$ for a finite extension L of $\mathbb{Q}_p$ we will write $\mathcal{R}_L = \mathcal{R}_{L,K}$ for (the global sections of) this sheaf. This sheaf of rings is endowed with a continuous $\mathcal{O}_X$-linear ring homomorphism $\varphi : \mathcal{R}_X \to \mathcal{R}_X$ and a continuous $\mathcal{O}_X$-linear action of the group Γ. Recall that a (φ, Γ)-module over a rigid space X consists of an $\mathcal{R}_X$-module D that is locally on X finite free over $\mathcal{R}_X$ together with a φ-linear isomorphism $\Phi : D \to D$ and a semi-linear Γ-action commuting with Φ.

Let us write $\mathbb{U}_L$ for the open unit disc over a p-adic field L and $\mathbb{U}_{r,L} \subset \mathbb{U}_L$ for the admissible open subspace of points of absolute value $\geq r$ for some $r \in p^{\mathbb{Q}} \cap [0,1)$. Given such an r we write $\mathcal{R}_X^r$ for the sheaf

$$X \supset U \longmapsto \Gamma(U \times \mathbb{U}_{r,K_0}, \mathcal{O}_{U \times \mathbb{U}_{r,K_0}})$$

and we write $\mathcal{R}_X^+$ for the sheaf $\mathcal{R}_X^0$ of functions converging on the product $X \times \mathbb{U}_{K_0}$.

Given a family of $\mathcal{G}_K$-representations $\mathcal{V}$ over a rigid space X, the work of Berger-Colmez [BeCo] and Kedlaya-Liu [KL] associates to $\mathcal{V}$ a (φ, Γ)-module $\mathbf{D}^\dagger_{\mathrm{rig}}(\mathcal{V})$ over $\mathcal{R}_X$.

[1]The sheaf $\mathcal{R}_X$ is denoted by $\mathscr{B}^\dagger_{X,\mathrm{rig}}$ in [He1]

Given a (φ,Γ)-module D over X, we write $H^*_{\varphi,\Gamma}(D)$ for the cohomology of the complex

$$C^\bullet_{\varphi,\Gamma}(D) = [D^\Delta \xrightarrow{\varphi-\mathrm{id},\gamma-\mathrm{id}} D^\Delta \oplus D^\Delta \xrightarrow{(\mathrm{id}-\gamma)\oplus(\varphi-\mathrm{id})} D^\Delta],$$

where $\Delta \subset \Gamma$ is the p-torsion subgroup of Γ and $\gamma \in \Gamma/\Delta$ is a topological generator. It is known that the cohomology sheaves $H^i_{\varphi,\Gamma}(D)$ are coherent $\mathcal{O}_X$-modules for $i = 0, 1, 2$ see [KPX, Theorem 4.4.5].

2.1. **The parameters.** In this section, we recall the construction of the space (φ,Γ)-modules of rank 1 over $\mathcal{R}$ essentially following [Co]. This is first step toward a construction of the *space of trianguline representations*.

Let $\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}_K^\times, \mathbb{G}_m(-))$ be the *weight space* of K. This functor on the category of rigid analytic spaces is representable by the generic fiber of Spf $\mathbb{Z}_p[\![\mathcal{O}_K^\times]\!]$. Further let $\mathcal{T} = \mathrm{Hom}_{\mathrm{cont}}(K^\times, \mathbb{G}_m(-))$. There is a natural projection $\mathcal{T} \to \mathcal{W}$ given by restriction to $\mathcal{O}_K^\times$. The choice of the uniformizer ϖ gives rise to a section of this projection and identifies $\mathcal{T}$ with $\mathbb{G}_m \times \mathcal{W}$ via $\delta \mapsto (\delta(\varpi), \delta|_{\mathcal{O}_K^\times})$. Especially $\mathcal{T}$ is representable by a rigid space.

We recall how the (φ,Γ)-modules of rank 1 over a rigid space X are classified by $\mathcal{T}(X)$, see [KPX, Theorem 6.1.10] (and also [Na1, 1.4] for the case $X = \mathrm{Sp}\, L$ in the context of B-pairs).

Let X be a rigid space over $\mathbb{Q}_p$ and let D be a rank 1 family of K-filtered φ-modules over X. Recall that this is a coherent $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0$-module that is locally on X free of rank 1 together with an $\mathrm{id}\otimes\varphi$-linear automorphism $\Phi : D \to D$ and a filtration $\mathrm{Fil}^\bullet$ on $D_K = D \otimes_{K_0} K$ by $\mathcal{O}_X \otimes_{\mathbb{Q}_p} K$ submodules that are locally on X direct summands as $\mathcal{O}_X$-modules.

Assume that X is defined over the normalization K^{norm} of K inside $\bar{\mathbb{Q}}_p$ and assume that D is free. Then such a K-filtered φ-module may be described as follows. There exists a uniquely determined $a \in \Gamma(X, \mathcal{O}_X^\times)$ and uniquely determined $k_\sigma \in \mathbb{Z}$ for each embedding $\sigma : K \hookrightarrow K^{\mathrm{norm}}$ such that $D \cong D(a; (k_\sigma)_\sigma)$ where $\Phi^{[K_0:\mathbb{Q}_p]}$ acts on $D(a; (k_\sigma)_\sigma)$ via multiplication with $a \otimes \mathrm{id} \in \Gamma(X, \mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0)^\times$ and

$$(\mathrm{gr}_i\, D_K) \otimes_{\mathcal{O}_X \otimes_{\mathbb{Q}_p} K, \mathrm{id}\otimes\sigma} \mathcal{O}_X \cong \begin{cases} 0 & i \neq k_\sigma \\ \mathcal{O}_X & i = k_\sigma \end{cases} \tag{2.1}$$

for all embeddings $\sigma : K \hookrightarrow \bar{\mathbb{Q}}_p$.

Given $k_\sigma \in \mathbb{Z}$ for each embedding $\sigma : K \hookrightarrow \bar{\mathbb{Q}}_p$ we consider the following special K-filtered φ-module $D((k_\sigma)_\sigma)$ over $L = K^{\mathrm{norm}}$ whose filtration is given by (2.1) and which has a basis on which φ-acts via multiplication with $\prod_\sigma \sigma(\varpi)^{k_\sigma}$.

Let X be a rigid space defined over K^{norm} and let D be a K-filtered φ-module over X. Associated to D there is a (φ,Γ)-module $\mathcal{R}_X(D)$ of rank 1 as follows. We write

$$D = D(a; (0)_\sigma) \otimes_{K_0} D((k_\sigma)_\sigma)$$

for some $k_\sigma \in \mathbb{Z}$ and $a \in \Gamma(X, \mathcal{O}_X^\times)$ and define

$$\mathcal{R}_X(D(a; (0)_\sigma)) = D(a; (0)_\sigma) \otimes_{\mathcal{O}_X \otimes_{\mathbb{Q}_p} K_0} \mathcal{R}_X,$$

where φ acts diagonally and Γ acts by acting on the second factor. Further

$$\mathcal{R}_X(D((k_\sigma)_\sigma)) = \prod\nolimits_\sigma t_\sigma^{k_\sigma}\mathcal{R}_X \subset \mathcal{R}_X\left[\tfrac{1}{t}\right]$$

with action of φ and Γ inherited from $\mathcal{R}_X[1/t]$. Here $t = \log([(1,\epsilon_1,\epsilon_2,\dots)]) \in \mathcal{R}^+_{\mathbb{Q}_p}$ is the usual period of the cyclotomic character and

$$t_\sigma = (t_{\sigma,\sigma'})_{\sigma'} \in \mathcal{R}^+_{K^{\mathrm{norm}}} = \prod_{\sigma'} \Gamma(\mathbb{U}_{K^{\mathrm{norm}}}, \mathcal{O}_{\mathbb{U}_{K^{\mathrm{norm}}}})$$

is defined via

$$t_{\sigma,\sigma'} = \begin{cases} 1 & \sigma \neq \sigma' \\ \sigma(t) & \sigma = \sigma'. \end{cases}$$

Finally we set

$$\mathcal{R}_X(D) = \mathcal{R}_X(D(a;(0)_\sigma)) \otimes_{\mathcal{R}_X} \mathcal{R}_X(D((k_\sigma)_\sigma)).$$

More generally, let $\delta : K^\times \to \Gamma(X, \mathcal{O}_X^\times)$ be a continuous character. Then there is a (φ,Γ)-module $\mathcal{R}_X(\delta)$ of rank 1 associated to δ as follows, cf. [KPX, Construction 6.1.4]. Write $\delta = \delta_1\delta_2$ with $\delta_1|_{\mathcal{O}_K^\times} = 1$ and such that δ_2 extends to a character of $\mathcal{G}_K$. Then we set

$$\mathcal{R}_X(\delta) = \mathcal{R}_X(D(\delta_1(\varpi),(0)_\sigma)) \otimes_{\mathcal{R}_X} \mathbf{D}^\dagger_{\mathrm{rig}}(\delta_2).$$

We write $\delta(D)$ for the character of $K^\times$ such that $\mathcal{R}_X(\delta(D)) = \mathcal{R}_X(D)$. Further, given $k_\sigma \in \mathbb{Z}$, we write $\delta((k_\sigma)_\sigma) = \delta(D(k_\sigma))$ for the character $z \mapsto \prod_\sigma \sigma(z)^{k_\sigma}$ and $\delta_{\mathcal{W}}((k_\sigma)_\sigma)$ for its restriction to $\mathcal{O}_K^\times$. Finally we write $\varepsilon = \delta(1,\dots,1)|\delta(1,\dots,1)|$ for the cyclotomic character (seen as a character of $K^\times$ or of $\mathcal{G}_K$).

Lemma 2.1. *Let $\delta \in \mathcal{T}(L)$ for a local field $L \supset K^{\mathrm{norm}}$. Then*

$$H^0_{\varphi,\Gamma}(\mathcal{R}(\delta)) \neq 0 \iff \delta = \delta((-k_\sigma)_\sigma) \text{ for some } (k_\sigma)_\sigma \in \prod\nolimits_{\sigma:K\hookrightarrow L}\mathbb{Z}_{\geq 0},$$
$$H^2_{\varphi,\Gamma}(\mathcal{R}(\delta)) \neq 0 \iff \delta = \varepsilon\cdot\delta((k_\sigma)_\sigma) \text{ for some } (k_\sigma)_\sigma \in \prod\nolimits_{\sigma:K\hookrightarrow L}\mathbb{Z}_{\geq 0}.$$

Especially $H^1_{\varphi,\Gamma}(\mathcal{R}(\delta))$ has L-dimension $[K:\mathbb{Q}_p]$ if and only if

$$\delta \notin \left\{\delta((-k_\sigma)_\sigma), \varepsilon\cdot\delta((k_\sigma)_\sigma) \mid (k_\sigma) \in \prod\nolimits_\sigma \mathbb{Z}_{\geq 0}\right\}.$$

Proof. We only need to prove the first statement, as the other statements follow by duality resp. by use of the Euler characteristic formula. The proof is the same as the proof of [Co, Proposition 2.1]. Compare also [Na1, Proposition 2.14].

As $\mathcal{R}_X \subset (\prod_\sigma t_\sigma^{-k_\sigma})\mathcal{R}_X$ for $k_\sigma \geq 0$ the one implication is obvious. For the other implication note that

$$\mathcal{R}^+_L = \left(\bigoplus_{\sigma,i_\sigma=0}^{k-1} (\prod_\sigma t_\sigma^{i_\sigma})L\right) \oplus t^k\mathcal{R}^+_L.$$

As the (φ,Γ)-cohomology is known to be finite dimensional over L, there are no invariants under φ and Γ in $t^k\mathcal{R}_L(\delta)$ for $k \gg 0$. On the other hand $\varphi(\prod_\sigma t_\sigma^{i_\sigma}) = (\prod_\sigma \sigma(\varpi)^{i_\sigma})\prod_\sigma t_\sigma^{i_\sigma}$ and hence $\prod t_\sigma^{i_\sigma}L$ contributes to the φ-invariants in $\mathcal{R}_L(\delta)$ if and only if $\delta(\varpi) = \prod_\sigma \sigma(\varpi)^{i_\sigma}$. ∎

Notation 2.2. (i) Let us write $\mathcal{T}_{\mathrm{reg}} \subset \mathcal{T}$ for the set of *regular characters*, i.e the characters

$$\delta \notin \left\{\delta((-k_\sigma)_\sigma), \varepsilon \cdot \delta((k_\sigma)_\sigma) \mid (k_\sigma) \in \prod\nolimits_\sigma \mathbb{Z}_{\geq 0}\right\}$$

(ii) Let $d > 0$ be an integer. We define the set of *regular parameters* $\mathcal{T}^d_{\mathrm{reg}} \subset \mathcal{T}^d$ to be the set of $(\delta_1, \dots, \delta_d) \in \mathcal{T}^d$ such that $\delta_i/\delta_j \in \mathcal{T}_{\mathrm{reg}}$ for $i \leq j$. Note that by construction $\mathcal{T}^d_{\mathrm{reg}} \neq (\mathcal{T}_{\mathrm{reg}})^d$.

(iii) A weight $\delta \in \mathcal{W}(\bar{\mathbb{Q}}_p)$ is *algebraic of weight* $(k_\sigma)_\sigma$ if $\delta = \delta_{\mathcal{W}}((k_\sigma)_\sigma)$.

(iv) We say that $\delta \in \mathcal{W}(\bar{\mathbb{Q}}_p)$ is *locally algebraic* of weight $(k_\sigma)_\sigma$ if $\delta \otimes \delta_{\mathcal{W}}((-k_\sigma)_\sigma)$ becomes trivial after restricting to some open subgroup of $\mathcal{O}_K^\times$.

(v) An element $\mathbf{k} = (k_{\sigma,i})_\sigma \in \prod_\sigma \mathbb{Z}^d$ is called *regular* if $k_{\sigma,1} < k_{\sigma,2} < \dots < k_{\sigma,d}$ for all σ.

(vi) Let $\mathbf{k} \in \prod_\sigma \mathbb{Z}^d$. We say that $(\delta_1, \dots, \delta_d) \in \mathcal{W}^d(\bar{\mathbb{Q}}_p)$ is *algebraic of weight* $\mathbf{k}$ if δ_i is algebraic of weight $(k_{\sigma,i})_\sigma$. An element $\delta = (\delta_1, \dots, \delta_d) \in \mathcal{W}^d$ is called *locally algebraic of weight* $\mathbf{k}$ if δ_i is locally algebraic of weight $(k_{\sigma,i})_\sigma$. The set of weight that are locally algebraic of weight $\mathbf{k}$ is denoted by $\mathcal{W}^d_{\mathbf{k},\mathrm{la}} \subset \mathcal{W}^d$.

2.2. **The space of trianguline (φ, Γ)-modules.** Following the idea of Chenevier [Ch1] we construct a space of trianguline (φ, Γ)-modules with regular parameters.

Let d be a positive integer and consider the functor $\mathcal{S}^\square_d$ that assigns to a rigid space X the isomorphism classes of quadruples $(D, \mathrm{Fil}_\bullet(D), \delta, \nu)$, where D is a (φ, Γ)-module over $\mathcal{R}_X$ and $\mathrm{Fil}_\bullet(D)$ is a filtration of D by sub-$\mathcal{R}_X$-modules that are stable under the action of φ and Γ and that are locally on X direct summands as $\mathcal{R}_X$-modules. Further $\delta \in \mathcal{T}^d_{\mathrm{reg}}(X)$ and $\nu = (\nu_1, \dots, \nu_d)$ is a collection of trivializations

$$\nu_i : \mathrm{Fil}_{i+1}(D)/\mathrm{Fil}_i(D) \xrightarrow{\cong} \mathcal{R}_X(\delta_i).$$

Similarly, we consider a variant of this functor parametrizing non-split extensions, cf. [He2], that is, the functor $\mathcal{S}^{\mathrm{ns}}_d$ that assigns to X the quadruples $(D, \mathrm{Fil}_\bullet(D), \delta, \nu_d)$ where D and $\mathrm{Fil}_\bullet(D)$ are as above and $\delta \in \mathcal{T}^d_{\mathrm{reg}}$ such that locally on X there exist short exact sequences

$$0 \longrightarrow \mathrm{Fil}_i(D) \longrightarrow \mathrm{Fil}_{i+1}(D) \longrightarrow \mathcal{R}_X(\delta_i) \longrightarrow 0$$

that are non split at every geometric point $x \in X$ as a sequence of (φ, Γ)-modules. Finally ν_d is a trivialization

$$\nu_d : \mathrm{Fil}_{d+1}(D)/\mathrm{Fil}_d(D) \xrightarrow{\cong} \mathcal{R}_X(\delta_d).$$

Proposition 2.3. *Let $\delta = (\delta_1, \dots, \delta_d) \in \left(\mathcal{T}_{\mathrm{reg}}\right)^d(X)$ for some rigid space X and let D be a successive extension of the $\mathcal{R}_X(\delta_i)$. Then $H^1_{\varphi,\Gamma}(D)$ is a locally free $\mathcal{O}_X$-module of rank $d[K : \mathbb{Q}_p]$.*

Proof. It follows from [KPX, Theorem 4.4.5] that the cohomology is a coherent sheaf and it suffices to compute its rank at all points. We proceed by induction on d. The rank 1 case is settled by Lemma 2.1. Consider the short exact sequence

$$0 \longrightarrow \mathcal{R}_X(\delta_1) \longrightarrow D \longrightarrow D' \longrightarrow 0. \tag{2.2}$$

Then, by induction hypothesis, $H^1_{\varphi,\Gamma}(D')$ is locally free of rank $(d-1)[K:\mathbb{Q}_p]$ and hence the Euler-characteristic formula [KPX, Theorem 4.4.5 (2)] implies that $H^0_{\varphi,\Gamma}(D') = H^2_{\varphi,\Gamma}(D') = 0$. The claim now follows from the long exact sequence associated to (2.2) and the fact that $H^1_{\varphi,\Gamma}(\mathcal{R}_X(\delta_1))$ is locally free of rank $[K:\mathbb{Q}_p]$ and $H^2_{\varphi,\Gamma}(\mathcal{R}_X(\delta_1)) = 0$ by Lemma 2.1. $\square$

Theorem 2.4. (i) *The functors* $\mathcal{S}_d^\square$ *and* $\mathcal{S}_d^{\mathrm{ns}}$ *are representable by rigid spaces.*
(ii) *The map* $\mathcal{S}_d^\square \to \mathcal{T}_{\mathrm{reg}}^d$ *is smooth of relative dimension* $\frac{d(d-1)}{2}[K:\mathbb{Q}_p]$.
(iii) *The map* $\mathcal{S}_d^{\mathrm{ns}} \to \mathcal{T}_{\mathrm{reg}}^d$ *is smooth and proper and*

$$\dim \mathcal{S}_d^{\mathrm{ns}} = 1 + [K:\mathbb{Q}_p]\left(\tfrac{d(d+1)}{2}\right)$$

Proof. The proof is the same as the proof of [Ch1, Theorem 3.3] resp. [He2, Proposition 2.3]. For the convenience of the reader we give a short sketch. The case $d=1$ is settled by $\mathcal{S}_1^\square = \mathcal{S}_1^{\mathrm{ns}} = \mathcal{T}$. Now assume that $\mathcal{S}_{d-1}^\square$ and $\mathcal{S}_{d-1}^{\mathrm{ns}}$ are constructed with universal objects $\mathcal{D}_{d-1}^\square$ resp. $\mathcal{D}_{d-1}^{\mathrm{ns}}$. Let $U \subset \mathcal{T} \times \mathcal{S}_{d-1}^\square$ resp. $V \subset \mathcal{S}_{d-1}^{\mathrm{ns}} \times \mathcal{T}$ be the preimage of $\mathcal{T}_{\mathrm{reg}}^d \subset \mathcal{T} \times \mathcal{T}_{\mathrm{reg}}^{d-1}$ under the canonical projection. Then Proposition 2.3 implies that

$$\mathscr{E}xt^1_{\mathcal{R}_U}(\mathcal{R}_U(\delta_1), \mathcal{D}_{d-1}^\square) = H^1_{\varphi,\Gamma}(\mathcal{D}_{d-1}^\square(\delta_1^{-1}))$$

resp.

$$\mathscr{E}xt^1_{\mathcal{R}_V}(\mathcal{R}_V(\delta_1), \mathcal{D}_{d-1}^{\mathrm{ns}}) = H^1_{\varphi,\Gamma}(\mathcal{D}_{d-1}^{\mathrm{ns}}(\delta_1^{-1}))$$

are vector bundles of rank $(d-1)[K:\mathbb{Q}_p]$. As the Tate-duality is a perfect pairing [KPX, Theorem 4.4.5] we find that also

$$\mathcal{M}_U = \mathscr{E}xt^1_{\mathcal{R}_U}(\mathcal{D}_{d-1}^\square, \mathcal{R}_U(\delta_1))$$

resp.

$$\mathcal{M}_V = \mathscr{E}xt^1_{\mathcal{R}_V}(\mathcal{D}_{d-1}^{\mathrm{ns}}, \mathcal{R}_V(\delta_1))$$

are vector bundles of rank $(d-1)[K:\mathbb{Q}_p]$. Now $\mathcal{S}_d^\square = \underline{\mathrm{Spec}}_U(\mathrm{Sym}^\bullet \mathcal{M}_U^\vee)$ is the geometric vector bundle over U associated to $\mathcal{M}_U$ while $\mathcal{S}_d^{\mathrm{ns}} = \mathbb{P}_V(\mathcal{M}_V^\vee)$ is the projective bundle associated to $\mathcal{M}_V$. Here $\underline{\mathrm{Spec}}$ is the relative spectrum in the sense of [Con, 2.2] and given a vector bundle $\mathcal{E}$ the projective bundle $\mathbb{P}(\mathcal{E}) = \underline{\mathrm{Proj}}(\mathrm{Sym}^\bullet \mathcal{E})$ is the relative Proj in the sense of [Con, 2.3].

The universal object $\mathcal{D}_d^\square$ then is the universal extension

$$0 \longrightarrow \mathcal{R}(\delta_1) \longrightarrow \mathcal{D}_d^\square \longrightarrow \mathcal{D}_{d-1}^\square \longrightarrow 0$$

over $\mathcal{S}_d^\square$. In the non-split context consider the geometric vector bundle $\tilde{\mathcal{S}}_d^{\mathrm{ns}} = \underline{\mathrm{Spec}}_V(\mathrm{Sym}^\bullet \mathcal{M}_V^\vee)$ over V associated to $\mathcal{M}_V$. Then there is a universal extension

$$0 \longrightarrow \mathcal{R}(\delta_1) \longrightarrow \tilde{\mathcal{D}}_d^{\mathrm{ns}} \longrightarrow \mathcal{D}_{d-1}^{\mathrm{ns}} \longrightarrow 0$$

over $\tilde{\mathcal{S}}_d^{\mathrm{ns}}$. Consider the open subspace $\tilde{\mathcal{S}}_d^{\mathrm{ns}} \backslash V \subset \tilde{\mathcal{S}}_d^{\mathrm{ns}}$ where the image of the zero section $0: V \hookrightarrow \tilde{\mathcal{S}}_d^{\mathrm{ns}}$ is removed. This space carries a natural action of $\mathbb{G}_m$ and this action lifts to an action on the restriction of $\tilde{\mathcal{D}}_d^{\mathrm{ns}}$ to $\tilde{\mathcal{S}}_d^{\mathrm{ns}} \backslash V$ by acting on $\mathcal{R}(\delta_1)$. Hence $\tilde{\mathcal{D}}_d^{\mathrm{ns}}$ descends to a (φ,Γ)-module $\mathcal{D}_d^{\mathrm{ns}}$ over $\mathbb{P}(\mathcal{M}_V^\vee) = (\tilde{\mathcal{S}}_d^{\mathrm{ns}} \backslash V)/\mathbb{G}_m$.

The computation of the dimension follows from the construction as well as the fact that $\mathcal{S}_d^\square$ is smooth over $\mathcal{T}_{\mathrm{reg}}^d$ and $\mathcal{S}_d^{\mathrm{ns}}$ is smooth and proper over $\mathcal{T}_{\mathrm{reg}}^d$. $\square$

Let $r \in p^{\mathbb{Q}} \cap [0,1)$ and consider the ring $\mathcal{R}^r = \mathcal{R}^r_{\mathbb{Q}_p}$. If $n \gg 0$, then there is a morphism $\mathcal{R}^r \to K_n[\![t]\!]$ where the ring $K_n[\![t]\!]$ is viewed as the complete local ring at the point of $\mathbb{U}_{r,K_0}$ corresponding to (the $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K_0)$-orbit of) $1-\epsilon_n$. If D_r is a (φ,Γ)-module defined over $\mathcal{R}^r_L$ for some p-adic field L and some $r \in p^{\mathbb{Q}} \cap [0,1)$ and if $D = D_r \otimes_{\mathcal{R}^r_L} \mathcal{R}_L$, then we define

$$D_{\mathrm{dR}}(D) = (K_\infty \otimes_{K_n} K_n(\!(t)\!) \otimes_{\mathcal{R}^r_{\mathbb{Q}_p}} D_r)^\Gamma$$
$$\mathrm{Fil}^i D_{\mathrm{dR}}(D) = (K_\infty \otimes_{K_n} t^i K_n[\![t]\!] \otimes_{\mathcal{R}^r_{\mathbb{Q}_p}} D_r)^\Gamma.$$

If L contains K^{norm}, then $D_{\mathrm{dR}}(D)$ splits up into a product $D_{\mathrm{dR}}(D) = \prod_\sigma D_{\mathrm{dR},\sigma}(D)$ and $\mathrm{Fil}^i D_{\mathrm{dR}}(D) = \prod_\sigma \mathrm{Fil}^i_\sigma D_{\mathrm{dR}}(D)$ splits up into filtrations $\mathrm{Fil}^i_\sigma D_{\mathrm{dR}}(D)$ of the $D_{\mathrm{dR},\sigma}(D)$.
As usual we can extend the notions of being crystalline or de Rham to (φ,Γ)-modules.

Definition 2.5. Let L be a finite extension of $\mathbb{Q}_p$ and let D be a (φ,Γ)-module of rank d over $\mathcal{R}_L = \mathcal{R}_{L,K}$. Assume that $D = D_r \otimes_{\mathcal{R}^r_L} \mathcal{R}_L$ for some (φ,Γ)-module D_r defined over $\mathcal{R}^r_L$ and some $r < 1$.
(i) The (φ,Γ)-module D is called *de Rham* if $D_{\mathrm{dR}}(D)$ is a free $L \otimes_{\mathbb{Q}_p} K$-module of rank d.
(ii) The module D is called *crystalline* if $D_{\mathrm{cris}}(D) = D[1/t]^\Gamma$ is free of rank d over $L \otimes_{\mathbb{Q}_p} K_0$.
(iii) The module D is called *crystabeline* if $D \otimes_{\mathcal{R}_{L,K}} \mathcal{R}_{L,K'}$ is crystalline for some abelian extension K' of K.

The following proposition is the generalization of [BeCh, Proposition 2.3.4] to our context and its proof is essentially the same as in the case $K = \mathbb{Q}_p$.

Proposition 2.6. *Let L be a finite extension of $\mathbb{Q}_p$ containing K^{norm} and let D be a (φ,Γ)-module of rank d over $\mathcal{R}_L$ that is a successive extension of rank 1 objects $\mathcal{R}_L(\delta_i)$. Assume that $(\delta_1|_{\mathcal{O}_K^\times},\dots,\delta_d|_{\mathcal{O}_K^\times})$ is locally algebraic of weight $-\mathbf{k} = (-k_{\sigma,i})$ for some regular weight $\mathbf{k}$. Then D is de Rham with labeled Hodge-Tate weights $-\mathbf{k}$.*

Proof. Write $R = \bigcup_n (L \otimes_{\mathbb{Q}_p} K_n[\![t]\!])$ for the moment. We proceed by induction on d. The case $d = 1$ easily follows form the fact that we may twist by characters δ such that $\delta|_{\mathcal{O}_K^\times} = 1$ and the fact that the claim is true for characters of $\mathcal{G}_K^{\mathrm{ab}} = \hat{\mathbb{Z}} \times \mathcal{O}_K^\times$ by the definition of locally algebraic weights.

For simplicity we only treat the case $p \neq 2$. In this case the group Γ is pro-cyclic. In the case $p = 2$ one concludes similarly after taking invariants under the 2-power torsion subgroup Δ of Γ.

Let $\gamma \in \Gamma$ be a topological generator and let $\Gamma_0 = \langle \gamma \rangle \subset \Gamma$. We will prove by induction on $1 \leq j \leq d$ that for r big enough, $(\prod_\sigma t^{k_{\sigma,j}} R \otimes_{\mathcal{R}^r_{\mathbb{Q}_p}} \mathrm{Fil}_j(D)_r)^{\Gamma_0} \neq 0$. Suppose we have the result for $j \leq d-1$ and consider the short exact sequence

$$0 \to \prod_\sigma \mathrm{Fil}^{k_{\sigma,d}}_\sigma D_{\mathrm{dR}}(\mathrm{Fil}_{d-1}(D)) \to \prod_\sigma \mathrm{Fil}^{k_\sigma}_\sigma D_{\mathrm{dR}}(D) \to \prod_\sigma \mathrm{Fil}^{k_\sigma}_\sigma \mathcal{R}_L(\delta_d)$$
$$\to H^1\big(\Gamma_0, \mathrm{Fil}_{d-1}(D)_r \otimes_{\mathcal{R}^r_{\mathbb{Q}_p}} \big(\prod_\sigma t_\sigma^{k_{\sigma,d}}\big) R\big).$$

Hence it suffices to show that

$$H^1\big(\Gamma_0, \mathrm{Fil}_{d-1}(D)_r \otimes_{\mathcal{R}^r_{\mathbb{Q}_p}} \big(\prod_\sigma t_\sigma^{k_{\sigma,d}}\big)R\big) = 0.$$

To do so we are reduced to compute the first cohomology of $(\prod_\sigma t_\sigma^{k_\sigma})R \otimes_L \delta_i$ for $i \leq d-1$ which vanishes as $\prod_\sigma t^{k_{\sigma,d}} R \otimes \delta_j \simeq \prod_\sigma t^{k_{\sigma,d}-k_{\sigma,j}} R \otimes \delta$ with δ a finite order character, $k_{\sigma,d} - k_{\sigma,j} > 0$ for all σ and

$$H^1\left(\Gamma_0, (\prod_\sigma t_\sigma^{i_\sigma})R\right) = (\prod_\sigma t_\sigma^{i_\sigma})R \Big/ (\gamma-1)(\prod_\sigma t_\sigma^{i_\sigma})R = 0$$

if $i_\sigma > 0$ for all embeddings σ. It follows that D has to be de Rham. □

Let $\omega_d^\square : \mathcal{S}_d^\square \to \mathcal{W}^d$ resp. $\omega_d : \mathcal{S}_d^{\mathrm{ns}} \to \mathcal{W}^d$ denote the projection to the weight space.

Corollary 2.7. (i) *Let $w \in \mathcal{W}^d_{\mathrm{reg,alg}}$ be a regular algebraic weight. Then there is a non-empty Zariski-open subset $Z_{\mathrm{cris}}(w) \subset \omega_d^{-1}(w)$ such that all points of $Z_{\mathrm{cris}}(w)$ are crystalline (φ,Γ)-modules.*
(ii) *Let $\mathbf{k} \in \prod_\sigma \mathbb{Z}^d$ be regular and let $w \in \mathcal{W}^d_{\mathbf{k},\mathrm{la}}$ be a locally algebraic weight. Then there is a non-empty Zariski-open subset $Z_{\mathrm{pcris}}(w) \subset \omega_d^{-1}(w)$ such that all points of $Z_{\mathrm{pcris}}(w)$ are crystabeline.*

Proof. The proof is identical to the one of [Ch1, Theorem 3.14].
(i) As $w = (w_1, \dots, w_d) \in \mathcal{W}^d$ is algebraic we may write $\mathcal{R}(\delta_i) = \mathcal{R}(D(\delta_i))$ for any character $\delta_i \in \mathcal{T}$ restricting to w_i on $\mathcal{O}_K^\times$. We write $D(\delta_i) = D(a_i, (k_\sigma)_\sigma)$ with $a_i = \delta_i(\sigma)\prod_\sigma \sigma(\varpi)^{-k_\sigma}$ and let

$$Z_{\mathrm{cris}}(w) = \left\{(D, \mathrm{Fil}_\bullet(D), \delta, \nu_d) \in \omega_d^{-1}(w) \mid \tfrac{a_i}{a_j} \neq p^{\pm[K_0:\mathbb{Q}_p]} \text{ for } i<j\right\}.$$

Let D be a (φ,Γ)-module associated to some point in $Z_{\mathrm{cris}}(w)$ then D is de Rham by Proposition 2.6 above and hence potentially semi-stable. As D is a successive extension of crystalline (φ,Γ)-modules it has to be semi-stable and we have to assure that the monodromy acts trivial. However the monodromy operator maps the Φ^f-eigenspace with eigenvalue λ to the Φ^f-eigenspace with eigenvalue $p^f\lambda$, where $f = [K_0 : \mathbb{Q}_p]$. As the possible eigenvalues of Φ^f are given by the a_i the monodromy has to be trivial.

(ii) Let $w = (w_1, \dots, w_n)$ and let K' be the abelian extension of K corresponding to $\bigcap \ker w_i \subset \mathcal{O}_K^\times \hookrightarrow \mathcal{G}_K^{\mathrm{ab}}$. Then the same argument as above yields a Zariski-open subset $Z_{\mathrm{pcris}}(w) \subset \omega_d^{-1}(w)$ whose points are (φ,Γ)-modules that become crystalline over K'. □

Remark 2.8. In the case $d=2$ the second claim of the corollary above especially applies to the weight $\mathbf{k} = ((0,1)_\sigma)$ that is to potentially Barsotti-Tate representations. If $d > 2$ a corresponding statement for potentially Barsotti-Tate representations can not hold true any longer. There are no regular weights for potentially Barsotti-Tate representations in this case and the dimension of the flag variety parametrizing the Hodge-filtrations for non-regular weights will be strictly smaller than the dimension of the space of extensions of (φ,Γ)-modules.

Lemma 2.9. *Let $L \subset \bar{\mathbb{Q}}_p$ be a finite extension of the Galois closure K^{norm} of K inside $\bar{\mathbb{Q}}_p$ and let V be a crystalline representation of $\mathcal{G}_K$ on a d-dimensional*

L-vector space with labeled Hodge-Tate weights $-\mathbf{k} = (-k_{\sigma,i})$ such that $\mathbf{k}$ is regular. Let $D = D_{\mathrm{cris}}(V)$ and assume that the $[K_0 : \mathbb{Q}_p]$-th power of the crystalline Frobenius Φ_{cris} on $\mathrm{WD}(D) = D \otimes_{L\otimes_{\mathbb{Q}_p} K_0} \bar{\mathbb{Q}}_p$ is semi-simple. Let $\lambda_1, \dots, \lambda_d$ be an ordering of its eigenvalues and assume that for all σ one has

$$\begin{aligned} \tfrac{[K:\mathbb{Q}_p]}{[K_0:\mathbb{Q}_p]}\mathrm{val}(\lambda_1) &< k_{\sigma,2} + \sum\nolimits_{\sigma'\neq\sigma} k_{\sigma',1} \\ \tfrac{[K:\mathbb{Q}_p]}{[K_0:\mathbb{Q}_p]}\mathrm{val}(\lambda_1 \dots \lambda_i) &< k_{\sigma,i+1} + \sum\nolimits_{\sigma'\neq\sigma} k_{\sigma',i} + \sum\nolimits_{\sigma'} \sum\nolimits_{j=1}^{i-1} k_{\sigma',j}. \end{aligned} \tag{2.3}$$

Then there is a triangulation $0 = D_0 \subset D_1 \subset \dots \subset D_d = D^\dagger_{\mathrm{rig}}(V)$ such that $D_i/D_{i-1} \cong D(\delta_i)$ with $\delta_i : K^\times \to L^\times$ given by

$$\begin{aligned} \delta_i|_{\mathcal{O}_K^\times} &: z \longmapsto \prod\nolimits_\sigma \sigma(z)^{-k_{\sigma,i}} \\ \delta_i(\varpi) &= \lambda_i \prod\nolimits_\sigma \sigma(\varpi)^{k_{\sigma,i}}. \end{aligned}$$

Proof. Let $D_i \subset D^\dagger_{\mathrm{rig}}(V)$ be the filtration induced by a filtration $0 = D'_0 \subset D'_1 \subset \dots \subset D'_d = D = D_{\mathrm{cris}}(V)$ by Φ_{cris}-stable subspaces such that the restriction of $\Phi_{\mathrm{cris}}^{[K_0:\mathbb{Q}_p]}$ to $\mathrm{WD}(D_i)$ has eigenvalues $\lambda_1, \dots, \lambda_i$. Then D_i is stable under φ and Γ and we need to compute the graded pieces. One easily sees that the graded pieces are as claimed if the filtration $D'_\bullet$ is in general position with all the Hodge filtrations $\mathrm{Fil}^\bullet_\sigma$ which is to say

$$\big(D'_i \otimes_{K_0\otimes L,\sigma\otimes\mathrm{id}} \bar{\mathbb{Q}}_p\big) \oplus \big(\mathrm{Fil}^{k_{\sigma,i+1}} D_K \otimes_{K\otimes L,\sigma\otimes\mathrm{id}} \bar{\mathbb{Q}}_p\big) = D \otimes_{K_0\otimes L,\sigma\otimes\mathrm{id}} \bar{\mathbb{Q}}_p = \mathrm{WD}(D).$$

However, one easily sees that this is assured by weak admissibility and condition (2.3). □

2.3. **Construction of Galois-representations.** Let $\bar\rho : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ be an absolutely irreducible continuous representation. Write $R_{\bar\rho}$ for the universal deformation ring of $\bar\rho$ and $\mathfrak{X}_{\bar\rho}$ for the generic fiber of $\mathrm{Spf}\, R_{\bar\rho}$ in the sense of Berthelot.

Let X be a rigid space and let $T : \mathcal{G}_K \to \Gamma(X, \mathcal{O}_X)$ be a continuous pseudo-character of dimension d. We say that T has *residual type* $\bar\rho$ if for all $x \in X$ the semi-simple representation $\rho_x : \mathcal{G}_K \to \mathrm{GL}_d(\mathcal{O}_{\bar{\mathbb{Q}}_p})$ with $\mathrm{tr}\,\rho_x = (T \otimes k(x)) \otimes_{k(x)} \bar{\mathbb{Q}}_p$ (which is uniquely determined up to conjugation) reduces to (the isomorphism class of) $\bar\rho$ modulo the maximal ideal of $\mathcal{O}_{\bar{\mathbb{Q}}_p}$.

Then the rigid space $\mathfrak{X}_{\bar\rho}$ represents the functor that assigns to a rigid space X the pseudo-characters $T : \mathcal{G}_K \to \Gamma(X, \mathcal{O}_X)$ of dimension d and residual type $\bar\rho$.

By [He1, Theorem 5.2] there exists a natural rigid space $\mathcal{S}_d^{\mathrm{ns,adm}}$ which is étale[2] over $\mathcal{S}_d^{\mathrm{ns}}$ and a vector bundle $\mathcal{V}$ on $\mathcal{S}_d^{\mathrm{ns,adm}}$ together with a continuous representation $\rho : \mathcal{G}_K \to \mathrm{GL}(\mathcal{V})$ such that $\mathbf{D}^\dagger_{\mathrm{rig}}(\mathcal{V})$ is the restriction of the universal trianguline (φ, Γ)-module. Let us write $\mathcal{S}(\bar\rho) \subset \mathcal{S}_d^{\mathrm{ns,adm}}$ for the open and closed subspace where the pseudo-character $\mathrm{tr}\rho$ has residual type $\bar\rho$. Then we obtain a canonical map

$$\pi_{\bar\rho} : \mathcal{S}(\bar\rho) \longrightarrow \mathfrak{X}_{\bar\rho} \times \mathcal{T}^d_{\mathrm{reg}}.$$

Theorem 2.10. *The map $\pi_{\bar\rho}$ is finite and injective.*

[2]In the set up of adic spaces the spaces $\mathcal{S}_d^{\mathrm{ns,adm}}$ is an open subspace of $\mathcal{S}_d^{\mathrm{ns}}$.

Proof. The proof is the same as the proof of [He2, Theorem 3.1]. □

Let $X^{\mathrm{reg}}(\bar\rho) = \mathrm{Im}(\pi_{\bar\rho}) \subset \mathfrak{X}_{\bar\rho} \times \mathcal{T}^d_{\mathrm{reg}}$ which is a Zariski-closed subset and let $X(\bar\rho)$ denote the Zariski-closure of $X^{\mathrm{reg}}(\bar\rho)$ in $\mathfrak{X}_{\bar\rho} \times \mathcal{T}^d$. The space $X(\bar\rho)$ is called the *finite slope space* in the following. The next proposition verifies [He2, Conjecture 3.15 (i)].

Proposition 2.11. *The inclusion $X^{\mathrm{reg}}(\bar\rho) \hookrightarrow X(\bar\rho)$ is an open immersion. Further the Galois representations $\rho \in X(\bar\rho)$ are trianguline.*

Proof. Let us write $M = |X^{\mathrm{reg}}(\bar\rho)|$ for the underlying point set of $X^{\mathrm{reg}}(\bar\rho)$ and let $x = (\rho, \delta_1, \dots, \delta_d) \in M$. Then $\mathbf{D}^{\dagger}_{\mathrm{rig}}(\rho)$ is strictly trianguline with ordered parameters $\delta_1, \dots, \delta_d$ in the sense of [KPX, Definition 6.1.13]. Let $X_1 = X^{\mathrm{reg}}(\bar\rho)$ denote the Zariski-closure of M in $\mathfrak{X}_{\bar\rho} \times \mathcal{T}^d_{\mathrm{reg}}$ and $X_2 = X(\bar\rho)$ denote its closure in $\mathfrak{X}_{\bar\rho} \times \mathcal{T}^d$. Further we write ρ^{un} for the pullback of the universal $\mathcal{G}_K$-representation on $\mathfrak{X}_{\bar\rho}$ to X_2 and $\delta_1, \dots, \delta_d$ for the pullback of the universal characters of $K^\times$ on $\mathcal{T}^d$ to X_2. By [KPX, Corollary 6.2.9], there exists a proper birational morphism $p: X_2' \to X_2$ such that there is a (unique) increasing filtration Fil_i on $\mathbf{D}^{\dagger}_{\mathrm{rig}}(p^*\rho^{\mathrm{un}})$ by (φ, Γ)-submodules which is a strictly trianguline filtration with ordered parameters $p^*\delta_1, \dots, p^*\delta_d$ over a Zariski-open and dense subset $U \subset X_2'$ containing $p^{-1}(M)$. As the formation of X_2' commutes with flat base change we are reduced to show that $U \cap (X_2' \times_{\mathcal{T}^d} \mathcal{T}^d_{\mathrm{reg}})$ maps to X_1. However this is obvious as X_1 by construction is exactly the set of all trianguline representations which are strictly trianguline with ordered parameters $(\delta_1, \dots, \delta_d) \in \mathcal{T}^d_{\mathrm{reg}}$.

The last claim follows from [KPX, Theorem 6.2.12]. □

3. Application of eigenvarieties

In this section, we recall some facts on eigenvarieties attached to definite unitary groups and prove a density statement about them which will be used in the proof of the main theorem.

3.1. **The eigenvarieties.** The eigenvarieties that we are going to use are studied in Chenevier's paper [Ch3]. The result that we need is the analogue of the results in [He2], where the corresponding eigenvarieties were studied in [BeCh]. We recall the set up of Chenevier's paper.

Notation 3.1.

(i) We choose a totally real field F such that $[F:\mathbb{Q}]$ is even and let E be a CM quadratic extension of F. We write c for the complex conjugation of E over F and assume that there is a place v_0 of F dividing p unramified in E, such that $v_0 = w_0 w_0^c$ splits in E and such that $F_{v_0} = E_{w_0} \cong K$. We fix such an isomorphism and view the uniformizer ϖ of K a an uniformizer of F_{v_0}.

(ii) We fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and embeddings $\iota_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Let $I_\infty = \mathrm{Hom}(F, \mathbb{C}) = \mathrm{Hom}(F, \mathbb{R})$ denote the set of infinite places of F. Given a place v of F dividing p the set $I(v) = \mathrm{Hom}(F_v, \bar{\mathbb{Q}}_p)$ is identified with a subset $I_\infty(v) \subset I_\infty$ via our choice of embeddings ι_∞ and ι_p.

(iii) Let $d \geq 1$ be an integer and let us write G for the unique unitary group in d variables defined over F which splits over E, is quasi-split at all finite places and compact at all infinite places. The existence of such a group can be deduced from the considerations of section 2 of [Cl].

(iv) As v_0 splits in E, there exists an isomorphism $G(F_{v_0}) \cong \mathrm{GL}_d(K)$ that we fix for the following. We write S_p for the set of places v of F dividing p and $S'_p = S_p \backslash \{v_0\}$[3].

(v) Let T denote the diagonal torus in $\mathrm{GL}_d(K)$ and denote by T^0 its maximal compact subgroup. Further we fix the Borel $B \subset \mathrm{GL}_d(K)$ of upper triangular matrices in order to have a notion of dominant weights. Let $L \subset \bar{\mathbb{Q}}_p$ be a subfield containing $\sigma(F_{v_0})$ for all $\sigma \in I(v_0)$. We define the weight space for the automorphic representations to be

$$\mathcal{W}^{\mathrm{aut}} = \mathrm{Hom}_{\mathrm{cont}}(T^0, \mathbb{G}_m(-)),$$

as rigid space over L. Especially we have a canonical identification $\mathcal{W}^{\mathrm{aut}} \cong \mathcal{W}_L^d$.

(vi) Fix a finite set S of finite places of F containing S_p and all places such that $G(F_v)$ ramifies and fix a compact open subgroup $H = \prod_v H_v \subset G(\mathbb{A}_{F,f})$ such that H_v is maximal hyperspecial for all $v \notin S$ and such that H_{v_0} is $\mathrm{GL}_d(\mathcal{O}_K)$. Write $S' = S \backslash \{v_0\}$. We define $H' = \prod_v H'_v$ such that $H'_v = H_v$ is $v \neq v_0$ and H'_{v_0} is the Iwahori-subgroup I of $\mathrm{GL}_d(\mathcal{O}_K)$ of matrices whose reduction modulo ϖ_K are upper triangular. Further let $\mathcal{H}^{\mathrm{un}} = \mathcal{O}_L[G(\mathbb{A}_{F,f}^S) /\!\!/ H^S]$ denote the spherical Hecke-algebra outside of S. Furthermore, we ask that H is small enough, ie for $g \in G(\mathbb{A}_{F,f})$,

$$G(F) \cap gHg^{-1} = 1.$$

(vii) For each place $v \in S'$ we fix an idempotent element e_v in the Hecke-algebra $\mathcal{O}_L[G(F_v) /\!\!/ H_v]$ and write $e = (\otimes_{v \in S'} e_v) \otimes 1_{\mathcal{H}^{\mathrm{un}}}$ for the resulting idempotent element of the Hecke algebra $\mathcal{O}_L[G(\mathbb{A}_{F,f}^{v_0}) /\!\!/ H^{v_0}]$.

(viii) For $1 \leq i \leq d$, let $t_i = \mathrm{diag}(1, \dots, 1, \varpi, 1 \dots, 1) \in T$, where the uniformizer is at the i-th diagonal entry. Let $T^- \subset T$ denote the set of $\mathrm{diag}(x_1, \dots, x_d) \in T$ such that $\mathrm{val}(x_1) \geq \dots \geq \mathrm{val}(x_d)$. We regard $\mathbb{Z}[T/T^0]$ as a subring of the Iwahori-Hecke algebra of $G(F_{v_0})$ with coefficients in $\mathbb{Z}[1/p]$ by means of $t \mapsto 1_{H_{v_0} t H_{v_0}}$. This subalgebra is generated by the Hecke-operators $1_{H_{v_0} t H_{v_0}}$ for $t \in T^-$ and their inverses. Finally let $\mathcal{H} = \mathcal{H}^{\mathrm{un}} \otimes_{\mathbb{Z}} \mathbb{Z}[T/T^0]$, which is a subalgebra of $\mathcal{O}_L[G(\mathbb{A}_{F,f}) /\!\!/ H']$.

Let W_∞ be an irreducible algebraic representation of $\prod_{v \in S'_p, w \in I_\infty(v)} G(F_w)$ and let $\mathcal{A} = \mathcal{A}(W_\infty, S, e)$ denote the set of all irreducible automorphic representations Π of $G(\mathbb{A}_F)$ such that $\bigotimes_{v \in S'_p, w \in I_\infty(v)} \Pi_w$ is isomorphic to W_∞ and $e(\Pi_f)^{H'_{v_0}} \neq 0$. Further define the set of classical points to be

$$\mathcal{Z} = \left\{ (\Pi, \chi) \;\middle|\; \begin{array}{c} \Pi \in \mathcal{A}, \chi : T/T^0 \to \bar{\mathbb{Q}}_p^\times \text{ continuous} \\ \text{such that } \Pi_{v_0} \text{ is a sub-object of } \mathrm{Ind}_B^{\mathrm{GL}_d(K)} \chi \end{array} \right\} \tag{3.1}$$

where the parabolic induction is normalized.

[3] Let just remark that here S_p is not exactly the same as in [Ch3]

Associated to these data there is an *eigenvariety*, that is a reduced rigid analytic space $Y(W_\infty, S, e)$ over L together with a morphism

$$\kappa : Y(W_\infty, S, e) \longrightarrow \mathcal{W}^{\mathrm{aut}}$$

and

$$\psi = \psi^{\mathrm{un}} \otimes \psi_{v_0} : \mathcal{H} \longrightarrow \Gamma(Y(W_\infty, S, e), \mathcal{O}_{Y(W_\infty,S,e)})$$

a morphism of algebras such that $Y(W_\infty, S, e)$ contains a set Z as a Zariski-dense accumulation[4] subset. These data are due to the property that there is a bijection between Z and $\mathcal{Z}$ sending a point $z \in Z$ on the pair $(\Pi_z, \chi_z) \in \mathcal{Z}$ according to the following rule.

The evaluation $\psi^{\mathrm{un}}(z) : \mathcal{H}^{\mathrm{un}} \to k(z)$ is the character of the spherical Hecke-algebra associated to the representation Π_z^S. For $w \in I_\infty(v_0)$, let $\kappa_{\Pi_{z,w}}$ denote the algebraic character of T_{v_0} obtained from $\Pi_{z,w}$ following the rule of [Ch3, §1.4]. Then, $\kappa(z) = \prod_{w\in I_\infty(v_0)} \kappa_{\Pi_{z,w}}$. Let $\kappa_\varpi(z)$ be the unique charater $T/T^0 \to \bar{\mathbb{Q}}_p^\times$ such that $\kappa_\varpi(z)(t) = \kappa(z)(t)$ when t is a diagonal matrix whose entries are powers of ϖ_K. Finally the component ψ_{v_0} of the morphism ψ is given by

$$\psi_{v_0}(z)|_{T_{v_0}^-} : 1_{H_{v_0}tH_{v_0}} \longmapsto \chi_z(t)\cdot \delta_{B_v}^{-1/2}(t)|\det(t)|^{\frac{d-1}{2}}\kappa_\varpi(z)(t),$$

where δ_{B_v} is the modulus character.

In what follows, we fix the data (W_∞, S, e) and write simply Y for $Y(W_\infty, S, e)$.

In [Ch3, §2], Chenevier constructs a space of overconvergent p-adic automorphic forms. More precisely, if V is an open affinoid of $\mathcal{W}^{\mathrm{aut}}$, one defines a certain $r_V \geq 1$, and constructs for each $r \geq r_V$ an $\mathcal{O}(V)$-Banach space denoted $e\mathcal{S}(V, r)$ with a continuous action of $\mathcal{H}$ such that the operators U_{v_0} corresponding to $\mathrm{diag}(\varpi_K^{d-1}, \varpi_K^{d-2}, \ldots, 1) \in \mathbb{Z}[T/T^0] \subset \mathcal{H}$ acts as a compact operator. We say that a charater of $\mathcal{H}$ is U_{v_0}-finite if the image of U_{v_0} is non zero. Then we have the following interpretation of points of Y, which is a consequence of Buzzard's construction of eigenvarieties [Bu, §5].

Proposition 3.2. *Let $t \in \mathcal{W}(\bar{\mathbb{Q}}_p)$. Then there is a natural bijection between $\bar{\mathbb{Q}}_p$-points of Y mapping to t and $\bar{\mathbb{Q}}_p$-valued U_{v_0}-finite system of eigenvalues of $\mathcal{H}$ on $\varinjlim_{V,r} e\mathcal{S}(V, r) \otimes_{\mathcal{O}(V)} k(t)$.*

3.2. **The map to the finite slope space.** In the above section we have recalled the construction of an eigenvariety $Y \to \mathcal{W}^{\mathrm{aut}}$. As above we write $Z \subset Y$ for the set of classical points (3.1). Let $(\Pi, \chi) \in Z$ and let $\pi = \bigotimes'_v \mathrm{BC}(\Pi_v)$ be the representation of $\mathrm{GL}_d(\mathbb{A}_E)$ defined by local base change for GL_d. Then by [Ch3, Theorem 3.2, 3.3] there are Galois-representations $\rho_\Pi : \mathcal{G}_E \to \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$ attached to the automorphic representations $\Pi \in Z$ such that the Weil-Deligne representation attached to $\rho_\Pi|_{\mathcal{G}_v}$ equals the Langlands parameter of $\pi_v|\cdot|^{(1-d)/2}$, where $\mathcal{G}_v \subset \mathcal{G}_E$ is the decomposition group at v for v not dividing p.

Let $\mathbb{B} \subset \mathbb{G} = \mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GL}_d$ denote the Weil restriction of the Borel subgroup of upper triangular matrices and let $\mathbb{T} \subset \mathbb{B}$ denote the Weil restriction of the diagonal torus. Using the canonical isomorphism $\mathbb{G}_{\bar{\mathbb{Q}}_p} \cong \prod_\sigma \mathrm{GL}_{d,\bar{\mathbb{Q}}_p}$ an algebraic weight $\mathbf{n}$

[4] Recall that a subset $A \subset Y$ of a rigid space accumulates at a point $x \in Y$ if $A \cap U$ is Zariski-dense in U for every connected affinoid neighborhood U of x in Y.

of $(\mathbb{G}_{\bar{\mathbb{Q}}_p}, \mathbb{T}_{\bar{\mathbb{Q}}_p})$ that is dominant with respect to $\mathbb{B}_{\bar{\mathbb{Q}}_p}$ can be identified with a tuple $(n_{\sigma,1},\dots,n_{\sigma,d})_{\sigma\in I(v_0)} \in \prod_{\sigma\in I(v_0)}\mathbb{Z}^d$ such that $n_{\sigma,1}\geq\dots\geq n_{\sigma,d}$ for all σ. Note that this algebraic weight is already canonically defined over the reflex field $E_{\mathbf{n}}$ of the weight $\mathbf{n}$, i.e. over the subfield of $\bar{\mathbb{Q}}_p$ defined by

$$\mathrm{Gal}(\bar{\mathbb{Q}}_p/E_{\mathbf{n}}) = \{\psi\in\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \mid n_{\sigma,i} = n_{\psi\circ\sigma,i} \text{ for allembeddings } \sigma\}$$

and hence especially over our fixed field L. Especially $\mathbf{n}$ defines an L-valued point of $\mathcal{W}^d$.

Let $z = (\Pi,\chi)\in Z$ and for $\sigma\in I(v_0)$ let $n_{\sigma,1}\geq\dots\geq n_{\sigma,d}$ denote the highest weight of $\Pi_{v(\sigma)}$, where $v(\sigma) = \iota_\infty\iota_p^{-1}\sigma\in I_\infty(v_0)$. We say that z is regular (with respect to v_0) if $n_{\sigma,1}>\dots>n_{\sigma,d}$ for all $\sigma\in I(v_0)$ and if

$$\tfrac{\lambda_i}{\lambda_j}\notin\{1, p^{\pm[K_0:\mathbb{Q}_p]}\},$$

where we set

$$\lambda_i = \psi_{v_0}(t_{v_0,i})\prod\nolimits_\sigma \sigma(\varpi)^{-n_{\sigma,i}}.$$

We further say that $z=(\Pi,\chi)$ is uncritical if in addition condition (2.3) holds with $\lambda'_i = \lambda_{d+1-i}$ and $k_{\sigma,i} = n_{\sigma,d+1-i}+i-1$. We write $Z^{\mathrm{reg}}\subset Z$ for the set of regular points and $Z^{\mathrm{un}}\subset Z$ for the set of uncritical regular points.

Lemma 3.3. *The subsets Z^{reg} and Z^{un} are Zariski-dense in the eigenvariety Y and accumulate at all classical points $z\in Z$.*

Proof. The proof is the same as the usual proof of density of classical points. Let us denote by $Y_0\subset\mathcal{W}^d\times\mathbb{G}_m$ the Fredholm hypersurface cut out by the Fredholm determinant of $U_{v_0} = \mathrm{diag}(\varpi^{d-1},\dots,\varpi,1)$. Let $z\in Z\subset Y$ be a classical point and let $U\subset Y$ be a connected affinoid neighborhood. After shrinking U we may assume that there is an affinoid open subset $V\subset\mathcal{W}^d$ such that $U\to V$ is finite and torsion free. As U is quasi-compact, there exist $C_1,\dots,C_d$ such that

$$C_i\geq \mathrm{val}_x(\psi_{v_0}(t_{v_0,1}\dots t_{v_0,i})(x))+1$$

for all $x\in U$, where val_x is the valuation on $k(x)$ normalized by $\mathrm{val}_x(p)=1$. Let us write $A\subset V$ for the set of dominant algebraic weights $n_{\sigma,1}\geq\dots\geq n_{\sigma,d}$ such that $C_i < n_{\sigma,i}-n_{\sigma,i+1}+1$ for all i and $\sigma: K\hookrightarrow\bar{\mathbb{Q}}_p$ and

(3.2)
$$\begin{aligned}\tfrac{[K:\mathbb{Q}_p]}{[K_0:\mathbb{Q}_p]}\cdot C_1 - \sum\nolimits_{\sigma'} n_{\sigma',1} &< n_{\sigma,d-1}+1+\sum\nolimits_{\sigma'\neq\sigma} n_{\sigma',d} \text{ for all embeddings } \sigma\\ \tfrac{[K:\mathbb{Q}_p]}{[K_0:\mathbb{Q}_p]}C_i - \sum_{j=1}^{i}\sum\nolimits_{\sigma'} n_{\sigma',j} &< n_{\sigma,d-i}+\sum\nolimits_{\sigma'\neq\sigma} n_{\sigma',d+1-i}+\sum\nolimits_{\sigma'}\sum_{j=1}^{i-1} n_{\sigma',d+1-j}\\ &\quad +1+\tfrac{1}{2}[K:\mathbb{Q}_p](i-1)(i+2) \text{ for all } \sigma, i.\end{aligned}$$

Then one easily sees that A accumulates at the point $\kappa(z)$. It follows from [Ch3, Theorem 1.6 (vi)] that the points $z'\in U$ such that $\kappa(z')\in A$ are classical, whereas (3.2) assures that these points lie in Z^{un}, as

$$\tfrac{[K:\mathbb{Q}_p]}{[K_0:\mathbb{Q}_p]}\mathrm{val}(\lambda_1) < \tfrac{[K:\mathbb{Q}_p]}{[K_0:\mathbb{Q}_p]}\big(C_1-\sum\nolimits_\sigma n_{\sigma,i}\mathrm{val}_x(\sigma(\varpi))\big) = \tfrac{[K:\mathbb{Q}_p]}{[K_0:\mathbb{Q}_p]}C_1 - \sum\nolimits_\sigma n_{\sigma,1}$$

and

$$k_{\sigma,2}+\sum\nolimits_{\sigma\neq\sigma'} k_{\sigma',1} = n_{\sigma,d-1}+1+\sum\nolimits_{\sigma\neq\sigma'} n_{,\sigma',d}$$

for all σ and similarly for the second required inequation. The claim now follows from this as the map $U \to V$ is finite and torsion free. □

Let us fix an identification of the decomposition group $\mathcal{G}_{w_0}$ of $\mathcal{G}_E$ at w_0 with the local Galois group $\mathcal{G}_K$.

Proposition 3.4. *Let $\Pi = \Pi_z$ for some $z \in Z^{\mathrm{reg}}$. For an infinite place $v \in I$ let $n_{v,1} \geq \cdots \geq n_{v,d}$ denote the highest weight of Π_v. Then the representation $\rho_\Pi|_{\mathcal{G}_K}$ is crystalline with Hodge-Tate weights $k_{\sigma,i} = n_{v(\sigma),d+1-i} + i - 1$, where $v(\sigma) = \iota_\infty \iota_p^{-1}\sigma$. Moreover the Frobenius $\Phi_{\mathrm{cris},\Pi}$ that is the $[K_0 : \mathbb{Q}_p]$-th power of the crystalline Frobenius on*

$$\mathrm{WD}(\rho_\Pi|_{\mathcal{G}_K}) = D_{\mathrm{cris}}(\rho_\Pi|_{\mathcal{G}_K}) \otimes_{K_0 \otimes_{\mathbb{Q}_p} \bar{\mathbb{Q}}_p} \bar{\mathbb{Q}}_p$$

is semi-simple, its eigenvalues are distinct and given by

$$\lambda_i = \psi_{v_0}(t_{v_0,i}) \prod\nolimits_\sigma \sigma(\varpi)^{-n_{\sigma,i}} \in k(z) \subset \bar{\mathbb{Q}}_p.$$

Proof. It follows from [Ch3, Theorem 3.2] that the representation is semi-stable with Hodge-Tate weights and Frobenius eigenvalues as described above. The condition

$$\tfrac{\lambda_i}{\lambda_j} \neq p^{\pm[K_0:\mathbb{Q}_p]}$$

assures that the monodromy operator has to vanish and hence the representation is crystalline. Further the condition $\lambda_i/\lambda_j \neq 1$ assures that the Frobenius has distinct eigenvalues and is apriori semi-simple. □

By [Ch3, Corollary 3.9] there is a pseudo-character $T_Y : \mathcal{G}_{E,S} \to \Gamma(Y, \mathcal{O}_Y)$ such that for all $\Pi \in Z^{\mathrm{reg}}$ one has $T \otimes k(\Pi) = \mathrm{tr}\,\rho_\Pi$. Let us write $\mathcal{G}_{E,S}$ for the Galois group of the maximal extension E_S inside $\bar{\mathbb{Q}}$ that is unramified outside S and fix a continuous residual representation $\bar\rho : \mathcal{G}_{E,S} \to \mathrm{GL}_d(\mathbb{F})$ with values in a finite extension $\mathbb{F}$ of $\mathbb{F}_p$ such that the restriction $\bar\rho_{w_0} = \bar\rho|_{\mathcal{G}_{w_0}}$ is absolutely irreducible. We write $R_{\bar\rho,S}$ reps. $R_{\bar\rho_{w_0}}$ for the universal deformation rings of $\bar\rho$ resp. $\bar\rho_v$ and let $\mathfrak{X}_{\bar\rho,S}$ resp. $\mathfrak{X}_{\bar\rho_v}$ denote their rigid analytic generic fibers. As we assume $\bar\rho_{w_0}$ (and hence also $\bar\rho$) to be absolutely irreducible [Ch2, Theorem A and Theorem B] implies that the universal deformation rings $R_{\bar\rho_{w_0}}$ and $R_{\bar\rho,S}$ agree with the universal deformation rings of the corresponding pseudo-characters $\mathrm{tr}\,\bar\rho_{w_0}$ resp. $\mathrm{tr}\,\bar\rho$.

Let $Y_{\bar\rho} \subset Y$ denote the open and closed subset where the pseudo-character T_Y has residual type $\bar\rho$. Then the restriction to the decomposition group $\mathcal{G}_K \cong \mathcal{G}_{w_0} \subset \mathcal{G}_{E,S}$ at w_0 induces a map $f_{\bar\rho} : Y_{\bar\rho} \to \mathfrak{X}_{\bar\rho_{w_0}}$. Let $N_{K/\mathbb{Q}_p} : K^\times \to \mathbb{Q}_p^\times$ denote the norm map of K. We define $g_i : Y \to \mathbb{G}_m$ by

$$z \longmapsto \psi_{v_0}(t_{v_0,d+1-i}) \cdot N_{K/\mathbb{Q}_p}(\varpi)^{i-1}.$$

Further we define a morphism

$$\omega_Y = (\omega_{Y,i})_i : Y \longrightarrow \mathcal{W}^d$$

by setting $\omega_{Y,i} = \iota(\kappa_{v_0,d+1-i})\delta((1-i,\dots,1-i))$, where $\iota : \mathcal{W} \to \mathcal{W}$ denotes the morphism induced by $\iota(\delta)(z) = \delta(z)^{-1}$.

Theorem 3.5. *The map*

$$f = (f_{\bar\rho}, \omega_Y, (g_i)_i) : Y_{\bar\rho} \longrightarrow \mathfrak{X}_{\bar\rho_{w_0}} \times \mathcal{W} \times \mathbb{G}_m^d = \mathfrak{X}_{\bar\rho_{w_0}} \times \mathcal{T}^d$$

factors over the finite slope space $X(\bar\rho_{w_0}) \subset \mathfrak{X}_{\bar\rho_{w_0}} \times \mathcal{T}^d$ *and fits into the commutative diagram*

$$\begin{array}{ccc} Y_{\bar\rho} & \xrightarrow{f} & X(\bar\rho_{w_0}) \\ & \searrow^{\omega_Y} & \downarrow{\scriptstyle \omega_d} \\ & & \mathcal{W}^d \end{array}$$

Proof. The subset $X(\bar\rho_{w_0}) \subset \mathfrak{X}_{\bar\rho_{w_0}} \times \mathcal{T}^d$ is Zariski-closed an hence it suffices to check that $f(z) \in X(\bar\rho_{w_0})$ for all $z = (\Pi_z, \chi_z) \in Z^{\rm un} \cap Y(\bar\rho)$, as this subset is Zariski-dense by Lemma 3.3. However this amounts to say that for $z \in Z^{\rm un}$ the representation $\rho_{\Pi_z}|_{D_{w_0}}$ is trianguline with graded pieces $\mathcal{R}(\delta_i)$, where $\delta_i : K^\times \to \bar{\mathbb{Q}}_p^\times$ is the character

$$\delta_i|_{\mathcal{O}_K^\times} : z \longmapsto \prod_\sigma \sigma(z)^{-n_{v(\sigma),d+1-i}+1-i}$$
$$\delta_i(\varpi) = \psi_{v_0}(z)(t_{v_0,d+1-i}) \prod_\sigma \sigma(\varpi)^{i-1}.$$

where as above $v(\sigma) = \iota_\infty \iota_p^{-1}\sigma$ and we write $(n_{v(\sigma),i})$ for the highest weight of $\Pi_{z,v(\sigma)}$. By our choice of $Z^{\rm un}$ this follows from Lemma 2.9 and Proposition 3.4. □

3.3. **Variation on the density of trianguline representations.** In order to deduce density statements in the generic fiber of a local deformation ring from density statements in the space of trianguline representations one needs to show that the image of the space of trianguline representations is dense. This density result is included in the work of Chenevier [Ch1] and Nakamura [Na3]. In our case the situation will be a bit more restrictive: we only can make a statement about the components of the space of trianguline representation that are met by the eigenvariety. Hence we need to sharpen these density result a bit. In order to do so we need to compare the fibers of the space of trianguline representations over dominant algebraic weights with Kisin's crystalline deformation rings.

Let $\mathbf{k} = (k_{\sigma,i})$ be an algebraic weight. We assume that $k_{\sigma,1} < k_{\sigma,2} < \dots < k_{\sigma,d}$ for all σ. By the main result of [Ki3] there exists a closed subspace $\mathfrak{X}^{\rm cris}_{\bar\rho_{w_0},\mathbf{k}} \subset \mathfrak{X}_{\bar\rho_{w_0}}$ parametrizing the representations that are crystalline of labeled Hodge-Tate weight $(-k_{\sigma,i})$. Let $U^{\rm cris}_{\bar\rho_{w_0},\mathbf{k}} \subset \mathfrak{X}_{\bar\rho_{w_0},\mathbf{k}}$ denote the subset of points where the eigenvalues of the $[K_0 : \mathbb{Q}_p]$-th power of the crystalline Frobenius are pairwise distinct. Recall further that we have a map $\omega : \mathcal{S}^{\rm ns}(\bar\rho_{w_0}) \to \mathcal{W}^d$ to the weight space.

Proposition 3.6. *The map* $\omega^{-1}(\mathbf{k}) \to \mathfrak{X}_{\bar\rho_{w_0}}$ *induces a map* $g_{\mathbf{k}} : \omega^{-1}(\mathbf{k}) \to \mathfrak{X}_{\bar\rho_{w_0}}$ *which is étale over* $U^{\rm cris}_{\bar\rho_{w_0},\mathbf{k}}$*. Further* $g_{\mathbf{k}}^{-1}(U^{\rm cris}_{\bar\rho_{w_0},\mathbf{k}})$ *is open and dense in* $\omega^{-1}(\mathbf{k})$*.*

Proof. By Corollary 2.7 the map $\omega^{-1}(\mathbf{k}) \to \mathfrak{X}_{\bar\rho_{w_0}}$ generically factors over $\mathfrak{X}^{\rm cris}_{\bar\rho_{w_0},\mathbf{k}}$. As $\mathfrak{X}^{\rm cris}_{\bar\rho_{w_0},\mathbf{k}} \subset \mathfrak{X}_{\bar\rho_{w_0}}$ is Zariski-closed the first claim follows. It is further easy to see that the preimage of $U^{\rm cris}_{\bar\rho_{w_0},\mathbf{k}}$ is open and dense. It remains to prove the claim on étaleness which we prove by verifying the infinitesimal lifting criterion. Consider an affinoid algebra A with ideal $I \subset A$ satisfying $I^2 = 0$ and the diagram

$$\begin{array}{ccc} \mathrm{Sp}(A/I) & \longrightarrow & \omega^{-1}(\mathbf{k}) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(A) & \longrightarrow & U^{\mathrm{cris}}_{\bar\rho_{w_0},\mathbf{k}}. \end{array}$$

This diagram gives rise to a family of filtered φ-modules over A and we write (D, Φ_D) for the associated family of Weil-Deligne representations on $\mathrm{Sp}(A)$. Further the upper arrow in the diagram gives us a filtration

$$0 \subsetneq \overline{\mathrm{Fil}}^1 \subsetneq \cdots \subsetneq \overline{\mathrm{Fil}}^{d-1} \subset \overline{\mathrm{Fil}}^d = \overline{D} = D/I$$

by subspaces that are locally on $\mathrm{Sp}(A)$ direct summands and stable under the Frobenius $\Phi_{\overline{D}} = \Phi_D \mod I$ and we have to prove that this filtration uniquely lifts to a Φ_D-stable filtration $\mathrm{Fil}^\bullet$ of D such that locally on $\mathrm{Sp}(A)$ the Fil^i are direct summands of D. After localizing on $\mathrm{Sp}(A)$ we may assume that D is free and that all the $\overline{\mathrm{Fil}}^i$ are direct summands of $\overline{D}$. We show that we can lift $\overline{\mathrm{Fil}}^1$ uniquely to a Φ-stable direct summand of D. The rest will follow by induction. Let us chose a basis $\bar e_1, \dots, \bar e_d$ of $\overline{D}$ such that $\overline{\mathrm{Fil}}^i$ is generated by $\bar e_1, \dots, \bar e_i$ and take arbitrary lifts e_j of the $\bar e_j$ in D. We write $A = (a_{ij})$ for the matrix of Φ in this basis. Then the above implies that $a_{ij} \in I$ for $i > j$ and that $a_{ii} \neq a_{jj}$ for all $i \neq j$, as this is true modulo all maximal ideals of A by definition of $U^{\mathrm{cris}}_{\bar\rho_{w_0},\mathbf{k}}$. We have to show that there exists uniquely determined $\lambda_2 \dots, \lambda_d \in I$ and a $\mu \in A^\times$ such that

$$\Phi(e_1 + \sum_{j=2}^{d} \lambda_j e_j) = \mu(e_1 + \sum_{j=2}^{d} \lambda_j e_j).$$

However, this comes down to showing that

$$(A - \mu E)e_1 + \sum_{j=2}^{d} \lambda_j (A - \mu E) e_j = 0$$

has (up to scalar) a unique solution with $\lambda_i \in I$ which is an easy consequence of $a_{ij} \in I$ for $i > j$ and $a_{ii} \neq a_{jj}$ for $i \neq j$. □

Corollary 3.7. *Let $X(\bar\rho, W_\infty, e) \subset X(\bar\rho_{w_0})$ denote the Zariski-closure of those connected components of $X(\bar\rho_{w_0})^{\mathrm{reg}}$ that are met by the image of the map f defined in Theorem 3.5. Then the image of $X(\bar\rho, W_\infty, e)$ in $\mathfrak{X}_{\bar\rho_{w_0}}$ is Zariski-dense in a union of irreducible components of $\mathfrak{X}_{\bar\rho_{w_0}}$.*

Proof. Let us write X for the Zariski-closure of the image of $X(\bar\rho, W_\infty, e)$ for the moment. Following the proof of [Ch1, 4.5] and [Na3, Theorem. 4.3], we are reduced to show that there exists a crystalline point $\rho \in X^{\mathrm{sm}}$ such that $X(\bar\rho, W_\infty, e)$ contains all possible triangulations of the representation ρ. Here $X^{\mathrm{sm}} \subset X$ is the smooth locus which is Zariski-open and dense in X.

We first claim that the image of the eigenvariety produces many crystalline points such that all possible triangulations are contained in $X(\bar\rho, W_\infty, e)$. Let $z = f(y) \in X(\bar\rho, W_\infty, e)$ for some $y = (\Pi, \chi) \in Z \cap Y_{\bar\rho} \subset Y_{\bar\rho}$ such that the image ρ_z of z in $\mathfrak{X}_{\bar\rho_{w_0}}$ is regular crystalline and uncritical in the sense of (2.3). Then the triangulations are in bijection with the orderings of the Frobenius eigenvalues as the representation is regular and hence there are exactly $d!$ such triangulations. We claim that all these possible triangulations of ρ lie in $f(Y_{\bar{\rho}})$. Indeed, as Π_{v_0} is unramified, there are $d!$ distinct characters $\chi_i : T_{v_0}/T^0_{v_0}$ such that Π_{v_0} appears in the parabolic induction $\mathrm{Ind}_B^{\mathrm{GL}_d(K)}\chi_i$ and as the representation is uncritical it follows that the parameters of the triangulation are prescribed by the character χ_i.

Let us write $\mathbf{k}_z = \omega(z)$ and $z_1, \dots, z_{d!} \in X(\bar{\rho}, W_\infty, e)$ for the distinct points in the preimage of the ρ_z. Then there exist open neighborhoods $U_i \subset X(\bar{\rho}, W_\infty, e) \cap \omega^{-1}(\mathbf{k})$ of the z_i such that $U_i \cap U_j = \emptyset$. As the map $g_{\mathbf{k}_z}$ from Proposition 3.6 is étale at all the z_i it follows that the image of the U_i in $U^{\mathrm{cris}}_{\bar{\rho}_{w_0},\mathbf{k}_z}$ is open and after shrinking the U_i we may assume that $U_z = g_{\mathbf{k}_z}(U_i) = g_{\mathbf{k}_z}(U_j)$ for all i, j. It follows that all the crystalline representations $\rho \in U_z$ have the property that $X(\bar{\rho}, W_\infty, e)$ contains all their possible triangulations.

The claim now follows from the observation that $\bigcup U_z$ as z runs over all classical regular, uncritical crystalline points of the eigenvariety is Zariski-dense in X and hence contains a point of the smooth locus X^{sm}. □

3.4. **A density result for the space of p-adic automorphic forms.** Now we introduce the Banach space of p-adic automorphic forms of tame level H^{v_0} and prove that a an element of $R_{\bar{\rho},S}$ vanishing on this space, vanishes on the eigenvariety $Y(W_\infty, S, e)_{\bar{\rho}}$ too.

Recall that we have fixed a finite extension L of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformizer ϖ_L. If $H = \prod_v H_v \subset G(\mathbb{A}_{F,f})$ is a compact open subgroup such that $H_v \subset \mathrm{GL}_d(F_v)$ for $v|p$, we can define, for W_0 a finite $\mathcal{O}$-module with a continuous action of $\mathrm{GL}_d(\mathcal{O}_F \otimes \mathbb{Z}_p)$ and define

$$S_{W_0}(H, \mathcal{O}) = \{f : G(F)\backslash G(\mathbb{A}_F^\infty) \to W_0 \mid f(gh) = h^{-1}f(g) \text{ for all } h \in H\}.$$

If $H^{v_0} = \prod_{v \neq v_0} H^v$, we can define

$$S_{W_0}(H^{v_0}, \mathcal{O}) = \varinjlim_{H_{v_0} \subset G(\mathcal{O}_{F_{v_0}})} S_{W_0}(H^{v_0}H_{v_0}, \mathcal{O}),$$

where the limit is taken over all compact open subgroups of $G(\mathcal{O}_{F_{v_0}})$.

Let $\hat{S}_{W_0}(H^{v_0}, \mathcal{O})$ be the ϖ_L-adic completion of $S_{W_0}(H^{v_0}, \mathcal{O})$. When W_0 is the trivial representation, we omit it from the notation.

If $\mathbf{k}$ is a dominant algebraic weight, we write $\mathbb{W}_{\mathbf{k}}$ for the irreducible representation of $\mathbb{G}_{\bar{\mathbb{Q}}_p} = (\mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GL}_d)_{\bar{\mathbb{Q}}_p}$ of highest weight $\mathbf{k}$ relatively to our choice of Borel subgroup. Note that this representation is already canonically defined over the reflex field $E_{\mathbf{k}}$ of the weight $\mathbf{k}$ and hence especially over our fixed field L, because we assumed that L contains all the $\sigma(K)$. Finally we write $W_{\mathbf{k}}$ for the representation of $\mathrm{GL}_d(K)$ or $\mathrm{GL}_d(\mathcal{O}_K)$ given by composing the embedding

$$\begin{aligned} \mathrm{GL}_d(K) &\longrightarrow \prod\nolimits_\sigma \mathrm{GL}_d(L) = (\mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GL}_d)(L) \\ x &\longmapsto (\sigma(x))_\sigma. \end{aligned}$$

with the evaluation of $\mathbb{W}_{\mathbf{k}}$ on L-valued points (and similar for its restriction to $\mathrm{GL}_d(\mathcal{O}_K)$).

Recall that W_∞ is the representation of $\prod_{v\in S'_p}\prod_{w\in I_\infty(v)} G(F_v)$ fixed in section 3.1. Using ι_p and ι_∞ and choosing L big enough, we can view W_∞ as a representation of $\prod_{v\in S'_p} G(F_v)$ and put an L-structure on it. Let us write $\hat{S}_{\mathbf{k}}(H^{v_0}, L) = \hat{S}_{W^0}(H^{v_0}, \mathcal{O}) \otimes_{\mathcal{O}} L$, where W^0 is a stable $\mathcal{O}_L$-lattice of the representation $W \otimes_{\mathcal{O}_L} W_\infty$ of $G(\mathcal{O}_F \otimes \mathbb{Z}_p)$.

If H is a compact open subgroup of $G(\mathbb{A}_f)$ we write $\mathcal{H}(H)$ for the image of $\mathcal{H}^{\mathrm{un}}$ in $\mathrm{End}(\hat{S}_{\mathbf{k}}(H, L))$.

Now we fix H as in section 3.1, and assume moreover now that all places $v|p$ are split in E and H_v is maximal at theses places[5]. Recall that we fixed a Galois representation $\bar\rho$ which is *automorphic* of level H, i.e. there exists $z \in Z$ such that $\bar\rho$ is isomorphic to the reduction mod ϖ_L of ρ_{Π_z}. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{H}^{\mathrm{un}}$ such that for $v \notin S$, the conjugacy class of $\bar\rho(\mathrm{Frob}_v)$ coincides via the Satake correspondence with the morphism $\mathcal{H}(G(F_v), H_v) = \mathcal{O}_L[G(F_v)/\!\!/H_v] \to \mathcal{H}(H)/\mathfrak{m} \simeq k_L$. Let $H_{v_0} \subset G(\mathcal{O}_{F_{v_0}})$ be a compact open subgroup and $\mathcal{H}_{\mathfrak{m}}(H_{v_0})$ be the image of $\mathcal{H}^{\mathrm{un}}_{\mathfrak{m}}$ in $\mathrm{End}(S(H^{v_0}H_{v_0}, \mathcal{O})_{\mathfrak{m}})$.

Then there is a unique continuous map $\theta : R_{\bar\rho} \to \mathcal{H}_{\mathfrak{m}}(H_{v_0})$ with the following property: Given a morphism $\psi : \mathcal{H}_{\mathfrak{m}}(H_{v_0}) \to \bar{\mathbb{Q}}_p$ of $\mathcal{O}_L$-algebras, the deformations ρ of $\bar\rho$ corresponding to $\psi\circ\theta$ are such that for $v \notin S$, the conjugacy class of $\rho(\mathrm{Frob}_v)$ coincides via the Satake correspondence with the morphism

$$\mathcal{H}(G(F_v), H_v) \longrightarrow \mathcal{H}_{\mathfrak{m}}(H_{v_0}) \xrightarrow{\psi} \bar{\mathbb{Q}}_p.$$

By unicity, these maps glue into a map

$$\theta : R_{\bar\rho} \longrightarrow \varprojlim_{H_{v_0}} \mathcal{H}_{\mathfrak{m}}(H_{v_0})$$

giving a continuous action of $R_{\bar\rho}$ on $\hat{S}(H^{v_0}, \mathcal{O})_{\mathfrak{m}}$.

Now we can fix an idempotent e as in section 3.1 such that $eS_{\mathbf{k}}(H, L)_{\mathfrak{m}} \neq 0$. We will prove that if an element $t \in R_{\bar\rho}$ vanishes on $e\hat{S}(H^{v_0}, L)_{\mathfrak{m}}$, then it vanishes on $Y_{\bar\rho} = Y(W_\infty, S, e)_{\bar\rho}$ too.

Lemma 3.8. *There exists a* $\mathrm{GL}_d(\mathcal{O}_K) \times \mathcal{H}^{\mathrm{un}}$*-equivariant homeomorphism*

$$\hat{S}_{\mathbf{k}}(H^{v_0}, L)_{\mathfrak{m}} \simeq W_{\mathbf{k}} \otimes_L \hat{S}(H^{v_0}, L)_{\mathfrak{m}},$$

supposing that $\mathcal{H}^{\mathrm{un}}$ *acts trivially on* $W_{\mathbf{k}}$.

Proof. It is sufficient to prove it before the localization in $\mathfrak{m}$, by $\mathcal{H}^{\mathrm{un}}$-equivariance. Then we can use the following list of $\mathrm{GL}_d(\mathcal{O}_K) \times \mathcal{H}^{\mathrm{un}}$-isomorphisms

$$\begin{aligned} W^0_{\mathbf{k}} \otimes_{\mathcal{O}_L} \hat{S}(H^{v_0}, \mathcal{O}) &= \varprojlim_n (W^0_{\mathbf{k}}/\varpi_L^n) \otimes_{\mathcal{O}_L} S(H^{v_0}, \mathcal{O}/\varpi_L^n)) \\ &= \varprojlim_n (W^0_{\mathbf{k}}/\varpi_L^n) \otimes_{\mathcal{O}_L} (\varinjlim_{H_{v_0}} S(H_{v_0}H^{v_0}, \mathcal{O}/\varpi_L^n)) \\ &= \varprojlim_n \varinjlim_{H_{v_0}} (S_{W^0_{\mathbf{k}}/\varpi_L^n}(H_{v_0}H^{v_0}, \mathcal{O}/\varpi_L^n) \\ &= \hat{S}_{\mathbf{k}}(H^{v_0}, \mathcal{O}_L). \end{aligned}$$

[5]This restriction is only here to be able to apply the idempotent e at the spaces $S_W(H, L)$.

□

Proposition 3.9. *The* $\mathrm{GL}_d(\mathcal{O}_K)$*-representation* $\hat{S}(H^{v_0}, \mathcal{O})_{\mathfrak{m}}$ *is isomorphic to a direct factor of* $C(G(\mathcal{O}_K), L)^r$ *for some* $r \geq 0$*, where* $C(G(\mathcal{O}_K), L)$ *denotes the space of continuous* L*-valued functions on* $G(\mathcal{O}_K)$*.*

Proof. Using Lemma 3.8, it is sufficient to prove it when $\mathbf{k} = \mathbf{0}$. Then we remark that the Banach space $\hat{S}_{\mathbf{0}}(H^{v_0}, L)$ is the Banach space of continuous functions $G(F)\backslash G(\mathbb{A}_{F,f})/H^{v_0} \to W_\infty$. Let $g_1, \dots, g_{r'}$ a list of representative elements of $G(F)\backslash G(\mathbb{A}_{F,f})/H$, we have $G(F) \cap g_i H g^{-1} = \{1\}$ for each i, proving that $G(F)\backslash G(\mathbb{A}_{F,f})/H^{v_0}$ is isomorphic to $\mathrm{GL}_d(\mathcal{O}_K)^{r'}$. This proves that $S_{\mathbf{0}}(H^{v_0}, L)$ is $\mathrm{GL}_d(\mathcal{O}_K)$-equivariantly isomorphic to $C(\mathrm{GL}_d(\mathcal{O}_K), L)^r$ with $r = r' \dim W_\infty$. Using the fact that $\varprojlim_{H_{v_0}} \mathcal{H}(H_{v_0}, \mathcal{O}_L)$ and its action on $\hat{S}(H^{v_0}, L)$ commutes to $\mathrm{GL}_d(\mathcal{O}_K)$, we can conclude that $\hat{S}_{\mathbf{0}}(H^{v_0}, L)_{\mathfrak{m}}$ is isomorphic to a direct factor of $C(\mathrm{GL}_d(\mathcal{O}_K), L)^r$. □

From Lemma 3.8, there is an $\mathcal{H}^{\mathrm{un}}$-equivariant isomorphism

$$S_{\mathbf{k}}(H, L)_{\mathfrak{m}} \simeq \mathrm{Hom}_{\mathrm{GL}_d(\mathcal{O}_K)}(W_{\mathbf{k}}^*, \hat{S}(H^{v_0}, L)_{\mathfrak{m}}).$$

This implies that if $t \in R_{\bar{\rho}}$ vanishes on $\hat{S}_{\mathbf{k}}(H^{v_0}, L)_{\bar{\rho}}$ it will vanish at each point of $Z \subset Y_{\bar{\rho}}$. These points being Zariski-dense in Y, the function t vanishes on $Y_{\bar{\rho}}$.

3.5. **A density result for the eigenvariety.** Now we fix $\mathbf{k}$ a regular weight. We say that a closed point $y \in Y_{\bar{\rho}}$ is crystabelline of Hodge-Tate weights $\mathbf{k}$ if its image in $\mathfrak{X}(\bar{\rho})$ is crystalline on an abelian extension of K and its Hodge-Tate weights are given by $\mathbf{k}$. The purpose of this section is to prove that if $t \in R_{\bar{\rho}}$ is zero on the subset of points of $Y_{\bar{\rho}}$ which are crystabelline of Hodge-Tate weights $\mathbf{k}$, then t induces the null function on $Y_{\bar{\rho}}$.

Recall that we have fixed a Borel subgroup $B \subset \mathrm{GL}_d(K)$ and let us write $N \subset B$ for its unipotent radical. Further we write $N_0 = N \cap \mathrm{GL}_d(\mathcal{O}_K)$.

Definition 3.10. Let Π be an irreducible smooth representation of $G(F_{v_0})$. We say that Π has finite slope if the operator U_{v_0} has a non zero eigenvalue on the space Π^{N_0}. If Π is an irreducible automorphic representation of $G(\mathbb{A}_F)$, we say that Π has finite slope if Π_{v_0} has finite slope as a smooth representation of $G(F_{v_0})$.

The following result tells us that finite slope automorphic representations of $G(\mathbb{A}_F)$ give rise to closed points of $Y_{\bar{\rho}}$.

Proposition 3.11. *Let* Π *be an irreducible automorphic representation of* $G(\mathbb{A}_F)$ *of finite slope. Then there exists a point* $z \in Y(\bar{\mathbb{Q}}_p)$ *such that* $\psi_z|_{\mathcal{H}^{\mathrm{un}}} = \psi_\Pi|_{\mathcal{H}^{\mathrm{un}}}$*. Moreover, if* $\bigotimes_{w \in I_\infty(v_0)} \Pi_w$ *is isomorphic to* $W_{\mathbf{k}}$*, then the Galois-representation attached to* z *becomes semi-stable of weight* $\mathbf{k}$*, when restricted to the Galois group of an abelian extension of* K*. If moreover, for* $\eta_i = \omega_{Y,i}(z)\delta_{\mathcal{W}}((k_{\sigma,i}))$*, we have* $\eta_i \neq \eta_j$ *for* $i \neq j$ *then this Galois representation is even potentially crystalline of weight* $\mathbf{k}$*.*

Proof. By assumption, there exists $f \in \hat{S}(H^{v_0}, L)_{\mathfrak{m}}^{N_0}$ which is an eigenvector of $\mathcal{H} \times L[T^0]$ such that the character of $\mathcal{H}$ is ψ_Π and the eigenvalue of U_{v_0} is non zero.

Let χ be the character of T^0 giving the action of T^0 on f. By [Loe, Proposition 3.10.1], we have

$$e\hat{S}(H^{v_0}, L)_{\mathfrak{m}}^{N_0}[\chi] \simeq \varinjlim_r e\mathcal{S}(\chi, r)$$

By Proposition 3.2, there exists a point z of $Y(\bar{\mathbb{Q}}_p)$ such that $\psi_z|_{\mathcal{H}^{\mathrm{un}}} = \psi_\Pi|_{\mathcal{H}^{\mathrm{un}}}$. The claim about becoming semi-stable after an abelian extension follows easily using the map to the finite slope space and the fact that the fibers over regular locally algebraic characters have this property. By Theorem 3.5, the character $(\eta_i)_i$ gives the action of the inertia on the Weil-Deligne module of this Galois representation, which is non monodromic under the last assumption of the proposition. □

This proposition shows us that if we want to prove that an element $t \in \mathcal{H}^{\mathrm{un}}$ vanishing on all crystabelline points of type $\mathbf{k}$ of $Y_{\bar{\rho}}$ is zero, it is sufficient to prove that an element $t \in \mathcal{H}^{\mathrm{un}}$ vanishing on all $S_{\mathbf{k}}(H^{v_0}, L)_{\mathfrak{m}}[\Pi_f]$ with Π an irreducible automorphic representation of finite slope such that $e\Pi_f \neq 0$, then $t = 0$ on $e\hat{S}_{\mathbf{k}}(H^{v_0}, L)_{\mathfrak{m}}$.

To produce sufficiently automorphic finite slope representations we can use the following result. Now we write I_n for the level n Iwahori subgroup of $\mathrm{GL}_d(\mathcal{O}_K)$ i.e. elements of $\mathrm{GL}_d(\mathcal{O}_K)$ such that the entries below the diagonal are divisible by ϖ^n, and $B_0 = B \cap I_n$ and $N_0 = N \cap B_0$. Recall that the level of a smooth character $\chi : \mathcal{O}_K^\times \to \mathbb{C}^\times$ is the least integer n such that $1 + \varpi^{n+1}\mathcal{O}_K$ is contained in $\ker(\chi)$.

Proposition 3.12. *Let $\chi = \bigotimes_{i=1}^d \chi_i$ be a smooth character of T^0 such that for $1 \leq i \leq n-1$, the level of χ_i is stricly bigger than the level of χ_{i+1}, then there exists an open subgroup $I(\chi)$ such that $I(\chi) = (I(\chi) \cap \bar{N})T^0(I(\chi) \cap N)$, $I(\chi) \cap B = B^0$ and if we write χ for the composite $I(\chi) \to T^0 \to \mathbb{C}^\times$, then the pair $(I(\chi), \chi)$ is a type for the inertial conjugacy class of (T, χ), more precisely, if π is a smooth irreducible representation of $\mathrm{GL}_d(K)$, then*

$$\mathrm{Hom}_{I(\chi)}(\chi, \pi) \neq 0 \Longleftrightarrow \pi \cong \mathrm{Ind}_B^{\mathrm{GL}_d(K)}(\eta)$$

with η a character of T such that $\eta|_T = \chi$. Moreover, in this case, π has finite slope.

Proof. Let n_i be the level of χ_i and define $I(\chi)$ as the subgroup of I of matrices $(a_{i,j})_{1 \leq i,j \leq d}$ such that $\varpi^{n_j} | a_{i,j}$ for $j < i$. It is immediate to check that χ can be extended in a character of $I(\chi)$. It is enough to prove that $(I(\chi), \chi)$ is a type for the $\mathrm{GL}_d(K)$-inertial equivalence class of (T, χ). In this aim, we use the characterization of part 2 of the introduction of [BK]. The only non trivial condition is (iii). We follow closely the arguments of [BK] where the situation is much more general. Let z be the element of T whose diagonal entries are $(\varpi^{n-1}, \varpi^{n-2}, \ldots, \varpi, 1)$ and f_z the element of $\mathcal{H}(G, \chi)$ with support $I(\chi)zI(\chi)$ such that $f_z(z) = 1$. We only have to prove that f_z is an invertible element of $\mathcal{H}(G, \chi)$. Let $f_{z^{-1}}$ be the element of support $I(\chi)z^{-1}I(\chi)$ such that $f_{z^{-1}}(z^{-1}) = 1$. We want to prove that $g = f_{z^{-1}} * f_z$ has support in $I(\chi)$. The support of g is contained in $I(\chi)z^{-1}I(\chi)zI(\chi)$. Now remark that $I(\chi)zI(\chi) = \coprod_{u \in I(\chi)/(I(\chi) \cap zI(\chi)z^{-1})} uzI(\chi)$ and that each class of $I(\chi)$ modulo $I(\chi) \cap zI(\chi)z^{-1}$ contains an element of $N \cap I(\chi) = N_0$, so that $I(\chi)z^{-1}I(\chi)zI(\chi) = I(\chi)z^{-1}N_0zI(\chi) \subset I(\chi)NI(\chi)$. By [BH, Prop. 11.1.2.], it is then sufficient to check that if an element $u \in N$ intertwines the character χ, then $u \in N_0$. We can restrict us to the case $d=2$. Let $u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. Suppose that $x \notin \mathcal{O}_K$ and choose $n = n_1 - v(x)$ with n_i the level of χ_i. If u intertwines χ, an easy computation shows us that we must have $\chi_1(a + x\varpi^n c)\chi_2(d - x\varpi^n c) = \chi_1(a)\chi_2(d)$ for each $(a,d,c) \in \mathcal{O}_K^\times \times \mathcal{O}_K^\times \times \mathcal{O}_K$. As $n_1 \geq n_2 + 1$, we have $\chi_2(d - x\varpi^n c) = \chi_2(d)$, so that we have $\chi_1(1 + x\varpi^n c) = 1$ for all $c \in \mathcal{O}_K$, which contradicts the fact that $n_1 = n + v(x)$ is the level of χ_1. □

Let $\tilde{\mathcal{T}}^0$ be the set of smooth characters $T^0 \to \mathbb{C}_p$ of the form $\chi_1 \otimes \cdots \otimes \chi_d$ such that the level of χ_i is strictly bigger than the level of χ_{i+1} for $1 \leq i \leq d-1$.

Proposition 3.13. *Let B be the Banach space of continuous function $\mathcal{O}_K^\times \to \mathbb{C}_p$ and for $n \in \mathbb{N}$, let $B_{\geq n}$ denote the subspace generated by the characters $\mathcal{O}_K^\times \to \mathbb{C}_p^\times$ of finite level bigger than n. Then $B_{\geq n}$ is dense in B.*

Proof. As the space of smooth functions from $\mathcal{O}_K^\times \to \mathbb{C}_p$ is dense in B and a basis of this space is given by the set of all characters of finite level as a basis, the closure of $B_{\geq n}$ in B is a subspace of finite codimension. If it is strictly included in B, there exists a continuous map $\lambda : B \to \mathbb{C}_p$ which is U-equivariant for some open subgroup $U \subset \mathcal{O}_K^\times$ acting trivially on $\mathbb{C}_p$. Then λ gives rise to a non trivial Haar measure on U which can not exist. □

Corollary 3.14. *Let $C(T^0, \mathbb{C}_p)$ denote the space of continuous $\mathbb{C}_p$-valued functions on T^0. Then the subspace of $C(T^0, \mathbb{C}_p)$ generated by the elements of $\tilde{\mathcal{T}}^0$ is dense.*

Proposition 3.15. [6] *Let $C(N_0 \backslash \mathrm{GL}_d(\mathcal{O}_K), \mathbb{C}_p)$ denote the space of $\mathbb{C}_p$-valued continuous functions on $\mathrm{GL}_d(\mathcal{O}_K)$ which are left invariant under N_0. Then the subspace*

$$\sum_{\chi \in \mathcal{T}_d} \mathrm{Ind}_{I(\chi)}^{\mathrm{GL}_d(\mathcal{O}_K)}(\chi) \subset C(N_0 \backslash \mathrm{GL}_d(\mathcal{O}_K), \mathbb{C}_p)$$

is dense.

Proof. If $\chi \in \tilde{\mathcal{T}}^0$, the character χ of T^0 uniquely extends to a character χ of $I(\chi)$ which is trivial on $I_n \cap N$ and $I_n \cap \bar{N}$. Let's name such a function a *character function* for the moment. More generally for $g \in \mathrm{GL}_d(\mathcal{O}_K)$, the function $\chi(\cdot g)$ of support $I(\chi)g^{-1}$ is named a *right translated character function*. For $\chi \in \tilde{\mathcal{T}}^0$, the space $\mathrm{Ind}_{I(\chi)}^{\mathrm{GL}_d(\mathcal{O}_K)}(\chi)$ is exactly the subspace of $C(N_0 \backslash \mathrm{GL}_d(\mathcal{O}_K, \mathbb{C}_p))$ generated by the right translated character functions. Let $f : \mathrm{GL}_d(\mathcal{O}_K) \to \mathbb{C}_p$ be a continuous function, invariant on the left under N_0. We have to prove that we can approximate f by right translated character functions. Let $g_1, \dots, g_r$ a system of representatives of the quotient $I_1 \backslash \mathrm{GL}_d(\mathcal{O}_K)$. Let $f_i = f(\cdot\ g_i^{-1})|_{I_1}$, so that $f = \sum_{i=1}^r f_i(\cdot g_i)$. Now fix $1 \leq i \leq r$. Fix $\epsilon > 0$. As I_1 is compact, we can find $n \geq 1$, such that for $h \in I_n \cap \bar{N}$, we have $||f_i - f_i(\cdot\ h)|| < \epsilon$. Let $h_1, \dots, h_s \in I_1$ a system of representatives of $(I_n \cap \bar{N}) \backslash (I_1 \cap \bar{N})$, which is also a system of representatives of $I_n \backslash I_1$, and define $f_{i,j} = f_i(\cdot\ h_j^{-1})|_{I_n}$. Let $f'_{i,j}$ be the function on I_n defined by $f'_{i,j}(nt\bar{N}) = f_{i,j}(t)$ for $(n, t, \bar{N}) \in (N \cap I_n) \times T^0 \times (\bar{N} \cap I_n)$. As $(\bar{N} \cap I_n)$ is a normal subgroup of $(\bar{N} \cap I_1)$, we have $||f_{i,j} - f'_{i,j}|| < \epsilon$. Using Corollary 3.14, for each

[6] V. Paskunas informed us that he has more general versions of this result in his forthcoming work with M. Emerton

$(i,j) \in [1,r] \times\ [1,s]$, we can find elements $\tilde{f}_{i,j} \in \mathcal{T}_d$, such that $||f'_{i,j}|_{T^0} - \tilde{f}_{i,j}|| < \epsilon$. Now we can write each $\tilde{f}_{i,j}$ as $\sum_k a_{i,j,k}\chi_{i,j,k}$ with $I(\chi_{i,j,k}) \subset I_n$. We extend each $\chi_{i,j,k}$ to $I(\chi_{i,j,k})$ as previously described. As $I(\chi_{i,j,k}) \subset I_n$, we can write $\tilde{f}_{i,j}$ as a finite sum of right translated character functions. If $\tilde{f} = \sum_{i,j} \tilde{f}_{i,j}(\cdot\ h_j g_i)$, we have $||f - \tilde{f}|| < \epsilon$, and $\tilde{f}$ is a finite sum of right translated character functions. □

Now we can conclude the proof.

Proposition 3.16. *Let $t \in R_{\bar{\rho}}$ such that t is zero on each $eS_{\mathbf{k}}(H^{v_0}I(\chi), \mathbb{C}_p)_{\mathfrak{m}}[\chi]$ such that $\chi \in \tilde{\mathcal{T}}^0$, then $t = 0$ on $e\hat{S}(H^{v_0}, L)_{\mathfrak{m}}^{N_0}$.*

Proof. We know that $\sum_\chi \mathrm{Ind}_{I(\chi)}^{\mathrm{GL}_n(\mathcal{O}_K)}(\chi)$ is dense in $C(N_0\backslash \mathrm{GL}_n(\mathcal{O}_K), \mathbb{C}_p)$ and that $\hat{S}_{\mathbf{k}}(H^{v_0}, L)_{\mathfrak{m}}$ is isomorphic to a direct summand of $C(\mathrm{GL}_d(\mathcal{O}_K), L)^r$ for some r. It follows that $S_1 = \hat{S}_{\mathbf{k}}(H^{v_0}, L)_{\mathfrak{m}}\hat{\otimes}_L\mathbb{C}_p$ is isomorphic to a direct summand of $C(\mathrm{GL}_d(\mathcal{O}_K), \mathbb{C}_p)^r$. We can write $C(\mathrm{GL}_d(\mathcal{O}_K), \mathbb{C}_p)^r = S_1 \oplus S_2$.

As the functor $F = \bigoplus_\chi \mathrm{Hom}_{I(\chi)}(\chi, -)$ commutes with finite direct sums, we know that $F(S_1) \oplus F(S_2)$ is dense in $[C(\mathrm{GL}_d(\mathcal{O}_K), \mathbb{C}_p)^r]^{N_0}$. As the functor of N_0-invariants commutes with direct sums, we conclude that $F(S_1) \subset S_1^{N_0}$ must be dense. By assumption, t vanishes on $F(S_1)$, hence on $S_1^{N_0}$, which contains $\hat{S}_{\mathbf{k}}(H^{v_0}, L)_{\mathfrak{m}}^{N_0}$. Finally we conclude by remarking that

$$\hat{S}_{\mathbf{k}}(H^{v_0}, L)_{\mathfrak{m}}^{N_0} = W_{\mathbf{k}} \otimes_L \hat{S}(H^{v_0}, L)_{\mathfrak{m}}^{N_0}.$$

□

Corollary 3.17. *Let $f \in R_{\bar{\rho}}$ be a function vanishing on all points of $Y_{\bar{\rho}}$ which are crystabelline of Hodge-Tate weights $\mathbf{k}$. Then the image of f in $\Gamma(Y_{\bar{\rho}}, \mathcal{O}_Y)$ is zero.*

3.6. **Conclusion.** Let us summarize what we have proven so far using eigenvarieties. The following definition will be useful.

Definition 3.18. Let X be a rigid space and R be a ring together with a ring homomorphism $\psi : R \to \Gamma(X, \mathcal{O}_X)$.
(i) A subset $Z \subset X$ is called *R-closed* if $Z = \{x \in X \mid \psi(f)(x) = 0 \text{ for all } f \in I\}$ for some ideal $I \subset R$.
(ii) A subset $U \subset X$ is called *R-open* if its complement is R-closed.

Further we have an obvious notion of the R-closure of some subset $Z \subset X$ and a notion of R-density.

Let $\mathbf{k} = (k_{\sigma,i})_\sigma \in \prod_\sigma \mathbb{Z}^d$ be a regular algebraic weight and $X_{\mathbf{k}}(\bar{\rho}_{w_0})$ denote the $R_{\bar{\rho}_{v_0}}$-closure of the set of potentially crystalline points of $X(\bar{\rho}_{w_0})$ which have labelled Hodge-Tate weights $\mathbf{k}$. We have finally proved the following result.

Theorem 3.19. *The image of $Y(W_\infty, S, e)_{\bar{\rho}}$ is contained in $X_{\mathbf{k}}(\bar{\rho}_{w_0})$.*

4. The main theorem

Let us fix a continuous representation $\bar{r} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ fulfilling the condition of Conjecture $A.3$ of [EG], which is automatic if $d = 2$. We also assume that p does not divide $2d$. We need to embed our local situation into a global one. For this we use the results of the appendix of [EG].

Corollary $A.7$ of [EG] tells us that we can find F a totally real field, E a totally imaginary quadratic extension of F and a continuous irreducible representation $\bar{\rho}: \mathcal{G}_F \to \mathcal{G}_d(\bar{\mathbb{F}}_p)$ (see for example [EG, §5.1] for the definition of $\mathcal{G}_d$) such that

- $4|[F:\mathbb{Q}]$;
- each place $v|p$ of F splits in E and $F_v \simeq K$;
- for each place $v|p$ of F, there is a place $\tilde{v}$ of E dividing p and such that $\bar{\rho}|_{\mathcal{G}_{F_{\tilde{v}}}} \simeq \bar{r}$;
- $\bar{\rho}$ is unramified outside of p ;
- $\bar{\rho}^{-1}(\mathrm{GL}_d(\bar{\mathbb{F}}_p) \times \mathrm{GL}_1(\bar{\mathbb{F}}_p)) = \mathcal{G}_E$;
- $\bar{\rho}(\mathcal{G}_{E(\zeta_p)})$ is adequate (in the sense of [Tho, §2])
- $\bar{\rho}$ is automorphic, we will explain now what this means.

Let v_1 be a place[7] of F which is prime with p and satisfies the same hypothesis as in [EG, §5.3]. We define the compact open subgroup $H = \prod_v H_v \subset G(\mathbb{A})$ so that $H_v \simeq \mathrm{GL}_d(\mathcal{O}_K)$ if $v|p$, $H_v \subset G(F_v)$ is maximal hyperspecial if $v \nmid p$ and $v \neq v_1$, and H_{v_1} an open pro-ℓ-subgroup of $G(F_{v_1})$ for ℓ the residual characteristic of v_1. We say that $\bar{\rho}$ is automorphic if $S_W(H,L)_{\bar{\rho}} \neq 0$ for some irreducible locally algebraic representation W of $G(\mathcal{O}_F \otimes \mathbb{Z}_p)$.

Lemma 4.1. *There exists an irreducible algebraic representation W_∞ of the group $\prod_{v\in S'_p}\prod_{w\in I_\infty(v)} G(F_v)$ such that $\hat{S}(H^{v_0},L)_{\bar{\rho}} \neq 0$.*

Proof. By definition, we know that there exists an irreducible locally algebraic representation W of $G(\mathcal{O}_F \otimes \mathbb{Z}_p)$ such that $S_W(H,L)_{\bar{\rho}} \neq 0$. Let $\hat{S}(H^p,L)_{\bar{\rho}} = \varprojlim_n \varinjlim_{H_p \subset G(\mathcal{O}_F\otimes\mathbb{Z}_p)} S(H^pH_p, \mathcal{O}_L/\varpi_L^n)_{\bar{\rho}}$. We see, as in section 3.4 that $S_W(H,L)_{\bar{\rho}} \simeq \mathrm{Hom}_{G(\mathcal{O}_F\otimes\mathbb{Z}_p)}(W^*, \hat{S}(H^p,L)_{\bar{\rho}})$ and that, as a $G(\mathcal{O}_F\otimes\mathbb{Z}_p)$-representation, $S(H^p,L)_{\bar{\rho}}$ is isomorphic to a non zero direct factor of $C(G(\mathcal{O}_F\otimes\mathbb{Z}_p),L)^r$ for some $r\geq 1$. We can then find an irreducible algebraic representation W^{v_0} of $\prod_{v\in S'_p} G(F_v)$ such that $\mathrm{Hom}_{\prod_{v\in S'_p} G(F_v)}(W^{v_0}, \hat{S}(H^p,L)_{\bar{\rho}} \neq 0$ and choose for W_∞ the irreducible algebraic representation of $\prod_{v\in S'_p}\prod_{w\in I_\infty(v)} G(F_v)$ associated to W^{v_0} as explained in section 3.4, then we have $\hat{S}(H^{v_0},L)_{\bar{\rho}} \simeq \mathrm{Hom}_{\prod_{v\in S'_p} G(F_v)}(W^{v_0}, \hat{S}(H^p,L)_{\bar{\rho}}$. □

4.1. **A result of density in crystalline deformation spaces.** Fix $\mathbf{k}$ a regular algebraic weight such that $\mathrm{Hom}_{\mathrm{GL}_d(\mathcal{O}_K)}(W^*_{\mathbf{k}}, \hat{S}(H^{v_0},L)_{\bar{\rho}}) \neq 0$. In this section we will use patching techniques to prove that the $R_{\bar{r}}$-closure of automorphic points of $\mathfrak{X}^{\mathrm{cris}}_{\bar{r},\mathbf{k}}$ is a union of connected components of $\mathfrak{X}^{\mathrm{cris}}_{\bar{r},\mathbf{k}}$.

There exists a complete noetherian local ring $R^\square_{\bar{r}}$ pro-representing the functor $\mathcal{D}_{\bar{r}}$ on the category of local Artinian $W(\mathbb{F})$-algebras $(A,\mathfrak{m}_A)$ with residue field $\mathbb{F}$, where $\mathcal{D}_{\bar{r}}(A)$ is the set of continuous homomorphisms $r: \mathcal{G}_K \to \mathrm{GL}_d(A)$ reducing to $\bar{r}$ modulo the maximal ideal of A. Let $r^\square$ be the universal homomorphism $r^\square: \mathcal{G}_K \to \mathrm{GL}_d(R^\square_{\bar{r}})$. By Kisin's result [Ki3] there exists a reduced p-torsion

[7]The introduction of this auxiliary place is only needed for the patching construction which will be used in the forthcoming lines

free quotient $R^{\text{cris},\square}_{\bar{r},\mathbf{k}}$ of $R^{\square}_{\bar{r}}$ such that a continuous $\zeta : R^{\square}_{\bar{r}} \to \bar{\mathbb{Q}}_p$ factors through $R^{\text{cris},\square}_{\bar{r},\mathbf{k}}$ if and only if $\zeta \circ r^{\square}$ is crystalline of Hodge-Tate weight $\mathbf{k}$. If $\bar{r}$ is absolutely irreducible, the natural map $R^{\text{cris}}_{\bar{r},\mathbf{k}} \to R^{\text{cris},\square}_{\bar{r},\mathbf{k}}$ is formally smooth. We say that a $\bar{\mathbb{Q}}_p$-point of $\text{Spec}(R^{\text{cris},\square}_{\bar{r},\mathbf{k}})$ is automorphic if the corresponding deformation of $\bar{r}$ is automorphic. Further an irreducible component of $\text{Spec}(R^{\text{cris},\square}_{\bar{r},\mathbf{k}}[1/p])$ is called automorphic if it contains an automorphic point. Our goal in this section is to use the usual patching construction to prove the following result.

Theorem 4.2. *Let X be an automorphic component of* $\text{Spec}(R^{\text{cris},\square}_{\bar{r},\mathbf{k}}[1/p])$. *Then the set of automorphic points in X is Zariski dense.*

Let $\mathfrak{X}^{\text{aut},\square}_{\bar{r},\mathbf{k}}$ be the union the components of the rigid analytic generic fiber $\mathfrak{X}^{\text{cris},\square}_{\bar{r},\mathbf{k}}$ of $\text{Spf}\, R^{\text{cris},\square}_{\bar{r},\mathbf{k}}$ that correspond to automorphic components of $\text{Spec}(R^{\text{cris},\square}_{\bar{r},\mathbf{k}}[1/p])$. When $\bar{r}$ is absolutely irreducible, we can define in the same way, $\mathfrak{X}^{\text{aut}}_{\bar{r},\mathbf{k}}$ inside $\mathfrak{X}^{\text{cris}}_{\bar{r},\mathbf{k}}$.

Corollary 4.3. *The set automorphic points in $\mathfrak{X}^{\text{aut},\square}_{\bar{r},\mathbf{k}}$ is $R_{\bar{r}}$-dense in $\mathfrak{X}^{\text{aut},\square}_{\bar{r},\mathbf{k}}$. If $\bar{r}$ is absolutely irreducible, the set automorphic points in $\mathfrak{X}^{\text{aut}}_{\bar{r},\mathbf{k}}$ is $R_{\bar{r}}$-dense in $\mathfrak{X}^{\text{aut}}_{\bar{r},\mathbf{k}}$.*

Given a rigid space $\mathfrak{X}$ over $\mathbb{Q}_p$ we write $|\mathfrak{X}|$ for the underlying point set of $\mathfrak{X}$. Similarly we write $|X|$ for the set of closed points of a $\mathbb{Q}_p$-scheme X. Let R be a complete local Noetherian $\mathbb{Z}_p$-algebra with finite residue field and let $\mathfrak{X}$ denote the generic fiber of R in the sense of Berthelot. Further let $X = \text{Spec}\, R[1/p]$. Then we have $|X| = |\mathfrak{X}|$ as if we write $R = \mathbb{Z}_p[\![T_1, \dots, T_n]\!]/(f_1, \dots, f_m)$ then both sets are identified with the $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$-orbits in

$$\{x = (x_1, \dots, x_n) \in \bar{\mathbb{Q}}_p \mid |x_i| < 1 \text{ and } f_j(x) = 0\},$$

compare also [dJ, Lemma 7.1.9]. Further a subset $Z \subset |X| = |\mathfrak{X}|$ is dense in X if and only it is R-dense in $\mathfrak{X}$. This proves that the theorem implies the corollary.

Proof of the Theorem. Define R_{v_1} as the universal ring pro-representing the functor of lifts of $\bar{\rho}|_{G_{v_1}}$. Let $R^{\text{loc}} = R^{\square}_{v_1} \hat{\otimes}_{v \in S_p} R^{\text{cris},\square}_{\tilde{v},\mathbf{k}}$ and $R_{\infty,g} = R^{\text{loc}}[\![x_1, \dots, x_g]\!]$. Then the patching construction (see [Tho, §6]) gives us, for g big enough, a $R_{\infty,g}$-module M_∞ of finite type whose support in $\text{Spec}\, R_{\infty,g}$ is a union of irreducible components. Let R be the quotient of R^{loc} corresponding to one of these irreducible components. Then M_∞ is a faithful $R[\![x_1, \dots, x_g]\!]$-module. Moreover, M_∞ is constructed as an inverse limit of spaces M_n, where M_n is a quotient of a space of automorphic forms, on which the action of $R_{\infty,g}$ factors through a ring of Hecke operators.
Lemma 3.4.12 in [Ki2] shows that the irreducible components of $\text{Spec}\, R^{\text{loc}}[\![x_1, \dots, x_g]\!]$ are of the form $\text{Spec}(\bigotimes_{v \in S_p \cup \{v_1\}} R'_v)$ where R'_v is an irreducible component of $R^{\text{cris},\square}_{\tilde{v},\mathbf{k}}$ if $v|p$ and of $R^{\square}_{v_1}$ of $v = v_1$. Now fix $v|p$, and R'_v an automorphic irreducible component of $R^{\text{cris},\square}_{\bar{r},\mathbf{k}}$. This means that there exists an irreducible component $R'[\![x_1, \dots, x_g]\!]$ of $R^{\text{loc}}[\![x_1, \dots, x_g]\!]$ containing R'_v. If $t \in R^{\text{cris},\square}_{\bar{r},\mathbf{k}}$ vanishes on each automorphic point, then t acts trivially on the spaces M_n, and so on M_∞. Now M_∞ being a faithful R-module, the image of t in R is zero, which implies that $t = 0$ in R'_v since R'_v is p-torsion free. $\square$

4.2. **End of the proof.** Fix a regular weight[8] $\mathbf{k} = (k_{\sigma,i})_\sigma \in \prod_\sigma \mathbb{Z}^d$ and recall the subset $\mathcal{W}^d_{\mathbf{k},\mathrm{la}} \subset \mathcal{W}^d$. In this section we prove the following theorem which will imply the desired result on the density of potentially crystalline representations of fixed weight. The data of F, E, G, H, $\bar{\rho}$ are checking the properties of section 3.1, so that we can consider the eigenvariety $Y_{\bar{\rho}} = Y(W_\infty, S, e)_{\bar{\rho}}$ with $S = S_p \cup \{v_1\}$ and e a well suited idempotent.

Theorem 4.4. *Let $X \subset X^{\mathrm{reg}}(\bar{r})$ be the union of connected components intersecting the image of the eigenvariety $Y_{\bar{\rho}}$. Then the subset $\omega^{-1}(\mathcal{W}^d_{\mathbf{k},\mathrm{la}}) \cap X$ is $R_{\bar{r}}$-dense in X.*

Proof. Let us write $X' \subset X$ for the $R_{\bar{r}}$-closure of $\omega^{-1}(\mathcal{W}^d_{\mathbf{k},\mathrm{la}}) \cap X$. It then follows from Corollary 3.17 (resp. Theorem 3.19) that

$$f(Y_{\bar{\rho}}) = f(Y_{\bar{\rho}}) \cap X \subset X'.$$

Given any regular algebraic weight $\mathbf{k}'$ in the image of ω_Y let us write $X_{\mathbf{k}'}$ for the intersection of $\omega^{-1}(\mathbf{k}')$ with X. Let $y \in Y_{\bar{\rho}}$ be a classical regular point mapping to $x \in X_{\mathbf{k}'}$ and let Z denote the connected component of $X_{\mathbf{k}'}$ containing x and let $Z^{\mathrm{cris}} \subset Z$ be the Zariski-open (and dense) subset of crystalline points.

By construction Z maps under the projection to $\mathfrak{X}_{\bar{r}}$ to a connected component Z' of $\mathfrak{X}^{\mathrm{cris}}_{\bar{r},\mathbf{k}'}$. Further the component Z' is an automorphic component, as by assumption the representation defined by x extends to an automorphic Galois representation.

Let $t \in R_{\bar{r}}$ be an element vanishing on X' and consider its image, still denoted by t, in R' the quotient of $R^{\mathrm{cris}}_{\bar{r},\mathbf{k}'}$ corresponding to Z'. Let $z' \in Z'$ be an automorphic point corresponding to a crystalline $\mathcal{G}_K$-representation $r_{z'}$. By definition, we can find an irreducible cuspidal automorphic representation Π of an unitary group G, of some level $\tilde{H} \subset H$ as in section 3.1 such that $(\rho_\Pi)_{w_0} \simeq r_{z'}$. Then there exists a triple (W_∞, S, e) and a character χ of T/T^0 such $(\Pi, \chi) \in \mathcal{Z}$ the set of classical points of the eigenvariety $Y(W_\infty, S, e)$ and hence Theorem 3.19 implies that the image of $(\Pi, \chi) \in Y(W_\infty, S, e)_{\bar{\rho}_\Pi}$ in $X(\bar{r})$ is in fact contained in X'. By Corollary 4.3 the image t in R' is zero. As the map of Z in $\mathfrak{X}_{\mathrm{r}}$ factors through Z', the elements t vanishes on Z, which implies that $Z \subset X'$.

We can now conclude. There is a quasi-conpact neighborhood U of $f(y)$ inside X such that U is isomorphic to a product of an open subset $U_1 \subset \mathcal{W}^d$ with a rigid space U_2 which we may chose to be connected. This is true, as (by construction) locally around y the space $X^{\mathrm{reg}}(\bar{r})$ looks like a product of an affine space with $\mathbb{G}_m^d$. After shrinking U_1 we may also assume that $Y_{\bar{\rho}} \cap U$ surjects onto U_1. As U is quasi-compact, there exist $C_1, \dots, C_d$ such that

$$C_i \geq \mathrm{val}_x(\psi_{v_0}(t_{v_0,1} \dots t_{v_0,i})(x)) + 1$$

for all $x \in U$, where val_x is the valuation on $k(x)$ normalized by $\mathrm{val}_x(p) = 1$. Let us write $Z_1 \subset U_1$ for the set of dominant algebraic weights $n_{\sigma,1} \geq \dots \geq n_{\sigma,d}$ such that $C_i < n_{\sigma,i} - n_{\sigma,i+1} + 1$ for all i and $\sigma : K \hookrightarrow \bar{\mathbb{Q}}_p$. Then Z_1 is Zariski dense in U_1 and hence $\omega^{-1}(Z_1) \cap U$ is Zariski dense in U.

[8]not necessarily as in preceding section

It remains to show that X' contains $\omega^{-1}(Z_1)\cap U$. A point $x \in f(Y_{\bar{\rho}})\cap U$ mapping to $\mathbf{k}' \in Z_1$ is classical by the choice of C_i and [Ch3, Theorem 1.6 (vi)]. Applying what preceeds with the point x and the component of $X_{\mathbf{k}'}$ containing $\{\mathbf{k}'\} \times U_2$ we find that $\{\mathbf{k}'\} \times U_2 \subset X'$. This implies that X' contains $\omega^{-1}(Z_1) \cap U$ which is Zariski dense in U, hence $U \subset X'$.

We have proven that X' contains all connected components of X that meet $f(Y_{\bar{\rho}})$. Changing the eigenvariey $Y_{\bar{\rho}}$ we find that $X' = X$. □

Theorem 4.5. *Let $p \nmid 2d$ and let K be a finite extension of $\mathbb{Q}_p$. Let $\bar{r} : \mathcal{G}_K \to \mathrm{GL}_d(\mathbb{F})$ be an absolutely irreducible continuous representation which has a potentially diagonalizable lift and let $R_{\bar{r}}$ be its universal deformation ring. Assume that $\bar{r} \ncong \bar{r}(1)$. Let $\mathbf{k} = (k_{i,\sigma}) \in \prod_{\sigma:K\hookrightarrow\bar{\mathbb{Q}}_p} \mathbb{Z}^d$ be a regular weight. Then the representations that are crystabeline of labeled Hodge-Tate weight $\mathbf{k}$ are Zariski-dense in* $\mathrm{Spec}\, R_{\bar{r}}[1/p]$.

Proof. The assumptions that $\bar{r}$ is absolutely irreducible and $\bar{r} \ncong \bar{r}(1)$ imply that $Z = \mathrm{Spec}\, R_{\bar{r}}[1/p]$ is smooth and irreducible.

Let $X \subset X(\bar{r})$ be the subset defined in Theorem 4.4. Our assumptions imply that the X is non-empty and hence Corollary 3.7 implies that X has dense image in Z. Let $t \in R_{\bar{\rho}}$ be a function vanishing on all crystabelline points of weight $\mathbf{k}$. Then Corollary 2.7 implies that it vanishes on $X \cap \omega^{-1}(\mathcal{W}^d_{\mathbf{k},\mathrm{la}})$ and hence by Theorem 4.4 is vanishes on X. The claim follows as X has dense image in Z. □

As already mentioned above the extra assumptions on $\bar{r}$ are known to be true in the 2-dimensional case (see Remark A.4 in [EG]). This gives the following result.

Corollary 4.6. *Let $p \neq 2$ and let K be a finite extension of $\mathbb{Q}_p$. Let $\bar{r} : \mathcal{G}_K \to \mathrm{GL}_2(\mathbb{F})$ be an absolutely irreducible continuous representation such that $\bar{r} \ncong \bar{r}(1)$. Let $R_{\bar{r}}$ be its universal deformation ring and let $\mathbf{k} = (k_{i,\sigma}) \in \prod_{\sigma:K\hookrightarrow\bar{\mathbb{Q}}_p} \mathbb{Z}^2$ be a regular weight. Then the representations that are crystabeline of labeled Hodge-Tate weight $\mathbf{k}$ are Zariski-dense in* $\mathrm{Spec}\, R_{\bar{r}}[1/p]$.

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