A free surface model for incompressible pipe and open channel flow.

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Abstract

We present the full derivation of a free surface pipe or open channel model including friction, changes of section and changes of slope/topography. This 1D free surface model (also named **FS**-model) is obtained from the incompressible 3D Navier-Stokes equation under shallow water assumptions with well-suited boundary conditions. It was introduced in the general framework of unsteady mixed flows in closed water pipes and is largely used by the engineers community.

Keywords: free surface flow, incompressible Navier-Stokes, shallow water approximation, hydrostatic approximation, closed water pipe, open channel, friction, viscosity
AMS Subject classification : 65M08, 65M75, 76B07, 76M12, 76M28, 76N15

1 Introduction

Simulation of free surface flow in open channel or pipe as well as pressurized flow (when considering pipe flow) plays an important role in many engineering applications such as storm sewers, waste or supply pipes in hydroelectric installations ....

The free surface flow are described by a Newtonian, viscous an incompressible fluid through the 3D incompressible Navier-Stokes equations. Thus from a numerical viewpoint, the use of the full 3D Navier-Stokes equations can lead to time-consuming simulations. For specific application such as under shallow water assumption, one can reduce the Navier-Stokes equations to the so-called Saint-Venant equations (shallow water equations) by averaging technique.

Introducing reduced models preserving some of the main physical features is one of the most challenging issues that we address with the obvious consequence to decrease the computational time. During these last years, a great amount of works was devoted to the modeling and the simulation of free surface and pressurized flow (see for instance [21, 25, 24, 15, 13, 20, 19, 18, 22], [7, 2, 1, 16, 3, 4] and the reference therein).

The classical shallow water equations are usually obtained from the 3D Navier Navier-Stokes equations or the 2D Navier Navier-Stokes equations by vertical averaging leading to a two dimensional or a one dimensional model. The derivation of the shallow water equations from the Navier-Stokes equations are now classical. Gerbeau and Perthame [20] study the full derivation of the one dimensional shallow water equations from the 2D Navier-Stokes equations including a small friction term on a flat bottom. In particular, they derive the so-called "motion by slices" meaning that the horizontal velocity does not depend upon the vertical coordinate. They also propose a parabolic correction to get a viscous shallow water equations. In the same spirit, Marche [22] propose a two dimensional viscous and non viscous shallow water equations in a rotating framework with varying topography including Coriolis force, friction (linear and quadratic), and capillary effects. These source terms, in particular the capillary effects, is a great utility to prove the global existence of weak solutions for the viscous model as done in [11, 12, 10]. One can found a variant of this work including surface wind effects in [18]. These works are mainly based on a wall-law condition at the

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bottom while Boutounet *et al* [9] derive a shallow water equations with a no-slip condition at the bottom and include capillary effects.

A similar model can be also formally obtained from the 3D Euler equations by section averaging leading to a one dimensional shallow water model called **FS**-model by the author. This formal derivation is now classical (see for instance [7, 8] and [16, 4]) but in the same spirit of [20], this is unsatisfactory as pointed out by Bernardi and Pironneau [14]. Nevertheless, let us note that numerical experiment versus laboratory experiment was presented [2, 1, 3]) and the proposed schemes for the **FS**-model produce results that are in a very good agreement. Under the hypothesis

$$u(t,x,y,z) \approx \overline{u}(t,x)$$

the 1D **FS**-model is an accurate approximation of the 3D Euler equations in a shallow water framework. The so-called **FS**-model is:

$$\left\{ \begin{array}{lcl} \partial_t A + \partial_x Q & = & 0, \\ \partial_t Q + \partial_x \left(\dfrac{Q^2}{A} + gI_1(x,A)\cos\theta \right) & = & -gA\sin\theta + gI_2(x,A)\cos\theta \end{array} \right. \tag{1}$$

where A is the wet area, $\overline{u}$ is the mean water velocity, $Q = A\overline{u}$ is the discharge, I_1 is the usual hydrostatic pressure, I_2 the presssure source term and $\sin\theta$ is the "slope variation". To take into account the friction, a quadratic term $-\rho g C_f u|u|$ was added to the system (1). The friction factor C_f is defined by

$$C_f = \frac{1}{K_s^2 R_h(A)^{4/3}} \ ,$$

where K_s is the Strickler coefficient of roughness depending on the material, $R_h(A) = A/P_m$ is the hydraulic radius and P_m is the wet perimeter of the wet surface area (length of the part of the channel's section in contact with the water).

Thus, following the full derivation proposed by Ersoy [17] for the pressurized model (see [16, 4]), we study the full derivation including viscosity, friction with varying section and slope/topography for pipe/open channel flow. In particular, we justify the free surface model as an approximation at order $O(\varepsilon)$ of the full 3D incompressible Navier-Stokes system where ε is the aspect ratio assumed to be small

$$\varepsilon = \frac{D}{L} = \frac{l}{L} \ll 1 \ .$$

D is the mean pipe diameter (mean height of the open channel), l is the mean pipe width (mean width of the open channel) and L its length.

In particular, we show that the "motion by slices" assumption,

$$u \approx \overline{u} \ ,$$

used to formally derive the free surface model [4], is exact at first order and

$$u(t,x,y,z) = \overline{u}(t,x) + O(\varepsilon) \ .$$

Moreover, denoting $H(t,x,y)$ the local water elevation from the surface $z = 0$, H should satisfies:

$$H(t,x,y) = H(t,x,0) + O(\varepsilon)$$

meaning that the free surface has negligible oscillations in the y-direction. Let us mention that this asymptotic behavior is essential to reduce the 3D incompressible Navier-Stokes equations to the 1D free surface model.

The paper is organized as follows. In Section 2, we recall the incompressible Navier-Stokes equations with suitable boundary conditions including friction and we fix the notations. In Section 3, we deduce the so-called motion by slices and other assumptions required to derive the free surface model under shallow water assumptions from the hydrostatic approximation of the incompressible Navier-Stokes equations. Then, these equations are integrated following the section orthogonal to the main flow direction and the 1D free surface model is obtained. Finally, we end with Section 4 where we discuss about the paraboloid correction of the horizontal velocity.

2 The incompressible Navier-Stokes equation and its closure

In this section, we fix the notations of the geometrical quantities involved to describe the thin domain representing a pipe or an open channel. Then, we recall the incompressible Navier-Stokes system written in the Cartesian coordinates with a gravity. We define the wall-law boundary conditions including a general friction law with a no penetration condition. The free surface boundary conditions are also defined. For the sake of simplicity, we will consider only an infinitely rigid pipe with circular sections since the derivation is the same for open channel.

2.1 Notations and settings

Let us consider an incompressible fluid confined in a three dimensional domain $\mathcal{P}$, a pipe of length L oriented following the $\mathbf{i}$ vector,

$$\mathcal{P} := \big\{(x,y,z) \in \mathbb{R}^3;\ x \in [0,L],\ (y,z) \in \Omega(x)\big\}$$

where the section $\Omega(x)$, $x \in [0,L]$, is

$$\Omega(x) = \{(y,z) \in \mathbb{R}^2;\ y \in [\alpha(x,z), \beta(x,z)],\ z \in [0, 2R(x)]\}$$

as displayed on figure 1(a).

The main flow direction is supposed to be in the $\mathbf{i}$ direction. Thus the wet section

$$\Omega(t,x) = \Omega(x) \cap \{(y,z) \in \mathbb{R}^2; 0 \leqslant z \leqslant H(t,x,y)\}, \quad t > 0, x \in [0,L]\ ,$$

is assumed to be orthogonal to the main flow direction where $H(t,x,y)$ is the local water elevation from the surface $z = 0$ in the $\Omega(x)$-plane (as displayed on figure 1(b)).

Here $R(x)$ stands for the radius of the pipe section $S(x) = \pi R^2(x)$, $\alpha(x,z)$ (resp. $\beta(x,z)$) is the left (resp. the right) boundary point at elevation $0 \leqslant z \leqslant 2R(x)$.

In the Ω-plane, we define the coordinate of a point $\mathbf{m} \in \partial\Omega(x) := \Gamma_b(t,x)$, $x \in [0,L]$, by $(y, \varphi(x,y))$ where $\varphi(x,y) = \sqrt{R(x)^2 - y^2}$ for $y > 0$ and $\varphi(x,y) = -\sqrt{R(x)^2 - y^2}$ for $y < 0$. The wet boundary (part of the boundary in contact with water) is defined as follows:

$$\Gamma_b(t,x) = \{(y,z) \in \mathbb{R}^2;\ z = \varphi(x,y) \leqslant H(t,x,y)\}\ .$$

The point $\mathbf{m}$ stands for the vector $\omega\mathbf{m}$ where $w(x,0,b(x))$ defines the main slope elevation of the pipe with $b'(x) = \sin\theta(x)$. Then, we note $\mathbf{n} = \dfrac{\mathbf{m}}{|\mathbf{m}|}$ the outward unit vector at the point $\mathbf{m} \in \partial\Omega(x)$, $x \in [0,L]$ as represented on figure 1(b).

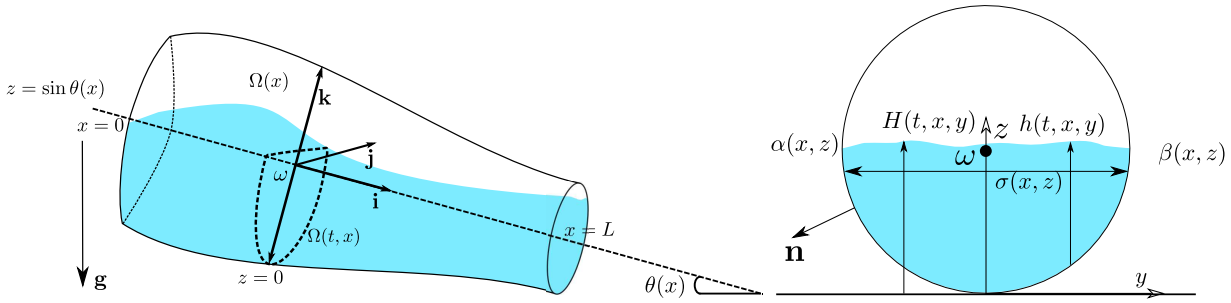
In the Ω-plane, we also define the coordinate of a point $\mathbf{m} \in \partial\Omega(x) := \Gamma_{fs}(t,x)$, $x \in [0,L]$, by $(y, H(t,x,y))$. The free surface boundary writes:

$$\Gamma_{fs}(t,x) = \{(y,z) \in \mathbb{R}^2;\ z = H(t,x,y)\}\ .$$

Finally, we note

$$h(t,x,y) = H(t,x,y) - \varphi(x,y)$$

the local elevation of the water (see figure 1(b)).



(a) Configuration (b) Ω-plane

Figure 1: Geometric characteristics of the pipe

Remark 2.1. For the sake of simplicity, we consider here only pipe with circular section. This work can be easily adapted to all type of realistic pipe or open channel by defining appropriately quantities involved in the definition of the geometry (see [16]). For instance, in case of "horseshoe" section as shown in figure 2(a), the section $\Omega(x)$, $x \in [0, L]$, is given by

$$\Omega(x) = \Omega_H(x) \cap \Omega_R(x)$$

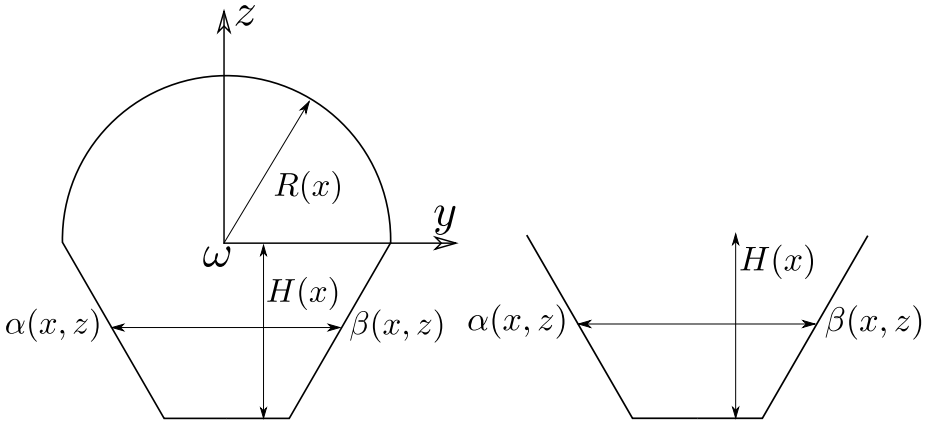
where

$$\Omega_H(x) = \left\{(y,z) \in \mathbb{R}^2; y \in [\alpha(x,z), \beta(x,z)],\ z \in [0, H(x)]\right\}$$

and

$$\Omega_R(x) = \{(y,z) \in \mathbb{R}^2;\ y \in [\alpha(x,z), \beta(x,z)],\ z \in [H(x), R(x)]\}\ .$$

H is the height of the trapezoidal basis and R is the radius of the upper part of the "horseshoe". A second example concerns an open channel as shown in figure 2(b).



(a) "horseshoe" section (b) Open channel

Figure 2: Pipe and open channel

Remark 2.2. For the sake of simplicity, the pressurized model is derived here in the Cartesian coordinates instead of the original one [4]. In the original work, the incompressible Euler equations was written in a local Serret-Frenet frame attached to the main pipe axis in order to take into account the local effects produced by the changes of section and the slope variation. This approach introduces naturally a curvature term in the derivation. Thus, as a minor difference, this term will not be present in this framework.

2.2 The incompressible Navier-Stokes model

We consider the incompressible Navier-Stokes equations in the domain $\mathcal{P}$ with suitable boundary conditions on the inner wall and on the free surface. These conditions are completed with inflows/outflows conditions at the upstream and downstream ends. The governing equations for the motion of an incompressible fluid in $[0,T]\times\mathcal{P}$, $T>0$ are given by

$$\left\{\begin{array}{rcl} \mathrm{div}(\rho_0\mathbf{u}) & = & 0\ , \\ \partial_t(\rho_0\mathbf{u})+\mathrm{div}(\rho_0\mathbf{u}\otimes\mathbf{u})-\mathrm{div}\sigma-\rho_0 F & = & 0\ , \end{array}\right. \tag{2}$$

where $\mathbf{u}=\begin{pmatrix} u \\ \mathbf{v}\end{pmatrix}$ is the velocity fields with u the **i**-component and $\mathbf{v}=\begin{pmatrix} v \\ w\end{pmatrix}$ the Ω-component, ρ_0 is the density of the fluid at atmospheric pressure and $F=g\begin{pmatrix}\sin\theta(x)\\ 0\\ -\cos\theta(x)\end{pmatrix}$ is the external gravity force.

The total stress tensor is

$$\sigma=\begin{pmatrix} -p+2\mu\partial_x u & R(\mathbf{u})^t \\ R(\mathbf{u}) & -pI_2+2\mu D_{y,z}(\mathbf{v})\end{pmatrix} \tag{3}$$

where I_2 is the identity matrix, μ is the dynamical viscosity and $R(\mathbf{u})$ is defined by $R(\mathbf{u})=\mu\left(\nabla_{y,z}u+\partial_x\mathbf{v}\right)$. Here, $\nabla_{y,z}u=\begin{pmatrix}\partial_y u\\ \partial_z u\end{pmatrix}$ is the gradient of u with respect to (y,z). Noting $\cdot^t$ the transpose of $\cdot$, we define the strain tensor $D_{y,z}(\mathbf{v})$ with respect to the variable (y,z):

$$2D_{y,z}(\mathbf{u})=\nabla_{y,z}\mathbf{v}+\nabla^t_{y,z}\mathbf{v}\ .$$

2.3 The boundary conditions

The Navier-Stokes system (2)–(3) is completed with suitable boundary conditions to introduce the border friction term on the wet inner wall of the pipe. We also assume a no-stress condition on the free surface.

As pointed out in [17], for pipe flow calculations, the Darcy-Weisbach equation, valid for laminar as well as turbulent flows, is generally adopted while the Manning formula is widely used for open channel (in turbulent regime). This factor C_f depends upon several parameter such as the Reynolds number, the relative roughness, ... Because of the extreme complexity of the rough surfaces of mean length δ, most of the advances in understanding have been developed around experiments leading to charts such as the Moody-Stanton diagram, expressing C_f as a function of the Reynolds number R_e, the relative roughness and some geometrical parameters depending on the material. Several formula such as Colebrook formula, Manning, Chézy, Swamee and Jain... have been introduced to represents natural pipe trends. For large Reynolds number, the friction factor C_f is almost function of the relative roughness $\dfrac{\delta}{D_h}$ where D_h is the hydraulic diameter. More precisely, one can show from the Colebrook law that $C_f=C_t+\dfrac{C_l}{R_e}$ where C_l, C_t are some quantity depending on the relative roughness $\dfrac{\delta}{D_h}$ and hydraulic parameters of the pipe. As a consequence if the relative roughness is negligible, the friction factor can be expressed only in function of the Reynolds number. We refer to [26, 28] for more details.

Thus, **on the inner wall** $\partial\Omega(x)$, $\forall x\in(0,L)$, we assume a wall-law condition including a general friction law:

$$(\sigma(\mathbf{u})n_b)\cdot\tau_{b_i}=(\rho_0 k(\mathbf{u})\mathbf{u})\cdot\tau_{b_i},\ \ t>0,\ x\in(0,L),\ (y,z)\in\Gamma_b(t,x)$$

where $k(\mathbf{u})=C_f|\mathbf{u}|$ and C_f is the friction factor depending on the material. The vector τ_{b_i} is the i^{th} vector of the tangential basis.

Let us detail the wall-law boundary conditions for the sake of completeness. Let us first write this condition as follows:

$$\sigma(\mathbf{u})n_b - (\sigma(\mathbf{u})n_b \cdot n_b)\, n_b = \rho_0 k(\mathbf{u})\mathbf{u},\,, \;\; t>0,\; x\in(0,L),\; (y,z)\in\Gamma_b(t,x)\;.$$

Here n_b stands for the unit outward normal vector:

$$n_b = \frac{1}{\sqrt{(\partial_x\varphi)^2 + \mathbf{n}\cdot\mathbf{n}}}\begin{pmatrix}-\partial_x\varphi\\ \mathbf{n}\end{pmatrix}$$

where $\mathbf{n} = \begin{pmatrix}-\partial_y\varphi\\ 1\end{pmatrix}$ is the outward normal vector in the Ω-plane. Thus, the wall-law boundary conditions are written:

$$R(\mathbf{u})\cdot\mathbf{n}\;(\mathbf{n}\cdot\mathbf{n} - (\partial_x\varphi)^2) + 2\mu\partial_x\varphi\,(D_{y,z}(\mathbf{v})\mathbf{n}\cdot\mathbf{n} - \partial_x u\;(\mathbf{n}\cdot\mathbf{n})) = \left(\mathbf{n}\cdot\mathbf{n} + (\partial_x\varphi)^2\right)^{3/2}\rho_0 k(u)u\;, \tag{4}$$

$$2\mu(\partial_x\varphi)^2\,(D_{y,z}(\mathbf{v})\mathbf{n} - \mathbf{n}) + \partial_x\varphi R(\mathbf{u})\;(\mathbf{n}\cdot\mathbf{n} - (\partial_x\varphi)^2) = \left(\mathbf{n}\cdot\mathbf{n} + (\partial_x\varphi)^2\right)^{3/2}\rho_0 k(\mathbf{v})\mathbf{v}\;. \tag{5}$$

We complete it with a no-penetration condition:

$$\mathbf{u}\cdot n_b = 0,\;\; t>0,\; x\in(0,L),\; (y,z)\in\Gamma_b(t,x)$$

i.e.

$$u\partial_x\varphi = \mathbf{v}\cdot\mathbf{n},\;\; t>0,\; x\in(0,L),\; (y,z)\in\Gamma_b(t,x)\;. \tag{6}$$

On the free surface, for the sake of simplicity we assume a no-stress condition:

$$\sigma(\mathbf{u})n_{fs} = 0,\;\; t>0,\; x\in(0,L),\; (y,z)\in\Gamma_{fs}(t,x)$$

where

$$n_{fs} = \frac{1}{\sqrt{(\partial_x H)^2 + \mathbf{n}_{fs}\cdot\mathbf{n}_{fs}}}\begin{pmatrix}-\partial_x H\\ \mathbf{n}_{fs}\end{pmatrix}\;\text{ where }\mathbf{n}_{fs} = \begin{pmatrix}-\partial_y H\\ 1\end{pmatrix}$$

is the outward normal vector to the free surface.

Remark 2.3. The range of validity of the modeling problem can be extended for pipe (respectively open channel) flow when the air entrainment (respectively the wind stress) has an influence (see for instance Ferrari and Saleri [18]). Let us also mention that the two-layer model proposed by Bourdarias *et al* [5] enters in this framework. In particular, one can justify such a model from the present work and Ersoy [17]. Moreover, one can also consider, as done in Marche [22], the surface tension at the fluid/air interface including capillary effects.

As done above for the wall-law condition, let us detail the free surface boundary conditions:

$$(p - 2\mu\partial_x u)\partial_x H + R(u)\cdot\mathbf{n}_{fs} = 0\;, \tag{7}$$

$$R(u)\partial_x H + (p - 2\mu D_{y,z}(\mathbf{v}))\mathbf{n}_{fs} = 0\;. \tag{8}$$

Following the idea developed in [20], we introduce an indicator function which allows to define the fluid region at time t:

$$\Phi(t,x,y,z) = \begin{cases}1 & \text{if } \varphi(x,y)\leqslant z\leqslant H(t,x,y)\;,\\ 0 & \text{otherwise}\end{cases}$$

Thanks to the divergence free condition, the equation of the conservation of mass becomes:

$$\partial_t\Phi + \partial_x(\Phi u) + \mathrm{div}_{y,z}(\Phi\mathbf{v}) = 0\;. \tag{9}$$

3 The averaged model

In a thin-layer domain (respectively open channel), the flow follows the main pipe (respectively channel) axis and thus it is almost unidirectional. For instance, in the supply pipes in hydroelectric installations (respectively long open channels), pipes (respectively channels) are such that the average thickness D (respectively the average width which is of the same order of a characteristic water elevation) is small in front of the characteristic length L. Besides the fact that the aspect ratio is small

$$\varepsilon = \frac{D}{L} \ll 1 \ ,$$

we should assume that the characteristic speed in the Ω-plane $\boldsymbol{V} = (V, W)$ is small compared to the horizontal one U:

$$\frac{W}{U} \approx \frac{V}{U} \ll 1$$

in order to get a unidirectional flow. These physical considerations define the so-called shallow water assumption.

3.1 The adimensionnalised Navier-stokes equations

Thus, in the sequel we consider the non-dimensional form of the Navier-Stokes system using the shallow water assumption by introducing a "small" parameter

$$\varepsilon = \frac{D}{L} = \frac{W}{U} = \frac{V}{U} \ll 1.$$

We then introduce a characteristic time T and a characteristic pressure P such that $T = \dfrac{L}{U}$ and $P = \rho_0 U^2$. The dimensionless quantities of time $\tilde{t}$, coordinate $(\tilde{x}, \tilde{y}, \tilde{z})$ and velocity field $(\tilde{u}, \tilde{v}, \tilde{w})$, noted temporarily by a $\tilde{\cdot}$, are defined by $\tilde{x} = \dfrac{x}{X}$. Thus, one has

$$\tilde{t} = \frac{t}{T}, \quad (\tilde{x}, \tilde{y}, \tilde{z}) = \left(\frac{x}{L}, \frac{y}{D}, \frac{z}{D}\right), \quad (\tilde{u}, \tilde{v}, \tilde{w}) = \left(\frac{u}{U}, \frac{v}{W}, \frac{w}{W}\right) \ .$$

We finally define the modified friction factor C_f/U that we write C_f.

We introduce the non-dimensional numbers:

$$\begin{array}{lll} F_r & \text{Froude number following the } \Omega\text{-plane} & : \quad F_r = \dfrac{U}{\sqrt{gD}} \, , \\ F_L & \text{Froude number following the } \mathbf{i}\text{-direction} & : \quad F_L = \dfrac{U}{\sqrt{gL}} \, , \\ R_e & \text{Reynolds numbers with respect to } \mu & : \quad R_\mu = \dfrac{\rho_0 U L}{\mu} \, . \end{array}$$

Dropping the $\tilde{\cdot}$ in equations (2), the non-dimensional incompressible Navier-Stokes system becomes:

$$\left\{ \begin{array}{rcl} \partial_x(u) + \mathrm{div}_{y,z}(\mathbf{v}) & = & 0 \\ \partial_t(u) + \partial_x(u^2) + \mathrm{div}_{y,z}(u\mathbf{v}) + \partial_x p & = & G_u \\ \varepsilon^2 \left(\partial_t(\mathbf{v}) + \partial_x(u\mathbf{v}) + \mathrm{div}_{y,z}(\mathbf{v} \otimes \mathbf{v})\right) + \nabla_{y,z} p & = & G_{\mathbf{v}} \end{array} \right. \tag{10}$$

where the source terms are given by

$$\begin{array}{rcl} G_u & = & -\dfrac{\sin\theta(x)}{F_L^2} + \partial_x \left(2R_e^{-1}\partial_x u\right) + \mathrm{div}_{y,z}\left(\dfrac{R_\varepsilon(\mathbf{u})}{\varepsilon}\right) \\ G_{\mathbf{v}} & = & \begin{pmatrix} 0 \\ -\dfrac{\cos\theta(x)}{F_r^2} \end{pmatrix} + \partial_x \left(\varepsilon R_\varepsilon(\mathbf{u})\right) + \mathrm{div}_{y,z}\left(2R_e^{-1} D_{y,z}(\mathbf{v})\right) \end{array} \tag{11}$$

with the notations

$$\mathrm{div}_{y,z}U = \partial_y U + \partial_z U, \quad \mathrm{div}U = \partial_x U + \mathrm{div}_{y,z}U \quad \text{and} \quad R_\varepsilon(\mathbf{u}) = R_e^{-1}\left(\frac{1}{\varepsilon}\nabla_{y,z}u + \varepsilon\partial_x \mathbf{v}\right) .$$

The non-dimensional wall-law boundary conditions (4)-(5) on the wet boundary Γ_b becomes:

$$R_\varepsilon(\mathbf{u})\cdot\mathbf{n}\ (\mathbf{n}\cdot\mathbf{n} - \varepsilon^2(\partial_x\varphi)^2) + 2\varepsilon R_e^{-1}\partial_x\varphi\ (D_{y,z}(\mathbf{v})\mathbf{n}\cdot\mathbf{n} - \partial_x u\ (\mathbf{n}\cdot\mathbf{n})) = \left(\mathbf{n}\cdot\mathbf{n} + \varepsilon^2(\partial_x\varphi)^2\right)^{3/2}\frac{k(u)}{U}u\ , \quad (12)$$

$$2\varepsilon^2 R_e^{-1}\partial_x\varphi^2\ (D_{y,z}(\mathbf{v})\mathbf{n} - \mathbf{n}) + \partial_x\varphi R_\varepsilon(\mathbf{u})\ (\mathbf{n}\cdot\mathbf{n} - \varepsilon^2(\partial_x\varphi)^2) = \varepsilon\left(\mathbf{n}\cdot\mathbf{n} + \varepsilon^2(\partial_x\varphi)^2\right)^{3/2}\frac{k(\mathbf{v})}{U}\mathbf{v}\ . \quad (13)$$

while the no-penetration condition (6) is not modified in its dimensionless form.

The unit outward normal vector on the wet boundary Γ_b is now

$$n_b = \frac{1}{\sqrt{\mathbf{n}\cdot\mathbf{n} + \varepsilon^2\partial_x\varphi}}\begin{pmatrix} -\varepsilon\partial_x\varphi \\ \mathbf{n}\end{pmatrix} .$$

The non-dimensional boundary conditions (7)-(8) on the free surface Γ_{fs} becomes:

$$(p - 2\mu\partial_x u)\partial_x H + \frac{R_\varepsilon(u)}{\varepsilon}\cdot\mathbf{n}_{fs} = 0\ , \quad (14)$$

$$\varepsilon R_\varepsilon(u)\partial_x H + (p - 2\mu D_{y,z}(\mathbf{v}))\mathbf{n}_{fs} = 0 \quad (15)$$

where the unit outward normal vector on the free surface Γ_{fs} is now

$$n_{fs} = \frac{1}{\sqrt{\mathbf{n}_{fs}\cdot\mathbf{n}_{fs} + \varepsilon^2\partial_x H}}\begin{pmatrix} -\varepsilon\partial_x H \\ \mathbf{n}_{fs}\end{pmatrix} .$$

To go further in the derivation of the free surface model, we rearrange the terms with respect to ε to bring out the so-called hydrostatic approximation of System (10):

$$\begin{aligned}
\mathrm{div}(\mathbf{u}) &= 0 && (16)\\
\partial_t(u) + \partial_x(u^2) + \mathrm{div}_{y,z}(u\mathbf{v}) + \partial_x p &= -\frac{\sin\theta(x)}{F_L^2} + \mathrm{div}_{y,z}\left(\frac{R_e^{-1}}{\varepsilon^2}\nabla_{y,z}u\right) + R_{\varepsilon,1}(\mathbf{u}) && (17)\\
\nabla_{y,z}p &= \begin{pmatrix} 0 \\ -\dfrac{\cos\theta(x)}{F_r^2}\end{pmatrix} + R_{\varepsilon,2}(\mathbf{u}) && (18)
\end{aligned}$$

where

$$R_{\varepsilon,1}(\mathbf{u}) = R_e^{-1}\left(\partial_x\left(2\partial_x u\right) + \mathrm{div}_{y,z}\left(\partial_x\mathbf{v}\right)\right) = O(R_e^{-1})$$

and

$$\begin{aligned}
R_{\varepsilon,2}(\mathbf{u}) &= R_e^{-1}\left(\partial_x\left(\nabla_{y,z}u + \varepsilon^2\partial_x\mathbf{v}\right) + \mathrm{div}_{y,z}\left(2D_{y,z}(\mathbf{v})\right)\right)\\
&\quad -\varepsilon^2\left(\partial_t(\mathbf{v}) + \partial_x(u\mathbf{v}) + \mathrm{div}_{y,z}(\mathbf{v}\otimes\mathbf{v})\right)\ ,\\
&= R_e^{-1}\left(\partial_x\left(\nabla_{y,z}u\right) + \mathrm{div}_{y,z}\left(2D_{y,z}(\mathbf{v})\right)\right) + \boldsymbol{O}(\varepsilon^2)\ ,\\
&= \boldsymbol{O}(R_e^{-1}) + \boldsymbol{O}(\varepsilon^2)\ .
\end{aligned}$$

The first component of the wall-law boundary condition (12) becomes:

$$\begin{aligned}
\frac{R_e^{-1}}{\varepsilon}\nabla_{y,z}u\cdot\mathbf{n} &= \frac{\left(\mathbf{n}\cdot\mathbf{n} + \varepsilon^2(\partial_x\varphi)^2\right)^{3/2}\frac{k(u)}{U}u}{(\mathbf{n}\cdot\mathbf{n} - \varepsilon^2(\partial_x\varphi)^2)} - \varepsilon R_e^{-1}\left(\frac{2\partial_x\varphi\left(D_{y,z}(\mathbf{v})\mathbf{n}\cdot\mathbf{n} - \partial_x u\ (\mathbf{n}\cdot\mathbf{n})\right)}{(\mathbf{n}\cdot\mathbf{n} - \varepsilon^2(\partial_x\varphi)^2)} + \partial_x\mathbf{v}\cdot\mathbf{n}\right)\ ,\\
&= K(u) + O(\varepsilon) + O(\varepsilon R_e^{-1})
\end{aligned} \quad (19)$$

where we make use of the notations

$$K(u) = \sqrt{\mathbf{n}\cdot\mathbf{n}}\frac{k(u)}{U}u \quad \text{and} \quad \nabla_{y,z}u\cdot\mathbf{n} := \partial_{\mathbf{n}}u$$

which are respectively the friction term and the normal derivative of u in the Ω-plane.

The second component of wall-law boundary condition (13) becomes:

$$\begin{aligned} R_e^{-1}\nabla_{y,z}u &= \frac{\varepsilon^2\left(\mathbf{n}\cdot\mathbf{n}+\varepsilon^2(\partial_x\varphi)^2\right)^{3/2}\rho_0\frac{k(\mathbf{v})}{U}\mathbf{v}}{\partial_x\varphi(\mathbf{n}\cdot\mathbf{n}-\varepsilon^2(\partial_x\varphi)^2)} - \frac{2\varepsilon^3R_e^{-1}\partial_x\varphi^2\left(D_{y,z}(\mathbf{v})\mathbf{n}-\mathbf{n}\right)}{\partial_x\varphi(\mathbf{n}\cdot\mathbf{n}-\varepsilon^2(\partial_x\varphi)^2)} - \varepsilon^2\partial_x\mathbf{v}\cdot\mathbf{n}\ , \\ &= \boldsymbol{O}(\varepsilon^2)+\boldsymbol{O}(\varepsilon^3R_e^{-1}) \end{aligned} \tag{20}$$

On the free surface, the boundary conditions (14)-(15) become:

$$R_e^{-1}\nabla_{y,z}u\cdot\mathbf{n}_{fs} = -\varepsilon^2\left((p-2R_e^{-1}\partial_x u)\partial_x H + R_e^{-1}\partial_x\mathbf{v}\cdot\mathbf{n}_{fs}\right) = O(\varepsilon^2)\ , \tag{21}$$

$$(p-2R_e^{-1}D_{y,z}(\mathbf{v}))\mathbf{n}_{fs} = -\left(R_e^{-1}\nabla_{y,z}u+\varepsilon^2R_e^{-1}\partial_x\mathbf{v}\right)\partial_x H\ . \tag{22}$$

As a consequence, from the relation (21) and the relation (22), the pressure on the free surface satisfies

$$p\ (\mathbf{n}_{fs}\cdot\mathbf{n}_{fs}) - 2R_e^{-1}D_{y,z}(\mathbf{v})\mathbf{n}_{fs}\cdot\mathbf{n}_{fs} = \varepsilon^2\left(\partial_x H\right)^2(p-2R_e^{-1}\partial_x u) = O(\varepsilon^2) \tag{23}$$

On the basis of physical assumptions and mathematical considerations, following [20] and [17], we shall assume the following asymptotic regime

$$R_e^{-1} = \varepsilon\mu_0, \quad K = \varepsilon K_0\ . \tag{24}$$

3.2 First order approximation

Assuming shallow water assumption, the nearly unidirectional flow induces small vertical accelerations that the Archimedes principle is applicable. As a consequence, one can drop all terms of order $O(\varepsilon^2)$ in equations (16)–(18). Moreover, in view of the asymptotic assumption (24), taking the formal limit as ε vanishes, we deduce the hydrostatic approximation

$$\partial_x(u_\varepsilon)+\mathrm{div}_{y,z}(\boldsymbol{v}_\varepsilon) = 0 \tag{25}$$

$$\partial_t(u_\varepsilon)+\partial_x(u_\varepsilon^2)+\mathrm{div}_{y,z}(u_\varepsilon\boldsymbol{v}_\varepsilon)+\partial_x p_\varepsilon = -\frac{\sin\theta(x)}{F_L^2}+\mathrm{div}_{y,z}\left(\frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon\right)+O(\varepsilon) \tag{26}$$

$$\nabla_{y,z}p_\varepsilon = \begin{pmatrix}0\\ -\dfrac{\cos\theta(x)}{F_r^2}\end{pmatrix}+\boldsymbol{O}(\varepsilon) \tag{27}$$

Let us emphasize that even if this system results from a formal limit of Equations (16)–(18) as ε goes to 0, we note its solution $(p_\varepsilon, u_\varepsilon, \boldsymbol{v}_\varepsilon)$ due to the explicit dependency on ε. Indeed, notice that we cannot neglect the terms $\dfrac{1}{\varepsilon}$ in Equation (26) since we are interested in computing a result at zeroth order.

Keeping in mind the above remark,

- the wall-law boundary conditions (19)-(20) become

$$\frac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon\cdot\mathbf{n} = \rho_\varepsilon K_0(u)+O(\varepsilon) \quad \text{and} \quad \mu_0\nabla_{y,z}u_\varepsilon = \boldsymbol{O}(\varepsilon), \quad t>0,\ x\in(0,L),\ (y,z)\in\Gamma_b(t,x)\ . \tag{28}$$

- the free surface conditions (21)-(22) are:

$$\mu_0\nabla_{y,z}u_\varepsilon\cdot\boldsymbol{n}_\varepsilon^{fs} = O(\varepsilon) \quad \text{and} \quad \mu_0\nabla_{y,z}u_\varepsilon = \boldsymbol{O}(\varepsilon), \quad t>0,\ x\in(0,L),\ (y,z)\in\Gamma_{fs}(t,x)\ . \tag{29}$$

Thus, from Equation (26), the wall-law boundary conditions on Γ_b (28) and the free surface conditions on Γ_{fs} (29), the so-called "motion by slices" is obtained by solving the Neumann problem for $t > 0$, $x \in (0, L)$

$$\left\{ \begin{array}{rcll} \mathrm{div}_{y,z}\left(\dfrac{\mu_0}{\varepsilon}\nabla_{y,z}u_\varepsilon\right) & = & O(\varepsilon)\ , & (y,z) \in \Omega(t,x) \\ \mu_0 \partial_{\mathbf{n}} u_\varepsilon & = & O(\varepsilon)\ , & (y,z) \in \partial\Omega(t,x) \end{array} \right.$$

where the constant is fixed through the second component of the wall-law condition (20), i.e.

$$\mu_0 \nabla_{y,z} u_\varepsilon = \boldsymbol{O}(\varepsilon)\ ,\ \ t > 0,\ \ x \in (0, L),\ \ (y,z) \in \Gamma_b(t,x)\ .$$

Thus, we get at first order

$$u_\varepsilon(t,x,y,z) \quad = \quad \overline{u_\varepsilon}(t,x)$$

since

$$u_\varepsilon(t,x,y,z) \quad = \quad \overline{u_\varepsilon}(t,x) + O(\varepsilon)\ . \tag{30}$$

Consequently, at first order, we get

$$\overline{u_\varepsilon^2} = \overline{u_\varepsilon}^2\ . \tag{31}$$

Using Equations (27), one has

$$\partial_z p_\varepsilon = -\frac{\cos\theta}{F_r^2} + O(\varepsilon).$$

Fixing y and integrating the previous relation between z and $H(t,x,y)$, keeping in mind the relation (23), we obtain

$$p_\varepsilon(t,x,y,z) = \frac{\cos\theta}{F_r^2}(H_\varepsilon(t,x,y) - z) + O(\varepsilon)\ .$$

Moreover, from the first component of Equation (27), we deduce

$$H_\varepsilon(t,x,y) = H_\varepsilon(t,x,0) + O(\varepsilon)\ . \tag{32}$$

As a consequence, at first order, we find the usual hydrostatic pressure law,

$$p_\varepsilon(t,x,y,z) \quad = \quad \frac{\cos\theta}{F_r^2}(H_\varepsilon(t,x,0) - z)\ ,$$

i.e.

$$p_\varepsilon(t,x,y,z) \quad = \quad \frac{\cos\theta}{F_r^2}(H_\varepsilon(t,x,0) - z) + O(\varepsilon)\ . \tag{33}$$

In view of the the definition of the water elevation H_ε (32), the wet area is now modified at first order as follows, $t > 0, x \in [0, L]$:

$$\Omega_\varepsilon(t,x) = \{(y,z) \in \mathbb{R}^2; \alpha(x,z) \leqslant y \leqslant \beta(x,z) \text{ and } 0 \leqslant z \leqslant H_\varepsilon(t,x,0)\} \tag{34}$$

where the outward unit normal vector to the free surface $\mathbf{n}_{fs}$ reads at first order as displayed on figure 3

$$\boldsymbol{n}_\varepsilon^{fs} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ .$$

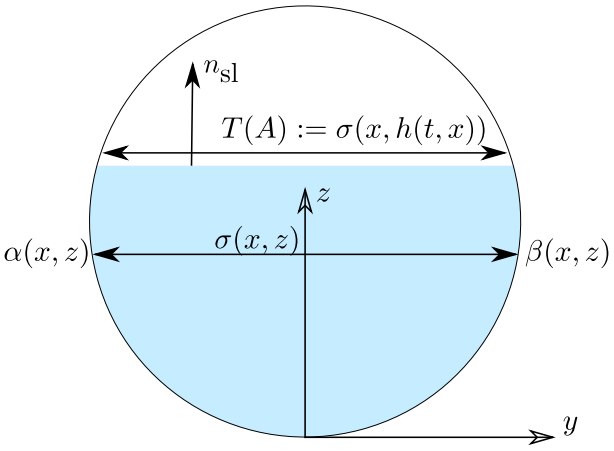


Figure 3: First order approximation of the wet area

Remark 3.1. At first order, one can deduce the vector $\boldsymbol{v}_\varepsilon$ from Equation (25) and the no-penetration condition (6).

3.3 The free surface model

By virtue of the relations (30)–(34), dropping all terms of order $O(\varepsilon)$ and integrating Equation (25)–(27) over the cross-section $\Omega(t,x)$, following [4] we proceed as follows.

Let us first recall that $\mathbf{m} = (y, \varphi(x,y)) \in \partial\Omega(x)$ stands for the vector $\omega\mathbf{m}$ and $\mathbf{n} = \dfrac{\mathbf{m}}{|\mathbf{m}|}$ for the outward unit normal vector to the boundary Γ_b at the point $\mathbf{m}$ in the Ω-plane (as displayed on figure 1(b)).

Then, let us introduce $A(t,x)$ and $Q(t,x)$ the conservative variables of wet area and discharge defined by the following relations:

$$A(t,x) = \int_{\Omega_\varepsilon(t,x)} dydz \tag{35}$$

and

$$Q(t,x) = A(t,x)\overline{u_\varepsilon}(t,x) \tag{36}$$

where

$$\overline{u_\varepsilon}(t,x) = \frac{1}{A(t,x)} \int_{\Omega_\varepsilon(t,x)} u(t,x,y,z)\, dydz$$

is the mean speed of the fluid over the section $\Omega_\varepsilon(t,x)$.

Kinematic boundary conditions.

Let $\mathbf{v}$ be the vector field $\begin{pmatrix} v \\ w \end{pmatrix}$. Integrating the equation of conservation of the mass (9) on the set:

$$\overline{\Omega}(x) = \{(y,z);\, \alpha(x,z) \leqslant y \leqslant \beta(x,z),\, 0 \leqslant z \leqslant \infty\},$$

we get the following equation:

$$\int_{\overline{\Omega}(x)} \partial_t(\phi) + \partial_x(\phi u_\varepsilon) + \mathrm{div}_{y,z}(\phi \boldsymbol{v}_\varepsilon)\, dydz = \partial_t A + \partial_x Q + \int_{\partial\Omega_\varepsilon(t,x)} (u_\varepsilon \partial_x \mathbf{m} - \boldsymbol{v}_\varepsilon) \cdot \mathbf{n}\, ds \tag{37}$$

where A and Q are given by (35) and (36).

Now, integrating the equation (9) on $\Omega_\varepsilon(t,x)$, we get:

$$\int_0^{H_\varepsilon(t,x)} \partial_t \int_{\alpha(x,z)}^{\beta(x,z)} dydz + \partial_x Q + \int_{\partial\Omega_\varepsilon(t,x)} (\boldsymbol{v}_\varepsilon - u_\varepsilon \partial_x \mathbf{m}) \cdot \mathbf{n}\, ds = 0$$

where

$$\int_0^{H_\varepsilon(t,x)} \partial_t \int_{\alpha(x,z)}^{\beta(x,z)} dydz = \partial_t A - \sigma(x, H_\varepsilon(t,x))\partial_t h$$

with $\sigma(x, H_\varepsilon(t,x))$ the width at the free surface elevation as displayed on figure 3.

Then, one has:

$$\partial(A) + \partial_x(Q) + \int_{\Gamma_\varepsilon^{fs}(t,x)} (\partial_t \mathbf{m} + u_\varepsilon \partial_x \mathbf{m} - \boldsymbol{v}_\varepsilon) \cdot \boldsymbol{n}_\varepsilon^{fs}\, ds + \int_{\Gamma_b(t,x)} (u_\varepsilon \partial_x \mathbf{m} - \boldsymbol{v}_\varepsilon) \cdot \mathbf{n}\, ds = 0. \tag{38}$$

Keeping in mind the no penetration condition (6) and by comparing equations (37) and (38), we finally get the kinematic condition at the free surface:

$$\int_{\Gamma_\varepsilon^{fs}(t,x)} (\partial_t \mathbf{m} + u_\varepsilon \partial_x \mathbf{m} - \boldsymbol{v}_\varepsilon) \cdot \boldsymbol{n}_\varepsilon^{fs}\, ds = 0. \tag{39}$$

which is nothing else than the usual kinematic boundary condition:

$$\partial_t H_\varepsilon + u_\varepsilon(z = H_\varepsilon)\partial_x H_\varepsilon - w_\varepsilon(z = H_\varepsilon) = 0\ .$$

Finally, we deduced from (38) and (39) the following equation of the conservation of the mass:

$$\partial_t(A) + \partial_x(Q) = 0. \tag{40}$$

Equation of the conservation of the momentum.

In order to get the equation of the conservation of the momentum of the free surface model, we integrate each terms of (26) over sections $\Omega_\varepsilon(t,x)$ as follows:

$$\int_{\Omega_\varepsilon(t,x)} \underbrace{\partial_t(u_\varepsilon)}_{a_1} + \underbrace{\partial_x(u_\varepsilon^2)}_{a_2} + \underbrace{\mathrm{div}_{y,z}\,(u_\varepsilon \boldsymbol{v}_\varepsilon)}_{a_3} + \underbrace{\partial_x p_\varepsilon}_{a_4}\, dydz = \int_{\Omega_\varepsilon(t,x)} \underbrace{-\frac{\sin\theta}{F_L^2}}_{a5}\, dydz + \int_{\Omega_\varepsilon(t,x)} \underbrace{\mathrm{div}_{y,z}\,(\mu_0 \nabla_{y,z} u_\varepsilon)}_{a6}\, dydz\ .$$

By virtue of relations (30), (31) and (33), we successively get:

Computation of the term $\int_{\Omega_\varepsilon(t,x)} a_1\, dydz$.

The pipe being non-deformable, only the integral at the free surface is relevant:

$$\int_{\Gamma_b(t,x)} u_\varepsilon\, \partial_t \mathbf{m} \cdot \mathbf{n}\, ds = 0.$$

So, we get:

$$\int_{\Omega_\varepsilon(t,x)} \partial_t(u_\varepsilon)\, dydz \quad = \quad \partial_t \int_{\Omega_\varepsilon(t,x)} u_\varepsilon\, dydz - \int_{\Gamma_\varepsilon^{fs}(t,x)} u_\varepsilon\, \partial_t \mathbf{m} \cdot \boldsymbol{n}_\varepsilon^{fs}\, ds.$$

Computation of the term $\int_{\Omega_\varepsilon(t,x)} a_2\, dydz$.

$$\int_{\Omega_\varepsilon(t,x)} \partial_x(u_\varepsilon^2)\, dydz \quad = \quad \partial_x \int_{\Omega_\varepsilon(t,x)} u_\varepsilon^2\, dydz - \int_{\Gamma_\varepsilon^{fs}(t,x)} u_\varepsilon^2 \partial_x \mathbf{m} \cdot \boldsymbol{n}_\varepsilon^{fs}\, ds - \int_{\Gamma_b(t,x)} u_\varepsilon^2 \partial_x \mathbf{m} \cdot \mathbf{n}\, ds.$$

Computation of the term $\int_{\Omega_\varepsilon(t,x)} a_3\, dydz$.

$$\int_{\Omega_\varepsilon(t,x)} \mathrm{div}_{y,z}\left(u_\varepsilon \boldsymbol{v}_\varepsilon\right)\, dydz \quad = \quad \int_{\Gamma_\varepsilon^{fs}(t,x)} u_\varepsilon \mathbf{v}\cdot \boldsymbol{n}_\varepsilon^{fs}\, ds + \int_{\Gamma_b(t,x)} u_\varepsilon \boldsymbol{v}_\varepsilon\cdot \mathbf{n}\, ds.$$

Summing the result of the previous step $a_1 + a_2 + a_3$, we get:

$$\int_{\Omega_\varepsilon(t,x)} a_1 + a_2 + a_3\, dydz = \partial_t(Q) + \partial_x\left(\frac{Q^2}{A}\right) \tag{41}$$

where A and Q are given by (35) and (36).

Computation of the term $\int_{\Omega_\varepsilon(t,x)} a_4\, dydz$.

For $\psi = p$, p given by the relation (33), (t,x) fixed, we have:

$$\begin{aligned}
\int_{\Omega_\varepsilon(t,x)} \partial_x\psi\, dydz &= \int_0^{H_\varepsilon(t,x)}\int_{\alpha(x,z)}^{\beta(x,z)} \partial_x\psi\, dydz \\
&= \int_0^{H_\varepsilon(t,x)} \partial_x \int_{\alpha(x,z)}^{\beta(x,z)} \psi\, dydz \\
&\quad - \left(\int_0^{H_\varepsilon(t,x)} \partial_x\beta(x,z)\,\psi_{|y=\beta(x,z)} - \partial_x\alpha(x,z)\psi_{|y=\alpha(x,z)}\, dz\right) \\
&= \partial_x \int_{\Omega_\varepsilon(t,x)} \psi\, dydz \\
&\quad - \left(\int_0^{H_\varepsilon(t,x)} \partial_x\beta(x,z)\,\psi_{|y=\beta(x,z)} - \partial_x\alpha(x,z)\psi_{|y=\alpha(x,z)}\, dz\right) \\
&\quad -\partial_x H_\varepsilon(t,x) \int_{\alpha_{|z=H_\varepsilon(t,x)}}^{\beta_{|z=H_\varepsilon(t,x)}} \psi_{|z=H_\varepsilon(t,x)}\, dy
\end{aligned}$$

Finally, we have:

$$\int_{\Omega_\varepsilon(t,x)} \partial_x p\, dydz \quad = \quad \partial_x(gI_1(x,A(t,x))\cos\theta(x)) - gI_2(x,A)\cos\theta(x) \tag{42}$$

where I_1 is the hydrostatic pressure:

$$I_1(x,A) = \int_0^{H_\varepsilon(A)} (H_\varepsilon(A) - z)\sigma(x,z)\, dz.$$

The term I_2 is the pressure source term:

$$I_2(x,A) = \int_0^{H_\varepsilon(A)} (H_\varepsilon(A) - z)\partial_x\sigma(x,z)\, dz.$$

It takes into account of the section variation via the term $\partial_x\sigma(x,\cdot)$.

Computation of the term $\int_{\Omega_\varepsilon(t,x)} a_5\, dydz$.

We have:

$$\int_{\Omega_\varepsilon(t,x)} g\sin\theta\, dydz = gA\sin\theta. \tag{43}$$

Computation of the term $\int_{\Omega_\varepsilon(t,x)} a_6\, dydz$.

We have:

$$\int_{\Omega_\varepsilon(t,x)} \mathrm{div}_{y,z}\left(\mu_0 \nabla_{y,z} u_\varepsilon\right)\, dydz = \int_{\Gamma_\varepsilon^{fs}(t,x)} \mu_0 \nabla_{y,z} u_\varepsilon \cdot \boldsymbol{n}_\varepsilon^{fs}\, ds + \int_{\Gamma_b(t,x)} \mu_0 \nabla_{y,z} u_\varepsilon \cdot \mathbf{n}\, ds \tag{44}$$

where $\int_{\Gamma_\varepsilon^{fs}(t,x)} \mu_0 \nabla_{y,z} u_\varepsilon \cdot \boldsymbol{n}_\varepsilon^{fs}\, ds = 0$ due to the boundary condition (29). Using the boundary condition (28) and (30), at first order, the second integral

$$\int_{\Gamma_b(t,x)} \mu_0 \nabla_{y,z} u_\varepsilon \cdot \mathbf{n}\, ds = \int_{\Gamma_b(t,x)} K_0(u_\varepsilon)\, ds = AK(\overline{u_\varepsilon})$$

where

$$K(x, \overline{u_\varepsilon}) = K_0(\overline{u_\varepsilon}) \frac{\int_{\Gamma_b(t,x)} ds}{A} ds$$

where $\int_{\Gamma_b(t,x)} ds$ is the wet perimeter P_m (i.e. the portion of the perimeter where the wall is in contact with the fluid) and thus $\dfrac{A}{\int_{\Gamma_b(t,x)} ds}$ is nothing else than the so-called hydraulic radius. This quantity was introduce by engineers as a length scale for non-circular ducts in order to use the analysis derived for the circular pipes (see for instance [26, 27]). Let us outline that this factor is naturally obtained in the derivation of the averaged model.

Then, gathering results (40) and (41)–(44), we get the equation of the conservation of the momentum. Finally, multiplying by $\rho_0 U^2/L$, the shallow water equations for free surface flows introduced in [4] are:

$$\left\{ \begin{array}{lcl} \partial_t(A) + \partial_x(Q) & = & 0 \\ \partial_t(Q) + \partial_x\left(\dfrac{Q^2}{A} + gI_1 \cos\theta\right) & = & -gA\sin\theta + gI_2\cos\theta - gAK(x, Q/A) \end{array} \right. \tag{45}$$

This model takes into account the slope variation, change of section and the friction due to roughness on the inner wall of the pipe. This system was formally introduced by the author in [16] and [4] in the context of unsteady mixed flows in closed water pipes assuming the motion by slices that we have now justified here.

We have proposed a Finite volume discretisation of the free surface model introducing a new kinetic solver in [1] based on the kinetic scheme of Perthame and Simeoni [23]. We have also proposed a new well-balanced VFRoe scheme. These numerical scheme have been validated in [6] in a channel with varying width on a trans-critical steady state shock. Several test cases have been passed with success through comparison with an exact solution or a code to code comparison, see for instance [2, 3].

4 Further extensions and concluding remarks

One can also formally increase the order of accuracy by determining the first order correction on (y, z) in the asymptotic expansion of $u_\varepsilon(t, x, y, z)$. It corresponds to a paraboloid correction. To do so, let us come back to the equation (17) assuming $\theta = cst$ and write:

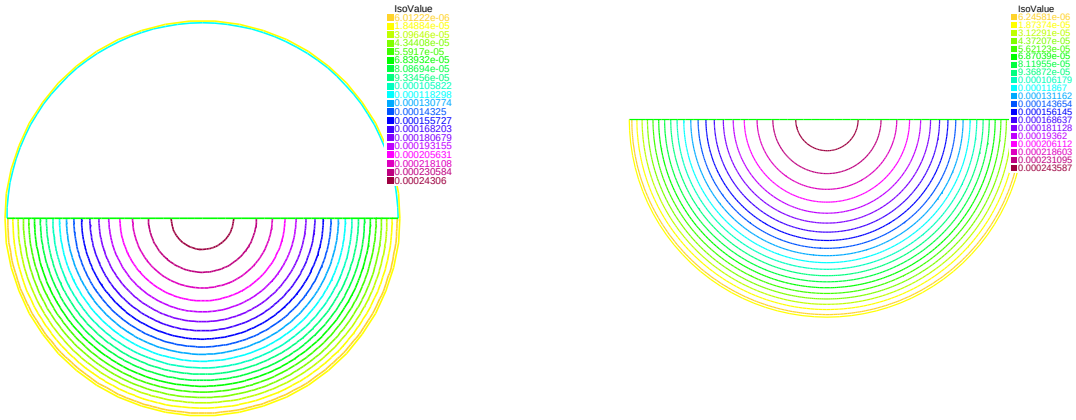
$$\begin{array}{rcl} \mathrm{div}_{y,z}\left(\dfrac{\mu_0}{\varepsilon}\nabla_{y,z} u_\varepsilon\right) & = & \partial_t(u_\varepsilon) + \partial_x(u_\varepsilon^2) + \mathrm{div}_{y,z}(u_\varepsilon \boldsymbol{v}_\varepsilon) + \partial_x p_\varepsilon + \dfrac{\sin\theta(x)}{F_L^2} + O(\varepsilon) \\ & = & \partial_t(u_\varepsilon) + \boldsymbol{u}_\varepsilon \cdot \nabla(u_\varepsilon) + \partial_x p_\varepsilon + \dfrac{\sin\theta(x)}{F_L^2} + O(\varepsilon) \\ & = & \partial_t(\overline{u_\varepsilon}) + \overline{\boldsymbol{u}_\varepsilon} \cdot \nabla(\overline{u_\varepsilon}) + \partial_x p_\varepsilon + \dfrac{\sin\theta(x)}{F_L^2} + O(\varepsilon) \\ & = & -K(\overline{u_\varepsilon}) + O(\varepsilon) \end{array}$$

Let us numerically illustrate the paraboloid correction of u_ε. We consider three type of geometry (section) for pipe: circular, rectangular and "horseshoe" (see figure 2(a)) and three one for open channel: semi-circular, open rectangular and trapezoidal. We consider the following settings

$$\varepsilon = 10^{-3}, \quad \cos\theta = 1, \quad K(\overline{u_\varepsilon}) = 1, \quad \text{and} \quad \overline{u_\varepsilon} = 0 \ .$$

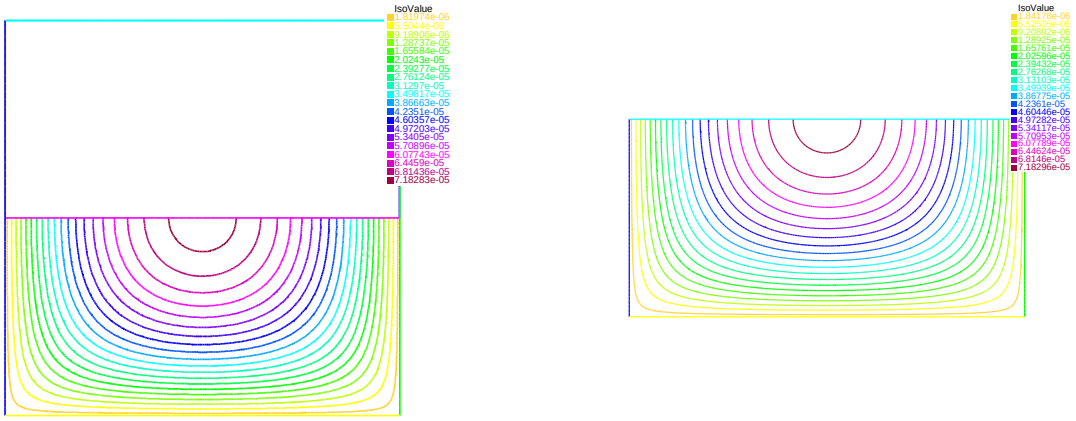
The results are represented on figures 4(a), 5(a) and 6(a) for pipe flow and on figures 4(b), 5(b) and 6(b) for open channel flow. Let us note that the scale of the paraboloid is of order ε as predicted by the asymptotic expansion of u_ε, in particular the expression (30).

Finally, we have performed an asymptotic analysis of the 3D incompressible Navier-Stokes equation with wall-law and no-penetration conditions, and suitable free surface boundary conditions in the shallow water limit. We have considered the incompressible hydrostatic approximation with friction boundary conditions and free surface boundary conditions and we have integrated these equations along the Ω sections to get the free surface model. In particular, we have shown that the free surface model (45) is an approximation of $O(\varepsilon)$ of the hydrostatic approximation (25)–(27) and therefore of the incompressible Navier-Stokes equations (10).



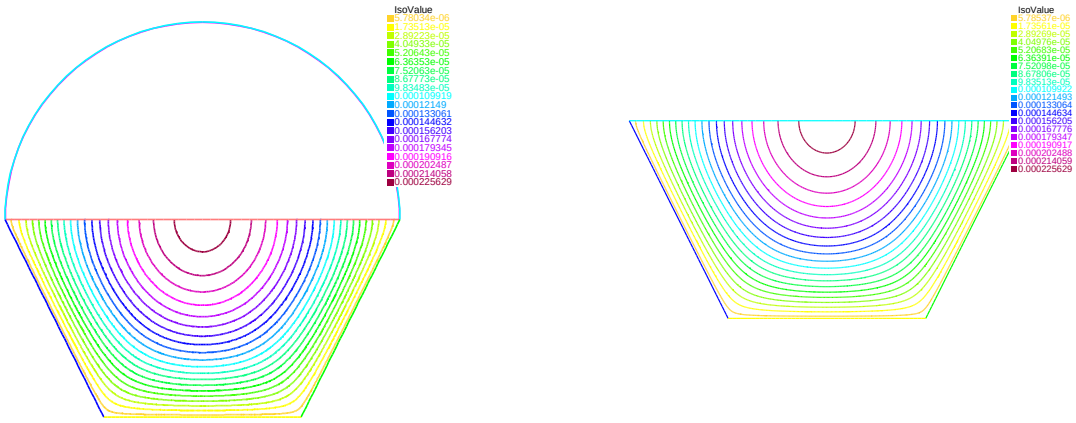
(a) pipe (b) open channel

Figure 4: Circular basis



(a) pipe (b) open channel

Figure 5: Rectangular basis



(a) basis (b) open channel

Figure 6: "horseshoe" basis

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