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VISCO-POTENTIAL FLOWS IN ELECTROHYDRODYNAMICS

MATTHEW HUNT* AND DENYS DUTYKH

ABSTRACT. In this study we consider the problem of the interface motion under the capillary–gravity and an external electric forces. The infinitely deep fluid layer is assumed to be viscous, perfectly conducting and the flow to be incompressible. The weak viscous effects are introduced using the Helmholtz–Leray decomposition and the visco-potential flow approach. The electric charge distributions above and on the free surface are considered. Finally, we derive some linearized analytical solutions for the free surface elevation shape under the localised pressure distribution and the combined action of the forces mentioned hereinabove.

Key words and phrases: electro-hydrodynamics; free surface flows; viscous dissipation; potential flow

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1. Introduction

Liquid thin films are a common occurrence in the fields of biology and engineering under the guise of coating flows, and have been subject to an intensive study. Instabilities and the ensuing dynamics associated of the liquid film then can be caused by a number of effects including interfacial instabilities due to surface tension variations (e.g. Marangoni

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instabilities), gravitational instabilities (such as Rayleigh–Taylor) as well as instabilities caused by external fields like electric or magnetic ones. Instabilities of the interface can also be due to topography or a moving pressure distribution (as it is considered in this study), but an external electric field may lead to the stabilization of the interface. More precisely, depending on the asymptotic form of the electric field in question, its effect can be either stabilizing [24] or destabilizing [22].

The study of electro-capillary waves was first initiated by G.I. Taylor *et al.* [22] for 3D waves with a prescribed disturbance of the interface. On the other hand, in modern investigations the disturbance is determined according to an applied external electric field and then the free surface profile is calculated by solving the electro-hydrodynamic formulation [20]. The work carried out by Hunt [11] focused on forced waves in electrohydrodynamics using the methods set out in [20, 27] to examine linear and weakly nonlinear free surface flows in 2D and weakly 3D (Kadomtsev–Petvishvili-type models). The method proposed in the present study allows to take into account for the viscous effects [25, 26, 23, 24] by keeping the *simplicity* of the potential flow approach. For the sake of the clarity of the exposition, we illustrate this method in 2D infinite depth case.

The theory of visco-potential flows probably originates from the pioneering work of J. Boussinesq (1895) [1] who estimated the water wave amplitude decay due to the effect of viscosity in the linear approximation. Then, this research has been continued in 70's in the context of nonlinear long wave models [19, 14]. Later, Kit & Shemer (1989) [15] developed a theoretical model which allowed to estimate the wave energy dissipation which included the friction effect at the bottom and lateral walls in a rectangular wave tank. The potential flows of viscous fluids were also thoroughly investigated theoretically by D. Joseph and his collaborators [13, 12]. The visco-potential formulation in the deep water case was derived by Dias *et al.* (2008) [5] and generalized to the finite depth case in [17] and later in [8, 7]. This formulation was validated experimentally and numerically for the practically important case of solitary wave propagation in [18]. More recently, the damping rates for various dissipative operators were investigated numerically in [2, 21]. Finally, the asymptotic long time behaviour of some visco-potential models were found and justified analytically in [4, 3, 9].

The present manuscript is organized as follows. After a brief introduction, in Section 2 we present the derivation of the electro-hydrodynamic visco-potential formulation. An analytical solution to the linearized formulation is shown in Section 3 for several values of problem parameters. A particular case of the electro-hydrodynamic problem where the charge is distributed on the interface is considered in Appendix A. Finally, the main conclusions and perspectives of this study are outlined in Section 4.

2. Mathematical model

Consider an infinitely deep channel of a perfectly conducting weakly viscous and incompressible fluid (referred to as the heavy fluid) and is in the region $\{(x, y) | x \in \mathbb{R}, y \leq \eta(x, t)\}$. The flow is assumed to be exactly two-dimensional. A Cartesian coordinate system (x, y) is introduced, with y pointing vertically upwards. The interface between two fluids is given by $y = \eta(x, t)$. The sketch of the physical domain is given on Figure 1. Below we will

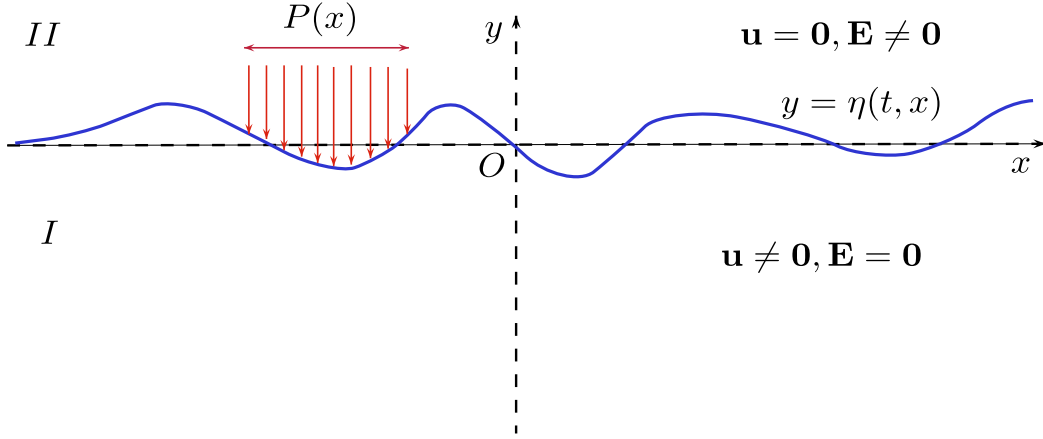


Figure 1. Sketch of the physical domain considered in this study.

adopt the so-called free surface assumption and the light fluid will enter into equations only through the electric forces.

The velocity field in the heavy fluid in region I is then given as $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + 0\mathbf{k}$. We assume that there is an electric field in region II $\{(x, y) | x \in \mathbb{R}, y > \eta(x, t)\}$. The electric field \mathbf{E} satisfies the equations $\nabla \times \mathbf{E} = \mathbf{0}$ and, thus, can be written in the following potential form:

$$\mathbf{E} \equiv -\nabla V.$$

Since there are no free charges in the light fluid, the Gauss's law states that $\nabla \cdot \mathbf{E} = 0$, and hence the governing equation for the electric potential reads:

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad y > \eta(x, t).$$

This equation is completed by appropriate boundary conditions. At infinity we require the following asymptotic behaviour of the solution $V(x, y)$:

$$V(x, y) \rightarrow -E_0 y \quad \text{as} \quad y \rightarrow +\infty$$

In the heavy fluid the velocity field $\mathbf{u} = (u, v)$ is governed by the linear incompressible 2-D Navier–Stokes equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2.1)$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (2.2)$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - g + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \quad (2.3)$$

$$(2.4)$$

where p is the pressure, g is the gravity acceleration and ν is the kinematic viscosity of the heavy fluid.

The main idea behind the visco-potential formulation consists in writing the velocity field \mathbf{u} in the form of the Helmholtz decomposition [5, 8]:

$$\mathbf{u} = \nabla\varphi + \nabla \times \mathcal{A}. \quad (2.5)$$

We will assume also that the flow is essentially irrotational, i.e. $\|\varphi\| \ll \|\mathcal{A}\|$. Under this assumption, equations (2.2), (2.3) can be rewritten in the following vectorial form which is valid for irrotational flows:

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u}, \quad \mathbf{g} := (0, -g).$$

Taking the curl of the last equation we obtain:

$$\frac{\partial}{\partial t} (\nabla \times \mathbf{u}) = \nu \nabla^2 (\nabla \times \mathbf{u}) \quad (2.6)$$

After substituting the representation (2.5) into (2.6) we obtain the following governing equation for the vortical component \mathcal{A} :

$$\frac{\partial \mathcal{A}}{\partial t} = \nu \nabla^2 \mathcal{A}.$$

The vector \mathcal{A} is known as the *streamline vector* and in 2D, it reduces just to a single component $\mathcal{A} = (0, 0, \psi)$, which satisfies the following scalar equation:

$$\frac{\partial \psi}{\partial t} = \nu \nabla^2 \psi. \quad (2.7)$$

Moreover, in 2D the velocity field \mathbf{u} can be simply expressed as:

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}.$$

The continuity equation (2.1) leads the Laplace equation for the potential part inside the fluid domain:

$$\nabla \cdot \mathbf{u} = 0 \implies \nabla^2 \varphi = 0.$$

By inserting the representation (2.5) into the linearized Navier–Stokes equations (2.2), (2.3) we obtain

$$\frac{\partial}{\partial t} (\nabla \varphi + \nabla \times \mathcal{A}) = -\nabla p + \nu \nabla^2 (\nabla \varphi + \nabla \times \mathcal{A}).$$

By using the property that the velocity potential φ is a harmonic function, the vortical component \mathcal{A} satisfies the diffusion equation (2.7) and the commutativity of differential operators lead to

$$\nabla \left(\frac{\partial \varphi}{\partial t} + \frac{p}{\rho} + gy \right) = 0.$$

Integration of the last identity yields the so-called linearized Bernoulli equation:

$$\frac{\partial \varphi}{\partial t} + \frac{p}{\rho} + gy = B(t) \quad (2.8)$$

The arbitrary function $B(t)$ (an integration constant) may be removed from the right hand side of (2.8) by the following transformation [16]:

$$\varphi \mapsto \varphi + \int B(s) ds,$$

to give finally the resulting Bernoulli equation:

$$\frac{\partial \varphi}{\partial t} + \frac{p}{\rho} + gy = 0. \quad (2.9)$$

2.1. Dynamic boundary conditions

In constrast to some previous studies which considered the linear case, in this section we will perform the derivations on the fully equations.

The capillary and electric effects come into the problem through the Young– Laplace jump condition at the interface between two media:

$$[\hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}]_1^2 = \sigma \nabla \cdot \hat{\mathbf{n}}, \quad (2.10)$$

where the brackets $[\cdot]$ denote the jump across the interface $y = \eta(x, t)$ and the unit normal $\hat{\mathbf{n}}$ (pointing from the lower fluid to the upper fluid) is given by

$$\hat{\mathbf{n}} = \frac{(-\eta_x, 1)}{\sqrt{1 + \eta_x^2}} \quad (2.11)$$

The stress tensor take on two different forms depending upon which region is under consideration. In region I,

$$T_{ij} = -p\delta_{ij} + \tau_{ij},$$

where p is the pressure as defined above and δ_{ij} is the Kronecker delta symbol. and the tensor τ_{ij} corresponds to the viscosity. The stress tensor in region II is given by

$$T_{ij} = -P\delta_{ij} + \Sigma_{ij},$$

where P is the pressure distribution *on* the interface and Σ_{ij} corresponds to the electric field in region II. Two constitutive components of the viscous and electric stresses are correspondingly:

- $\tau_{ij} = \mu(\partial_j u_i + \partial_i u_j),$
- $\Sigma_{ij} = \varepsilon_p \left(E_i E_j - \frac{1}{2} \delta_{ij} E_k E_k \right),$

where $\mu = \rho\nu$ is the dynamic viscosity and ε_p is the Let us recall that the Young– Laplace condition (2.10) involves the projection of the stresses onto the normal direction $\hat{\mathbf{n}}$ to the interface. In order to avoid cumbersome expressions, we will perform the computations of $\hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{n}}$ by parts. The viscous part of the stress tensor \mathbf{T} gives:

$$\begin{aligned} \hat{\mathbf{n}} \cdot \boldsymbol{\tau} \cdot \hat{\mathbf{n}} &= \hat{n}_i \tau_{ij} \hat{n}_j = \hat{n}_1^2 \tau_{11} + \hat{n}_2^2 \tau_{22} + 2\hat{n}_1 \hat{n}_2 \tau_{12} = \\ &= \frac{\nu}{1 + \eta_x^2} \left[\eta_x^2 (\varphi_{xx} + \psi_{xy}) - 2\eta_x (2\varphi_{xy} + \psi_{yy} - \psi_{xx}) + 2(\varphi_{yy} - \psi_{xy}) \right]. \end{aligned}$$

The same expansion can be also done for the Faraday stress tensor Σ :

$$\hat{\mathbf{n}} \cdot \Sigma \cdot \hat{\mathbf{n}} = \hat{n}_1^2 \Sigma_{11} + \hat{n}_2^2 \Sigma_{22} - 2\hat{n}_1 \hat{n}_2 \Sigma_{12},$$

where

$$\Sigma_{11} = \frac{\varepsilon_p}{2} (V_x^2 - V_y^2), \quad \Sigma_{12} = \varepsilon_p V_x V_y, \quad \Sigma_{22} = -\frac{\varepsilon_p}{2} (V_x^2 - V_y^2).$$

These expressions can be substituted into the Young–Laplace condition (2.10) to give the following expanded form:

$$\begin{aligned} \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} &= [p]_2^1 + \left[\frac{\eta_x^2}{1 + \eta_x^2} \Sigma_{11} \right]_2^1 + 2 \left[\frac{\eta_x}{1 + \eta_x^2} \Sigma_{12} \right]_2^1 + \left[\frac{1}{1 + \eta_x^2} \Sigma_{22} \right]_2^1 \\ &\quad + \frac{\nu}{1 + \eta_x^2} \left(\eta_x^2 [\varphi_{xx} + \psi_{xy}]_2^1 - \eta_x [2\varphi_{xy} + \psi_{yy} - \psi_{xx}]_2^1 + 2[\varphi_{yy} - \psi_{xy}]_2^1 \right) \end{aligned}$$

Since we adopt the free surface approximation, all hydrodynamic quantities vanish in the upper fluid except the external pressure distribution:

$$p_2 = P(x, t).$$

Consequently, the Young–Laplace condition allows us to find the pressure at the interface:

$$\begin{aligned} p_1 &= P(x, t) + \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} - \frac{1}{1 + \eta_x^2} \left[\eta_x^2 \Sigma_{11} + 2\eta_x \Sigma_{12} + \Sigma_{22} \right] \\ &\quad + \frac{\nu}{1 + \eta_x^2} \left(\eta_x^2 (\varphi_{xx} + \psi_{xy}) - \eta_x (2\varphi_{xy} + \psi_{yy} - \psi_{xx}) + 2(\varphi_{yy} - \psi_{xy}) \right) \end{aligned}$$

Inserting the last expression for the pressure p_1 into (2.9) and evaluating at the free surface $y = \eta(x, t)$ leads the following dynamic boundary condition at the free surface:

$$\begin{aligned} \frac{\partial \varphi}{\partial t} + g\eta + \frac{P}{\rho} - \frac{1}{\rho(1 + \eta_x^2)} \left[\eta_x^2 \Sigma_{11} + 2\eta_x \Sigma_{12} + \Sigma_{22} \right] + \frac{\nu}{1 + \eta_x^2} \left(\eta_x^2 (\varphi_{xx} + \psi_{xy}) \right. \\ \left. - \eta_x (2\varphi_{xy} + \psi_{yy} - \psi_{xx}) + 2(\varphi_{yy} - \psi_{xy}) \right) = \frac{\sigma}{\rho} \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}, \quad \text{at } y = \eta(x, t). \end{aligned}$$

The last equation is also referred to in the literature as the Bernoulli condition. Below in Section 2.3 the vortical part ψ will be neglected under two following constitutive assumptions of visco-potential flows:

$$\|\psi\| \ll \|\varphi\| \quad \text{and} \quad \nu \ll 1.$$

2.2. Kinematic boundary condition

The derivation of the corrected kinematic boundary condition in the framework of the visco-potential free surface flows uses the linearized form of the tangential stress conditions:

$$\hat{\mathbf{n}} \cdot \mathbf{T} \cdot \hat{\mathbf{t}} = 0, \quad \text{at } y = \eta(x, t),$$

where $\hat{\mathbf{t}}$ is the Since the electric component of the stress tensor \mathbf{T} is nonlinear, after the linearisation this effect disappears. Consequently, we can transpose directly the results of the visco-potential flow theory [17, 5, 8] into the electro-hydrodynamic context:

$$\frac{\partial \eta}{\partial t} = \frac{\partial \varphi}{\partial y} + 2\nu \frac{\partial^2 \eta}{\partial x^2}.$$

2.3. Fully nonlinear formulation

Now we can write down the set of fully nonlinear equations which govern the motion of the viscous fluid under the action of an exterior electric force:

$$\begin{aligned}
\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} &= 0, & -\infty < y \leq \eta(x, t) \\
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= 0, & \eta(x, t) < y < +\infty \\
\frac{\partial \eta}{\partial t} + \frac{\partial \varphi}{\partial x} \frac{\partial \eta}{\partial x} &= \frac{\partial \varphi}{\partial y} + 2\nu \frac{\partial^2 \eta}{\partial x^2}, & y = \eta(x, t) \\
\frac{\partial \varphi}{\partial t} + g\eta + \frac{P}{\rho} - \frac{1}{\rho(1 + \eta_x^2)} & \left[\eta_x^2 \Sigma_{11} + 2\eta_x \Sigma_{12} + \Sigma_{22} \right] \\
+ \frac{\nu}{1 + \eta_x^2} \left(\eta_x^2 \varphi_{xx} - 2\eta_x \varphi_{xy} + 2\varphi_{yy} \right) &= \frac{\sigma}{\rho} \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}}, & \text{at } y = \eta(x, t). \\
\frac{\partial V}{\partial x} + \frac{\partial \eta}{\partial x} \frac{\partial V}{\partial y} &= 0, & y = \eta(x, t) \\
V &\rightarrow -E_0 y & y \rightarrow +\infty, \\
\frac{\partial \varphi}{\partial y} &\rightarrow 0, & y \rightarrow -\infty
\end{aligned} \tag{2.12}$$

This set of equations will be used below to derive some linearised solutions to the electro-hydrodynamic problem of weakly viscous fluids. A particular case of this problem when the charge is distributed on the interface is discussed in Appendix A.

Remark 1. *The formulation presented above was done in deep water approximation. The generalization to the finite depth case can be done relatively easily. The usual bottom impermeability condition is modified to include a nonlocal term in time which is due to the presence of a boundary layer at the bottom. the details can be found in [17, 8, 7]. For the same reasons, the dispersion relation of the visco-electro-hydrodynamic problem coincides with the classical visco-potential formulation [7, 6].*

2.4. Linear wave theory on a uniform current

We will assume that there is a current flowing with a speed U , and choose the asymptotic expansion as follows:

$$\begin{aligned}
\varphi &= Ux + \varepsilon \varphi_1 + o(\varepsilon) \\
\eta &= \varepsilon \eta_1 + o(\varepsilon) \\
V &= -E_0 y + \varepsilon V_1 + o(\varepsilon) \\
p &= \varepsilon p_1 + o(\varepsilon)
\end{aligned}$$

The linearisation of the system of equations presented in Section 2.3 reads:

$$\begin{aligned}
\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial y^2} &= 0 & -\infty < y \leq 0 \\
\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} &= 0 & 0 \leq y < +\infty \\
U \frac{\partial \eta_1}{\partial x} &= \frac{\partial \varphi_1}{\partial y} + 2\nu \frac{\partial^2 \eta_1}{\partial x^2} & y = 0 \\
U \frac{\partial \varphi_1}{\partial x} + \frac{p_1}{\rho} + g\eta_1 + \frac{\varepsilon_p E_0^2}{\rho} \frac{\partial V_1}{\partial y} - 2\nu \frac{\partial^2 \varphi_1}{\partial y^2} &= \frac{\sigma}{\rho} \frac{\partial^2 \eta_1}{\partial x^2} & y = 0 \\
\frac{\partial V_1}{\partial x} &= E_0 \frac{\partial \eta_1}{\partial x} \\
\frac{\partial \varphi_1}{\partial y} &\rightarrow 0 & y \rightarrow -\infty \\
\frac{\partial V_1}{\partial y} &\rightarrow 0 & y \rightarrow +\infty
\end{aligned} \tag{2.13}$$

The obtained linear system will be understood using the Fourier analysis:

$$\begin{aligned}
\varphi_1(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varphi}_1 e^{ikx} dk, & V_1(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{V}_1 e^{ikx} dk, \\
\eta_1(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\eta}_1 e^{ikx} dk, & p_1(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{p}_1 e^{ikx} dk.
\end{aligned}$$

The solutions for $\varphi_1(x, y)$ and $V_1(x, y)$ can be easily obtained:

$$\varphi_1(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} A(k) e^{|k|y} e^{ikx} dk, \quad V_1(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} B(k) e^{-|k|y} e^{ikx} dk.$$

By satisfying the boundary conditions one can find the unknown functions $A(k)$ and $B(k)$:

$$A(k) = (iU + 2\nu k) \operatorname{sgn}(k) \hat{\eta}_1(k), \quad B(k) = E_0 \hat{\eta}_1(k).$$

Inserting all the elements into equation (2.13) allows us to find the Fourier transform of the free surface elevation:

$$\hat{\eta}_1(k) = \frac{\hat{p}_1(k)}{\rho U^2} \left[k \operatorname{sgn}(k) - \frac{4ik^2 \nu \operatorname{sgn}(k)}{U} + \frac{g}{U^2} - \frac{\varepsilon_p E_0^2}{\rho U^2} k - \frac{4\nu^2 k^3 \operatorname{sgn}(k)}{U^2} + \frac{\sigma}{\rho U^2} k^2 \right]^{-1}.$$

For the sake of illustration, we will take the following pressure distribution at the free surface:

$$p_1(x) = \frac{\rho U^2}{2} e^{-\frac{5g^2 x^2}{U^4}}.$$

The expression for the free surface elevation in the physical space under this pressure distribution is then given by:

$$\eta_1(x) = \frac{U^2}{4g\sqrt{5\pi}} \operatorname{Re} \int_0^\infty \frac{e^{-\frac{k^2 U^4}{20g^2}} e^{ikx}}{k - \frac{4i\nu k^2}{U} - \frac{g}{U^2} + \frac{\varepsilon_p E_0^2 k}{\rho U^2} - \frac{4\nu^2 k^3}{\rho U^2} - \frac{\sigma k^2}{\rho U^2}} dk. \tag{2.14}$$

This expression can be further simplified by choosing new dimensionless variables:

$$l := \frac{U^2}{g}k, \quad \hat{x} := \frac{g}{U^2}x, \quad \zeta := \frac{g}{U^2}\eta_1.$$

With these new variables the solution (2.14) becomes:

$$\zeta(\hat{x}) = \frac{1}{4\sqrt{5}\pi} \operatorname{Re} \int_0^\infty \frac{e^{-\frac{l^2}{20}} e^{i\hat{x}l}}{(1+\beta)l - 1 - i\delta l^2 - \gamma l^3 - \alpha l^2} dl, \quad (2.15)$$

where

$$\alpha := \frac{\sigma g}{\rho U^4}, \quad \beta := \frac{\varepsilon_p E_0^2}{\rho U^2}, \quad \gamma := \frac{4\nu^2 g^2}{\rho U^6}, \quad \delta := \frac{4\nu g}{U^3}.$$

For instance, we can notice that the coefficient $\gamma = \mathcal{O}(\delta^2) = \mathcal{O}(\nu^2) \ll \nu \ll 1$ and, thus, can be neglected. The integrand in (2.15) is complex. The complex part plays the role of the Rayleigh dispersion as mentioned in [27]. From analytical point of view, the complex terms remove the singularity. Consequently, formula (2.15) will always define a valid solution in L_2 thanks to this complex denominator. It is expected also that $\zeta(\hat{x}) \rightarrow 0$ as $\hat{x} \rightarrow \pm\infty$.

3. Results

In order to illustrate the analytical result derived in the previous Section, we plot on Figure 2 the free surface elevation shape predicted analytically by solution (2.15). The parameters were chosen to be $\alpha = 0.3$, $\beta = 0.15$ and $\delta = 0.01$. The profile for these base values is given in Figure 2.

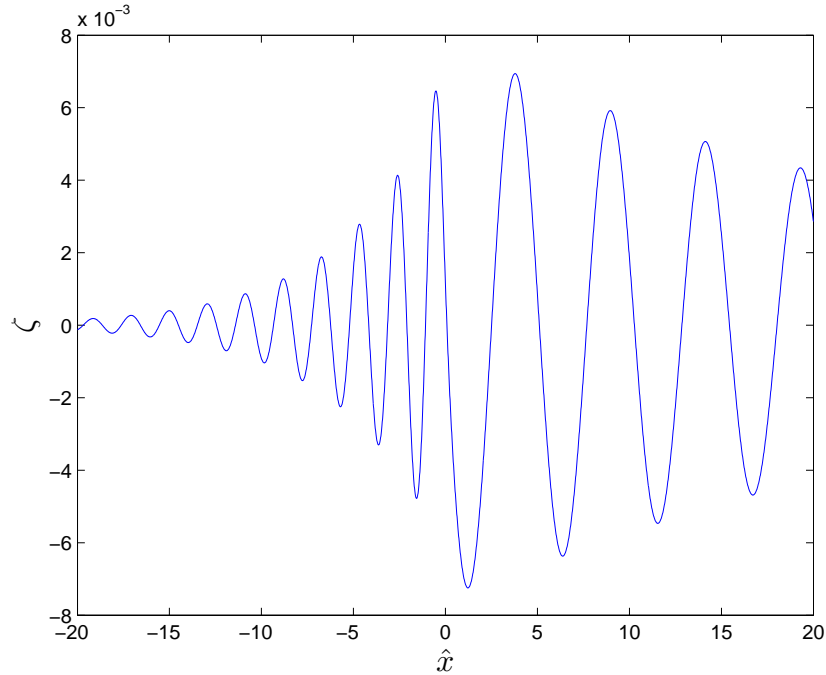


Figure 2. Typical free surface profile predicted by solution (2.15).

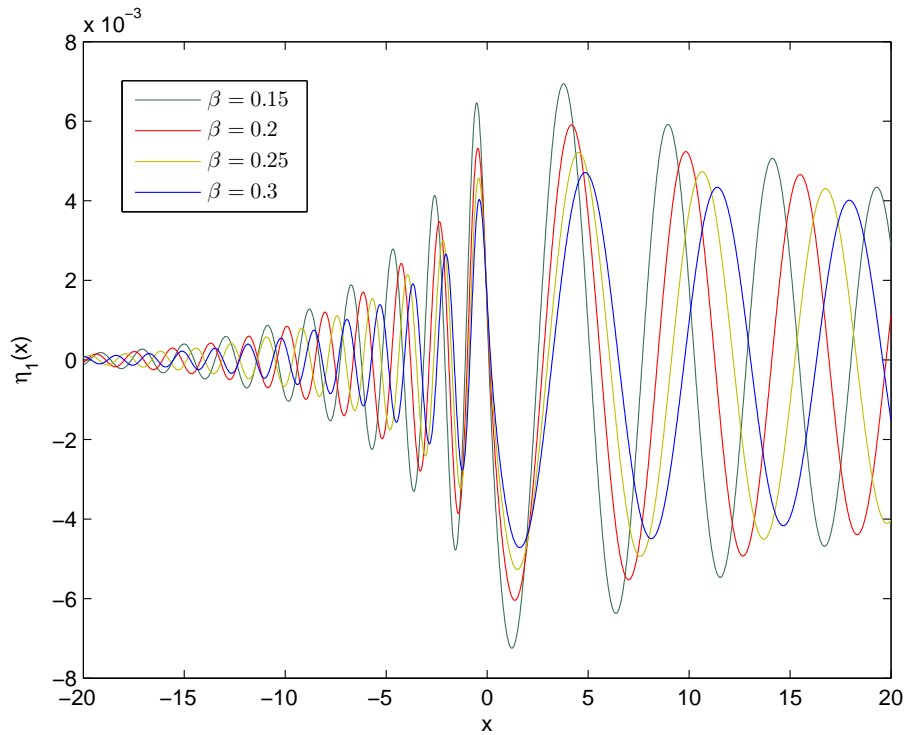


Figure 3. Free surface elevation shapes for various values of the parameter β .

In order to illustrate how the free surface elevation depends on the parameter β (which measures the relative importance of the electric force to the inertia), we represent on Figure 3 the same solution (2.15) for several values of β (all other parameters being fixed). The same result is plotted on Figure 4 also for the parameter δ , which measures the relative magnitude of dissipative effects. The increase in parameters leads the slight decrease of the amplitude, as it can be expected from the analytical solution. On the other hand, Figure 3 shows that the wavelength increases with the magnitude of the electric field $\beta \sim E_0^2$.

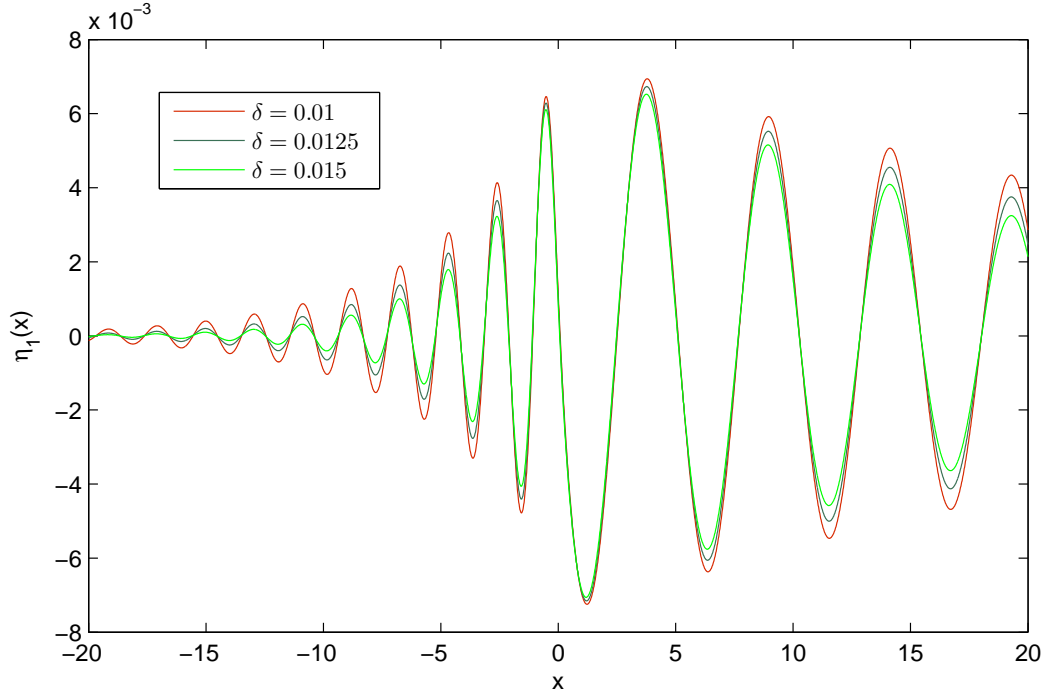


Figure 4. Free surface elevation shapes for various values of the parameter δ .

4. Conclusions & Perspectives

In this study we considered the problem of a free surface flow description in the presence of electric forces. Moreover, the fluid is assumed to be weakly viscous. By using the visco-potential flow theory, the classical potential formulation was modified to take into account weak dissipative effects. The derivations presented in this study were performed in the deep water approximation. However, the generalization to the finite depth case does not represent any major difficulties.

The free surface shape in the presence of a uniform current and a localized interfacial pressure distribution was computed analytically. The dependence of this shape on two important dimensionless parameters was studied numerically as well.

This study opens a certain number of directions for future investigations. For instance, the present analysis was only linear. Nonlinearities have to be taken into account as well as the finite depth effects. Moreover, long wave asymptotics can be also performed to investigate the shallow water regime.

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A. Electric charge at the interface

This section details the analysis of the problem where there is an interfacial surface charge. In the inviscid case, as there were no shearing forces, there could be no dynamic charge on the surface and it was only possible to compute an *induced* charge on the interface given by:

$$\Sigma_Q = \epsilon_p \frac{\partial V}{\partial n}$$

Then the definition leads to the same result as in the inviscid case, that the induced surface charge is just the Hilbert transform of the derivative of the free surface profile. For the *dynamics* surface charge equation (2.12) is replaced with two following equations [10]:

$$\hat{\mathbf{n}} \cdot [\epsilon_p \mathbf{E}]_1^2 = q,$$

$$\frac{\partial q}{\partial t} + \nabla_T \cdot (q\mathbf{u}) + \hat{\mathbf{n}} [\sigma_q \mathbf{E}]_1^2 = 0.$$

The operator ∇_T is the covariant derivative defined as

$$\nabla_T = \nabla - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \nabla).$$

Below we will perform the linear analysis of this problem as well. The expansions in this case are given by:

$$\begin{aligned} \varphi &= Ux + \varepsilon\varphi_1 + o(\varepsilon) \\ \eta &= \varepsilon\eta_1 + o(\varepsilon) \\ V &= -E_0x + \varepsilon V_1 + o(\varepsilon) \\ q &= \varepsilon q_1 + o(\varepsilon) \\ p &= \varepsilon p_1 \end{aligned}$$

The covariant derivative on the free surface with the normal defined by equation (2.11) is expressed as

$$\nabla_T = \left(\frac{\partial}{\partial x}, 0 \right) + \varepsilon \left(\frac{\partial \eta_1}{\partial x} \frac{\partial}{\partial y}, \frac{\partial \eta_1}{\partial x} \frac{\partial}{\partial x} \right) + o(\varepsilon)$$

The equations reduce to the following [10]:

$$\begin{aligned} U \frac{\partial q_1}{\partial x} + \sigma_q E_0 \frac{\partial \eta_1}{\partial x} + \sigma_q \frac{\partial V_1}{\partial y} &= 0 \\ -\varepsilon_p E_0 \frac{\partial \eta_1}{\partial x} + \varepsilon_p \frac{\partial V_1}{\partial y} &= q_1 \end{aligned}$$

Two last equations can be combined into one:

$$-\varepsilon_p E_0 U \frac{\partial^2 \eta_1}{\partial x^2} + U \varepsilon_p \frac{\partial^2 V_1}{\partial x \partial y} + \sigma_q E_0 \frac{\partial \eta_1}{\partial x} + \sigma_q \frac{\partial V_1}{\partial y} = 0.$$

The rest of linearised equations in this case is exactly the same as in the previous section. The analytical expression for the linearized free surface elevation in the Fourier space can be derived in a similar way:

$$\hat{\eta}_1(k) = \left[k \operatorname{sgn}(k) - \frac{4i\nu k^2 \operatorname{sgn}(k)}{U} - \frac{g}{U^2} + \frac{\varepsilon_p E_0^2}{\rho U^2} k f(k) - \frac{\sigma}{\rho U^2} k^2 \right]^{-1} \frac{\hat{p}_1(k)}{\rho U^2}$$

Then, the free surface elevation in the physical space can be easily obtained:

$$\eta_1(x) = \frac{1}{2\pi\rho U^2} \int_{\mathbb{R}} \frac{\hat{p}_1(k)e^{ikx}}{k \operatorname{sgn}(k) - \frac{4i\nu k^2 \operatorname{sgn}(k)}{U} - \frac{g}{U^2} + \frac{\varepsilon_p E_0^2}{\rho U^2} k f(k) - \frac{\sigma}{\rho U^2} k^2} dk, \quad f(k) = \frac{\varepsilon_p U k + i\sigma_q}{\sigma_q + ikU\varepsilon_p}$$

Using the same scalings and pressure form as before, the integral can be reduced to the following dimensionless form:

$$\zeta(\hat{x}) = \frac{1}{4\sqrt{5}\pi} \operatorname{Re} \int_0^\infty \frac{e^{-\frac{l^2}{20}} e^{i\hat{x}l}}{l - i\delta l^2 - 1 + \beta l \tilde{f}(l) - \alpha l^2} dl,$$

where

$$\tilde{f} = \frac{\mu l + i}{i\mu l + 1}, \quad \mu = \frac{\varepsilon_p g}{U\sigma_q},$$

and coefficients α , β and δ are defined as in the previous section.

References

- [1] J. Boussinesq. Lois de l'extinction de la houle en haute mer. *C. R. Acad. Sci. Paris*, 121:15–20, 1895. [2](#)
- [2] J.-P. Chehab and G. Sadaka. Numerical study of a family of dissipative KdV equations. *Communications on Pure and Applied Analysis*, 12(1):519–546, Sept. 2013. [2](#)
- [3] M. Chen, S. Dumont, L. Dupaigne, and O. Goubet. Decay of solutions to a water wave model with a nonlocal viscous dispersive term. *Discrete Contin. Dyn. Syst.*, 27(4):1473–1492, 2010. [2](#)
- [4] M. Chen and O. Goubet. Long-time asymptotic behaviour of dissipative Boussinesq systems. *Discrete and Continuous Dynamical Systems*, 17:61–80, 2007. [2](#)
- [5] F. Dias, A. I. Dyachenko, and V. E. Zakharov. Theory of weakly damped free-surface flows: a new formulation based on potential flow solutions. *Physics Letters A*, 372:1297–1302, 2008. [2](#), [4](#), [6](#)
- [6] D. Dutykh. Group and phase velocities in the free-surface visco-potential flow: new kind of boundary layer induced instability. *Physics Letters A*, 373:3212–3216, 2009. [7](#)
- [7] D. Dutykh. Visco-potential free-surface flows and long wave modelling. *Eur. J. Mech. B/Fluids*, 28:430–443, 2009. [2](#), [7](#)
- [8] D. Dutykh and F. Dias. Viscous potential free-surface flows in a fluid layer of finite depth. *C. R. Acad. Sci. Paris, Ser. I*, 345:113–118, 2007. [2](#), [4](#), [6](#), [7](#)
- [9] O. Goubet and G. Warnault. Decay of solutions to a linear viscous asymptotic model for water waves. *Chinese Annals of Mathematics, Series B*, 31(6):841–854, Oct. 2010. [2](#)
- [10] P. W. Hammerton and A. P. Bassom. The effect of surface stress on interfacial solitary wave propagation. *Q. J. Mechanics Appl. Math.*, 66(3):395–416, July 2013. [12](#)
- [11] M. Hunt. *Linear and Nonlinear Free Surface Flows in Electrohydrodynamics*. Phd, University College London, 2013. [2](#)
- [12] D. D. Joseph. Potential Flow of Viscous Fluids: Historical Notes. *Int. J. Multiphase Flow*, 32:285–310, 2006. [2](#)
- [13] D. D. Joseph and T. Y. Liao. Potential flows of viscous and viscoelastic fluids. *J. Fluid Mech.*, 265:1–23, 1994. [2](#)
- [14] T. Kakutani and K. Matsuuchi. Effect of viscosity on long gravity waves. *J. Phys. Soc. Japan*, 39:237–246, 1975. [2](#)
- [15] E. Kit and L. Shemer. On Dissipation Coefficients in a Rectangular Wave Tank. *Acta Mechanica*, 77:171–180, 1989. [2](#)
- [16] H. Lamb. *Hydrodynamics*. Cambridge University Press, 1932. [4](#)
- [17] P.-F. Liu and A. Orfila. Viscous effects on transient long-wave propagation. *J. Fluid Mech.*, 520:83–92, 2004. [2](#), [6](#), [7](#)

- [18] P.-F. Liu, G. Simarro, J. Vandever, and A. Orfila. Experimental and numerical investigation of viscous effects on solitary wave propagation in a wave tank. *Coastal Engineering*, 53:181–190, 2006. [2](#)
- [19] E. Ott and R. N. Sudan. Damping of Solitary Waves. *Phys. Fluids*, 13:1432–1434, 1970. [2](#)
- [20] D. T. Papageorgiou and J.-M. Vanden-Broeck. Large-amplitude capillary waves in electrified fluid sheets. *J. Fluid Mech.*, 508:71–88, June 2004. [2](#)
- [21] G. Sadaka and J.-P. Chehab. On damping rates of dissipative KdV equations. *Discrete and Continuous Dynamical Systems - Series S*, 6(6):1487–1506, Apr. 2013. [2](#)
- [22] G. I. Taylor and A. D. McEwan. The stability of a horizontal fluid interface in a vertical electric field. *J. Fluid Mech.*, 22:1–15, Mar. 1965. [2](#)
- [23] D. Tseluiko, M. G. Blyth, D. T. Papageorgiou, and J.-M. Vanden-Broeck. Electrified viscous thin film flow over topography. *J. Fluid Mech.*, 597:449–475, Feb. 2008. [2](#)
- [24] D. Tseluiko, M. G. Blyth, D. T. Papageorgiou, and J.-M. Vanden-Broeck. Viscous Electrified Film Flow over Step Topography. *SIAM J. Appl. Math.*, 70(3):845–865, Jan. 2009. [2](#)
- [25] D. Tseluiko and D. T. Papageorgiou. Wave evolution on electrified falling films. *J. Fluid Mech.*, 556:361–386, May 2006. [2](#)
- [26] D. Tseluiko and D. T. Papageorgiou. Nonlinear Dynamics of Electrified Thin Liquid Films. *SIAM J. Appl. Math.*, 67(5):1310–1329, Jan. 2007. [2](#)
- [27] J.-M. Vanden-Broeck. *Gravity-Capillary Free-Surface Flows*. Cambridge University Press, Cambridge, 2010. [2](#), [9](#)

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