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## To cite this version:

Pierre Andreoletti, Alexis Devulder. Localization and number of visited valleys for a transient diffusion in random environment. 2013. hal-00908626v1

HAL Id: hal-00908626
https://hal.science/hal-00908626v1
Preprint submitted on 25 Nov 2013 (v1), last revised 9 Mar 2015 (v3)

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# LOCALIZATION AND NUMBER OF VISITED VALLEYS FOR A TRANSIENT DIFFUSION IN RANDOM ENVIRONMENT 

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#### Abstract

We consider a transient diffusion in a ( $-\kappa / 2$ )-drifted Brownian potential $W_{\kappa}$ with $0<\kappa<1$. We prove its localization before time $t$ in an a neighborhood of some random points depending only on the environment, which are the positive $h_{t}$-minima of the environment, for $h_{t}$ a bit smaller than $\log t$. We also prove an Aging phenomenon for the diffusion, and provide a central limit theorem for the number of valleys visited up to time $t$.

The proof relies on a Williams' decomposition of the trajectory of $W_{\kappa}$ in the neighborhood of local minima, with the help of results of Faggionato [19], and on a precise analysis of exponential functionals of $W_{\kappa}$ and of 3-dimensional ( $-\kappa / 2$ )-drifted Bessel processes.


## 1. Introduction and notation

1.1. Presentation of the model. We are interested in a diffusion $(X(t), t \geq 0)$ in a random càdlàg potential $(V(x), x \in \mathbb{R})$, defined informally by $X(0)=0$ and

$$
\mathrm{d} X(t)=d \beta(t)-\frac{1}{2} V^{\prime}(X(t)) \mathrm{d} t
$$

where $\beta$ is a Brownian motion independent of $V$. More rigorously, $X$ is defined by its conditional generator given $V$, which is

$$
\frac{1}{2} e^{V(x)} \frac{\mathrm{d}}{\mathrm{~d} x}\left(e^{-V(x)} \frac{\mathrm{d}}{\mathrm{~d} x}\right)
$$

These diffusions in random potentials are considered as a continuous time analogues of random walks in random environment (RWRE) (see e.g. P. Révész, [31], B.D. Hughes [22], Z. Shi [35] and O. Zeitouni [42] for reviews on RWRE).

The study of such a process starts with a choice for $V$. A classic one, originally introduced by S. Schumacher [33] and T. Brox [5], is to take for $V$ a Lévy process. In fact only a few papers deal with the discontinuous case, see for example P. Carmona [6] or A. Singh [36, 37], and most of the results concern continuous $V$, that is to say

$$
V(x)=W_{\kappa}(x):=W(x)-\frac{\kappa}{2} x, \quad x \in \mathbb{R},
$$

with $\kappa \in \mathbb{R}$ and $W$ a two sided Brownian motion. We denote by $P$ the probability measure associated to $W_{\kappa}($.$) . The probability conditionally on the potential W_{\kappa}$ is denoted by $\mathbb{P}^{W_{\kappa}}$ and is called the quenched probability. We also define the annealed probability as

$$
\mathbb{P}(.):=\int \mathbb{P}^{W_{\kappa}}(.) P\left(W_{\kappa} \in \mathrm{d} \omega\right) .
$$

We denote respectively by $\mathbb{E}^{W_{\kappa}}, \mathbb{E}$, and $E$ the expectancies with regard to $\mathbb{P}^{W_{\kappa}}, \mathbb{P}$ and $P$.

[^0]In the case $\kappa=0, X$ is recurrent and [5] shows that it is sub-diffusive with asymptotic behavior in $(\log t)^{2}$, moreover $X$ is localized, at time $t$, in the neighborhood of a random point $b_{\log t}$ depending only on $t$ and $W$. This result can be written:
Theorem 1.1. (Brox [5]) Assume $\kappa=0$, then for all $\varepsilon>0$

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \mathbb{P}\left[X(t) \in\left[b_{\log t}-\varepsilon(\log t)^{2}, b_{\log t}+\varepsilon(\log t)^{2}\right]\right]=1 . \tag{1.1}
\end{equation*}
$$

The limit law of $b_{\log t} /(\log t)^{2}$ and therefore of $X(t) /(\log t)^{2}$ were made explicit independently by H. Kesten [25] and A. O. Golosov [23]. For recent results for this case see for example [1] and [11].

In the case $\kappa>0$, the diffusion $X$ is a.s. transient, with a wide range of limiting behaviors, depending on the value of $\kappa$. It was first studied by K. Kawazu and H. Tanaka. Let us denote by $H(r)$ the hitting time of $r \in \mathbb{R}$ by $X$ :

$$
H(r):=\inf \{t>0, X(t)=r\} .
$$

Kawazu et al. [24] proved in particular that when $0<\kappa<1, H(r) / r^{1 / \kappa}$ converges in law to a stable distribution (see also Y. Hu et al. [21], and H. Tanaka [40]). More recently we mention the results for large and moderate deviations, by M. Taleb ([38] and [39]), A. Devulder [8] and G. Faraud [18].

In this paper we follow a different approach from [21] and [24]. Indeed we focus on a quenched study, which has attracted much interest for transient RWRE in the last few years, see for example the works of N. Enriquez et al. [14], [15], [16], [17], D. Dolgopyat et al. [12], and J. Peterson et al. [27], [28], [29]. Heuristically, the diffusion goes to locations where the potential is low, hence it goes to $+\infty$, but it is slowed by "valleys" of the potential, which trap the diffusion for some time.
1.2. Main results. The goals of this paper are to localize the diffusion $X$, when $0<\kappa<1$, in some valleys of the potential $W_{\kappa}$, to understand the difference with Brox's result given by (1.1), and to prove an Aging phenomenon, as was done in [16] for transient zero-speed RWRE. We moreover obtain a central limit theorem for the number of valleys visited up to time $t$. We also prove some intermediate results, which we think will be useful for obtaining new results about the maximum local time of $X$.

Let $t \mapsto \phi(t)$ a positive increasing function of $t$, such that $\phi(t)=o(\log t)$ and $\log \log t=o(\phi(t))$, where $f(t)=o(g(t))$ means $\lim _{t \rightarrow+\infty} f(t) / g(t)=0$. We prove the following aging phenomenon:

Proposition 1.2. Assume $0<\kappa<1$. For all $\alpha>1$, we have

$$
\lim _{t \rightarrow+\infty} \mathbb{P}(|X(\alpha t)-X(t)| \leq \phi(t))=\frac{\sin (\kappa \pi)}{\pi} \int_{0}^{1 / \alpha} u^{\kappa-1}(1-u)^{-\kappa} \mathrm{d} u
$$

This is actually a consequence of Theorem 1.3. Before stating it, we first introduce the notion of $h$-extrema, which were first introduced by Neveu et al. [26], and studied in the case of drifted Brownian motion by Faggionato [19]. For $h>0$, we say that $x \in \mathbb{R}$ is an $h$-minimum for a given process $V$ if there exist $u<x<v$ such that $V(y) \geq V(x)$ for all $y \in[u, v], V(u) \geq V(x)+h$ and $V(v) \geq V(x)+h$. Moreover, $x$ is an $h$-maximum for $V$ if $x$ is an $h$-minimum for $-V$, and $x$ is an $h$-extrema for $V$ iff it is an $h$-maximum or an $h$-minimum for $V$.
As we are studying the process $X$ until time $t$, we are more especially interested in the $h_{t}$-extrema of $W_{\kappa}$ where

$$
h_{t}:=\log t-\phi(t) .
$$

It is known (see [19]) that almost surely, the $h_{t}$-extrema of $W_{\kappa}$ form a sequence indexed by $\mathbb{Z}$, unbounded from below and above, and that the $h_{t}$-minima and $h_{t}$-maxima alternate. We denote respectively by $\left(m_{j}, j \in \mathbb{Z}\right)$ and $\left(M_{j}, j \in \mathbb{Z}\right)$ the increasing sequences of $h_{t}$-minima and of $h_{t}$-maxima of $W_{\kappa}$, such that $m_{0} \leq 0<m_{1}$ and $m_{j}<M_{j}<m_{j+1}$ for every $j \in \mathbb{Z}$.
We also define

$$
N_{t}:=\max \left\{k \in \mathbb{N}, \sup _{0 \leq s \leq t} X(s) \geq m_{k}\right\}
$$

so that $m_{N_{t}}$ is the largest $h_{t}$-minima visited by $X$ until time $t$. The main result of this paper concerns the localization of the diffusion. It is stated as follows:

Theorem 1.3. Assume $0<\kappa<1$. There exists a constant $\mathcal{C}_{1}>0$, such that

$$
\lim _{t \rightarrow+\infty} \mathbb{P}\left(\left|X(t)-m_{N_{t}}\right| \leq \mathcal{C}_{1} \phi(t)\right)=1
$$

We first recall that $X(t)$ is asymptotically of order $t^{\kappa}$ (see e.g. [21]). So, the size $\phi(t)$ of the intervals in which $X$ is localized, which can nearly be of the order of $\log \log t$, is small and is related to the minimum height $h_{t}$ of our valleys. We could not say however if it the best that can be obtained. The main difference with the result of Brox (1.1) is the appearance of the (random) integer $N_{t}$, which is the number of typical valleys of height $h_{t}$ visited before time $t$. In the recurrent case of Brox, the diffusion is, with a large probability, localized near the bottom of a unique valley of the potential, whereas in our transient case, the diffusion is localized near the bottom of one among several valleys of the potential.

We also prove a renewal theorem for the time to reach the last valley visited by $X$ before $t$ :
Proposition 1.4. We have the following convergence in law under the annealed probability $\mathbb{P}$, as $t \rightarrow+\infty$,

$$
\left(H\left(m_{N_{t}}\right) / t, H\left(m_{N_{t}+1}\right) / t\right) \xrightarrow{\mathcal{L}}[\kappa \sin (\pi \kappa) / \pi](y-x)^{-\kappa-1} x^{\kappa-1} \mathbb{1}_{[0,1]}(x) \mathbb{1}_{[1, \infty)}(y) \mathrm{d} x \mathrm{~d} y .
$$

As a consequence, we get the following results, which are useful for the proofs of Proposition 1.2 and Theorem 1.3:

Corollary 1.5. Assume $0<\kappa<1$ and let $0 \leq r<s \leq 1$. Then,

$$
\begin{align*}
\lim _{t \rightarrow+\infty} \mathbb{P}\left(1-s \leq \frac{H\left(m_{N_{t}}\right)}{t} \leq 1-r\right) & =\frac{\sin (\pi \kappa)}{\pi} \int_{1-s}^{1-r} x^{\kappa-1}(1-x)^{-\kappa} \mathrm{d} x  \tag{1.2}\\
\lim _{t \rightarrow+\infty} \mathbb{P}\left(H\left(m_{N_{t}+1}\right) / t \geq 1+s\right) & =\frac{\sin (\pi \kappa)}{\pi} \int_{s}^{+\infty}(1+x)^{-1} x^{-\kappa} \mathrm{d} x \tag{1.3}
\end{align*}
$$

Moreover, the total time spent in the last valley of height at least $h_{t}$ visited before time $t$ renormalized by $t$, that is $\left[H\left(m_{N_{t}+1}\right)-H\left(m_{N_{t}}\right)\right] / t$ converges in law under $\mathbb{P}$ to a r.v. with density $\sin (\pi \kappa) \pi^{-1} x^{-\kappa-1}\left[\left(1-(1-x)^{\kappa}\right) \mathbb{1}_{[0,1]}(x)+\mathbb{1}_{(1,+\infty)}(x)\right]$.

We will see in Section 4 and 5 , that this is due to the fact that for any integer $k \leq n_{t}$ with

$$
n_{t}:=\left\lfloor e^{\kappa \phi(t)(1+\delta)}\right\rfloor, \delta>0,
$$

$H\left(m_{k}\right)$ can approximated by a sum of i.i.d. random variables each of these random variables having the law of $\mathbf{U}$. We show that $\mathbf{U}$ is the product of a random variable depending only on the environment and $t$, and an independent variable with exponential law of parameter 2. This first random variable can itself be approximated by a product of sums of functionals of 3-dimensional $(-\kappa / 2)$ drifted Bessel processes and of $W_{\kappa}$ (see Proposition 4.3).

These results are in accordance with those obtained by Enriquez et al. ([14], [15] and [16]) for transient RWRE. The work we present here is self contained, in particular we present in this same
paper the technical study of the Laplace transform of the first exit time $\mathbf{U}$. The study of the environment only requires continuous arguments of stochastic calculus, starting by a Williams decomposition of the trajectory of $W_{\kappa}$ which mainly comes from the work of A. Faggionato [19].

The number $N_{t}$ of valleys visited goes to $+\infty$ as $t \rightarrow+\infty$, More precisely, we prove the following central limit theorem for $N_{t}$, with renormalization $e^{\kappa \phi(t)}$ :
Proposition 1.6. Assume $0<\kappa<1$. Then $N_{t} e^{-\kappa \phi(t)} \rightarrow_{t \rightarrow+\infty} \mathcal{N}$ in law under the annealed law $\mathbb{P}$, where $\mathcal{N}$ is a r.v. determined by its Laplace transform:

$$
\begin{equation*}
\forall u>0, \quad \mathbb{E}\left(e^{-u \mathcal{N}}\right)=\sum_{j=0}^{+\infty} \frac{1}{\Gamma(\kappa j+1)}\left(\frac{-u}{C_{\kappa}}\right)^{j} \tag{1.4}
\end{equation*}
$$

where $C_{\kappa}>0$ is explicitly known (see Proposition 4.7).
Moreover we expect that the results of this paper will be useful to study other properties for the diffusion. In particular, let ( $\left.\mathcal{L}_{X}(t, x), t \geq 0, x \in \mathbb{R}\right)$ be a bicontinuous version of the local time of $X$. It is known that the maximum local time of $X$ at time $t$, that is $\mathcal{L}_{X}^{*}(t):=\max _{x \in \mathbb{R}} \mathcal{L}_{X}(t, x)$, satisfies $\lim \sup _{t \rightarrow+\infty} \mathcal{L}_{X}^{*}(t) / t=+\infty$ a.s. in the cases $\kappa=0$ (see [34] and [11]) and even in the transient case $0<\kappa<1$ (see [10]). Hence the maximum local times of $X$ exhibits very interesting properties, very different from those of the maximum local time of RWRE at time $t$, which is naturally bounded by $t / 2$. We expect that the better understanding of the localization of $X$ and some intermediate results provided in this paper will be useful to prove new results about $\mathcal{L}_{X}^{*}$ (work in progress).

The rest of the paper is organized as follows. First, we give in Section 2 the main properties of the environment that will be useful for our study; in particular we present a Williams' decomposition of the trajectory of $W_{\kappa}$, close to the one detailed in [19]. Then, in Section 3, we approximate the trajectory of $X$ by a sum of i.i.d. random variables, and we study one of these random variable in Section 4. We study the asymptotic behavior of the Laplace transform of the first exit time $\mathbf{U}$ in Section 4.3. Finally, we prove the renewal results stated in Proposition 1.4, Theorem 1.3 and Proposition 1.2 in Section 5. Moreover, Sections 2, 3 and 4 start with basic facts on diffusion in random media and/or estimates on the drifted brownian motion and Bessel processes.

## 2. Williams' decomposition and Standard valleys

We use a Williams' like decomposition ([41]), based on the results of A. Faggionato [19].
2.1. Williams' decomposition in the neighborhood of $h_{t}$-minima, $\left(m_{i}, i\right)$. We now recall Williams' decomposition of the trajectory in a neighborhood of the local minima $m_{i}, i \in \mathbb{N}^{*}$.
Let $a>0$. For any process $\left(U(t), \quad t \in \mathbb{R}^{+}\right)$we denote by

$$
\tau^{U}(a):=\inf \{t>0, U(t)=a\},
$$

the first time this process hit $a$, with the convention $\inf \emptyset=+\infty$. We denote by $\mathcal{L}_{U}$ a bicontinuous version of the local time of $U$ when it exists. We also denote by $U^{a}$ the process $U$ starting from $a$, and by $P^{a}$ the law of $U^{a}$; with the notation $U=U^{0}$.

Definition 2.1. We recall the definition of a $(-\zeta / 2)$-drifted Brownian motion $W_{\zeta}$ Doob- conditioned to stay positive (see [3], Chapter VII.3), where $\zeta \neq 0$. For $z>0, P_{z}^{-\zeta / 2, \uparrow}$ is defined by $P_{z}^{-\zeta / 2, \uparrow}(\Lambda):=\frac{1}{1-e^{-\zeta z}} E^{z}\left[\left(1-\exp \left(-\zeta W_{\zeta}^{z}(t)\right)\right), \Lambda, t<\tau^{W_{\zeta}^{z}}(0)\right]$ for any $\Lambda \in \sigma\left(W_{\zeta}(u), 0 \leq u \leq\right.$ $t)=: \mathcal{F}_{t}, t \geq 0$, where $E^{z}$ and $P^{z}$ are the expectancy and probability related to $W_{\zeta}^{z}$. This induces
a unique probability measure $P_{z}^{-\zeta / 2, \uparrow}$ on $\sigma\left(W_{\zeta}(u), u \geq 0\right)$. Moreover, $P_{z}^{-\zeta / 2, \uparrow}$ converges weakly as $z \rightarrow 0^{+}$, in the space of Skorokhod $D\left(\mathbb{R}_{+}, \mathbb{R}_{+}\right)$(see [3] Prop. 14) and in $C\left(\mathbb{R}_{+}, \mathbb{R}_{+}\right)$(see [19]) to a probability measure denoted by $P_{0}^{-\zeta / 2, \uparrow}$. The canonical process, which we denote by $R$, is a Feller process for the family $\left(P_{z}^{-\zeta / 2, \uparrow}, z \geq 0\right)$; it takes values in $\mathbb{R}_{+}$, and its infinitesimal generator is given for every $x>0$ by (see [19] Lemma 6)

$$
\begin{equation*}
\frac{1}{2} \frac{d^{2}}{d x^{2}}+\frac{\zeta}{2} \operatorname{coth}\left(\frac{\zeta}{2} x\right) \frac{d}{d x} \tag{2.1}
\end{equation*}
$$

In the following, we call $R$ under $P_{z}^{-\zeta / 2, \uparrow}$ for $z \geq 0$ a 3 dimensional ( $-\zeta / 2$ )-drifted Bessel process starting from $z$. Of course this is a misuse of language as this process is not drifted directly but is obtained from a drifted process. We notice in particular that, by (2.1), the law of $R$ is the same if $\zeta$ is replaced by $-\zeta$, that is, 3 dimensional $(-\kappa / 2)-$ drifted Bessel processes have the same law as 3 dimensional ( $\kappa / 2$ )-drifted Bessel processes. Finally, when $\zeta<0$, we have $P_{z}^{-\zeta / 2, \uparrow}(\Lambda)=P^{z}\left(\Lambda \mid \tau^{W_{亏}^{z}}(0)=\infty\right)$ for every $\Lambda \in \mathcal{F}_{t}, t \geq 0$.

Let

$$
V^{(i)}(.):=W_{\kappa}(.)-W_{\kappa}\left(m_{i}\right), i \in \mathbb{N}^{*},
$$

which is the potential re-centered at the local minima $m_{i}$. We also define for $h>0$

$$
\tau_{i}^{-}(h):=\sup \left\{s<m_{i}, V^{(i)}(x)=h\right\}, \quad \tau_{i}(h):=\inf \left\{s>m_{i}, V^{(i)}(x)=h\right\} .
$$

We have,
Fact 2.2. (William's decomposition)
For any $i \in \mathbb{N}^{*}$, let $P_{1}^{(i)}$ be the truncated process $\left(V^{(i)}\left(m_{i}-s\right), 0 \leq s \leq m_{i}-\tau_{i}^{-}\left(h_{t}\right)\right), P_{2}{ }^{(i)}$ the truncated process $\left(V^{(i)}\left(m_{i}+s\right), 0 \leq s \leq \tau_{i}\left(h_{t}\right)-m_{i}\right)$ and $P_{3}^{(i)}$ the truncated process $\left(V^{(i)}(s+\right.$ $\left.\left.\tau_{i}\left(h_{t}\right)\right), 0 \leq s \leq \inf \left\{s>\tau_{i}\left(h_{t}\right), V^{(i)}(s)=a\right\}-\tau_{i}\left(h_{t}\right)\right)$, for $a<h_{t}$. Then the truncated processes $\left(\left(P_{1}^{(i)}, P_{2}^{(i)}\right), i \geq 1\right)$ are independent, and for a given $i, P_{1}^{(i)}, P_{2}^{(i)}$ and $P_{3}^{(i)}$ are independent as well.
Let us denote $(R, h)$ the killed process $\left(R(s), 0 \leq s \leq \tau^{R}(h)\right)$. Let $\left(R^{(1)}, h_{t}\right)$ and $\left(R^{(2)}, h_{t}\right)$ two independent copies of $\left(R, \tau\left(h_{t}\right)\right)$ and $\left(\tilde{W}_{\kappa}^{b}, a\right):=\left(\tilde{W}_{\kappa}^{b}(s), 0 \leq s \leq \tau^{\tilde{W}_{\kappa}}(a)\right) a(-\kappa / 2)$-drifted Brownian motion starting from $b$ and killed when it first hits $a<b$, and independent of $R^{(1)}$ and $R^{(2)}$. Then for all $i \geq 2, P_{1}^{(i)}$ is equal in law to $\left(R^{(1)}, h_{t}\right)$; for all $i \geq 1, P_{2}^{(i)}$ is equal in law to $\left(R^{(2)}, h_{t}\right)$, and $P_{3}^{(i)}$ is equal in law to $\left(\tilde{W}_{\kappa}^{h_{t}}, a\right)$, for $a<h_{t}$.

The results for the sequence $\left(\left(P_{1}^{(i)}, P_{2}^{(i)}\right), i \geq 1\right)$, comes from Theorems 1 and 2 in [19]. The result for $P_{3}^{(i)}$ comes from the fact that $\tau_{i}\left(h_{t}\right)$ is a stopping time. We treat the central slope that is to say $P_{1}^{(1)}$ in Remark 2.4.
2.2. Standard $h_{t}$-minima $\left(\tilde{m}_{i}, i\right)$. Among the $h_{t}$-minima $\left(m_{i}, i\right)$ only some of them are interesting for the analysis of the process $X$ : the ( $\left.\tilde{m}_{i}, i\right) h_{t}$-minima. Recall that $\delta>0$ is a positive real that can be chosen as small as needed (see the definition of $n_{t}$ just after Corollary 1.5). Let

$$
h_{t}^{+}:=(1+\kappa+2 \delta) h_{t} .
$$

We define $\tilde{L}_{0}^{+}:=0, \tilde{m}_{0}:=0$, and recursively for $i \geq 1$ (see Figure 1),

$$
\begin{aligned}
\tilde{L}_{i}^{\sharp} & :=\inf \left\{x>\tilde{L}_{i-1}^{+}, W_{\kappa}(x)=W_{\kappa}\left(\tilde{L}_{i-1}^{+}\right)-h_{t}^{+}\right\}, \\
\tilde{\tau}_{i}\left(h_{t}\right) & :=\inf \left\{x \geq \tilde{L}_{i}^{\sharp}, W_{\kappa}(x)-\inf _{\left[\tilde{L}_{i}^{\sharp}, x\right]} W_{\kappa}=h_{t}\right\}, \\
\tilde{m}_{i} & :=\inf \left\{x \geq \tilde{L}_{i}^{\sharp}, W_{\kappa}(x)=\inf _{\left[\tilde{L}_{i}^{\sharp}, \tilde{\tau}_{i}\left(h_{t}\right)\right]} W_{\kappa}\right\}, \\
\tilde{L}_{i}^{+} & :=\inf \left\{x>\tilde{\tau}_{i}\left(h_{t}\right), W_{\kappa}(x)=W_{\kappa}\left(\tilde{\tau}_{i}\left(h_{t}\right)\right)-h_{t}^{+}-h_{t}\right\}, \\
\tilde{\tau}_{i}^{-}(h) & :=\sup \left\{s<\tilde{m}_{i}, W_{\kappa}(x)-W_{\kappa}\left(\tilde{m}_{i}\right)=h\right\}, \quad h>0 .
\end{aligned}
$$



Figure 1. $h_{t}$ standard valleys

We also introduce the equivalent of $V^{(i)}$, for the $\left(\tilde{m}_{i}, i\right)$

$$
\tilde{V}^{(i)}(.):=W_{\kappa}(.)-W_{\kappa}\left(\tilde{m}_{i}\right)
$$

We call $i$ th valley, the re-centered truncated trajectory $\left(\tilde{V}^{(i)}(x), \tilde{L}_{i-1}^{+} \leq x<\tilde{L}_{i}^{+}\right)$.
The next step is to show that with an overwhelming probability the first $n_{t}$ positive $h_{t}$-minima $\left(m_{i}, 1 \leq i \leq n_{t}\right)$ coincide with $\left(\tilde{m}_{i}, 1 \leq i \leq n_{t}\right)$. We denote $\mathcal{V}_{t}$ the event $\left\{\left(m_{i}, 1 \leq i \leq n_{t}\right)=\right.$ $\left.\left(\tilde{m}_{i}, 1 \leq i \leq n_{t}\right)\right\}$.
In all the paper, $C_{+}$and $c_{+}$(resp. $C_{-}$and $c_{-}$) denote positive constants that may grow (resp. decrease) from line to line.
Lemma 2.3. For any $0<\delta<1$, and anyt large enough $\underset{\tilde{V}}{P}\left(\mathcal{V}_{t}\right) \geq 1-C_{1} w_{t}$, where $w_{t}:=n_{t} e^{-\kappa h_{t} / 2}$ and $C_{1}$ is a positive constant. Moreover, the sequence $\left(\left(\tilde{V}^{(i)}\left(x+\tilde{L}_{i-1}^{+}\right), 0 \leq x<\tilde{L}_{i}^{+}-\tilde{L}_{i-1}^{+}\right), i \geq\right.$ 1) is i.i.d.

Proof: We first notice that $\tilde{m}_{i}$ is a $h_{t}$-minimum for $W_{\kappa}$ for every $i \geq 1$, so $\left\{\tilde{m}_{i}, i \in \mathbb{N}^{*}\right\} \subset$ $\left\{m_{i}, i \in \mathbb{N}^{*}\right\}$. Hence on the complementary of $\mathcal{V}_{t}$, there would exist $1 \leq i \leq n_{t}$ and $1 \leq j \leq n_{t}$ such that $\tilde{m}_{i-1}<m_{j}<\tilde{m}_{i}$. If for such $i$ and $j, \tilde{L}_{i}^{\sharp}<m_{j}<\tilde{m}_{i}$, there would be a $v>m_{j}$ such that $W_{\kappa}\left(m_{j}\right)=\inf _{\left[m_{j}, v\right]} W_{\kappa}$ and $W_{\kappa}(v) \geq W_{\kappa}\left(m_{j}\right)+h_{t}$, so $m_{j}<\tilde{m}_{i} \leq \tilde{\tau}_{i}\left(h_{t}\right) \leq v$, then
$W_{\kappa}\left(m_{j}\right) \leq W_{\kappa}\left(\tilde{m}_{i}\right)$, which contradicts the definition of $\tilde{m}_{i}$. Hence, $\tilde{m}_{i-1}<m_{j} \leq \tilde{L}_{i}^{\sharp}$ and then $W_{\kappa}\left(m_{j}\right) \geq W_{\kappa}\left(\tilde{m}_{i-1}\right)-2 h_{t}^{+}$.

Consequently, $\overline{\mathcal{V}}_{t} \subset\left(\mathcal{E}_{1}^{2.3} \cap \mathcal{E}_{2}^{2.3}\right) \cup \overline{\mathcal{E}_{1}^{2.3}}$ where $\mathcal{E}_{1}^{2.3}:=\left\{W_{\kappa}\left(m_{1}\right)<0\right\} \cap \cap_{j=2}^{n_{t}}\left\{W_{\kappa}\left(m_{j}\right)<\right.$ $\left.W_{\kappa}\left(m_{j-1}\right)\right\}$ and $\mathcal{E}_{2}^{2.3}:=\left\{W_{\kappa}\left(m_{1}\right) \geq-2 h_{t}^{+}\right\} \cup \cup_{j=2}^{n_{t}}\left\{W_{\kappa}\left(m_{j}\right) \geq W_{\kappa}\left(m_{j-1}\right)-2 h_{t}^{+}\right\}$. For $j \geq 2$, $W_{\kappa}\left(m_{j}\right)-W_{\kappa}\left(m_{j-1}\right)$ is by ([19] Prop. 1, Thm 1 and the remark before (2.26)) equal in law to $\zeta_{+}-\zeta_{-}$, where $\zeta_{+}$and $\zeta_{-}$are independent exponential r.v, such that the mean of $\zeta_{ \pm}$is $2 \kappa^{-1} \sinh \left(\kappa h_{t} / 2\right) e^{\mp \kappa h_{t} / 2}$. So for $j \geq 2$,

$$
\mathbb{P}\left[W_{\kappa}\left(m_{j}\right) \geq W_{\kappa}\left(m_{j-1}\right)-2 h_{t}^{+}\right] \leq \mathbb{P}\left(\zeta_{+}>e^{\kappa h_{t} / 2}-2 h_{t}^{+}\right)+\mathbb{P}\left(\zeta_{-}<e^{\kappa h_{t} / 2}\right) \leq C_{+} e^{-\kappa h_{t} / 2}
$$

For $j=1$, we notice that either there is an $h_{t}$-maximum between 0 and $m_{1}$, with probability $\leq h_{t} e^{-\kappa h_{t}}$ by ([19], Thm. 1 and (2.25)), either $m_{1} \leq \tau_{1}^{*}\left(h_{t}\right)$ with

$$
\begin{equation*}
\tau_{1}^{*}\left(h_{t}\right):=\inf \left\{u \geq 0, W_{\kappa}(u)-\inf _{[0, u]} W_{\kappa} \geq h_{t}\right\} . \tag{2.2}
\end{equation*}
$$

In this case, $W_{\kappa}\left(m_{1}\right)=\inf _{\left[0, \tau_{1}^{*}\left(h_{t}\right)\right]} W_{\kappa}=: \beta^{*}$, where $-\beta^{*}$ is by ([19] Lemma 1) exponentially distributed with mean $2 \kappa^{-1} \sinh \left(\kappa h_{t} / 2\right) e^{\kappa h_{t} / 2}$. Hence $\mathbb{P}\left[W_{\kappa}\left(m_{1}\right) \geq-2 h_{t}^{+}\right] \leq h_{t} e^{-\kappa h_{t}}+\mathbb{P}\left[-\beta^{*} \leq\right.$ $\left.2 h_{t}^{+}\right] \leq C_{+} e^{-\kappa h_{t} / 2}$.

Hence, $\mathbb{P}\left(\overline{\mathcal{E}_{1}^{2.3}}\right) \leq \mathbb{P}\left(\mathcal{E}_{2}^{2.3}\right) \leq C_{+} n_{t} e^{-\kappa h_{t} / 2}$, and then $P\left(\mathcal{V}_{t}\right) \geq 1-C_{1} n_{t} e^{-\kappa h_{t} / 2}$.
Finally, the fact that the sequence $\left(\left(\tilde{V}^{(i)}\left(x+\tilde{L}_{i-1}^{+}\right), 0 \leq x<\tilde{L}_{i}^{+}-\tilde{L}_{i-1}^{+}\right), i \geq 1\right)$ is i.i.d. follows directly from the strong Markov property applied at times $\tilde{L}_{i-1}^{+}$, which are stopping times.

The following Remark will useful in the sequel
Remark 2.4. Lemma 2.3 implies that for all $1 \leq i \leq n_{t}, \tau_{i}^{-}\left(h_{t}^{+}\right) \geq 0$ with a probability larger than $1-C_{1} w_{t}$. Also by Fact 2.2, $\left(V^{(i)}\left(m_{i}-s\right), 0 \leq s \leq m_{i}-\tau_{i}^{-}\left(h_{t}\right)\right)$ is equal in law to $\left(R^{(1)}, h_{t}\right)$, and by Corollary 1 in [30] this result can be extended until $\tau_{i}^{-}\left(h_{t}^{+}\right)$, as long as $\tau_{i}^{-}\left(h_{t}^{+}\right) \geq 0$ that is to say $\left(V^{(i)}\left(m_{i}-s\right), 0 \leq s \leq m_{i}-\tau_{i}^{-}\left(h_{t}^{+}\right)\right)$on $\tau_{i}^{-}\left(h_{t}^{+}\right) \geq 0$ is equal in law to $\left(R^{(1)}, h_{t}^{+}\right)$. So for every event $\mathcal{A}$ which belongs to the $\sigma$-algebra generated by the truncated trajectory $\left(\tilde{V}^{(i)}(s), \quad \tilde{\tau}_{i}^{-}\left(h_{t}^{+}\right) \leq s \leq \tilde{m}_{i}\right), \mathcal{A} \equiv \mathcal{A}\left(\left(\tilde{V}^{(i)}(s), \tilde{\tau}_{i}^{-}\left(h_{t}^{+}\right) \leq s \leq \tilde{m}_{i}\right)\right)$ we can write $P\left(\mathcal{A}\left(R^{(1)}, h_{t}^{+}\right)\right)-C_{1} w_{t} \leq P(\mathcal{A}) \leq P\left(\mathcal{A}\left(R^{(1)}, h_{t}^{+}\right)\right)+C_{1} w_{t}$.

We also need the following intermediate random variables

$$
\begin{aligned}
\tilde{M}_{i} & :=\inf \left\{s>\tilde{m}_{i}, W_{\kappa}(s)=\max _{\tilde{m}_{i} \leq u \leq \tilde{L}_{i}^{+}} W_{\kappa}(u)\right\}, \\
\tilde{\tau}_{i}^{+}(h) & :=\inf \left\{s>\tilde{m}_{i}, W_{\kappa}(x)-W_{\kappa}\left(\tilde{m}_{i}\right)=h\right\}, \quad h>0, \\
\tilde{L}_{i}^{-} & :=\tilde{\tau}_{i}^{-}\left(h_{t}^{+}\right) .
\end{aligned}
$$

We have the following result for the distance between the points of a given valley:
Lemma 2.5. For all $0 \leq i \leq n_{t}$,

$$
\begin{align*}
& P\left(\tilde{m}_{i+1}-\tilde{M}_{i} \leq e^{\kappa h_{t}(1-\delta)}, \mathcal{V}_{t}\right) \leq C_{+} e^{-\kappa \delta h_{t}},  \tag{2.3}\\
& P\left(\tilde{\tau}_{i}^{+}(h)-\tilde{m}_{i}>8 h / \kappa, \mathcal{V}_{t}\right) \leq C_{+} e^{-\kappa h / 2 \sqrt{2}}, \quad 0 \leq h \leq h_{t},  \tag{2.4}\\
& P\left(\tilde{m}_{i+1}-\tilde{L}_{i+1}^{-}>8 h_{t}^{+} / \kappa, \mathcal{V}_{t}\right) \leq C_{+} e^{-\kappa h_{t}^{+} / 2 \sqrt{2}},  \tag{2.5}\\
& P\left(\tilde{L}_{i+1}^{+}-\tilde{L}_{i+1}^{-}>28 h_{t} / \kappa, \mathcal{V}_{t}\right) \leq C_{+} e^{-\kappa h_{t} / 24} . \tag{2.6}
\end{align*}
$$

Equation (2.4) is also valid for $0 \leq h \leq h_{t}^{+}$if $\tilde{\tau}_{i}^{+}(h)-\tilde{m}_{i}$ is replaced by $\tilde{m}_{i+1}-\tilde{\tau}_{i+1}^{-}(h)$.

Before giving the proof, we detail a basic result and its short proof:
Lemma 2.6. Let $0<\alpha<\omega$. For all $h$ large enough, we have

$$
\begin{equation*}
P\left(\tau^{W_{\kappa}}(-\alpha h) \geq 2 \omega h / \kappa\right) \leq e^{-\kappa(\omega-\alpha)^{2} h / 4 \omega} \tag{2.7}
\end{equation*}
$$

Proof: $P\left(\tau^{W_{\kappa}}(-\alpha h) \geq 2 \omega h / \kappa\right) \leq P(W(2 \omega h / \kappa) \geq(\omega-\alpha) h)=P(W(1)>\sqrt{\kappa h}(\omega-\alpha) / \sqrt{2 \omega}) \leq$ $e^{-\kappa(\omega-\alpha)^{2} h / 4 \omega}$, where we have used $\mathbb{P}(W(1) \geq x) \leq e^{-x^{2} / 2}$.
Proof of Lemma 2.5: Working on the event $\mathcal{V}_{t}$ allows us to write that $\left\{\tilde{m}_{i+1}-\tilde{M}_{i} \leq\right.$ $\left.e^{\kappa h_{t}(1-\delta)}, \mathcal{V}_{t}\right\} \subset\left\{m_{i+1}-M_{i} \leq e^{\kappa h_{t}(1-\delta)}\right\}$ and then use Faggionato's results. This idea is used several time in this proof and all along the paper. For $i \geq 1$, thanks to ([19] Thm 1 ), the law of $\left(W_{\kappa}\left(M_{i}+t\right)-W_{\kappa}\left(M_{i}\right), 0 \leq t \leq m_{i+1}-M_{i}\right)$ is $P_{-}^{\kappa / 2}$, which is defined in [19] p. 1769.
Applying ([19] Proposition 1 p. 1769 and especially Formula 2.14), $m_{i+1}-M_{i}$ has the same law as a r.v. called $\ell_{-}$, which Laplace transform is given by $E\left(e^{-\alpha \ell_{-}}\right)=\bar{\alpha} e^{-\kappa h_{t} / 2} /\left[\bar{\alpha} \cosh \left(\bar{\alpha} h_{t}\right)-\right.$ $\left.(\kappa / 2) \sinh \left(\bar{\alpha} h_{t}\right)\right]$ for $\alpha>0$, with $\bar{\alpha}:=\sqrt{2 \alpha+\kappa^{2} / 4}$. In particular, with a Markov inequality with $\alpha=e^{-\kappa h_{t}(1-\delta)}$, we get $P\left(m_{i+1}-M_{i} \leq e^{\kappa h_{t}(1-\delta)}\right) \leq \exp \left(\alpha e^{\kappa h_{t}(1-\delta)}\right) E\left(e^{-\alpha \ell_{-}}\right) \leq C_{+} e^{-\kappa \delta h_{t}}$. Applying ([19] Thm 1 and formula 2.3) and taking $\alpha=-\kappa^{2} / 16$, we get $E\left(e^{\kappa^{2}\left(\tau_{i}(h)-m_{i}\right) / 16}\right) \sim_{h \rightarrow+\infty}$ $e^{\kappa h(1-1 / \sqrt{2}) / 2} / \sqrt{2}$. Then, (2.4) follows from Markov inequality. We get (2.5) by the same way. To get the last inequality, we just need an upper bound for $\tilde{L}_{i}^{+}-\tilde{\tau}_{i}\left(h_{t}\right)$. Since $\tilde{L}_{i}^{+}$is a stopping time for $W_{\kappa}$ (in particular we do not need to work on $\mathcal{V}_{t}$ for this part), we have by using (2.7)

$$
P\left(\tilde{L}_{i}^{+}-\tilde{\tau}_{i}\left(h_{t}\right) \geq 12 h_{t} / \kappa\right)=P\left(\tau^{W_{\kappa}}\left(-h_{t}^{+}-h_{t}\right) \geq 12 h_{t} / \kappa\right) \leq e^{-\kappa h_{t} / 24}
$$

And we used the fact that $0<\delta<1$ hence $h_{t}^{+}+h_{t} \leq 4 h_{t}$. Combining these inequalities yields (2.6).

## 3. Quasi-Independence in the trajectories of $X$

In this section we show that the times to escape from the different valleys are asymptotically in $t$ independent under the annealed measure. Then we prove that the time spent by $X$ between the valleys is negligible.

We start with some basic facts about hitting times by $X, R$ and $W_{\kappa}$.
3.1. About hitting times. We first introduce some notation. Let

$$
A(r):=\int_{0}^{r} e^{W_{\kappa}(x)} \mathrm{d} x, \quad r \in \mathbb{R}
$$

and $A_{\infty}:=\lim _{r \rightarrow+\infty} A(r)<\infty$ a.s. As in Brox [5], there exists a Brownian motion $B$ such that $X(t)=A^{-1}\left[B\left(T^{-1}(t)\right)\right]$, where

$$
\begin{equation*}
T(r):=\int_{0}^{r} \exp \left\{-2 W_{\kappa}\left[A^{-1}(B(s))\right]\right\} \mathrm{d} s, \quad 0 \leq r \leq \tau^{B}\left(A_{\infty}\right) \tag{3.1}
\end{equation*}
$$

With these notations, we recall the following expression of $H(r)$, for all $r \geq 0$,

$$
H(r)=T\left[\tau^{B}(A(r))\right]=\int_{-\infty}^{r} e^{-W_{\kappa}(u)} \mathcal{L}_{B}\left[\tau^{B}(A(r)), A(u)\right] \mathrm{d} u
$$

We also need some estimates on hitting times by $W_{\kappa}$ and a $-(\kappa / 2)$-drifted Bessel process $R$ :

Lemma 3.1. Let $0<\gamma<\alpha<\omega$. For all $h$ large enough, we have

$$
\begin{align*}
P^{\alpha h}\left(\tau^{R}(\gamma h)<\tau^{R}(\omega h)\right) & \leq 2 e^{-\kappa(\alpha-\gamma) h},  \tag{3.2}\\
P\left(\tau^{R}(\omega h)-\tau^{R}(\alpha h) \leq 1\right) & \leq C_{+} e^{-c-[(\omega-\alpha) h]^{2}},  \tag{3.3}\\
P\left(\tau^{W_{\kappa}^{(h)}}(\gamma h) \geq 1\right) & \geq 1-2 e^{-[(1-\gamma) h-\kappa / 2]^{2} / 2}, \quad \gamma \neq 1,  \tag{3.4}\\
P\left(\int_{0}^{\tau^{R}(h)} e^{R(u)} \mathrm{d} u \geq e^{(1-\alpha) h}\right) & \geq 1-3 e^{-\kappa \alpha h / 2}, \quad 0<\alpha<1  \tag{3.5}\\
P\left(\tau^{R}(h)>8 h / \kappa\right) & \leq C_{+} e^{-\kappa h / 2 \sqrt{2}} . \tag{3.6}
\end{align*}
$$

Proof: We recall that $R$ has the same law as the ( $\kappa / 2$ )-drifted Brownian motion $W_{-\kappa}^{0}=W_{-\kappa}$ Doob conditioned to stay positive, and more precisely that $P(R \in \Lambda)=\lim _{x \downarrow 0} P^{x}\left(\Lambda \mid \tau^{W_{\kappa}}(0)=\right.$ $\infty$ ) for all $\Lambda \in \mathcal{G}_{t}$, where ( $\mathcal{G}_{t}, t \geq 0$ ) is the natural filtration (see [19] Lem 6. and the discussion before), and then for every $\Lambda \in \mathcal{G}_{\tau}$, where $\tau$ is an a.s. finite stopping time. Moreover, we know that a scale function of $W_{-\kappa}$ is given by $s_{\kappa}(u):=2\left(1-e^{-\kappa u}\right) / \kappa=4 e^{-\kappa u / 2} \sinh (\kappa u / 2) / \kappa, u \in \mathbb{R}$ (see [19], (5.1)), that is, $P\left(\tau^{W_{-\kappa}}(y)<\tau^{W_{-\kappa}}(-x)\right)=s_{\kappa}(x) / s_{\kappa}(x+y)$ for $x>0$ and $y>0$. Hence, since $\tau_{\omega h}^{W-\kappa}<\infty P^{x}-$ a.s.,

$$
\begin{align*}
& \text { LHS of }(3.2) \\
= & P^{\alpha h}\left[\tau^{W_{-\kappa}}(\gamma h)<\tau^{W_{-\kappa}}(\omega h) \mid \tau^{W_{-\kappa}}(\infty)<\tau^{W_{-\kappa}}(0)\right] \\
= & P^{\alpha h}\left[\tau^{W_{-\kappa}}(\gamma h)<\tau^{W_{-\kappa}}(\omega h)\right] P^{\gamma h}\left[\tau^{W_{-\kappa}}(\infty)<\tau^{W_{-\kappa}}(0)\right] / P^{\alpha h}\left[\tau^{W_{-\kappa}}(\infty)<\tau^{W_{-\kappa}}(0)\right] \\
= & \left(1-\frac{s_{\kappa}((\alpha-\gamma) h)}{s_{\kappa}((\omega-\gamma) h)}\right) \frac{s_{\kappa}(\gamma h)}{s_{\kappa}(\alpha h)} \\
= & \frac{\sinh (\kappa(\omega-\alpha) h / 2) \sinh (\kappa \gamma h / 2)}{\sinh (\kappa(\omega-\gamma) h / 2) \sinh (\kappa \alpha h / 2)}, \tag{3.7}
\end{align*}
$$

where LHS means left hand side. This gives (3.2) for large $h$.
We now turn to the proof of (3.4). We have, if $0<\gamma<1$,

$$
P\left(\tau^{W_{\kappa}^{h}}(\gamma h) \leq 1\right)=P\left(\inf _{[0,1]}\left(W(x)+h-\frac{\kappa}{2} x\right) \leq \gamma h\right) \leq P\left(\inf _{[0,1]} W \leq(\gamma-1) h+\frac{\kappa}{2}\right) .
$$

This yields to (3.4), since $\inf _{[0,1]} W \stackrel{\mathcal{L}}{=}|W(1)|$, where $\xlongequal{\mathcal{L}}$ denotes equality in law, and $\mathbb{P}(|W(1)| \geq$ $x) \leq 2 e^{-x^{2} / 2}$. The case $\gamma>1$ is treated similarly.
Now, we notice that the left hand side of (3.3) is less than

$$
\underset{x \downarrow 0}{\lim \sup } P^{x}\left[\tau^{W_{-\kappa}}(\alpha h)<\tau^{W_{-\kappa}}(0)\right] \times P^{\alpha h}\left[\tau^{W_{-\kappa}}(\omega h)-\tau^{W_{-\kappa}}(\alpha h) \leq 1\right] / P^{x}\left[\tau^{W_{-\kappa}}(0)=\infty\right] .
$$

Moreover

$$
P^{x}\left[\tau^{W-\kappa}(\omega h)-\tau^{W_{-\kappa}}(\alpha h) \leq 1\right]=P^{\alpha h}\left[\tau^{W_{-\kappa}}(\omega h) \leq 1\right] \leq 2 \exp \left(-[(\omega-\alpha) h-\kappa / 2]^{2} / 2\right),
$$

where the inequality is proved in the same way as (3.4). Since $\lim _{x \rightarrow 0} P^{x}\left[\tau^{W_{-\kappa}}(\alpha h)<\tau^{W_{-\kappa}}(0)\right] /$ $P^{x}\left[\tau^{W_{-\kappa}}(0)=\infty\right]=\left(1-e^{-\kappa \alpha h}\right)^{-1} \leq 2$ for large $h$, we get (3.3).

To prove (3.5), let $0<\alpha<1$. Notice that the probability of $\left\{\inf _{\tau^{R}[(1-\alpha / 2) h] \leq u \leq \tau^{R}(h)} R(u) \geq\right.$ $(1-\alpha) h\} \cap\left\{\tau^{R}[(1-\alpha / 2) h]-\tau^{R}(h) \geq 1\right\}$ is at least $1-3 e^{-\kappa \alpha h / 2}$ for large $h$ by (3.2) and (3.3). Moreover, we have on this event,

$$
\int_{0}^{\tau^{R}(h)} e^{R(u)} \mathrm{d} u \geq \int_{\tau^{R}[(1-\alpha / 2) h]}^{\tau^{R}(h)} e^{R(u)} \mathrm{d} u . \geq\left[\tau^{R}[(1-\alpha / 2) h]-\tau^{R}(h)\right] e^{(1-\alpha) h)} \geq e^{(1-\alpha) h)},
$$

which proves (3.5). (3.6) is obtained similarly as (2.4).
3.2. Independence in a trajectory of $X$. We prove that the sequence $\left(U_{i}:=H\left(\tilde{L}_{i}\right)-\right.$ $\left.H\left(\tilde{m}_{i}\right), i \geq 1\right)$ is "nearly" i.i.d under $\mathbb{P}$ for large $t$ in the following sense: First we need some more points which belongs to the standard valleys, for all $i \geq 1$

$$
\tilde{L}_{i}:=\inf \left\{x>\tilde{\tau}_{i}\left(h_{t}\right), W_{\kappa}(x)-W_{\kappa}\left(\tilde{m}_{i}\right)=h_{t} / 2\right\} .
$$

Proposition 3.2. There exists a constant $C_{2}>0$ such that for large $t$,

$$
\begin{equation*}
\forall \lambda>0,1 \leq n \leq n_{t}, \quad\left|\mathbb{E}\left(e^{-\lambda \sum_{i=1}^{n} U_{i}}\right)-\left[\mathbb{E}\left(e^{-\lambda \mathbf{U}}\right)\right]^{n}\right| \leq C_{2} u(t, n), \tag{3.8}
\end{equation*}
$$

where $u(t, n):=n e^{-\delta \kappa h_{t}}$ if $\delta$ is chosen small enough,

$$
\mathbf{U}:=\int_{\tilde{L}_{2}^{-}}^{\tilde{L}_{2}} e^{-\tilde{V}^{(2)}(u)} \mathcal{L}_{B}\left[\tau^{B}\left(\tilde{A}\left(\tilde{L}_{2}\right)\right), \tilde{A}(u)\right] \mathrm{d} u
$$

with $\tilde{A}(z):=\int_{\tilde{m}_{2}}^{z} e^{\tilde{V}^{(2)}(x)} d x$ for all $\tilde{L}_{2}^{-} \leq z \leq \tilde{L}_{2}$. Moreover for all $n \in \mathbb{N}^{*},[a, b] \subset[0,1]$ and $\alpha>0$,

$$
\begin{equation*}
\left|\mathbb{P}\left(\sum_{i=1}^{n-1} \frac{U_{i}}{t} \in[a, b], \sum_{i=1}^{n} \frac{U_{i}}{t} \geq \alpha\right)-\int_{a}^{b} \mathbb{P}\left(\sum_{i=1}^{n-1} \frac{U_{i}}{t} \in \mathrm{~d} x\right) \mathbb{P}(\mathbf{U} / t \geq \alpha-x)\right| \leq C_{+} e^{-\delta \kappa h_{t}} . \tag{3.9}
\end{equation*}
$$

Proof: Let $\mathcal{E}_{i}:=\left\{U_{i}<H_{i}^{\prime}\left(\tilde{L}_{i}^{-}\right)-H\left(\tilde{m}_{i}\right)\right\}$, where $H_{i}^{\prime}\left(\tilde{L}_{i}^{-}\right):=\inf \left\{s>H\left(\tilde{m}_{i}\right), X(s)=\tilde{L}_{i}^{-}\right\}$. We also introduce $\mathcal{F}_{t}:=\sigma\left(X(u), 0 \leq u \leq t, W_{\kappa}(x), x \in \mathbb{R}\right)$ for $t \geq 0$. We get

$$
\begin{aligned}
\mathbb{E}\left(e^{-\lambda \sum_{i=1}^{n} U_{i}} \prod_{i=1}^{n} \mathbb{1}_{\mathcal{E}_{i}}\right) & =E\left[\mathbb{E}^{W_{\kappa}}\left(\mathbb{E}^{W_{\kappa}}\left(e^{-\lambda U_{n}} \mathbb{1}_{\mathcal{E}_{n}} \mid \mathcal{F}_{H\left(\tilde{m}_{n}\right)}\right) e^{-\lambda \sum_{i=1}^{n-1} U_{i}} \prod_{i=1}^{n-1} \mathbb{1}_{\mathcal{E}_{i}}\right)\right] \\
& =E\left[\mathbb{E}^{W_{\kappa}}\left(\mathbb{E}_{\tilde{m}_{n}}^{W_{\kappa}}\left(e^{-\lambda H\left(\tilde{L}_{n}\right)} \mathbb{1}_{H\left(\tilde{L}_{n}\right)<H\left(\tilde{L}_{n}^{-}\right)}\right) e^{-\lambda \sum_{i=1}^{n-1} U_{i}} \prod_{i=1}^{n-1} \mathbb{1}_{\mathcal{E}_{i}}\right)\right],
\end{aligned}
$$

by the strong Markov property and since $\tilde{m}_{i}<\tilde{L}_{i}<\tilde{m}_{n}$ for $1 \leq i<n$. Hence we obtain by induction

$$
\begin{align*}
\mathbb{E}\left(e^{-\lambda \sum_{i=1}^{n} U_{i}} \prod_{i=1}^{n} \mathbb{1}_{\mathcal{E}_{i}}\right) & =E\left(\prod_{i=1}^{n} \mathbb{E}_{\tilde{m}_{i}}^{W_{\kappa}}\left(e^{-\lambda H\left(\tilde{L}_{i}\right)} \mathbb{1}_{H\left(\tilde{L}_{i}\right)<H\left(\tilde{L}_{i}^{-}\right)}\right)\right) \\
& =E\left(\prod_{i=1}^{n}\left[L_{\lambda}\left(H\left(\tilde{L}_{i}\right) \mathbb{P}^{W_{\kappa}}\left(\mathcal{E}_{i}\right)\right]\right),\right. \tag{3.10}
\end{align*}
$$

as $\mathbb{P}^{W_{\kappa}}\left(\mathcal{E}_{i}\right)=\mathbb{P}_{\tilde{m}_{i}}^{W_{n}}\left(H\left(\tilde{L}_{i}\right)<H\left(\tilde{L}_{i}^{-}\right)\right)$and whith $L_{\lambda}\left[H\left(\tilde{L}_{i}\right)\right]:=\mathbb{E}_{\tilde{m}_{i}}^{W_{k}}\left(e^{-\lambda H\left(\tilde{L}_{i}\right)} \mid H\left(\tilde{L}_{i}\right)<H\left(\tilde{L}_{i}^{-}\right)\right)$.
We now need to prove that $\mathbb{P}^{W_{\kappa}}\left(\mathcal{E}_{i}\right)$ is closed to one with a large probability, so the next step is to get a lower bound for this probability. We now work under $\mathcal{V}_{t}$ which allows us to use $V^{(i)}$ and its William's decomposition instead of $\tilde{V}^{(i)}$, we have

$$
E\left(\prod_{i=1}^{n}\left[L_{\lambda}\left(H\left(\tilde{L}_{i}\right) \mathbb{P}^{W_{\kappa}}\left(\mathcal{E}_{i}\right)\right]\right) \geq E\left(\prod_{i=1}^{n}\left[L_{\lambda}\left(H\left(\tilde{L}_{i}\right) \mathbb{P}^{W_{\kappa}}\left(\mathcal{E}_{i}\right) \mathbb{1}_{\tilde{m}_{i}=m_{i}}\right]\right) .\right.\right.
$$

Let us give an upper bound for $\mathbb{P}^{W_{\kappa}}\left(\mathcal{E}_{i}\right)$ on $\left\{\tilde{m}_{i}=m_{i}\right\}$. Recall that $A$ is a scale function of $X$ under $\mathbb{P}^{W_{\kappa}}$ (see e.g. [35] formula (2.2)), that is

$$
\begin{equation*}
\mathbb{P}^{W_{\kappa}}\left(\overline{\mathcal{E}_{i}}\right)=\mathbb{P}_{\tilde{m}_{i}}^{W_{\kappa}}\left(H\left(\tilde{L}_{i}\right)>H\left(\tilde{L}_{i}^{-}\right)\right)=\left(\int_{10}^{\tilde{m}_{i}} e^{\tilde{L}_{i}} e^{\tilde{V}^{(i)}(x)} \mathrm{d} x\right)\left(\int_{\tilde{L}_{i}^{-}}^{\tilde{L}_{i}} e^{\tilde{V}^{(i)}(u} \mathrm{d} x\right)^{-1}, \tag{3.11}
\end{equation*}
$$

so on $\left\{\tilde{m}_{i}=m_{i}\right\}$,

$$
(3.11) \leq\left(\left(\tilde{L}_{i}-\tilde{m}_{i}\right) e^{V^{(i)}\left(M_{i}\right)}\right) /\left(\int_{\tau_{i}^{-}\left(h_{t}^{+}\right)}^{\tau_{i}^{-}\left[h_{t}^{+}(1-\kappa / 4)\right]} e^{V^{(i)}(x)} \mathrm{d} x\right)=: Q_{i} / D_{i},
$$

recalling that $\tau_{i}^{-}(z)=\sup \left\{s<m_{i}, V^{(i)}(s) \geq z\right\}$ for $z>0$.
We start with the denominator $D_{i}$. First on $\left\{\tilde{m}_{i}=m_{i}\right\}, \tau_{i}^{-}\left(h_{t}^{+}\right) \geq 0$ so by Remark $2.4\left(V^{(i)}\left(m_{i}-\right.\right.$ $\left.x), m_{i}-\tau_{i}^{-}\left[h_{t}^{+}\right] \leq x \leq m_{i}-\tau_{i}^{-}\left[h_{t}^{+}(1-\kappa / 4)\right]\right)$ is equal in law to $\left(R^{(2)}(x), \tau^{R^{(2)}}\left[h_{t}^{+}(1-\kappa / 4)\right] \leq x \leq\right.$ $\left.\tau^{R^{(2)}}\left(h_{t}^{+}\right)\right)$. Then, formula (3.2) applied with $h=h_{t}^{+}, \gamma=1-\kappa / 2, \alpha=1-\kappa / 4$ and $\omega=1$ gives for $t$ large enough, with a probability larger than $1-C_{+} e^{-\kappa^{2} h_{t} / 4}$,

$$
D_{i} \mathbb{1}_{\tilde{m}_{i}=m_{i}} \geq e^{h_{t}^{+}(1-\kappa / 2)}\left[\tau_{i}^{-}\left(h_{t}^{+}\right)-\tau_{i}^{-}\left(h_{t}^{+}(1-\kappa / 4)\right)\right] \mathbb{1}_{\tilde{m}_{i}=m_{i}} .
$$

Moreover by (3.3), we have

$$
P\left[\tau_{1}^{R^{(2)}}\left(h_{t}^{+}\right)-\tau_{1}^{R^{(2)}}\left(h_{t}^{+}(1-\kappa / 4)\right) \geq 1\right] \geq 1-C_{+} e^{-c_{-} h_{t}^{2}}
$$

So finally, we get $D_{i} \mathbb{1}_{\tilde{m}_{i}=m_{i}} \geq e^{h_{t}^{+}(1-\kappa / 2)} \mathbb{1}_{\tilde{m}_{i}=m_{i}}$ with a probability larger than $1-C_{+} e^{-\kappa^{2} h_{t}}$.
For the numerator $Q_{i}$, first by (2.6) and Lemma $2.3 P\left(\tilde{L}_{i}-\tilde{L}_{i}^{-}>28 h_{t} / \kappa\right) \leq 2 e^{-\kappa h_{t} / 24}$. Also thanks to Williams' decomposition (Fact 2.2),

$$
P\left[V^{(i)}\left(M_{i}\right)>h_{t}(1+\delta)\right] \leq P\left(\sup _{s \geq 0} W_{\kappa}(s)>\delta h_{t}\right) \leq e^{-\delta \kappa h_{t}},
$$

where the last inequality comes from formula 1.1.4 (1) page 251 of [4]. Finally for $\delta$ small enough and $t$ large enough, with probability greater than $1-2 e^{-\delta \kappa h_{t}}, Q_{i} \mathbb{1}_{\tilde{m}_{i}=m_{i}} \leq 28 h_{t} \kappa^{-1} e^{h_{t}(1+\delta)} \mathbb{1}_{\tilde{m}_{i}=m_{i}}$ and $D_{i} \mathbb{1}_{\tilde{m}_{i}=m_{i}} \geq e^{h_{t}^{+}(1-\kappa / 2)} \mathbb{1}_{\tilde{m}_{i}=m_{i}}$, so

$$
\begin{equation*}
\mathbb{P}^{W_{\kappa}}\left(\mathcal{E}_{i}\right) \mathbb{1}_{\tilde{m}_{i}=m_{i}} \geq\left(1-C_{+} h_{t} e^{-(\kappa+2(1-\kappa) \delta) h_{t} / 2}\right) \mathbb{1}_{\tilde{m}_{i}=m_{i}} . \tag{3.12}
\end{equation*}
$$

Collecting what we did above, we get

$$
\begin{aligned}
& E\left[\prod_{i=1}^{n}\left(L_{\lambda}\left(H\left(\tilde{L}_{i}\right) \mathbb{P}^{W_{\kappa}}\left(\mathcal{E}_{i}\right) \mathbb{1}_{\tilde{m}_{i}=m_{i}}\right]\right)\right. \\
& \geq\left(1-C_{+} e^{-\delta \kappa h_{t}}\right)^{n} E\left(\mathbb{1}_{\mathcal{V}_{t}} \prod_{i=1}^{n} \mathbb{E}_{\tilde{m}_{i}}^{W_{k}}\left(e^{-\lambda H\left(\tilde{L}_{i}\right)} \mid H\left(\tilde{L}_{i}\right)<H\left(\tilde{L}_{i}^{-}\right)\right)\right) .
\end{aligned}
$$

Using Lemma 2.3 and considering (3.10),

$$
\mathbb{E}\left(e^{-\lambda \sum_{i=1}^{n} U_{i}} \prod_{i=1}^{n} \mathbb{1}_{\mathcal{E}_{i}}\right)-E\left(\prod_{i=1}^{n} \mathbb{E}_{\tilde{m}_{i}}^{W_{\kappa}}\left(e^{-\lambda H\left(\tilde{L}_{i}\right)} \mid H\left(\tilde{L}_{i}\right)<H\left(\tilde{L}_{i}^{-}\right)\right)\right) \geq-C_{+} n e^{-\delta \kappa h_{t}} .
$$

With similar ideas for the upper bound, we finally get

$$
\left|\mathbb{E}\left(e^{-\lambda \sum_{i=1}^{n} U_{i}}\right)-E\left(\prod_{i=1}^{n} \mathbb{E}_{\tilde{m}_{i}}^{W_{\kappa}}\left(e^{-\lambda H\left(\tilde{L}_{i}\right)} \mid H\left(\tilde{L}_{i}\right)<H\left(\tilde{L}_{i}^{-}\right)\right)\right)\right| \leq C_{+} n e^{-\delta \kappa h_{t}} .
$$

For every fixed $W_{\kappa}$, we have under $\mathbb{P}_{\tilde{m}_{i}}^{W_{\kappa}}$ on $\left\{H\left(\tilde{L}_{i}\right)<H\left(\tilde{L}_{i}^{-}\right)\right\}$,

$$
\begin{equation*}
H\left(\tilde{L}_{i}\right)=\int_{\tilde{L}_{i}^{-}}^{\tilde{L}_{i}} e^{-\tilde{V}^{(i)}(u)} \mathcal{L}_{\tilde{B}}\left[\tau^{\tilde{B}}\left(\tilde{A}_{i}\left(\tilde{L}_{i}\right)\right), \tilde{A}_{i}(u)\right] \mathrm{d} u, \tag{3.13}
\end{equation*}
$$

where for all $\tilde{L}_{i}^{-} \leq z \leq \tilde{L}_{j}, \tilde{A}_{i}(z):=\int_{\tilde{m}_{j}}^{z} e^{\tilde{V}^{(i)}(x)} \mathrm{d} x$ and $\tilde{B}$ is a standard Brownian motion. This and (3.11) show that the left hand side of (3.11) and $\underset{\tilde{V}^{(i}}{\tilde{m}_{i}}{ }^{W_{\kappa}}\left(e^{-\lambda H\left(\tilde{L}_{i}\right)} \mid H\left(\tilde{L}_{i}\right)<H\left(\tilde{L}_{i}^{-}\right)\right)$are measurable with respect to the $\sigma$-algebra generated by $\left(\tilde{V}^{(i)}(x), \tilde{L}_{i}^{-} \leq x \leq \tilde{L}_{i}\right)$.

Using (3.13), and the fact that $\left\{\left(\tilde{V}^{(i)}(x), \tilde{L}_{i}^{-} \leq x \leq \tilde{L}_{i}\right), i \leq n\right\}$ is i.i.d. by Lemma 2.3 give the upper bound of (3.8). Notice that we choose the second valley in the definition of $\mathbf{U}$ in order to avoid the central slope (see Fact 2.2) when working under $\mathcal{V}_{t}$.

For (3.9), we obtain

$$
\begin{aligned}
& \mathbb{P}\left(\sum_{i=1}^{n-1} \frac{U_{i}}{t} \in[a, b], \sum_{i=1}^{n} \frac{U_{i}}{t} \geq \alpha, \mathcal{E}_{n}\right)=\mathbb{E}\left[\mathbb{1}_{\left.\sum_{i=1}^{n-1} \frac{U_{i} \in[a, b]}{} \mathbb{E}^{W_{\kappa}}\left(\left.\mathbb{1}_{U_{n} \geq \alpha-\sum_{i=1}^{n-1} \frac{U_{i}}{t}} \mathbb{E}_{\mathcal{E}_{n}} \right\rvert\, \mathcal{F}_{H\left(\tilde{m}_{n}\right)}\right)\right]}\right. \\
&=\int_{a}^{b} \mathbb{P}\left(\sum_{i=1}^{n-1} \frac{U_{i}}{t} \in \mathrm{~d} x\right) \mathbb{P}\left(U_{n} / t \geq \alpha-x, \mathcal{E}_{n}\right),
\end{aligned}
$$

since $U_{i}$ is for $i \leq n-1$ measurable with respect to $\sigma\left(W_{\kappa}(x), x \leq \tilde{L}_{n-1}, X(u), u \leq H\left(\tilde{L}_{n-1}\right)\right)$, whereas $\mathbb{E}^{W_{\kappa}}\left(\mathbb{1}_{U_{n} \geq \alpha-x} \mathbb{1}_{\mathcal{E}_{n}} \mid \mathcal{F}_{H\left(\tilde{m}_{n}\right)}\right)$ is for every $x \in \mathbb{R}$ measurable with respect to $\sigma\left(W_{\kappa}(x+\right.$ $\left.\left.\tilde{m}_{n}\right)-W_{\kappa}\left(\tilde{m}_{n}\right), x \geq \tilde{L}_{n}^{-}-\tilde{m}_{n}, X\left(u+H\left(\tilde{m}_{n}\right)\right)-\tilde{m}_{n}, 0 \leq u \leq H\left(\tilde{L}_{n}\right)-H\left(\tilde{m}_{n}\right)\right)$. Since $U_{n}$ is equal in law to $\mathbf{U}$ and $\mathbb{P}\left(\overline{\mathcal{E}_{n}}\right) \leq C_{+} e^{-\delta h_{t}}$, we get (3.9).
3.3. Negligible parts in the trajectory of $X$. We now prove that the total time spent between the first $n_{t}$ large valleys is negligible compared to $t$.

We first need to give estimates concerning the hitting times of $\tilde{m}_{1}$ and $\tau_{1}\left(h_{t}\right)$. To this aim, notice that $H(r)=H_{-}(r)+H_{+}(r)$, where $H_{-}(r):=\int_{-\infty}^{0} e^{-W_{\kappa}(u)} \mathcal{L}_{B}\left[\tau^{B}(A(r)), A(u)\right] \mathrm{d} u$ and $H_{+}(r):=\int_{0}^{r} e^{-W_{\kappa}(u)} \mathcal{L}_{B}\left[\tau^{B}(A(r)), A(u)\right] \mathrm{d} u$. Actually, $H_{-}(r)\left(\right.$ resp. $\left.H_{+}(r)\right)$ is the time spent by $X$ in $\mathbb{R}_{-}$(resp. in $\mathbb{R}_{+}$) before it hits $r$ for the first time. We start with the following lemma about $H_{-}$; it comes from [10] and the proof is given for the sake of completeness:

Lemma 3.3. Let $\kappa>0$ and $H_{-}(+\infty):=\lim _{r \rightarrow+\infty} H_{-}(r)$. For $z$ large enough,

$$
\begin{equation*}
\mathbb{P}\left(H_{-}(+\infty)>z\right) \leq C_{+}[(\log z) / z]^{\kappa /(\kappa+2)} \tag{3.14}
\end{equation*}
$$

Proof: For $a>0, \alpha>0$ and $b>0$, let

$$
\begin{gathered}
\mathcal{E}_{1}^{3.3}:=\left\{\sup _{x<0} e^{-W_{\kappa}(x)} \leq a\right\}, \quad \mathcal{E}_{2}^{3.3}:=\left\{A_{\infty} \leq \alpha\right\}, \quad \mathcal{E}_{3}^{3.3}:=\left\{\sup _{y<0} \mathcal{L}_{B}\left[\tau^{B}(\alpha), y\right] \leq b\right\} \\
\mathcal{L}_{X}^{*-}(+\infty):=\sup _{r \geq 0} \sup _{x<0} \mathcal{L}_{X}(H(r), x)=\sup _{t \geq 0} \sup _{x<0} \mathcal{L}_{X}(t, x)
\end{gathered}
$$

We first prove an inequality with regards to $\mathcal{L}_{X}^{*-}(+\infty)$. We notice that

$$
\mathcal{L}_{X}^{*-}(+\infty)=\sup _{r \geq 0} \sup _{x<0}\left\{e^{-W_{\kappa}(x)} \mathcal{L}_{B}\left[\tau^{B}(A(r)), A(x)\right]\right\} \leq\left(\sup _{x<0} e^{-W_{\kappa}(x)}\right) \sup _{y<0} \mathcal{L}_{B}\left[\tau^{B}\left(A_{\infty}\right), y\right]
$$

By the first Ray-Knight theorem, there exist two Bessel processes $R_{2}$ and $R_{0}$, of dimensions 2 and 0 respectively, starting from 0 and $R_{2}(\alpha)$, such that $\mathcal{L}_{B}\left(\tau^{B}(\alpha), x\right)$ is equal to $R_{2}^{2}(\alpha-x)$ for $x \in[0, \alpha]$ and to $R_{0}^{2}(-x)$ for $x<0$. Hence, for $\alpha \leq b$,

$$
\mathbb{P}\left(\overline{\mathcal{E}_{3}^{3.3}}\right)=\mathbb{P}\left(R_{2}^{2}(\alpha)>b\right)+\int_{0}^{b} \mathbb{P}_{x}\left(\sup _{y>0} R_{0}^{2}(y)>b\right) \mathbb{P}\left(R_{2}^{2}(\alpha) \in[x, x+\mathrm{d} x]\right)
$$

Consequently,

$$
\mathbb{P}\left(\overline{\mathcal{E}_{3}^{3.3}}\right)=\exp \left(-\frac{b}{2 \alpha}\right)+\mathbb{E}\left(\frac{R_{2}^{2}(\alpha)}{b} \mathbf{1}_{\left\{R_{2}^{2}(\alpha) \leq b\right\}}\right) \leq\left[2+\mathbb{E}\left(R_{2}^{2}(1)\right)\right] \frac{\alpha}{b}=: c_{0} \frac{\alpha}{b},
$$

with $c_{0}>0$. Now, let $a:=z^{\frac{1}{\kappa+2}}, \alpha:=z^{\frac{1}{\kappa+2}}$ and $b:=z^{\frac{\kappa+1}{k+2}}$. Notice that on $\mathcal{E}_{1}^{3.3} \cap \mathcal{E}_{2}^{3.3} \cap \mathcal{E}_{3}^{3.3}$, $\mathcal{L}_{X}^{*-}(+\infty) \leq z$ and recall $A_{\infty} \stackrel{\mathcal{L}}{=} 2 / \gamma_{\kappa}$ (see Dufresne, [13]), where $\gamma_{\kappa}$ is a gamma variable of parameter $(\kappa, 1)$. We have for $z$ large enough,

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{L}_{X}^{*-}(+\infty)>z\right) \leq \mathbb{P}\left(\overline{\mathcal{E}_{1}^{3.3}}\right)+\mathbb{P}\left(\overline{\mathcal{E}_{2}^{3.3}}\right)+\mathbb{P}\left(\overline{\mathcal{E}_{3}^{3.3}}\right) \leq a^{-\kappa}+[2 / \alpha]^{\kappa} /(\kappa \Gamma(\kappa))+c_{0} \alpha / b \leq c_{1} z^{-\frac{\kappa}{\kappa+2}}, \tag{3.15}
\end{equation*}
$$

with $c_{1}>0$. We now turn back to $H_{-}(+\infty)$. Define for $c>0$,

$$
\mathcal{E}_{4}^{3.3}:=\left\{\min _{0 \leq s \leq \tau^{B}(\alpha)} B(s)>-\alpha z^{\frac{\kappa+1}{\kappa+2}}\right\}, \quad \mathcal{E}_{5}^{3.3}:=\left\{\left|A^{-1}(-z)\right| \leq c \log z\right\} .
$$

On $\mathcal{E}_{1}^{3.3} \cap \mathcal{E}_{2}^{3.3} \cap \mathcal{E}_{3}^{3.3} \cap \mathcal{E}_{4}^{3.3} \cap \mathcal{E}_{5}^{3.3}$, we have for $r \geq 0$,

$$
\begin{align*}
H_{-}(+\infty) & \left.\leq \lim _{r \rightarrow+\infty} \int_{A^{-1}\left(\min _{0 \leq s \leq \tau^{B}(A(r))}^{0}\right.}^{0} B(s)\right) \\
& \leq\left|A^{-1}\left(\min _{0 \leq s \leq \tau^{B}\left(A_{\infty}\right)}^{*-} B(s)\right)\right| \mathcal{L}_{X}^{*-}(+\infty) \mathrm{d} x \\
& \leq\left|A^{-1}(-z)\right| \mathcal{L}_{X}^{*-}(+\infty) \leq c z \log z . \tag{3.16}
\end{align*}
$$

Moreover, for $c>2 / \kappa$, and $\varepsilon>0$,

$$
\begin{align*}
\mathbb{P}\left(\overline{\mathcal{E}_{5}^{3.3}}\right)=\mathbb{P}\left(z>\int_{0}^{c \log z} e^{W(u)+\kappa u / 2} \mathrm{~d} u\right) & \leq \mathbb{P}\left[z>\exp \left(\inf _{0 \leq u \leq c \log z} W(u)\right) \frac{2}{\kappa}\left(z^{\kappa c / 2}-1\right)\right] \\
& \leq 2 z^{-\frac{1}{2 c}\left(\frac{\kappa c}{2}-1-\varepsilon\right)^{2}} \tag{3.17}
\end{align*}
$$

for all large $z$. Moreover, $\mathbb{P}\left(\overline{\mathcal{E}_{4}^{3.3}}\right)=\alpha /\left[\alpha+\alpha z^{\frac{\kappa+1}{\kappa+2}}\right] \leq z^{-\frac{\kappa+1}{\kappa+2}}$. Choosing $c$ large enough, this, together with (3.15), (3.16) and (3.17) gives (3.14).

Lemma 3.4. There exists a constant $C_{5}>0$ such that for every $h>0$,

$$
\begin{equation*}
\mathbb{E}\left[H_{+}\left(\tau_{1}^{*}(h)\right)\right] \leq C_{5} e^{h} \tag{3.18}
\end{equation*}
$$

where $\tau_{1}^{*}(h)$ is define in (2.2). Moreover,

$$
\begin{equation*}
\mathbb{P}\left[H_{-}\left(\tilde{m}_{1}\right) \geq t / \log h_{t}\right] \leq C_{+}\left[(\log t)^{2} / t\right]^{\kappa /(\kappa+2)} . \tag{3.19}
\end{equation*}
$$

Proof: We first remind that $\mathbb{E}\left[\mathcal{L}_{B}\left(\tau^{B}(1), y\right)\right]=2(1-y)$ for $0 \leq y \leq 1$; which is a consequence of the first Ray-Knight theorem (see e.g. [32]). We notice that by the scale property of $B$, recalling that $A(u) \geq 0$ for all $u \geq 0$ and $A$ is independent of $B$, we have for every $r \geq 0$, which can depend on the environment $W_{\kappa}$,

$$
\begin{align*}
\mathbb{E}^{W_{\kappa}}\left(H_{+}(r)\right) & =\mathbb{E}^{W_{\kappa}}\left(\int_{0}^{r} e^{-W_{\kappa}(u)} A(r) \mathcal{L}_{B}\left(\tau^{B}(1), A(u) / A(r)\right) \mathrm{d} u\right) \\
& =\int_{0}^{r} e^{-W_{\kappa}(u)} 2(A(r)-A(u)) \mathrm{d} u \\
& =2 \int_{0}^{r} \int_{u}^{r} e^{W_{\kappa}(v)-W_{\kappa}(u)} \mathrm{d} v \mathrm{~d} u \tag{3.20}
\end{align*}
$$

by Fubini. Hence, applying this to $r=\tau_{1}^{*}(h)$, we get

$$
\begin{aligned}
E\left[H_{+}\left(\tau_{1}^{*}(h)\right)\right] & =2 E\left(\int_{0}^{\tau_{1}^{*}(h)} \int_{u}^{\tau_{1}^{*}(h)} e^{W_{\kappa}(v)-W_{\kappa}(u)} \mathrm{d} v \mathrm{~d} u\right) \\
& \leq 2 E\left(\int_{0}^{\infty} \mathbb{1}_{u \leq \tau_{1}^{*}(h)} \int_{u}^{\tau_{1}^{*}(u, h)} e^{W_{\kappa}(v)-W_{\kappa}(u)} \mathrm{d} v \mathrm{~d} u\right)
\end{aligned}
$$

where $\tau_{1}^{*}(u, h):=\inf \left\{t>u, W_{\kappa}(t)-\inf _{[u, t]} W_{\kappa} \geq h\right\}$. Applying Fubini followed by the Markov property at time $u$, we get

$$
\begin{align*}
\mathbb{E}\left[H_{+}\left(\tau_{1}^{*}(h)\right)\right] & \leq 2 \int_{0}^{\infty} E\left(\mathbb{1}_{u \leq \tau_{1}^{*}(h)} \int_{0}^{\tau_{1}^{*}(u, h)-u} e^{W_{\kappa}(\alpha+u)-W_{\kappa}(u)} \mathrm{d} \alpha\right) \mathrm{d} u \\
& =2 \int_{0}^{\infty} E\left(\mathbb{1}_{u \leq \tau_{1}^{*}(h)}\right) E\left(\int_{0}^{\tau_{1}^{*}(h)} e^{W_{\kappa}(\alpha)} \mathrm{d} \alpha\right) \mathrm{d} u \\
& =2 \beta_{1}(h) \beta_{2}(h) \tag{3.21}
\end{align*}
$$

where, similarly as in Enriquez et al. ([15], Lem. 4.9),

$$
\beta_{1}(h):=E\left(\tau_{1}^{*}(h)\right), \quad \beta_{2}(h):=E\left(\int_{0}^{\tau_{1}^{*}(h)} e^{W_{\kappa}(u)} \mathrm{d} u\right)
$$

Using [19] (formula (2.3) and (2.7)) we have $\beta_{1}(h) \leq c_{1} e^{\kappa h}$, with $c_{1}>0$.
We now cut the integral which appears in the definition of $\beta_{2}(h)$ into several parts. to show that $\beta_{2}(h) \leq C_{+} e^{(1-\kappa) h}$ for $h$ large enough. To this aim, we introduce $e_{0}:=0$ and

$$
\begin{aligned}
e_{j} & :=\inf \left\{s>e_{j-1}, W_{\kappa}(s)=-j\right\}, \quad j \geq 1 \\
H_{j} & :=\sup _{e_{j} \leq u \leq v \leq e_{j+1}}\left(W_{\kappa}(v)-W_{\kappa}(u)\right), \quad j \in \mathbb{N}, \\
\Xi_{k} & :=\left\{e_{k} \leq \tau_{1}^{*}(h) \leq e_{k+1}\right\}, \quad k \in \mathbb{N} .
\end{aligned}
$$

We have,

$$
\begin{aligned}
\beta_{2}(h) & =E\left(\sum_{k=0}^{\infty} 1_{\Xi_{k}} \int_{0}^{\tau_{1}^{*}(h)} e^{W_{\kappa}(u)} \mathrm{d} u\right) \\
& =\sum_{k=0}^{\infty} E\left[1_{\left\{H_{0}<h\right\}} \times \cdots \times 1_{\left\{H_{k-1}<h\right\}} 1_{\left\{H_{k} \geq h\right\}}\left(\sum_{i=0}^{k-1} e^{W_{\kappa}\left(e_{i}\right)} J_{i}+e^{W_{\kappa}\left(e_{k}\right)} \bar{J}_{k}\right)\right]
\end{aligned}
$$

where $J_{i}:=\int_{e_{i}}^{e_{i+1}} e^{W_{\kappa}(u)-W_{\kappa}\left(e_{i}\right)} \mathrm{d} u$ and $\bar{J}_{k}:=\int_{e_{k}}^{\tau_{1}^{*}(h)} e^{W_{\kappa}(u)-W_{\kappa}\left(e_{k}\right)} \mathrm{d} u$. Hence, applying Markov at times $e_{i}$ and since $W_{\kappa}\left(e_{i}\right)=-i$, we get

$$
\begin{aligned}
\beta_{2}(h) & =\sum_{k=0}^{\infty}\left[\sum_{i=0}^{k-1} e^{-i}(1-q)^{k} q E\left(J_{0} \mid H_{0}<h\right)+(1-q)^{k} q e^{-k} E\left(\bar{J}_{0} \mid H_{0} \geq h\right)\right] \\
& \leq \sum_{k=0}^{\infty}\left[\left(1-e^{-1}\right)^{-1}(1-q)^{k} q E\left(J_{0} \mid H_{0}<h\right)\right]+\frac{q}{1-(1-q) e^{-1}} E\left(\bar{J}_{0} \mid H_{0} \geq h\right)
\end{aligned}
$$

with $q:=P\left(H_{0} \geq h\right)$. Hence $\beta_{2}(h) \leq C\left[E\left(J_{0} \mid H_{0}<h\right)+E\left(\bar{J}_{0} \mathbb{1}_{\left\{H_{0} \geq h\right\}}\right)\right]$. To finish first by 3.10.7 (a) page 317 in [4], taking $x=0, a=-1, \alpha=0$ and $b=h$, we easily get $E\left(J_{0} \mathbb{1}_{\left\{H_{0}<h\right\}}\right) \leq E\left(J_{0} \mathbb{1}_{\left\{\tau^{W_{\kappa}}(-1)<\tau^{W_{\kappa}}(h)\right\}}\right) \leq C e^{(1-\kappa) h}$. In the same way using formula 3.10 .7 (b) of the same reference with the same parameters except $b=h-1$, we also get $E\left(\overline{J_{0}} \mathbb{1}_{\left\{H_{0} \geq h\right\}}\right) \leq$
$E\left(\bar{J}_{0} \mathbb{1}_{\left\{\tau W_{\kappa}(-1)>\tau W_{\kappa}(h-1)\right\}}\right) \leq C e^{h(1-\kappa)}$. This, combined with $q \sim_{h \rightarrow+\infty} C e^{-\kappa h}$ gives $\beta_{2}(h) \leq$ $C e^{(1-\kappa) h}$ for large $h$, which together with (3.21) and $\beta_{1}(h) \leq c_{1} e^{\kappa h}$ gives (3.18).

The second inequality, that is (3.19), comes directly from (3.14).
We now have all the tools needed to bound the time spent between the deep valleys:
Lemma 3.5. For any $\delta$ small enough $\left(\delta<2^{-3 / 2}\right.$ and $\left.\kappa(1+2 \delta)<1\right)$ and $t$ large enough

$$
\mathbb{P}\left(H\left(\tilde{m}_{1}\right)<\tilde{v}_{t}\right) \geq \mathbb{P}\left(\bigcap_{k=1}^{n_{t}}\left\{0 \leq H\left(\tilde{m}_{k}\right)-\sum_{i=1}^{k-1} U_{i}<\tilde{v}_{t}\right\}\right) \geq 1-C_{3} v_{t}
$$

where $\tilde{v}_{t}:=2 t / \log h_{t}, v_{t}:=n_{t}\left(\log h_{t}\right) e^{-\phi(t)}=o(1), C_{3}>0$ and $\sum_{i=1}^{0} \cdots=0$ by convention.
Proof : We have, for every $1 \leq k \leq n_{t}$,

$$
\begin{equation*}
H\left(\tilde{m}_{k}\right)=H\left(\tilde{m}_{1}\right)+\sum_{i=1}^{k-1} U_{i}+\sum_{i=1}^{k-1}\left(H\left(\tilde{m}_{i+1}\right)-H\left(\tilde{L}_{i}\right)\right), \tag{3.22}
\end{equation*}
$$

so we just have to prove that $H\left(\tilde{m}_{1}\right)+\sum_{i=1}^{n_{t}-1}\left(H\left(\tilde{m}_{i+1}\right)-H\left(\tilde{L}_{i}\right)\right)$ is less than or equal to $\tilde{v}_{t}$ with large probability. We consider

$$
\tilde{L}_{i}^{*}:=\inf \left\{x \geq \tilde{\tau}_{i}\left(h_{t}\right), W_{\kappa}(x)-W_{\kappa}\left(\tilde{m}_{i}\right)=3 h_{t} / 4\right\} .
$$

We define $X_{i}(t):=X\left(t+H\left(\tilde{L}_{i}\right)\right)\left(\right.$ resp. $\left.X_{i}^{*}(t):=X\left(t+H\left(\tilde{L}_{i}^{*}\right)\right)\right), t \geq 0$, which is a diffusion in the environment $W_{\kappa}$, starting from $\tilde{L}_{i}$ (resp. $\tilde{L}_{i}^{*}$ ). We also denote by $H_{X_{i}}(r)$ the hitting time of $r$ by $X_{i}$, for $r \geq \tilde{L}_{i}$ and $A_{\infty}^{x}:=\int_{x}^{\infty} e^{W_{k}(u)-W_{\kappa}(x)} \mathrm{d} u$. We introduce the following events:

$$
\begin{aligned}
\mathcal{E}_{1}^{3.5} & :=\cap_{i=1}^{n_{t}-1}\left\{H_{X_{i}}\left(\tilde{m}_{i+1}\right)<H_{X_{i}}\left(\tilde{L}_{i}^{*}\right)\right\}, \\
\mathcal{E}_{2}^{3.5} & :=\cap_{i=1}^{n_{t}}\left\{A_{\infty}^{\tilde{L}_{i}} \leq e^{h_{t} / 16}, A_{\infty}^{\tilde{\tau}_{i+1}\left(h_{t}\right)} \leq e^{h_{t} / 16}, A_{\infty}^{\tilde{L}_{i}^{*}} \geq e^{-h_{t} / 16}\right\} .
\end{aligned}
$$

We recall that $A_{\infty}:=A_{\infty}^{0}$ has the same law as $2 / \gamma_{\kappa}$, where $\gamma_{\kappa}$ is a gamma variable of parameter $(\kappa, 1)$, with density $e^{-x} x^{\kappa-1} \mathbb{1}_{\mathbb{R}_{+}}(x) / \Gamma(\kappa)$ (see [13], or [4] IV.48). Hence, $P\left(A_{\infty} \geq y\right) \leq C y^{-\kappa}$ for $y>0$ and $C>0$, and $P\left(A_{\infty} \leq y\right) \leq e^{-1 / y}$ for small $y>0$. Moreover, since $\tilde{L}_{i}, \tilde{\tau}_{i+1}\left(h_{t}\right)$ and $\tilde{L}_{i}^{*}$ are stopping times for the natural filtration of $W_{\kappa}, A_{\infty}^{\tilde{L}_{i}}, A_{\infty}^{\tilde{\tau}_{i+1}\left(h_{t}\right)}$ and $A_{\infty}^{\tilde{L}_{i}^{*}}$ have the same law as $A_{\infty}$ under $P$ by the strong Markov property. Consequently,

$$
\begin{equation*}
P\left(\overline{\mathcal{E}_{2}^{3.5}}\right) \leq n_{t}\left(C e^{-\kappa h_{t} / 16}+e^{-e^{h_{t} / 16}}\right) . \tag{3.23}
\end{equation*}
$$

Moreover, we have for $1 \leq i \leq n_{t}-1$ on $\mathcal{E}_{2}^{3.5}, \mathbb{P}^{W_{\kappa}}\left[H_{X_{i}}\left(\tilde{m}_{i+1}\right)>H_{X_{i}}\left(\tilde{L}_{i}^{*}\right)\right]=Q_{i}^{*} / D_{i}^{*}$, where

$$
\begin{align*}
Q_{i}^{*} & :=\int_{\tilde{L}_{i}}^{\tilde{m}_{i+1}} e^{W_{\kappa}(x)} \mathrm{d} x \leq e^{W_{\kappa}\left(\tilde{L}_{i}\right)} A_{\infty}^{\tilde{L}_{i}} \leq \exp \left(W_{\kappa}\left(\tilde{m}_{i}\right)+h_{t} / 2+h_{t} / 16\right)  \tag{3.24}\\
D_{i}^{*} & :=\int_{\tilde{L}_{i}^{*}}^{\tilde{m}_{i+1}} e^{W_{\kappa}(x)} \mathrm{d} x=e^{W_{\kappa}\left(\tilde{L}_{i}^{*}\right)} A_{\infty}^{\tilde{L}_{i}^{*}}-\int_{\tilde{m}_{i+1}}^{\infty} e^{W_{\kappa}(x)} \mathrm{d} x
\end{align*}
$$

Moreover, we have on $\mathcal{E}_{2}^{3.5} \cap \mathcal{E}_{3}^{3.5}$, where $\mathcal{E}_{3}^{3.5}:=\cap_{j=1}^{n_{t}-1}\left\{\tilde{\tau}_{j+1}\left(h_{t}\right)-\tilde{m}_{j+1} \leq 8 h_{t} / \kappa\right\}$,

$$
\int_{\tilde{m}_{i+1}}^{\infty} e^{W_{\kappa}(x)} \mathrm{d} x \leq\left[\tilde{\tau}_{i+1}\left(h_{t}\right)-\tilde{m}_{i+1}+A_{\infty}^{\tilde{\tau}_{i+1}\left(h_{t}\right)}\right] e^{W_{\kappa}\left(\tilde{m}_{i+1}\right)+h_{t}} \leq\left(8 h_{t} / \kappa+e^{h_{t} / 16}\right) e^{W_{\kappa}\left(\tilde{m}_{i}\right)}
$$

for large $t$ since $W_{\kappa}\left(\tilde{m}_{i+1}\right) \leq W_{\kappa}\left(\tilde{L}_{i}^{+}\right) \leq W_{\kappa}\left(\tilde{m}_{i}\right)-h_{t}$. This yields to $D_{i}^{*} \geq e^{W_{\kappa}\left(\tilde{m}_{i}\right)+11 h_{t} / 16} / 2$ for large $t$. Hence,

$$
\begin{equation*}
\mathbb{P}\left(\overline{\mathcal{E}_{1}^{3.5}}\right) \leq \mathbb{P}\left(\overline{\mathcal{E}_{1}^{3.5}} \cap \mathcal{E}_{2}^{3.5} \cap \mathcal{E}_{3}^{3.5}\right)+P\left(\overline{\mathcal{E}_{2}^{3.5}}\right)+P\left(\overline{\mathcal{E}_{3}^{3.5}}\right) \leq 2 n_{t} e^{-h_{t} / 8}+P\left(\overline{\mathcal{E}_{2}^{3.5}}\right)+P\left(\overline{\mathcal{E}_{3}^{3.5}}\right) \tag{3.25}
\end{equation*}
$$

where $P\left(\overline{\mathcal{E}_{3}^{3.5}}\right) \leq C_{+} n_{t} e^{-\kappa h_{t} / 2^{3 / 2}}$ by (2.4) and Lemma 2.3.

Let $\tilde{\tau}_{i+1}^{*}\left(h_{t}\right):=\inf \left\{u \geq \tilde{L}_{i}^{*}, W_{\kappa}(u)-\inf _{\left[\tilde{L}_{i}^{*}, u\right]} W_{\kappa} \geq h_{t}\right\}$ for $i \geq 1$, and $\mathcal{E}_{4}^{3.5}:=\cap_{i=1}^{n_{t}-1}\left\{\tilde{\tau}_{i+1}^{*}\left(h_{t}\right)=\right.$ $\left.\tilde{\tau}_{i+1}\left(h_{t}\right)\right\}$. By definition of $\tilde{\tau}_{i}\left(h_{t}\right),\left\{\tilde{\tau}_{i+1}^{*}\left(h_{t}\right) \neq \tilde{\tau}_{i+1}\left(h_{t}\right)\right\}=\left\{\tilde{\tau}_{i+1}^{*}\left(h_{t}\right) \leq \tilde{L}_{i+1}^{\sharp}\right\}=\left\{\inf _{\left[\tilde{L}_{i}^{*}, \tilde{\tau}_{i+1}^{*}\left(h_{t}\right)\right]} W_{\kappa}-\right.$ $\left.W_{\kappa}\left(\tilde{L}_{i}^{*}\right) \geq-2 h_{t}^{+}-3 h_{t} / 4\right\}$. Since $\tilde{L}_{i}^{*}$ is a stopping time, using the strong Markov property, we obtain $P\left(\tilde{\tau}_{i+1}^{*}\left(h_{t}\right)=\tilde{\tau}_{i+1}\left(h_{t}\right)\right) \leq \mathbb{P}\left(\inf _{\left[0, \tau_{1}^{*}\left(h_{t}\right)\right]} W_{\kappa} \geq-2 h_{t}^{+}-3 h_{t} / 4\right) \leq C_{+} e^{-\kappa h_{t} / 2}$ as in the proof of Lemma 2.3. Then $P\left(\overline{\mathcal{E}_{4}^{3.5}}\right) \leq C_{+} n_{t} e^{-\kappa h_{t} / 2}$.

On $\mathcal{E}_{4}^{3.5}, H\left(\tilde{m}_{i+1}\right)-H\left(\tilde{L}_{i}\right)=H_{X_{i}}\left(\tilde{m}_{i+1}\right) \leq H_{X_{i}}\left(\tilde{\tau}_{i+1}\left(h_{t}\right)\right)=H_{X_{i}}\left(\tilde{\tau}_{i+1}^{*}\left(h_{t}\right)\right)$, which is, on $\mathcal{E}_{1}^{3.5}$, the total time spent by $X_{i}$ in $\left[\tilde{L}_{i}^{*},+\infty\right)$ before hitting $\tilde{\tau}_{i+1}\left(h_{t}\right)$. This last quantity is less than or equal to the total time spent in $\left[\tilde{L}_{i}^{*},+\infty\right)$ by $X_{i}^{*}$ before hitting $\tilde{\tau}_{i+1}^{*}\left(h_{t}\right)$, which has the same law as $H_{+}\left(\tau_{1}^{*}\left(h_{t}\right)\right)$ under the annealed probability $\mathbb{P}$, since $\tilde{L}_{i}^{*}$ is a stopping time for $W_{\kappa}$ and then $\left(W_{\kappa}\left(\tilde{L}_{i}^{*}+x\right)-W_{\kappa}\left(\tilde{L}_{i}^{*}\right), x \geq 0\right)$ is a standard Brownian motion. Consequently, $\mathbb{E}\left[\left(H\left(\tilde{m}_{i+1}\right)-H\left(\tilde{L}_{i}\right)\right) 1_{\mathcal{E}_{1}^{3.5} \cap \mathcal{E}_{4}^{3.5}}\right] \leq \mathbb{E}\left[H_{+}\left(\tau_{1}^{*}\left(h_{t}\right)\right)\right]$ for $1 \leq i \leq n_{t}-1$. This, together with Lemma 3.4 and a Markov inequality lead to

$$
\mathbb{P}\left(H_{+}\left(\tilde{m}_{1}\right)+\sum_{i=1}^{n_{t}-1}\left(H\left(\tilde{m}_{i+1}\right)-H\left(\tilde{L}_{i}\right)\right) \geq t /\left(\log h_{t}\right), \mathcal{E}_{1}^{3.5}, \mathcal{E}_{4}^{3.5}, \mathcal{E}_{5}^{3.5}, \mathcal{V}_{t}\right) \leq C_{+} n_{t} e^{h_{t}} \frac{\log h_{t}}{t}
$$

where $\mathcal{E}_{5}^{3.5}:=\left\{m_{1} \leq \tau_{1}^{*}\left(h_{t}\right)\right\}$. Recall that $\mathbb{P}\left(\overline{\mathcal{E}_{5}^{3.5}}\right) \leq \mathbb{P}\left(0 \leq M_{0}<m_{1}\right) \leq h_{t} e^{-\kappa h_{t}}$ as seen in the proof of Lemma 2.3, case $j=1$. This, combined with (3.19), (3.23), (3.25) and Lemma 2.3, concludes the proof.

## 4. Time spent in a standard valley

The aim of this section is to prove Proposition 4.6. First we need additional estimates given below.
4.1. Some technical estimates. Recall that $(R(s), s \geq 0)$ is a $(-\kappa / 2)$ drifted 3-dimensional Bessel process, and let $\left(\tilde{W}_{\kappa}^{b}(s), 0 \leq s \leq \tau^{\tilde{W}_{\kappa}^{b}}(a)\right)$ a $(-\kappa / 2)$-drifted Brownian motion starting from $b$ and killed when it first hits $a<b$. We now introduce

$$
F^{ \pm}(a):=\int_{0}^{\tau^{R}(a)} e^{ \pm R(s)} d s, \quad G^{ \pm}(a, b):=\int_{0}^{\tau^{\tilde{W}_{\kappa}^{b}}(a)} e^{ \pm \tilde{W}_{\kappa}^{b}(s)}(s) d s
$$

Lemma 4.1. There exists $C_{4}>0, M>0$ and $\eta_{1} \in(0,1)$ such that $\forall y>M, \forall \gamma \in\left(0, \eta_{1}\right]$,

$$
\begin{align*}
\left|E\left(e^{-\gamma F^{-}(y)}\right)-[1+2 \gamma /(\kappa+1)]^{-1}\right| & \leq C_{4} \max \left(e^{-\kappa y}, \gamma^{3 / 2}\right)  \tag{4.1}\\
\left|E\left(e^{-\gamma F^{+}(y) / e^{y}}\right)-[1-2 \gamma /(\kappa+1)]\right| & \leq C_{4} \max \left(e^{-\kappa y}, \gamma^{3 / 2}\right)  \tag{4.2}\\
\left|E\left(e^{-\gamma G^{+}(y / 2, y) / e^{y}}\right)-\left[1-\Gamma(1-\kappa)(2 \gamma)^{\kappa} / \Gamma(1+\kappa)\right]\right| & \leq C_{4} \max \left(\gamma^{\kappa} e^{-\kappa y / 2}, \gamma\right) \tag{4.3}
\end{align*}
$$

Moreover, there exists $c_{1}>0$, such that for all $y>0, E\left(F^{+}(y) / e^{y}\right) \leq c_{1}$.
Proof: We denote by $I_{\kappa}$ and $K_{\kappa}$ the modified Bessel functions, respectively of the first and second kind (see e.g. [4] p. 638). First, ([4], 2.10.3 page 302) with $\alpha=0, x=y, z=\omega y<y$, $\beta=1 / 2$, and $\mu=-\kappa / 2$ gives

$$
E\left(e^{-\gamma G^{+}(\omega y, y)}\right)=\frac{e^{\kappa(1-\omega) y / 2} K_{\kappa}\left(2 \sqrt{2 \gamma} e^{y / 2}\right)}{K_{\kappa}\left(2 \sqrt{2 \gamma} e^{\omega y / 2}\right)}, \quad \gamma>0
$$

Note that this expression can be deduced from the fact that $G^{+}(\omega y, y) \stackrel{\mathcal{L}}{=} \int_{0}^{\tau \tilde{W}_{\kappa}(\omega y)} e^{\tilde{W}_{\kappa}(x)} d x$, $A_{\infty} \stackrel{\mathcal{L}}{=} \int_{0}^{\tau^{\tilde{W}_{\kappa}}(\omega y)} e^{\tilde{W}_{\kappa}(x)} d x+e^{\omega y} A_{\infty}^{\prime}$, with $A_{\infty}^{\prime} \stackrel{\mathcal{L}}{=} A_{\infty}$ independent of $\int_{0}^{\tau^{\tilde{W}_{\kappa}}(\omega y)} e^{\tilde{W}_{\kappa}(x)} d x$ and [13].

Recall that $\mathbb{P}(R \in \Lambda)=\lim _{x \downarrow 0} P^{x}\left(\Lambda \mid \tau^{W_{\kappa}}(0)=\infty\right)$ for every $\Lambda \in \mathcal{G}_{\tau^{W_{\kappa}^{x}}(a)}$. Hence, we have for $\gamma>0$, by the strong Markox property,

$$
\begin{align*}
E\left(e^{-\gamma F^{ \pm}(y)}\right) & =\lim _{x \downarrow 0} E^{x}\left[\exp \left(-\gamma \int_{0}^{\tau_{W_{-\kappa}(y)}} e^{ \pm W_{-\kappa}(s)} \mathrm{d} s\right) \mathbb{1}_{\left\{T_{y}<T_{0}\right\}}\right] \frac{P^{y}\left(T_{0}=\infty\right)}{P^{x}\left(T_{0}=\infty\right)} \\
& =\frac{ \pm \kappa^{-1} \sinh (k y / 2)}{I_{\kappa}\left(2 \sqrt{2 \gamma} e^{ \pm y / 2}\right) K_{\kappa}(2 \sqrt{2 \gamma})-K_{\kappa}\left(2 \sqrt{2 \gamma} e^{ \pm y / 2}\right) I_{\kappa}(2 \sqrt{2 \gamma})} \tag{4.4}
\end{align*}
$$

where the last equality comes from ([4], 3.10.7(b) page 317) with $\alpha=0, a=0, b=a, \beta= \pm 1 / 2$, and $x>0$, since $P^{x}\left(T_{0}=\infty\right)=1-e^{-\kappa x}$ and $I_{\kappa}^{\prime}(u) K_{\kappa}(u)-I_{\kappa}(u) K_{\kappa}^{\prime}(u)=1 / u$ for $u>0$. Now, notice that when $\max \left(v^{2 \kappa}, u^{3}\right) \rightarrow 0$,

$$
\begin{equation*}
f(u, v):=\frac{-\kappa^{-1}\left(v^{-\kappa}-v^{\kappa}\right)}{I_{\kappa}(u v) K_{\kappa}(u)-K_{\kappa}(u v) I_{\kappa}(u)}=\left[1+u^{2} /(4(\kappa+1))\right]^{-1}+O\left(\max \left(v^{2 \kappa}, u^{3}\right)\right) . \tag{4.5}
\end{equation*}
$$

Hence, there exist $C_{4}^{\prime}>0$ and $\eta_{1}^{\prime}>0$ such that

$$
\max \left(v^{2 \kappa}, u^{3}\right)<\eta_{1}^{\prime} \Longrightarrow\left|f(u, v)-\left[1+u^{2} /(4(\kappa+1))\right]^{-1}\right| \leq C_{4}^{\prime} \max \left(v^{2 \kappa}, u^{3}\right) .
$$

Consequently, taking $u=2 \sqrt{2 \gamma}$ and $v=e^{-y / 2}$, there exist $M>0$ and $\eta_{1}>0$ satisfying (4.1). Similarly, we get as $u \rightarrow 0$ and $y \rightarrow+\infty$,

$$
\begin{aligned}
E\left(e^{-u F^{+}(y) / e^{y}}\right) & =1-2 u /(\kappa+1))+o\left(\max \left(u^{3 / 2}, e^{-\kappa y}\right)\right), \\
E\left(e^{-u G^{+}(y / 2, y) / e^{y}}\right) & =1-\Gamma(1-\kappa)(2 u)^{\kappa} / \Gamma(1+\kappa)+o\left(\max \left(u, u^{\kappa} e^{-\kappa y / 2}\right)\right),
\end{aligned}
$$

which yield respectively to (4.2) and (4.3). Finally we deduce the result for the mean of $F^{+}(y) / e^{y}$ computing the first derivative of (4.4).

Before proving Proposition 4.6, we still need to introduce the following technical lemma:
Lemma 4.2. $B$ being a standard two-sided Brownian motion, there exists a constant $c_{3}$ such that for every $0<\varepsilon<1,0<\delta<1$ and $x>0$,

$$
\begin{align*}
& \mathbb{P}\left(\sup _{u \in[-\delta, \delta]}\left|\mathcal{L}_{B}\left(\tau^{B}(1), u\right)-\mathcal{L}_{B}\left(\tau^{B}(1), 0\right)\right|>\varepsilon \mathcal{L}_{B}\left(\tau^{B}(1), 0\right)\right) \leq c_{3} \frac{\delta^{1 / 6}}{\varepsilon^{2 / 5}},  \tag{4.6}\\
& \mathbb{P}\left(\sup _{u \in[0,1]} \mathcal{L}_{B}\left(\tau^{B}(1), u\right) \geq x\right) \leq 4 e^{-x / 2},  \tag{4.7}\\
& \mathbb{P}\left(\sup _{u \leq 0} \mathcal{L}_{B}\left(\tau^{B}(1), u\right) \geq x\right) \leq 1 / x . \tag{4.8}
\end{align*}
$$

Proof: First, (4.7) and (4.8) follow respectively by [11] Lemma 2.3 and [39] Lemma 3.1 together with the first Ray-Knight theorem.

Let $0<\varepsilon<1,0<\delta<1$ and

$$
\mathcal{E}^{4.2}:=\left\{\sup _{u \in[-\delta, \delta]}\left|\mathcal{L}_{B}\left(\tau^{B}(1), u\right)-\mathcal{L}_{B}\left(\tau^{B}(1), 0\right)\right|>\varepsilon \mathcal{L}_{B}\left(\tau^{B}(1), 0\right)\right\} .
$$

We have, for $\alpha>0$ and $\beta>0$,

$$
\begin{align*}
\mathbb{P}\left(\mathcal{E}^{4.2}\right) & =\mathbb{P}\left(\mathcal{E}^{4.2} \cap\left\{\mathcal{L}_{B}\left(\tau^{B}(1), 0\right) \geq \alpha\right\}\right)+\mathbb{P}\left(\mathcal{E}^{4.2} \cap\left\{\mathcal{L}_{B}\left(\tau^{B}(1), 0\right)<\alpha\right\}\right) \\
& \leq \mathbb{P}\left(\sup _{u \in[-\delta, \delta]}\left|\mathcal{L}_{B}\left(\tau^{B}(1), u\right)-\mathcal{L}_{B}\left(\tau^{B}(1), 0\right)\right|>\varepsilon \alpha\right)+\mathbb{P}\left[\mathcal{L}_{B}\left(\tau^{B}(1), 0\right)<\alpha\right] \\
& \leq \mathbb{P}\left[\tau^{B}(1) \geq \beta\right]+\mathbb{P}\left(\sup _{u \in[-\delta, \delta], 0 \leq s \leq \beta}\left|\mathcal{L}_{B}(s, u)-\mathcal{L}_{B}(s, 0)\right|>\varepsilon \alpha\right)+\frac{\alpha}{2} \tag{4.9}
\end{align*}
$$

since $\mathcal{L}_{B}\left(\tau^{B}(1), 0\right)$ is an exponential variable with mean 2 . Now notice that

$$
\mathbb{P}\left[\tau^{B}(1) \geq \beta\right]=\mathbb{P}\left(\sup _{0 \leq u \leq \beta} B(u)<1\right)=\mathbb{P}(\sqrt{\beta}|B(1)|<1) \leq 2 / \sqrt{2 \pi \beta} .
$$

Let $0<\varepsilon_{0}<1 / 2$. The second term of (4.9) is less than or equal to

$$
\begin{align*}
& \mathbb{P}\left(\sup _{u \in[-\delta, \delta]-\{0\}, 0 \leq s \leq \beta} \frac{\left|\mathcal{L}_{B}(s, u)-\mathcal{L}_{B}(s, 0)\right|}{|u|^{1 / 2-\varepsilon_{0}}}>\frac{\varepsilon \alpha}{\delta^{1 / 2-\varepsilon_{0}}}\right) \\
\leq & \frac{\delta^{1 / 2-\varepsilon_{0}}}{\varepsilon \alpha} \mathbb{E}\left(\sup _{a \neq b, 0 \leq s \leq \beta} \frac{\left|\mathcal{L}_{B}(s, b)-\mathcal{L}_{B}(s, a)\right|}{|a-b|^{1 / 2-\varepsilon_{0}}}\right), \tag{4.10}
\end{align*}
$$

the last inequality being a consequence of Markov inequality. Now, applying Barlow and Yor ([2], (ii) p. 199 with $\gamma=1$ ) to the continuous martingale $B(. \wedge \beta)$, we can say that the expectancy in (4.10) is less than or equal to $C_{1, \varepsilon_{0}}(\sqrt{\beta})^{1 / 2+\varepsilon_{0}}$ where $C_{1, \varepsilon_{0}}>0$. Consequently,

$$
\mathbb{P}\left(\mathcal{E}^{4.2}\right) \leq c_{3} / \sqrt{\beta}+C_{1, \varepsilon_{0}}(\sqrt{\beta})^{1 / 2+\varepsilon_{0}} \delta^{1 / 2-\varepsilon_{0}}(\varepsilon \alpha)^{-1}+\alpha / 2 .
$$

Now, we choose $\alpha=\varepsilon^{-2 / 5} \delta^{1 / 5}, \beta=\varepsilon^{4 / 5} \delta^{-2 / 5}$ and $\varepsilon_{0}<1 / 36$; we get $\mathbb{P}\left(\mathcal{E}^{4.2}\right) \leq c_{3} \delta^{1 / 6} \varepsilon^{-2 / 5}$, which concludes the proof.

### 4.2. Approximation of the time to escape from a typical valley.

We now prove that a standard exit time can be approximated by product and sums of independent well known random variables. We recall that $\mathbf{U}$ is defined in Proposition 3.2.

Proposition 4.3. Let $\varepsilon_{t}:=3 e^{-(1-3 \delta) h_{t} / 6}$. There exist independent random variables $\mathcal{I}_{1}^{+}, \mathcal{I}_{2}^{+}, \mathcal{I}_{1}^{-}$ and $\mathcal{I}_{2}^{-}$, depending on $t$ and independent of $\mathbf{e}_{1}$ a random variable with exponential law with mean 2 , such that $\mathcal{I}_{1}^{+}$is distributed as $F^{+}\left(h_{t}\right), \mathcal{I}_{2}^{+}$as $G^{+}\left(h_{t} / 2, h_{t}\right)$, and $\mathcal{I}_{1}^{-}$and $\mathcal{I}_{2}^{-}$as $F^{-}\left(h_{t} / 2\right)$, such that for $t$ large enough, $\mathbb{P}\left(\mathcal{A}_{t}\right) \geq 1-C_{+} e^{-\delta^{2} c_{-} h_{t}}$, where

$$
\mathcal{A}_{t}:=\left\{\left|\mathbf{U}-\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right)\left(\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}\right) \mathbf{e}_{1}\right| \leq\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right)\left(\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}\right) \mathbf{e}_{1} \varepsilon_{t}\right\} .
$$

The proof of this Proposition involves 3 Lemmata, the first two are straightforward consequence of what we have already discussed or proved the last one is more technical.

First applying Williams decomposition (see Fact 2.2), we have the following
Lemma 4.4. We have

$$
\begin{align*}
& A_{1}^{+}:=\int_{m_{2}}^{\tau_{2}\left(h_{t}\right)} e^{V^{(2)}(x)} \mathrm{d} x \stackrel{\mathcal{L}}{=} F^{+}\left(h_{t}\right), \quad A_{1}^{-}:=\int_{m_{i}}^{\tau_{2}\left(h_{t} / 2\right)} e^{-V^{(2)}(x)} \mathrm{d} x \stackrel{\mathcal{L}}{=} F^{-}\left(h_{t} / 2\right)  \tag{4.11}\\
& A_{2}^{ \pm}:=\int_{\tau_{2}\left(h_{t}\right)}^{L_{2}} e^{ \pm V^{(2)}(x)} \mathrm{d} x \stackrel{\mathcal{L}}{=} G^{ \pm}\left(h_{t} / 2, h_{t}\right), \quad B_{1}^{ \pm}:=\int_{m_{2}}^{\tau_{2}^{-}\left(h_{t} / 2\right)} e^{ \pm V^{(2)}(x)} \stackrel{\mathcal{L}}{=} F^{ \pm}\left(h_{t} / 2\right),
\end{align*}
$$

with $L_{2}:=\inf \left\{x>\tau_{2}\left(h_{t}\right), W_{\kappa}(x)-W_{\kappa}\left(m_{2}\right)=h_{t} / 2\right\}$,
Recall that $\tilde{A}$ is defined just before (3.9). We have,
Lemma 4.5. For all $0<\zeta \leq 1,0<\varepsilon<1 / 2$ and $t$ large enough, we have

$$
\begin{equation*}
P\left[e^{\zeta h_{t}(1-\varepsilon)} \leq \tilde{A}\left(\tilde{\tau}_{1}\left(\zeta h_{t}\right)\right) \leq e^{\zeta h_{t}(1+\varepsilon)}\right] \geq 1-C_{+} e^{-\varepsilon \kappa \zeta h_{t}} . \tag{4.12}
\end{equation*}
$$

Proof: Working on $\mathcal{V}_{t}$, and considering the first expression in (4.11) (replacing $h_{t}$ by $\zeta h_{t}$ ), we know that the probability in 4.12 is larger than $P\left[e^{\zeta h_{t}(1-\varepsilon)} \leq F^{+}\left(\zeta h_{t}\right) \leq e^{\zeta h_{t}(1+\varepsilon)}\right]-P\left(\overline{\mathcal{V}}_{t}\right)$. Then by Markov inequality and Lemma 4.1 $P\left[F^{+}\left(\zeta h_{t}\right)>e^{\zeta h_{t}(1+\varepsilon)}\right] \leq c_{1} e^{-\epsilon \zeta h_{t}}$. For the lower
bound, taking $\gamma=\zeta(1-\varepsilon / 2), \alpha=\zeta(1-\varepsilon)$ and $\omega=\zeta$ in (3.2) we get with a probability larger than $1-C_{+} e^{-\varepsilon \kappa h_{t}}$,

$$
F^{+}\left(\zeta h_{t}\right) \geq e^{\zeta h_{t}(1-\varepsilon)}\left[\tau^{R}\left(\zeta h_{t}\right)-\tau^{R}\left(\zeta h_{t}(1-\varepsilon / 2)\right)\right] .
$$

Finally using (3.3) we get the result.
We first prove the following intermediate result
Lemma 4.6. For all $0<\varepsilon<\inf \left(2 / 27, \kappa^{2} / 2\right)$, and $t$ large enough,

$$
\mathbb{P}\left(\left|\mathbf{U}-\tilde{A}\left(\tilde{L}_{2}\right) \mathcal{I}^{-} \mathbf{e}_{1}\right| \leq 3 e^{-(1-3 \varepsilon) h_{t} / 6} \tilde{A}\left(\tilde{L}_{2}\right) \mathcal{I}^{-} \mathbf{e}_{1}\right) \geq 1-C_{+} e^{-c_{-} \varepsilon h_{t}},
$$

where $\mathcal{I}^{-}:=\int_{\tilde{\tau}_{2}^{-}\left(h_{t} / 2\right)}^{\tilde{\tau}_{2}\left(h_{t} / 2\right)} e^{-\tilde{V}^{(2)}(u)} \mathrm{d} u$. Also $\mathbf{e}_{1}$ is independent of $W_{\kappa}$.
Proof Let $\delta_{t}:=e^{-h_{t}(1-3 \varepsilon) / 2}$. We introduce $x_{0}^{+}:=\tilde{A}^{-1}\left(\delta_{t} \tilde{A}\left(\tilde{L}_{2}\right)\right)$ and $x_{0}^{-}:=\tilde{A}^{-1}\left(-\delta_{t} \tilde{A}\left(\tilde{L}_{2}\right)\right)$. We cut the integral defining $\mathbf{U}$ into three parts:

$$
\begin{equation*}
\mathbf{U}=\int_{\tilde{L}_{1}^{-}}^{x_{0}^{-}}+\int_{x_{0}^{-}}^{x_{0}^{+}}+\int_{x_{0}^{+}}^{\tilde{L}_{1}} e^{-\tilde{V}^{(2)}(u)} \tilde{A}\left(\tilde{L}_{2}\right) \mathcal{L}_{B^{\prime}}\left(\tau_{B^{\prime}}(1), \tilde{A}(u) / \tilde{A}\left(\tilde{L}_{2}\right)\right) \mathrm{d} u=: \mathcal{J}_{0}+\mathcal{J}_{1}+\mathcal{J}_{2}, \tag{4.13}
\end{equation*}
$$

where $B^{\prime}()=.B\left(\left[\tilde{A}\left(\tilde{L}_{2}\right)\right]^{2}.\right) / \tilde{A}\left(\tilde{L}_{2}\right)$, and $\left(B_{u}^{\prime}\right)_{u \geq 0}$ is a standard Brownian motion independent of $W_{\kappa}$, which we still denote by $B$ in the rest of this proof. In what follows, we show that the main contribution comes from $\mathcal{J}_{1}$.
For $\mathcal{J}_{2}$, we prove that for $t$ large enough

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{J}_{2} \geq \tilde{A}\left(\tilde{L}_{2}\right) c_{+} h_{t}^{2} e^{-(1-\varepsilon) h_{t} / 2}\right) \leq C_{+} e^{-\varepsilon c_{-} h_{t}} . \tag{4.14}
\end{equation*}
$$

To this aim, we first notice that $x_{0}^{+} \in\left(\tilde{m}_{2}, \tilde{L}_{2}\right)$ since $0 \leq \delta_{t} \leq 1$. Moreover, $\tilde{A}(x) \leq \delta_{t} \tilde{A}\left(\tilde{L}_{2}\right)$ iff $x \leq x_{0}^{+}$. We introduce $\left.\mathcal{E}_{1}^{4.6}:=\left\{x_{0}^{+} \geq \tilde{\tau}_{2}^{+}\left(h_{t} / 2\right)\right)\right\}$, and prove that $P\left(\overline{\mathcal{E}_{1}^{4.6}}\right) \leq C_{+} e^{-c_{-} \varepsilon h_{t}}$. First by (4.12)

$$
\begin{equation*}
\left.P\left(\tilde{A}\left(\tilde{\tau}_{2}^{+}\left(h_{t} / 2\right)\right)\right) \leq e^{h_{t}(1+\varepsilon) / 2}\right) \geq 1-C_{+} e^{-\varepsilon h_{t} / 2} . \tag{4.15}
\end{equation*}
$$

Working on $\mathcal{V}_{t}, \tilde{A}\left(\tilde{L}_{2}\right)=A_{1}^{+}+A_{2}^{+}$, so by Lemma $4.4 P\left(\tilde{A}\left(\tilde{L}_{2}\right) \geq e^{h_{t}(1-\varepsilon)}\right) \geq P\left(F^{+}\left(h_{t}\right) \geq\right.$ $\left.e^{h_{t}(1-\varepsilon)}\right)-P\left(\overline{\mathcal{V}}_{t}\right)$. Hence, we get $P\left(\tilde{A}\left(\tilde{L}_{2}\right) \geq e^{h_{t}(1-\varepsilon)}\right) \geq 1-4 e^{-\kappa \varepsilon h_{t} / 2}$ by (3.5). This, together with (4.15) gives

$$
\begin{equation*}
\left.P\left(\mathcal{E}_{1}^{4.6}\right)=P\left[\delta_{t} \geq \tilde{A}\left(\tilde{\tau}_{2}^{+}\left(h_{t} / 2\right)\right)\right) / \tilde{A}\left(\tilde{L}_{2}\right)\right] \geq 1-C_{+} e^{-c_{-} \varepsilon h_{t}} . \tag{4.16}
\end{equation*}
$$

We now turn back to $\mathcal{J}_{2}$. We have on $\mathcal{E}_{1}^{4.6}$,

$$
\begin{equation*}
\mathcal{J}_{2} \leq \sup _{u \in[0,1]} \mathcal{L}_{B}\left(\tau^{B}(1), u\right) \tilde{A}\left(\tilde{L}_{2}\right) \int_{\left.\tilde{\tau}_{2}^{+}\left(h_{t} / 2\right)\right)}^{\tilde{L}_{2}} e^{-\tilde{V}^{(2)}(u)} \mathrm{d} u . \tag{4.17}
\end{equation*}
$$

Let $\left.\mathcal{E}_{2}^{4.6}:=\left\{\inf _{\left.\tilde{\tau}_{2}^{+}\left(h_{t} / 2\right)\right) \leq u \leq \tilde{\tau}_{2}\left(h_{t}\right)} \tilde{V}^{(2)}(u)\right]>(1-\varepsilon) h_{t} / 2\right\}$ and $\mathcal{E}_{3}^{4.6}:=\left\{\tilde{L}_{2}^{+}-\tilde{L}_{2}^{-} \leq 28 h_{t} / \kappa\right\}$. Working on $\mathcal{V}_{t}, \mathcal{E}_{2}^{4.6}=\left\{\inf _{\tau_{2}\left(h_{t} / 2\right) \leq u \leq \tau_{2}\left(h_{t}\right)}\left[V^{(2)}(u)\right]>(1-\varepsilon) h_{t} / 2\right\}$, then Williams' decomposition, equation (3.2) with $\alpha=1 / 2, \gamma=1 / 2(1-\varepsilon)$ and $\omega=1$ give $P\left(\overline{\mathcal{E}_{2}^{4.6}}\right) \leq P\left(\overline{\mathcal{E}_{2}^{4.6}}, \mathcal{V}_{t}\right)+P\left(\overline{\mathcal{V}}_{t}\right) \leq$ $2 e^{-\kappa \varepsilon h_{t} / 2}+P\left(\overline{\mathcal{V}}_{t}\right)$. Moreover, $P\left(\overline{\mathcal{E}_{3}^{4.6}}\right) \leq C_{+} e^{-\kappa h_{t} / 24}$ by (2.6) and Lemma 2.3. We have on $\mathcal{E}_{2}^{4.6} \cap \mathcal{E}_{3}^{4.6}$,

$$
\left.\int_{\left.\tilde{\tau}_{2}^{+}\left(h_{t} / 2\right)\right)}^{\tilde{L}_{2}} e^{-\tilde{V}^{(2)}(u)} \mathrm{d} u \leq e^{-(1-\varepsilon) h_{t} / 2}\left(\tilde{L}_{2}-\tilde{\tau}_{2}^{+}\left(h_{t} / 2\right)\right)\right) \leq h_{t} \frac{28}{\kappa} e^{-(1-\varepsilon) h_{t} / 2} .
$$

This, together with (4.17) and (4.7) with $M=h_{t}$ proves (4.14).
For $\mathcal{J}_{0}$, we prove that for $t$ large enough

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{J}_{0} \geq C_{+} \tilde{A}\left(\tilde{L}_{2}\right) h_{t} e^{-(1 / 2-2 \varepsilon) h_{t}}\right) \leq C_{+} e^{-\varepsilon \delta c_{-} h_{t}} \tag{4.18}
\end{equation*}
$$

With the same arguments as those used for (4.16), we get $P\left(\mathcal{E}_{4}^{4.6}\right) \geq 1-C_{+} e^{-c_{-} \varepsilon h_{t}}$, where $\mathcal{E}_{4}^{4.6}:=\left\{x_{0}^{-} \leq \tilde{\tau}_{2}^{-}\left(h_{t} / 2\right)\right\}$.

We now define $\mathcal{E}_{5}^{4.6}:=\left\{x_{0}^{-} \geq \tilde{L}_{2}^{-}\right\}$and prove it has a large probability. In the one hand we have

$$
\begin{equation*}
P\left(\tilde{A}\left(\tilde{L}_{2}\right) \geq e^{h_{t}(1+\varepsilon / 2)}\right) \leq C_{+} e^{-c_{-} \varepsilon h_{t} / 2} . \tag{4.19}
\end{equation*}
$$

Indeed $\tilde{A}\left(\tilde{L}_{2}\right)=\tilde{A}\left(\tilde{\tau}_{2}\left(h_{t}\right)\right)+\tilde{A}\left(\tilde{L}_{2}\right)-\tilde{A}\left(\tilde{\tau}_{2}\left(h_{t}\right)\right)$. Thanks to Lemma 4.5 $P\left(\tilde{A}\left(\tilde{\tau}_{2}\left(h_{t}\right)\right)>2 e^{h_{t}(1+\varepsilon / 2)}\right)$ $\leq C_{+} e^{-\epsilon h_{t} / 2} / 2$. Also the strong Markov property gives $\tilde{A}\left(\tilde{L}_{2}\right)-\tilde{A}\left(\tilde{\tau}_{2}\left(h_{t}\right)\right) \stackrel{\mathcal{L}}{=} G^{+}\left(h_{t}, h_{t} / 2\right)$, then we notice that $G^{+}\left(h_{t}, h_{t} / 2\right) \stackrel{\mathcal{L}}{=} e^{h_{t}} G^{+}\left(0,-h_{t} / 2\right) \leq e^{h_{t}} A_{\infty}$, where $A_{\infty}$ is defined just above (3.1). So $P\left(\tilde{A}\left(\tilde{L}_{2}\right)-\tilde{A}\left(\tilde{\tau}_{2}\left(h_{t}\right)\right)>e^{h_{t}(1+\varepsilon / 2)} / 2\right) \leq P\left(A_{\infty}>e^{\left.\varepsilon h_{t} / 2\right)} / 2\right) \leq C_{+} e^{-\kappa \epsilon h_{t} / 2} / 2 \leq C_{+} e^{-c_{-} \epsilon h_{t} / 2} / 2$, as $A_{\infty} \stackrel{\mathcal{L}}{=} 2 / \gamma_{\kappa}$.

On the other hand $\left.P\left(-\tilde{A}\left(\tilde{L}_{2}^{-}\right) \geq e^{h_{t}(1+\kappa / 2)}\right) \geq P\left(F^{+}\left(h_{t}^{+}\right) \geq e^{h_{t}(1+\kappa / 2)}\right)\right)-P\left(\overline{\mathcal{L}}_{t}\right)$ (see Remark 2.4). So by (3.5) $P\left(-\tilde{A}\left(\tilde{L}_{2}^{-}\right) \geq e^{h_{t}(1+\kappa / 2)}\right) \geq 1-4 e^{-\kappa^{2} h_{t} / 4}$ for large $t$. This, together with (4.19), gives

$$
P\left(\mathcal{E}_{5}^{4.6}\right)=P\left(-\delta_{t} \geq \tilde{A}\left(\tilde{L}_{2}^{-}\right) / \tilde{A}\left(\tilde{L}_{2}\right)\right) \geq 1-C_{+} e^{-c_{-} \varepsilon h_{t} / 4}
$$

Let $\mathcal{E}_{6}^{4.6}:=\left\{\inf _{\tilde{L}_{2}^{-} \leq u \leq \tilde{\tau}_{2}^{-}\left(h_{t} / 2\right)}\left(\tilde{V}^{(2)}(u)\right) \geq e^{(1 / 2-\varepsilon) h_{t}}\right\}$ and $\mathcal{E}_{7}^{4.6}:=\left\{\sup _{s \leq 0} \mathcal{L}_{B}\left(\tau^{B}(1), s\right) \leq e^{\varepsilon h_{t}}\right\}$. Also by Remark 2.4, on $\left\{\tau_{2}^{-}\left(h_{t}^{+}\right) \geq 0\right\}$, $\left(V^{(2)}(u), \tau_{2}^{-}\left(h_{t}^{+}\right) \leq u \leq \tau_{2}^{-}\left(h_{t} / 2\right)\right)$ has the same law as $\left(R^{(2)}(s), \tau^{R^{(2)}}\left(h_{t} / 2\right) \leq s \leq \tau^{R^{(2)}}\left(h_{t}^{+}\right)\right)$. Hence (3.2) with $\gamma=1 / 2-\varepsilon, \alpha=1 / 2$ and $\omega=1+\kappa+2 \delta$ gives $P\left(\overline{\mathcal{E}_{6}^{4.6}}\right) \leq P\left(\overline{\mathcal{E}_{6}^{4.6}}, \mathcal{V}_{t}\right)+P\left(\overline{\mathcal{V}}_{t}\right) \leq 2 e^{-\kappa \varepsilon h_{t}}$. Moreover, by $(4.8), \mathbb{P}\left(\overline{\mathcal{E}_{7}^{4.6}}\right) \leq e^{-\varepsilon h_{t}}$.
Therefore, on $\cap{ }_{i=3}^{7} \mathcal{E}_{i}^{4.6} \cap \mathcal{V}_{t}$, i.e with a probability larger than $1-C_{+} e^{-c_{-} \varepsilon h_{t}}$, we obtain

$$
\begin{equation*}
\mathcal{J}_{0} \leq \tilde{A}\left(\tilde{L}_{2}\right) \sup _{s \leq 0} \mathcal{L}_{B}\left(\tau^{B}(1), s\right) \int_{\tilde{L}_{1}^{-}}^{\tilde{\tau}_{1}^{-}\left(h_{t} / 2\right)} e^{-\tilde{V}^{(1)}(u)} \mathrm{d} u \leq C_{+} h_{t} \tilde{A}\left(\tilde{L}_{2}\right) e^{-(1 / 2-2 \varepsilon) h_{t}} \tag{4.20}
\end{equation*}
$$

which yields (4.18).
For $\mathcal{J}_{1}$, we prove that for $t$ large enough,

$$
\begin{equation*}
\mathbb{P}\left[\mathcal{J}_{1} \leq \mathbf{e}_{1} \tilde{A}\left(\tilde{L}_{2}\right) e^{-\varepsilon h_{t}} / 2\right] \leq C_{+} e^{-c_{-} \varepsilon h_{t}}, \tag{4.21}
\end{equation*}
$$

where $\mathbf{e}_{\mathbf{1}}:=\mathcal{L}_{B}\left(\tau^{B}(1), 0\right)$ is an exponential r.v. with mean 2 by the first Ray-Knight theorem, and is independent of $W_{\kappa}$. First, let $\hat{\mathcal{J}}_{1}:=\int_{x_{0}^{-}}^{\tilde{m}_{1}}+\int_{\tilde{m}_{1}}^{x_{0}^{+}} e^{-\left(W_{\kappa}(u)-W_{\kappa}\left(\tilde{m}_{1}\right)\right)} \mathrm{d} u=: \hat{\mathcal{J}}_{1}^{-}+\hat{\mathcal{J}}_{1}^{+}$and

$$
\mathcal{E}_{8}^{4.6}:=\left\{\sup _{u \in\left[-\delta_{t}, \delta_{t}\right]}\left|\mathcal{L}_{B}\left(\tau^{B}(1), u\right)-\mathcal{L}_{B}\left(\tau^{B}(1), 0\right)\right| \leq \delta_{t}^{1 / 3} \mathcal{L}_{B}\left(\tau^{B}(1), 0\right)\right\}
$$

We know that $\mathbb{P}\left(\overline{\mathcal{E}_{8}^{4.6}}\right) \leq C_{+} \delta_{t}^{1 / 30}$ by (4.6). We have on $\mathcal{E}_{8}^{4.6}$,

$$
\begin{equation*}
\left|\mathcal{J}_{1}-\tilde{A}\left(\tilde{L}_{2}\right) \hat{\mathcal{J}}_{1} \mathbf{e}_{\mathbf{1}}\right| \leq \delta_{t}^{1 / 3} \tilde{A}\left(\tilde{L}_{2}\right) \hat{\mathcal{J}}_{1} \mathbf{e}_{\mathbf{1}} \tag{4.22}
\end{equation*}
$$

We finally need that $\hat{\mathcal{J}}_{1}$ is not too small, which we prove with a similar argument as before: first we have $\hat{\mathcal{J}}_{1} \geq \hat{\mathcal{J}}_{1}^{+} \geq e^{-\varepsilon h_{t}}\left(\tilde{\tau}_{2}^{+}\left(\varepsilon h_{t}\right)-\tilde{m}_{2}\right) \geq e^{-\varepsilon h_{t}}$ on $\mathcal{E}_{1}^{4.6} \cap \mathcal{E}_{9}^{4.6}$, where $\mathcal{E}_{9}^{4.6}:=\left\{\tilde{\tau}_{2}^{+}\left(\varepsilon h_{t}\right)-\tilde{m}_{2} \geq 1\right\}$ has a probability $P\left(\mathcal{E}_{9}^{4.6}\right) \geq P\left(\mathcal{E}_{9}^{4.6}, \mathcal{V}_{t}\right)-P\left(\overline{\mathcal{V}}_{t}\right) \geq 1-C_{+} e^{-c_{-} \varepsilon^{2} h_{t}^{2} / 4}-C_{1} n_{t} e^{-\kappa h_{t} / 2}$ by Lemma 2.3, Williams decomposition and (3.3). Hence $\mathcal{J}_{1} \geq\left(1-\delta_{t}^{1 / 3}\right) \tilde{A}\left(\tilde{L}_{2}\right) e^{-\varepsilon h_{t}} \mathbf{e}_{\mathbf{1}}$ on $\mathcal{E}_{1}^{4.6} \cap \mathcal{E}_{8}^{4.6} \cap \mathcal{E}_{9}^{4.6} \cap \mathcal{V}_{t}$, which gives (4.21) for large $t$.

End of the proof. To finish we have to prove that $\hat{\mathcal{J}}_{1}$ is nearly equal to $\mathcal{I}^{-}$.

On $\mathcal{E}_{1}^{4.6} \cap \mathcal{E}_{4}^{4.6} \cap \mathcal{E}_{5}^{4.6}$, we have $\left.\tilde{L}_{2}^{-} \leq x_{0}^{-} \leq \tilde{\tau}_{2}^{-}\left(h_{t} / 2\right) \leq \tilde{\tau}_{2}^{+}\left(h_{t} / 2\right)\right) \leq x_{0}^{+} \leq \tilde{L}_{2}$, and $\tilde{V}^{(2)}(x) \geq$ $(1-\varepsilon) h_{t} / 2$ for every $\left.x \in\left[\tilde{L}_{2}^{-}, \tilde{\tau}_{2}^{-}\left(h_{t} / 2\right)\right] \cap\left[\tilde{\tau}_{2}^{+}\left(h_{t} / 2\right)\right), \tilde{L}_{2}\right]$ on $\mathcal{E}_{2}^{4.6} \cap \mathcal{E}_{6}^{4.6}$, hence on $\cap_{i=4}^{9} \mathcal{E}_{i}^{4.6}$,

$$
0 \leq \hat{\mathcal{J}}_{1}-\mathcal{I}^{-} \leq \int_{\tilde{L}_{2}^{-}}^{\tilde{\tau}_{2}^{-}\left(h_{t} / 2\right)}+\int_{\left.\tilde{\tau}_{2}^{+}\left(h_{t} / 2\right)\right)}^{\tilde{L}_{2}} e^{-\tilde{V}^{(2)}(x)} \mathrm{d} x \leq e^{-(1-\varepsilon) h_{t} / 2}\left(\tilde{L}_{2}-\tilde{L}_{2}^{-}\right) \leq C_{+} h_{t} e^{-(1-\varepsilon) h_{t} / 2}
$$

Let $\mathcal{E}_{10}^{4.6}:=\left\{\mathbf{e}_{1} \geq e^{-\varepsilon h_{t} / 2}\right\}$, and observe that $\mathbb{P}\left(\mathcal{E}_{10}^{4.6}\right) \geq 1-e^{-\varepsilon h_{t} / 2}$. We now combine all the previous intermediate results. On $\cap_{i=1}^{10} \mathcal{E}_{i}^{4.6}$, for $t$ large enough, $\mathcal{J}_{1} \geq \tilde{A}\left(\tilde{L}_{2}\right) \mathbf{e}_{1} e^{-\varepsilon h_{t}} / 2 \geq$ $\tilde{A}\left(\tilde{L}_{2}\right) e^{-2 \varepsilon h_{t}} / 2$ and $\mathcal{J}_{0}+\mathcal{J}_{2} \leq \tilde{A}\left(\tilde{L}_{2}\right) e^{(-1 / 2+3 \varepsilon) h_{t}} \leq 2 \mathcal{J}_{1} e^{(-1 / 2+5 \varepsilon) h_{t}}$. This, together with (4.22), yields

$$
\begin{equation*}
\left(1-e^{-h_{t}(1-3 \varepsilon) / 6}\right) \tilde{A}\left(\tilde{L}_{2}\right) \hat{\mathcal{J}}_{1} \mathbf{e}_{\mathbf{1}} \leq \mathcal{J}_{1} \leq \mathbf{U} \leq\left(1+2 e^{(-1 / 2+5 \varepsilon) h_{t}}\right) \mathcal{J}_{1} \leq\left(1+2 e^{-h_{t}(1-3 \varepsilon) / 6}\right) \tilde{A}\left(\tilde{L}_{2}\right) \hat{\mathcal{J}}_{1} \mathbf{e}_{\mathbf{1}} \tag{4.23}
\end{equation*}
$$

Moreover, $\hat{\mathcal{J}}_{1} \geq \mathcal{I}^{-} \geq e^{-\varepsilon h_{t}}\left(\tilde{\tau}_{2}^{+}\left(\varepsilon h_{t}\right)-\tilde{m}_{1}\right) \geq e^{-\varepsilon h_{t}}$, and $\hat{\mathcal{J}}_{1}=\left(\hat{\mathcal{J}}_{1}-\mathcal{I}^{-}\right)+\mathcal{I}^{-} \leq C_{+} h_{t} e^{-(1-\varepsilon) h_{t} / 2}+$ $\mathcal{I}^{-} \leq\left(1+e^{(-1 / 2+2 \varepsilon) h_{t}}\right) \mathcal{I}^{-}$for large $t$. This and (4.23) give $\left(1-e^{-h_{t}(1-3 \varepsilon) / 6}\right) \tilde{A}\left(\tilde{L}_{2}\right) \mathcal{I}^{-} \mathbf{e}_{\mathbf{1}} \leq \mathbf{U} \leq$ $\left(1+3 e^{-h_{t}(1-3 \varepsilon) / 6}\right) \tilde{A}\left(\tilde{L}_{2}\right) \hat{\mathcal{J}}_{1} \mathbf{e}_{\mathbf{1}}$ which proves the lemma.

We are now ready prove the proposition :
Proof of Proposition 4.3: We define $\mathcal{I}_{1}^{-}:=\int_{m_{2}}^{\tau_{2}\left(h_{t} / 2\right)} e^{-V^{(2)}(s)} \mathrm{d} s=A_{1}^{-}$,
$\mathcal{I}_{2}^{-}:=\int_{\tau_{2}^{-}\left(h_{t} / 2\right)}^{m_{2}} e^{-V^{(2)}(s)} \mathrm{d} s=B_{1}^{-}, \mathcal{I}_{3}^{+}:=\int_{\tau_{2}\left(h_{t} / 2\right)}^{\tau_{2}\left(h_{t}\right)} e^{V^{(2)}(s)} \mathrm{d} s$, and $\mathcal{I}_{2}^{+}:=\int_{\tau_{2}\left(h_{t}\right)}^{L_{2}} e^{V^{(2)}(s)} \mathrm{d} s=A_{2}^{+}$. By Williams' decomposition and the strong Markov property, $\mathcal{I}_{1}^{-}, \mathcal{I}_{2}^{-}$and $\mathcal{I}_{2}^{+}$are independent, $\mathcal{I}_{1}^{-} \stackrel{\mathcal{L}}{=} F^{-}\left(h_{t} / 2\right), \mathcal{I}_{2}^{-} \stackrel{\mathcal{L}}{=} F^{-}\left(h_{t} / 2\right)$ and $\mathcal{I}_{2}^{+} \stackrel{\mathcal{L}}{=} G^{+}\left(h_{t} / 2, h_{t}\right)$, and they are independent of $\left(W_{\kappa}(s+\right.$ $\left.\left.\tau_{2}\left(h_{t} / 2\right)\right)-W_{\kappa}\left(m_{2}\right), 0 \leq s \leq \tau_{2}\left(h_{t}\right)-\tau_{2}\left(h_{t} / 2\right)\right)$ which has the same law as a ( $\kappa / 2$ )-drifted 3 dimensional Bessel process started from $h_{t} / 2$ and killed when it first hits $h_{t}$.

We now consider, possibly on an enlarged probability space, a process $\left(R^{(1)}(s), 0 \leq s \leq\right.$ $\tau^{R^{(1)}}\left(h_{t} / 2\right)$ ), independent of $W_{\kappa}$ and $\mathbf{e}_{1}$ and then independent of $\mathcal{I}_{1}^{-}, \mathcal{I}_{2}^{-}$and $\mathcal{I}_{2}^{+}$, and distributed as $\left(R(s), 0 \leq s \leq \tau^{R}\left(h_{t} / 2\right)\right)$. We now extend this process by setting $R^{(1)}(s):=W_{\kappa}[s-$ $\left.\tau^{R^{(1)}}\left(h_{t} / 2\right)+\tau_{2}\left(h_{t} / 2\right)\right]-W_{\kappa}\left(m_{2}\right)$ for $\tau^{R^{(1)}}\left(h_{t} / 2\right) \leq s \leq \tau^{R^{(1)}}\left(h_{t} / 2\right)+\tau_{2}\left(h_{t}\right)-\tau_{2}\left(h_{t} / 2\right)$. We also introduce $\mathcal{I}_{1}^{+}:=\int_{0}^{\tau^{R^{(1)}}\left(h_{t}\right)} e^{R^{(1)}(s)} \mathrm{d} s$. By the strong Markov property, $R^{(1)}$ (and hence also $\mathcal{I}_{1}^{+}$) is independent of $\mathcal{I}_{2}^{+}, \mathcal{I}_{1}^{-}, \mathcal{I}_{2}^{-}$and $\mathbf{e}_{\mathbf{1}}$.

Moreover, with the same notation as in Lemma 4.6, we have on $\mathcal{V}_{t}, \mathcal{I}^{-}=\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}$and $\tilde{A}\left(\tilde{L}_{2}\right)=A_{1}^{+}+\mathcal{I}_{2}^{+}$where $\mathcal{I}^{-}$is defined in Lemma 4.6.

We now prove that $A_{1}^{+}$can be approximated by $\mathcal{I}_{1}^{+}$. Since $A_{1}^{+}-\mathcal{I}_{3}^{+}=\int_{m_{2}}^{\tau_{2}\left(h_{t} / 2\right)} e^{V^{(2)}(s)} \mathrm{d} s \stackrel{\mathcal{L}}{=}$ $F^{+}\left(h_{t} / 2\right)$, and by Lemma 4.1

$$
P\left(\mathcal{I}_{1}^{+}-\mathcal{I}_{3}^{+}>e^{(1+\delta) h_{t} / 2}\right)=P\left(A_{1}^{+}-\mathcal{I}_{3}^{+}>e^{(1+\delta) h_{t} / 2}\right)=P\left(\frac{F^{+}\left(h_{t} / 2\right)}{e^{h_{t} / 2}}>e^{\delta h_{t} / 2}\right) \leq \frac{C_{+}}{e^{\delta h_{t} / 2}} .
$$

Moreover by (3.5), with a probability larger than $1-3 e^{-\kappa \delta h_{t} / 2}$ for large $t, \mathcal{I}_{1}^{+} \geq e^{h_{t}(1-\delta)}$. Therefore, with a probability greater than $1-4 e^{-\kappa \delta h_{t} / 2}$ for large $t, A_{1}^{+} \leq \mathcal{I}_{3}^{+}+e^{-h_{t}(1-3 \delta) / 2} \mathcal{I}_{1}^{+} \leq$ $\mathcal{I}_{1}^{+}\left(1+e^{-h_{t}(1-3 \delta) / 2}\right)$ and $A_{1}^{+} \geq \mathcal{I}_{1}^{+}-\left(\mathcal{I}_{1}^{+}-\mathcal{I}_{3}^{+}\right) \geq\left(1-e^{-h_{t}(1-3 \delta) / 2}\right) \mathcal{I}_{1}^{+}$, and then

$$
\left(1-e^{-h_{t}(1-3 \delta) / 2}\right) \mathcal{I}_{1}^{+} \leq A_{1}^{+} \leq\left(1+e^{-h_{t}(1-3 \delta) / 2}\right) \mathcal{I}_{1}^{+},
$$

which gives $\left(1-e^{-h_{t}(1-3 \delta) / 2}\right) \leq \tilde{A}\left(\tilde{L}_{2}\right) \mathcal{I}^{-} /\left[\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right)\left(\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}\right)\right] \leq\left(1+e^{-h_{t}(1-3 \delta) / 2}\right)$. To finish we use Lemma 4.6 with $\varepsilon=\delta$.

### 4.3. Second order Laplace transform of a standard exit time.

We are now ready to prove a key point of this paper which is the second order of the Laplace transform of $\mathbf{U} / t$ :

Proposition 4.7. Let $\lambda>0$. For $t$ large enough,

$$
e^{\kappa \phi(t)}\left(1-E\left(e^{-\frac{\lambda}{t} \mathbf{U}}\right)\right)=C_{\kappa} \lambda^{\kappa}+o(1)
$$

with $C_{\kappa}:=8^{\kappa}\left(C_{0}+\left|\Upsilon_{0}\right|\right)$, with $C_{0}:=\Gamma(1-\kappa) \Gamma(\kappa+2) /(1+\kappa)^{\kappa}$ and

$$
\Upsilon_{0}:=\int_{0}^{\infty} \int_{0}^{\infty} \frac{u^{\kappa} y^{-\kappa-1} e^{-u}}{\Gamma(\kappa)}\left[\frac{y^{\kappa}}{\left[\Gamma(\kappa+1) I_{\kappa}(\sqrt{y})\right]^{2}}-\left[1+\frac{y}{(\kappa+1)}\right]^{-2}\right] \mathrm{d} y \mathrm{~d} u .
$$

Proof: We have,

Hence by definition of $\mathcal{A}_{t}$ and Proposition 4.3, we get with $\lambda_{t}^{ \pm}:=2 \lambda\left(1 \pm \varepsilon_{t}\right) / t$,

$$
S_{0}^{+}-C_{+} e^{-\delta^{2} c_{-} h_{t}} \leq E\left(e^{-\frac{\lambda}{t} \mathbf{U}}\right) \leq S_{0}^{-}+C_{+} e^{-\delta^{2} c_{-} h_{t}},
$$

where $S_{0}^{ \pm}:=E\left(e^{-\left(\lambda_{t}^{ \pm} / 2\right)\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right)\left(\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}\right) \mathbf{e}_{1}}\right)$. Let $\varepsilon_{1}>0, \theta:=3 \kappa / 4+\varepsilon_{1}, \mathcal{B}_{1}:=\left\{\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right)>\right.$ $\left.e^{\log t-\theta \phi(t)}\right\}$, and $\mathcal{B}_{2}:=\left\{\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right) \leq e^{\log t-\theta \phi(t)}\right\}$. Since $\mathbf{e}_{\mathbf{1}}$ is an exponential r.v. with mean 2 , we get $S_{0}^{ \pm}=S_{1}^{ \pm}+S_{2}^{ \pm}$, where for $1 \leq i \leq 2$,

$$
\begin{align*}
S_{2}^{ \pm} & :=\int_{0}^{\infty} \mathrm{d} z e^{-z} E\left(\mathbb{1}_{\mathcal{B}_{i}} e^{-z \lambda_{t}^{ \pm}\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right)\left(\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}\right)}\right) \\
& =\int_{0}^{+\infty} \mathrm{d} z e^{-z} E\left(\mathbb{1}_{\mathcal{B}_{i}}\left[E\left(e^{-z \lambda_{t}^{ \pm}\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right) \mathcal{I}_{1}^{-}} \mid \mathcal{I}_{1}^{+}, \mathcal{I}_{2}^{+}\right)\right]^{2}\right) \tag{4.24}
\end{align*}
$$

since $\mathcal{I}_{1}^{-}$and $\mathcal{I}_{2}^{-}$are independent and independent of $\mathcal{I}_{1}^{+}$and $\mathcal{I}_{2}^{+}$and have the same law.
We start with $S_{2}^{ \pm}$. Let $a(t):=e^{-(3 / 4) \kappa \phi(t)}$. Now, consider $0 \leq z \leq \eta_{1} a(t) e^{\theta \phi(t)-\log t} / \lambda_{t}^{ \pm}=$ $\eta_{1} a(t) e^{\theta \phi(t)} /\left(2 \lambda\left(1 \pm \varepsilon_{t}\right)\right)$. On $\mathcal{B}_{2}$, we have $z \lambda_{t}^{ \pm}\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right) \leq \eta_{1} a(t)$, which gives by (4.1) applied to $\mathcal{I}_{1}^{-} \stackrel{\mathcal{L}}{=} F^{-}\left(h_{t} / 2\right)$ for $t$ so large that $h_{t} / 2 \geq M$,

$$
E\left(e^{-z \lambda_{t}^{ \pm}\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right) \mathcal{I}_{1}^{-}} \mid \mathcal{I}_{1}^{+}, \mathcal{I}_{2}^{+}\right) \leq Z(z)+C_{4} \max \left(e^{-\kappa h_{t} / 2},\left[z \lambda^{ \pm}\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right)\right]^{\frac{3}{2}}\right) .
$$

with $Z(x):=\left(1+2 \lambda_{t}^{ \pm} x\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right) /(\kappa+1)\right)^{-1}$ for any $x>0$.Therefore, for such $z$ and $t$, by (4.24),

$$
\begin{aligned}
S_{2}^{ \pm} & \leq \int_{0}^{\frac{\eta_{1} \alpha(t) e^{\theta \phi(t)}}{t \lambda_{t}}} \frac{\mathrm{~d} z}{e^{z}} E\left(\mathbb{1}_{\mathcal{B}_{2}} Z^{2}(z)+\mathbb{1}_{\mathcal{B}_{2}} 3 C_{4}\left(\left[z \lambda^{ \pm}\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right)\right]^{\frac{3}{2}}+e^{-\kappa h_{t} / 2}\right)\right) \\
& +\int_{\eta_{1} a(t) e^{\theta \phi(t)} /\left(t \lambda_{t}^{ \pm}\right)}^{+\infty} e^{-z} \mathrm{~d} z .
\end{aligned}
$$

We notice that $\int_{\eta_{1} a(t) e^{\theta \phi(t)} /\left(t \lambda_{t}^{ \pm}\right)}^{+\infty} e^{-z} \mathrm{~d} z \leq e^{-\phi(t)}$ for large $t$ since $a(t) e^{\theta \phi(t)} \geq \phi(t)$. Moreover, $\int_{0}^{\eta_{1} a(t) e^{\theta \phi(t)} /\left(t \lambda_{t}^{ \pm}\right)} \mathrm{d} z e^{-z} E\left(\mathbb{1}_{\mathcal{B}_{2}}\left[z \lambda_{t}^{ \pm}\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right)\right]^{3 / 2}\right) \leq\left[\eta_{1} a(t)\right]^{3 / 2}$. So for $t$ large,

$$
S_{2}^{ \pm} \leq \int_{0}^{\infty} \mathrm{d} z e^{-z} E\left(\mathbb{1}_{\mathcal{B}_{2}} Z^{2}(z)\right)+o\left(e^{-\kappa \phi(t)}\right) .
$$

We also define,

$$
S_{3}^{ \pm}:=\int_{0}^{+\infty} \mathrm{d} z e^{-z} E\left(\mathbb{1}_{\mathcal{B}_{1}} Z^{2}(z)\right)
$$

Recall that for any $\lambda>0$ and any random variable $X \geq 0$, we have by Fubini,

$$
E\left((1+\lambda X)^{-2}\right)=\int_{0}^{+\infty} \mathrm{d} u \int_{u}^{+\infty} E\left(e^{-x(1+\lambda X)}\right) \mathrm{d} x
$$

So, by independence of $\mathcal{I}_{1}^{+}$and $\mathcal{I}_{2}^{+}$, and defining $\rho_{\kappa}^{ \pm}:=\frac{4\left(1 \pm \varepsilon_{t}\right)}{\kappa+1}$

$$
\begin{equation*}
E\left(Z^{2}(z)\right)=\int_{0}^{+\infty} \mathrm{d} u \int_{u}^{+\infty} \mathrm{d} x e^{-x} E\left(e^{-\frac{\rho_{\kappa}^{ \pm} z x \lambda \mathcal{I}_{1}^{+}}{t}}\right) E\left(e^{-\frac{\rho_{\kappa}^{ \pm} z x \lambda \mathcal{I}_{2}^{+}}{t}}\right) \tag{4.25}
\end{equation*}
$$

Recall that $t=e^{h_{t}} e^{\phi(t)}$, so using (4.2), (4.3) and the laws of $\mathcal{I}_{1}^{+}$and $\mathcal{I}_{2}^{+}$, we notice than whenever $\rho_{\kappa}^{ \pm} z x \lambda / e^{\phi(t)} \leq \eta_{1}$ and $y \geq M$,

$$
\begin{aligned}
& E\left(e^{-\frac{\rho_{\kappa}^{ \pm} z x \lambda I_{1}^{+}}{t}}\right) \leq 1-\frac{2 \rho_{\kappa}^{ \pm} z x \lambda}{(\kappa+1) e^{\phi(t)}}+C_{4} \max \left(e^{-\kappa h_{t}},\left(\rho_{\kappa}^{ \pm} z x \lambda / e^{\phi(t)}\right)^{3 / 2}\right) \\
& E\left(e^{-\frac{\rho_{\kappa}^{ \pm} z x \lambda I_{2}^{+}}{t}}\right) \leq 1-\frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)} \frac{\left(2 \rho_{\kappa}^{ \pm} z x \lambda\right)^{\kappa}}{e^{\kappa \phi(t)}}+C_{4} \max \left(\left(\rho_{\kappa}^{ \pm} z x \lambda / e^{\phi(t)}\right)^{\kappa} e^{-\kappa h_{t} / 2}, \frac{\rho_{\kappa}^{ \pm} z x \lambda}{e^{\phi(t)}}\right)
\end{aligned}
$$

So, we get, $C$ being a positive constant,

$$
\begin{aligned}
(4.25) \leq & \int_{0}^{+\infty} \mathrm{d} u \int_{u}^{+\infty} \mathrm{d} x e^{-x}\left(1-\frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)} \frac{\left(2 \rho_{\kappa}^{ \pm} z x \lambda\right)^{\kappa}}{e^{\kappa \phi(t)}}\right) \mathbb{1}_{\rho_{\kappa}^{ \pm} z x \lambda / e^{\phi(t)} \leq \eta_{1}} \\
& +C \int_{0}^{+\infty} \mathrm{d} u \int_{u}^{+\infty} \mathrm{d} x e^{-x}\left(\left(\frac{\rho_{\kappa}^{ \pm} z x \lambda}{e^{\phi(t)}}\right)^{\kappa} e^{-\kappa h_{t} / 2}+\frac{\rho_{\kappa}^{ \pm} z x \lambda}{e^{\phi(t)}}+e^{-\kappa h_{t} / 2}\right) \mathbb{1}_{\rho_{\kappa}^{ \pm} z x \lambda / e^{\phi(t)} \leq \eta_{1}} \\
& +\int_{0}^{+\infty} \mathrm{d} u \int_{u}^{+\infty} e^{-x} \mathrm{~d} x \mathbb{1}_{\rho_{\kappa}^{ \pm} z x \lambda / e^{\phi(t)}>\eta_{1}} .
\end{aligned}
$$

Finally, notice that $\int_{0}^{+\infty} e^{-z} \mathrm{~d} z \int_{0}^{+\infty} \mathrm{d} u \int_{u}^{+\infty} \mathrm{d} x e^{-x} \rho_{\kappa}^{ \pm} z x \lambda / e^{\phi(t)}=2 \rho_{\kappa}^{ \pm} \lambda / e^{\phi(t)}$. Moreover, we have $\int_{0}^{+\infty} e^{-z} \mathrm{~d} z \int_{0}^{+\infty} \mathrm{d} u \int_{u}^{+\infty} \mathrm{d} x e^{-x}\left[\rho_{\kappa}^{ \pm} z x \lambda / e^{\phi(t)}\right]^{\kappa}=O\left(e^{-\kappa \phi(t)}\right)$, and furthermore $\int_{0}^{+\infty} \mathrm{d} z e^{-z}$ $\int_{0}^{+\infty} \mathrm{d} u \int_{u}^{+\infty} e^{-x} \mathrm{~d} x \mathbb{1}_{\rho_{\kappa}^{ \pm} z x \lambda / e^{\phi(t)}>\eta_{1}}=O\left(e^{-\phi(t)}\right)$. Hence, we get

$$
\begin{align*}
& \int_{0}^{+\infty} \mathrm{d} z e^{-z} E\left(Z^{2}(z)\right) \\
& \leq \int_{0}^{+\infty} \mathrm{d} z e^{-z} \int_{0}^{+\infty} \mathrm{d} u \int_{u}^{+\infty} \mathrm{d} x e^{-x}\left(1-\frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)} \frac{\left(2 \rho_{\kappa}^{ \pm} z x \lambda\right)^{\kappa}}{e^{\kappa \phi(t)}}\right) \mathbb{1}_{\rho_{\kappa}^{ \pm} z x \lambda / e^{\phi(t)} \leq \eta_{1}}+O\left(e^{-\phi(t)}\right) \\
& =1-\frac{\Gamma(1-\kappa)}{\Gamma(1+\kappa)} \frac{8^{\kappa} \lambda^{\kappa}}{(1+\kappa)^{\kappa} e^{\kappa \phi(t)}}\left(1 \pm \varepsilon_{t}\right)^{\kappa} \int_{0}^{+\infty} \mathrm{d} z e^{-z} z^{\kappa} \int_{0}^{+\infty} \mathrm{d} x e^{-x} x^{\kappa+1} \\
& +\frac{\Gamma(1-\kappa)\left(2 \lambda \rho_{\kappa}^{ \pm}\right)^{\kappa}}{\Gamma(1+\kappa) e^{\kappa \phi(t)}} \int_{0}^{+\infty} \mathrm{d} z e^{-z} \int_{0}^{+\infty} \mathrm{d} u \int_{u}^{+\infty} \mathrm{d} x e^{-x}(z x)^{\kappa} \mathbb{1}_{\rho_{\kappa}^{ \pm} z x \lambda / e^{\phi(t)}>\eta_{1}}+O\left(e^{-\phi(t)}\right)  \tag{4.26}\\
& =1-\frac{\Gamma(1-\kappa) \Gamma(\kappa+2) 8^{\kappa} \lambda^{\kappa}}{(1+\kappa)^{\kappa} e^{\kappa \phi(t)}}\left(1 \pm \varepsilon_{t}\right)^{\kappa}+o\left(e^{-\kappa \phi(t)}\right) \tag{4.27}
\end{align*}
$$

since, by the dominated convergence theorem, the integral in (4.26) goes to 0 as $t \rightarrow+\infty$ and then Line $(4.26)=o\left(e^{-\kappa \phi(t)}\right)$. Combining equations (4.24) to (4.27) we get $S_{2}^{ \pm}+S_{3}^{ \pm} \leq$ $1-C_{0} 8^{\kappa} \lambda^{\kappa} e^{-\kappa \phi(t)}+o\left(e^{-\kappa \phi(t)}\right)$. We prove similarly that $S_{2}^{ \pm}+S_{3}^{ \pm} \geq 1-C_{0} 8^{\kappa} \lambda^{\kappa} e^{-\kappa \phi(t)}+o\left(e^{-\kappa \phi(t)}\right)$.

We now turn to the estimation of $S_{1}^{ \pm}-S_{3}^{ \pm}$. We assume that $\phi(t) \geq 4 \log \log t, 0<\varepsilon<1 / 2$, $b(t)=-\theta \phi(t)$.
Step 1: Approximation of $\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}$. Since $\varepsilon e^{\left(1-3 \kappa / 4-\varepsilon_{1}\right) \phi(t)} \geq 8 h_{t} / \kappa$ for large $t$, we have $\mathbb{P}\left(\mathcal{I}_{1}^{+} \geq \varepsilon t e^{-\left(3 \kappa / 4+\varepsilon_{1}\right) \phi(t)}\right) \leq \mathbb{P}\left(\tau^{R}\left(h_{t}\right) \geq \varepsilon e^{\left(1-3 \kappa / 4-\varepsilon_{1}\right) \phi(t)}\right) \leq P\left(\tau^{R}\left(h_{t}\right) \geq 8 h_{t} / \kappa\right) \leq C_{+} e^{-\kappa h_{t} / 2 \sqrt{2}}$, since $F_{+}\left(h_{t}\right) \leq e^{h_{t}} \tau^{R}\left(h_{t}\right), \tau^{R}\left(h_{t}\right) \stackrel{\mathcal{L}}{=} \tau_{1}\left(h_{t}\right)-m_{1}$ by Fact 2.2, and thanks to (2.4).

Moreover, we can write $\mathcal{I}_{2}^{+}=e^{h_{t}} \widehat{A}_{\infty}-e^{h_{t} / 2} \tilde{A}_{\infty}$, where $\widehat{A}_{\infty} \stackrel{\mathcal{L}}{\underline{A}} \tilde{A}_{\infty} \stackrel{\mathcal{L}}{=} A_{\infty}$. We have $\mathbb{P}\left(e^{h_{t} / 2} \tilde{A}_{\infty} \geq \varepsilon t e^{-\left(3 \kappa / 4+\varepsilon_{1}\right) \phi(t)}\right) \leq \mathbb{P}\left(\tilde{A}_{\infty} \geq t^{1 / 3}\right) \leq C t^{-\kappa / 3}$ since $P\left(A_{\infty} \geq y\right) \leq C y^{-\kappa}$ (see Dufresne). Now, we have on $\mathcal{B}_{1} \cap\left\{\mathcal{I}_{1}^{+}<\varepsilon t e^{-\left(3 \kappa / 4+\varepsilon_{1}\right) \phi(t)}\right\} \cap\left\{e^{h_{t} / 2} \tilde{A}_{\infty}<\varepsilon t e^{-\left(3 \kappa / 4+\varepsilon_{1}\right) \phi(t)}\right\}$,

$$
t e^{-\left(3 \kappa / 4+\varepsilon_{1}\right) \phi(t)}<\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}=\mathcal{I}_{1}^{+}+e^{h_{t}} \widehat{A}_{\infty}-e^{h_{t} / 2} \tilde{A}_{\infty}<e^{h_{t}} \widehat{A}_{\infty}+\varepsilon t e^{-\left(3 \kappa / 4+\varepsilon_{1}\right) \phi(t)} .
$$

This yields $e^{h_{t}} \widehat{A}_{\infty} \geq(1-\varepsilon) t e^{-\left(3 \kappa / 4+\varepsilon_{1}\right) \phi(t)}$, and then $\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+} \leq\left(1+\frac{\varepsilon}{1-\varepsilon}\right) e^{h_{t}} \widehat{A}_{\infty} \leq(1+2 \varepsilon) e^{h_{t}} \widehat{A}_{\infty}$. Similarly, $\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+} \geq e^{h_{t}} \widehat{A}_{\infty}-\varepsilon t e^{-\left(3 \kappa / 4+\varepsilon_{1}\right) \phi(t)} \geq\left(1-\frac{\varepsilon}{1-\varepsilon}\right) e^{h_{t}} \widehat{A}_{\infty} \geq(1-2 \varepsilon) e^{h_{t}} \widehat{A}_{\infty}$. Consequently, replacing $\varepsilon$ by $\varepsilon / 2$,

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{B}_{1} \cap\left\{(1-\varepsilon) e^{h_{t}} A_{\infty} \leq \mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+} \leq(1+\varepsilon) e^{h_{t}} A_{\infty}\right\}^{c}\right)=O\left(e^{-\phi(t)}\right) . \tag{4.28}
\end{equation*}
$$

Step 2: Simplification. Thanks to (4.28), we get

$$
\left.\begin{array}{rl}
S_{1}^{ \pm}-S_{3}^{ \pm} \leq & \int_{0}^{\infty} E\left[\left(e^{-\lambda_{t}^{ \pm} u\left(\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}\right)\left(\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}\right)}-Z^{2}(u)\right) \mathbb{1}_{\left\{\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+} \geq t e^{b(t)}\right\}}\right. \\
& \left.\mathbb{1}_{\left\{(1-\varepsilon) e^{h} t \widehat{A}_{\infty} \leq \mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+} \leq(1+\varepsilon) e^{h} t\right.} \widehat{A}_{\infty}\right\}
\end{array}\right] e^{-u} \mathrm{~d} u+e^{-\phi(t)}, ~ S_{4}^{ \pm}+S_{5}^{ \pm}+e^{-\phi(t)},
$$

where

$$
\begin{aligned}
S_{4}^{ \pm}:= & \int_{0}^{\infty} E\left[\left(e^{-\lambda_{t}^{ \pm} u(1-\varepsilon) e^{h_{t}} \widehat{A}_{\infty}\left(\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}\right)}-\right) \mathbb{1}_{(1+\varepsilon) e^{h_{t}} \widehat{A}_{\infty} \geq t e^{b(t)}}\right. \\
& \left.\mathbb{1}_{\left\{(1-\varepsilon) e^{h} \widehat{A}_{\infty} \leq \mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+} \leq(1+\varepsilon) e^{h_{t}} \widehat{A}_{\infty}\right\}}\right] e^{-u} \mathrm{~d} u, \\
S_{5}^{ \pm}:= & \int_{0}^{\infty} E\left[\left(e^{-\lambda_{t}^{ \pm} u(1-\varepsilon) e^{h_{t}} \widehat{A}_{\infty}\left(\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}\right)}-Z_{\infty}^{2}(u)\right)\right. \\
& {\left[\mathbb{1}_{\left.\left.\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+} \geq t e^{b(t)}-\mathbb{1}_{\left.(1+\varepsilon) e^{h_{t}} \widehat{A}_{\infty} \geq t e^{b(t)}\right]}\right] \mathbb{1}_{\left\{(1-\varepsilon) e^{h_{t}} \widehat{A}_{\infty} \leq \mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+} \leq(1+\varepsilon) e^{h} \widehat{A}_{\infty}\right\}}\right] e^{-u} \mathrm{~d} u,} \quad\right.} \\
Z_{\infty}(u):= & {\left[1+2 \lambda_{t}^{ \pm} u(1+\varepsilon) e^{h_{t}} \widehat{A}_{\infty} /(\kappa+1)\right]^{-1} . }
\end{aligned}
$$

Using once more (4.28) leads to

$$
\begin{align*}
S_{4}^{ \pm} \leq & \int_{0}^{\infty} E\left[\left(e^{-\lambda_{t}^{ \pm} u(1-\varepsilon) e^{h t} \widehat{A}_{\infty}\left(\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}\right)}-Z_{\infty}^{2}(u)\right) \mathbb{1}_{(1+\varepsilon) e^{h} t \widehat{A}_{\infty} \geq t e^{b(t)}}\right] e^{-u} \mathrm{~d} u \\
& +2 e^{-\phi(t)} . \tag{4.30}
\end{align*}
$$

Now, for $t$ large enough so that $0 \leq \varepsilon_{t} \leq \varepsilon$, the integral in (4.30) can be written as

$$
\begin{align*}
& \int_{0}^{\infty} e^{-u} \mathrm{~d} u E\left[\left(E\left(e^{-\lambda_{t}^{ \pm} u(1-\varepsilon) e^{h t} \widehat{A}_{\infty} \mathcal{I}_{1}^{-}} \mid \widehat{A}_{\infty}\right)^{2}-Z_{\infty}^{2}(u)\right) \mathbb{1}_{\widehat{A}_{\infty} \geq \frac{e^{b(t)+\phi(t)}}{1+\varepsilon}}\right] \\
& \leq \int_{0}^{\infty} e^{-u} \mathrm{~d} u E\left[\left(E\left(e^{-4 \lambda(1-\varepsilon)^{2} u e^{-\phi(t)} \gamma_{\kappa}^{-1} \mathcal{I}_{1}^{-}} \mid \gamma_{\kappa}\right)^{2}-Z_{\infty}^{2}(u)\right) \mathbb{1}_{\left.\gamma_{\kappa} \leq \frac{2(1+\varepsilon)}{e^{(t(t)+\phi(t)}}\right]}\right] . \tag{4.31}
\end{align*}
$$

By Dufresne [13], since $\gamma_{\kappa}$ has density $x^{\kappa-1} e^{-x} / \Gamma(\kappa) \mathbb{1}_{\mathbb{R}_{+}}(x)$ and is, as $\widehat{A}_{\infty}$, independent of $\mathcal{I}_{1}^{-}$ then the RHS of (4.31) is equal to

$$
\begin{equation*}
\int_{0}^{\infty} e^{-u} \mathrm{~d} u \int_{0}^{\frac{2(1+\varepsilon)}{e^{b(t)+\phi(t)}}}\left[E\left(e^{-4 \lambda(1-\varepsilon)^{2} u e^{-\phi(t)} x^{-1} \mathcal{I}_{1}^{-}}\right)^{2}-\left(1+\frac{8 \lambda(1+\varepsilon)^{2} u}{(\kappa+1) e^{\phi(t)} x}\right)^{-2}\right] \frac{x^{\kappa-1} e^{-x}}{\Gamma(\kappa)} \mathrm{d} x \tag{4.32}
\end{equation*}
$$

With the change of variables $y=8 u \lambda e^{-\phi(t)} x^{-1}$, this is equal to $(8 \lambda)^{\kappa} e^{-\kappa \phi(t)} \Upsilon_{\epsilon}(t)$, with $\Upsilon_{\epsilon}(t):=$ $\int_{0}^{\infty} \int_{0}^{\infty} f_{t}(u, y) \mathrm{d} y \mathrm{~d} u$ and

$$
f_{t}(u, y):=\mathbb{1}_{\left\{y \geq \frac{4 \lambda u e^{-\theta \phi(t)}}{(1+\varepsilon)}\right\}} u^{\kappa} e^{-u}\left[E\left(e^{-(1-\varepsilon)^{2} y \mathcal{I}_{1}^{-} / 2}\right)^{2}-\left(1+\frac{(1+\varepsilon)^{2} y}{(\kappa+1)}\right)^{-2}\right] \frac{e^{-8 \lambda u y^{-1} e^{-\phi(t)}}}{\Gamma(\kappa) y^{\kappa+1}}
$$

Step 3: pointwise convergence.
Notice that thanks to (4.4), $\lim _{y \rightarrow+\infty} E\left(e^{-\gamma F^{-}(y)}\right)=\frac{(2 \gamma)^{\kappa / 2}}{\kappa \Gamma(\kappa) I_{\kappa}(2 \sqrt{2 \gamma})}$ for $\gamma>0$. Hence, for every $u>0$ and $y>0$,
$f_{t}(u, y) \rightarrow_{t \rightarrow+\infty} f(u, y):=u^{\kappa} e^{-u}\left[\frac{(1-\varepsilon)^{2 \kappa} y^{\kappa}}{\left[\Gamma(\kappa+1) I_{\kappa}(2(1-\varepsilon) \sqrt{y})\right]^{2}}-\left(1+\frac{(1+\varepsilon)^{2} y}{(\kappa+1)}\right)^{-2}\right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)}$.
Step 4: dominated convergence. We notice that $f_{t}(u, y)=a_{t}(u, y)+b_{t}(u, y)$ and $f(u, y)=$ $a(u, y)+b(u, y)$, where
$a_{t}(u, y):=\mathbb{1}_{\left\{y \geq \frac{4 \lambda u e^{-\theta \phi(t)}}{(1+\varepsilon)}\right\}} u^{\kappa} e^{-u}\left[E\left(e^{-(1-\varepsilon)^{2} y \mathcal{I}_{1}^{-} / 2}\right)^{2}-\left(1+\frac{(1-\varepsilon)^{2} y}{(\kappa+1)}\right)^{-2}\right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)} e^{-8 \lambda u y^{-1} e^{-\phi(t)}}$
$b_{t}(u, y):=\mathbb{1}_{\left\{y \geq \frac{4 \lambda u e^{-\theta \phi(t)}}{(1+\varepsilon)}\right\}} u^{\kappa} e^{-u}\left[\left(1+\frac{(1-\varepsilon)^{2} y}{(\kappa+1)}\right)^{-2}-\left(1+\frac{(1+\varepsilon)^{2} y}{(\kappa+1)}\right)^{-2}\right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)} e^{-8 \lambda u y^{-1} e^{-\phi(t)}}$ and their pointwise limits on $\left(\mathbb{R}_{+}^{*}\right)^{2}$ as $t \rightarrow+\infty$ are respectively

$$
\begin{aligned}
& a(u, y):=u^{\kappa} e^{-u}\left[\frac{(1-\varepsilon)^{2 \kappa} y^{\kappa}}{\left[\Gamma(\kappa+1) I_{\kappa}(2(1-\varepsilon) \sqrt{y})\right]^{2}}-\left(1+\frac{(1-\varepsilon)^{2} y}{(\kappa+1)}\right)^{-2}\right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)}, \\
& b(u, y)
\end{aligned}
$$

Since $\forall x>0, I_{\kappa}(x)>\frac{(x / 2)^{\kappa}}{\Gamma(\kappa+1)}+\frac{(x / 2)^{\kappa+2}}{\Gamma(\kappa+2)}$, we get $\frac{\gamma^{\kappa}}{\left[\Gamma(\kappa+1) I_{\kappa}(2 \sqrt{\gamma})\right]^{2}}-\left(1+\frac{\gamma}{(\kappa+1)}\right)^{-2}<0$ for every $\gamma>0$. Consequently, $\forall(u, t) \in \mathbb{R}_{+}^{2}, a(u, t)<0$. This, and the fact that $t \mapsto E\left(e^{-\gamma F^{-}\left(h_{t} / 2\right)}\right)$ is nonincreasing for $\gamma \geq 0$, lead to

$$
\begin{equation*}
\forall(u, y) \in\left(\mathbb{R}_{+}\right)^{2}, \quad a(u, y) \leq a_{t}(u, y) \leq u^{\kappa} e^{-u}\left[1-\left(1+\frac{(1-\varepsilon)^{2} y}{(\kappa+1)}\right)^{-2}\right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)}=: h(u, y) \tag{4.33}
\end{equation*}
$$

Hence, $\left|a_{t}(u, y)\right| \leq|a(u, y)|+|h(u, y)|$ and $\left|b_{t}(u, y)\right| \leq|b(u, y)|$. for every $(u, y) \in \mathbb{R}_{+}^{2}$. Moreover, since $0<\kappa<1,|h|,|b|$ and $|a|$ have finite integrals over $\mathbb{R}_{+}^{2}$ (notice for example that $e^{u} u^{-\kappa} a(u, y)=O\left(y^{-\kappa}\right)$ as $\left.y \rightarrow 0\right)$. Thus, by the dominated convergence theorem,

$$
\lim _{t \rightarrow+\infty} \Upsilon_{\epsilon}(t)=\int_{0}^{\infty} \int_{0}^{\infty} f(u, y) \mathrm{d} y \mathrm{~d} u=: \Upsilon_{\epsilon}
$$

Hence for small $\varepsilon$, $\limsup _{t \rightarrow+\infty}\left(\left(S_{4}^{ \pm}\right) e^{\kappa \phi(t)} \lambda^{-\kappa}\right) \leq(1-\varepsilon) 8^{\kappa} \Upsilon_{\epsilon}$, since $\lim _{x \rightarrow 0} J(x)=J(0)<0$.
Step 5: $S_{5}^{ \pm}$is negligible. First, we have

$$
\begin{align*}
\left|S_{5}^{ \pm}\right| \leq & \int_{0}^{\infty} E\left[\left|e^{-\lambda_{t}^{ \pm} u(1-\varepsilon) e^{h_{t}} A_{\infty}\left(\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}\right)}-Z_{\infty}^{2}(u)\right|\right. \\
& \left.\left.\mathbb{1}_{\left\{\mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+}<t e^{b(t)}\right\}} \mathbb{1}_{\left\{(1+\varepsilon) e^{h_{t}} A_{\infty} \geq t e^{b(t)}\right\}} \mathbb{1}_{\left\{(1-\varepsilon) e^{h_{t}} A_{\infty} \leq \mathcal{I}_{1}^{+}+\mathcal{I}_{2}^{+} \leq(1+\varepsilon) e^{h} t\right.} A_{\infty}\right\}\right] e^{-u} \mathrm{~d} u \\
\leq & \int_{0}^{\infty} E\left[\left|e^{-\lambda_{t}^{ \pm} u(1-\varepsilon) e^{h_{t}} A_{\infty}\left(\mathcal{I}_{1}^{-}+\mathcal{I}_{2}^{-}\right)}-Z_{\infty}^{2}(u)\right|\right. \\
& \left.\mathbb{1}_{\left\{(1-\varepsilon) e^{h_{t}} A_{\infty} \leq t e^{b(t)} \leq(1+\varepsilon) e^{h_{t}} A_{\infty}\right\}}\right] e_{25}^{-u} \mathrm{~d} u \tag{4.34}
\end{align*}
$$

where we used $\left|\left(\mathbb{1}_{E_{1}}-\mathbb{1}_{E_{2}}\right) \mathbb{1}_{E_{3}}\right|=\mathbb{1}_{\overline{E_{1}}} \mathbb{1}_{E_{2}} \mathbb{1}_{E_{3}}+\mathbb{1}_{E_{1}} \mathbb{1}_{\overline{E_{2}}} \mathbb{1}_{E_{3}}$, for events $E_{i}$ with $\mathbb{1}_{E_{1}} \mathbb{1}_{\overline{E_{2}}} \mathbb{1}_{E_{3}}=0$ in our case. The same calculations as for $S_{4}^{ \pm}$give $\left|S_{5}^{ \pm}\right| \leq \frac{(8 \lambda)^{\kappa}}{e^{\kappa \phi(t)}} \int_{0}^{\infty} \int_{0}^{\infty}\left(\left|F_{t}(u, y, \varepsilon)\right|+\left|F_{t}(u, y,-\varepsilon)\right| \mathrm{d} y \mathrm{~d} u\right.$ for large $t$, with
$F_{t}(u, y, z):=\mathbb{1}_{\left\{\frac{4 \lambda u e^{-\theta \phi(t)}}{(1+\varepsilon)} \leq y \leq \frac{4 \lambda u e^{-\theta \phi(t)}}{(1-\varepsilon)}\right\}} u^{\kappa} e^{-u}\left[E\left(e^{-(1-z)^{2} y \mathcal{I}_{1}^{-} / 2}\right)^{2}-\left(1+\frac{(1+z)^{2} y}{(\kappa+1)}\right)^{-2}\right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)}$.
Using the same domination as before, the dominated convergence theorem gives $S_{5}^{ \pm}=o\left(e^{-\kappa \phi(t)}\right)$.
Conclusion: Consequently, $\lim \sup _{t \rightarrow+\infty}\left[\left(S_{4}^{ \pm}+S_{5}^{ \pm}\right) e^{\kappa \phi(t)} \lambda^{-\kappa}\right] \leq(1-2 \varepsilon) 8^{\kappa} \Upsilon_{\epsilon}$, then

$$
\limsup _{t \rightarrow+\infty}\left[\left(S_{1}^{ \pm}-S_{3}^{ \pm}\right) e^{\kappa \phi(t)} \lambda^{-\kappa}\right] \leq(1-2 \varepsilon) 8^{\kappa} \Upsilon_{\epsilon}
$$

by (4.29). We prove with the same method that $\liminf _{t \rightarrow+\infty}\left[\left(S_{1}^{ \pm}-S_{3}^{ \pm}\right) e^{\kappa \phi(t)} \lambda^{-\kappa}\right] \geq(1+$ $2 \varepsilon) 8^{\kappa} \Upsilon_{-\epsilon}$. Moreover $\lim _{x \rightarrow 0} \Upsilon_{x}=\Upsilon_{0}$ which yields $\left(S_{1}^{ \pm}-S_{3}^{ \pm}\right) \sim_{t \rightarrow+\infty} 8^{\kappa} \Upsilon_{0} \lambda^{\kappa} e^{-\kappa \phi(t)}$. Since we already proved that $S_{2}^{ \pm}+S_{3}^{ \pm}=1-C_{0} 8^{\kappa} \lambda^{\kappa} e^{-\kappa \phi(t)}+o\left(e^{-\kappa \phi(t)}\right)$, we get

$$
1-E\left(e^{-\frac{\lambda}{t} \mathbf{U}}\right) \sim_{t \rightarrow+\infty} 8^{\kappa}\left(C_{0}-\Upsilon_{0}\right) \lambda^{\kappa} e^{-\kappa \phi(t)}
$$

which proves the proposition.

## 5. Proof of the main Results

5.1. The renewal results : In this section we prove Propositions 1.4, 1.6 and Corollary 1.5. We start with the following important intermediate result on the first exit time $\mathbf{U}$.
In what follows, we mainly use the same ideas as in [16] pages 18 to 22 , inspired from the book [20] pages 470-472. We first need a lemma and give its proof for the sake of completeness. Recall that for all $i \geq 1, U_{i}=H\left(\tilde{L}_{i}\right)-H\left(\tilde{m}_{i}\right)$.

Lemma 5.1. For $t>0$, let $\mu_{t}$ be the positive measure on $\mathbb{R}_{+}$such that for all $x \geq 0, \mu_{t}([0, x])=$ $e^{-\kappa \phi(t)} \sum_{j=1}^{n_{t}-2} \mathbb{P}\left(\sum_{i=1}^{j} U_{i} / t \leq x\right)$. Then, $\left(\mu_{t}\right)_{t}$ converges vaguely, as $t \rightarrow+\infty$, to the measure $\mu$ defined by $\mathrm{d} \mu(x):=\left(C_{\kappa} \Gamma(\kappa)\right)^{-1} x^{\kappa-1} \mathbb{1}_{(0,+\infty)}(x) \mathrm{d} x$. Moreover, on every compact subset in $(0,+\infty), x \mapsto e^{\kappa \phi(t)} \mathbb{P}(\mathbf{U} / t \geq x)$ converges uniformly when $t \rightarrow+\infty$ to $x \mapsto C_{\kappa} x^{-\kappa} / \Gamma(1-\kappa)$.

Proof: First, let us prove that for all $\lambda>0$, we have as $t \rightarrow+\infty$,

$$
\begin{align*}
\int_{0}^{+\infty} e^{-\lambda x} \mathrm{~d} \mu_{t}(x) & =\int_{0}^{+\infty} \frac{e^{-\lambda x} x^{\kappa-1}}{C_{\kappa} \Gamma(\kappa)} \mathrm{d} x+o(1)  \tag{5.1}\\
\int_{0}^{+\infty} e^{-\lambda x} e^{\kappa \phi(t)} \mathbb{P}(\mathbf{U} / t \geq x) \mathrm{d} x & =\int_{0}^{+\infty} e^{-\lambda x} \frac{C_{\kappa}}{\Gamma(1-\kappa) x^{\kappa}} \mathrm{d} x+o(1) \tag{5.2}
\end{align*}
$$

Let $\lambda>0$. First, we have, by Proposition 3.2,
$\int_{0}^{+\infty} e^{-\lambda x} \mathrm{~d} \mu_{t}(x)=\frac{1}{e^{\kappa \phi(t)}} \sum_{j=1}^{n_{t}-2} \mathbb{E}\left(e^{-\lambda \sum_{i=1}^{j} \frac{U_{i}}{t}}\right)=\sum_{j=1}^{n_{t}-2}\left(\frac{1}{e^{\kappa \phi(t)}}\left(\mathbb{E}\left(e^{-\lambda \frac{\mathrm{U}}{t}}\right)\right)^{j}\right)+O\left(\frac{n_{t} u\left(t, n_{t}\right)}{e^{\kappa \phi(t)}}\right)$.
We notice that, by Proposition 4.7, $\left[\mathbb{E}\left(e^{-\lambda \mathbf{U} / t}\right)\right]^{n_{t}-1}=o(1)$, since $n_{t} e^{-\kappa \phi(t)} \rightarrow_{t \rightarrow+\infty}+\infty$ and $C_{\kappa}>0$. Hence, again by Proposition 4.7, we get since $n_{t} u\left(t, n_{t}\right) e^{-\kappa \phi(t)} \rightarrow 0$ as $t \rightarrow+\infty$,

$$
\int_{0}^{+\infty} e^{-\lambda x} \mathrm{~d} \mu_{t}(x)=\frac{e^{-\kappa \phi(t)}(1+o(1))}{1-\mathbb{E}\left(e^{-\lambda \mathbf{U} / t}\right)}+o(1)=\frac{1}{C_{\kappa} \lambda^{\kappa}}+o(1)=\int_{0}^{+\infty} \frac{e^{-\lambda x} x^{\kappa-1}}{C_{\kappa} \Gamma(\kappa)} \mathrm{d} x+o(1)
$$

which gives (5.1). This implies the pointwise convergence of the Laplace transforms of $\left(\mu_{t}\right)_{t}$ to that of $\mu$, and then the vague convergence of $\left(\mu_{t}\right)_{t}$ to $\mu$. Now, we have by Fubini,

$$
\mathbb{E}\left(1-e^{-\frac{\lambda}{t} \mathbf{U}}\right)=\int_{0}^{\infty} \int_{0}^{u} \lambda e^{-\lambda x} \mathrm{~d} x \mathbb{P}(\mathbf{U} / t \in \mathrm{~d} u)=\lambda \int_{0}^{+\infty} e^{-\lambda x} \mathbb{P}(\mathbf{U} / t \geq x) \mathrm{d} x
$$

Since $\lambda^{\kappa}=\lambda \int_{0}^{+\infty} e^{-\lambda x} x^{-\kappa} \mathrm{d} x / \Gamma(1-\kappa)$ and using Proposition 4.7, we get (5.2). From this pointwise convergence of the Laplace transform of $e^{\kappa \phi(t)} \mathbb{P}(\mathbf{U} / t \geq x)$, we get the uniform convergence of $e^{\kappa \phi(t)} P(\mathbf{U} / t \geq x)$ when $t$ goes to infinity as in [16].

Proof of Proposition 1.4 and Corollary 1.5: Let $\tilde{N}_{t}$ be the unique integer such that $H\left(\tilde{m}_{\tilde{N}_{t}}\right) \leq t<H\left(\tilde{m}_{\tilde{N}_{t}+1}\right)$.

First, by Lemma 2.3 and (3.22), $\mathbb{P}\left(N_{t} \geq n_{t}\right) \leq \mathbb{P}\left(\sum_{j=1}^{n_{t}-1} U_{j} \leq H\left(\tilde{m}_{n_{t}}\right) \leq t\right)+C_{1} w_{t}$. By the exponential Markov inequality, this is $\leq e \mathbb{E}\left(e^{-\sum_{i=1}^{n_{t}-1} U_{i} / t}\right)+C_{1} w_{t}$. Finally, Propositions 3.2 and 4.7 give since $\phi(t)=o(\log t)$,

$$
\begin{equation*}
\mathbb{P}\left(N_{t} \geq n_{t}\right) \leq e\left(\mathbb{E}\left(e^{-\mathbf{U} / t}\right)\right)^{n_{t}}+C_{1} w_{t}+C_{2} u\left(t, n_{t}\right) \leq C_{+} e^{-c_{-} n_{t} e^{-\kappa \phi(t)}}+2 C_{2} u\left(t, n_{t}\right)=o(1) . \tag{5.3}
\end{equation*}
$$

Assume first that $0<r<s<1$, and $a>0$. Then, Lemmata 2.3, 3.5 and (5.3) yield to

$$
\begin{align*}
& \mathbb{P}\left[1-s \leq H\left(m_{N_{t}}\right) / t \leq 1-r, H\left(m_{N_{t}+1}\right) / t \geq 1+a\right]  \tag{5.4}\\
& \leq \sum_{j=1}^{n_{t}-1} \mathbb{P}\left(1-s \leq \frac{H\left(\tilde{m}_{\tilde{N}_{t}}\right)}{t} \leq 1-r, H\left(\tilde{m}_{\tilde{N}_{t}+1}\right) / t \geq 1+a, \tilde{N}_{t}=j, \mathcal{V}_{t}\right) \\
& \quad+\mathbb{P}\left(N_{t} \notin\left[1, n_{t}\right)\right)+P\left(\overline{\mathcal{V}}_{t}\right) \\
& \leq \sum_{j=1}^{n_{t}-1} \mathbb{P}\left(1-s_{t} \leq \sum_{l=1}^{j-1} U_{l} / t \leq 1-r, \sum_{l=1}^{j} U_{l} / t>1+a-2 / \log h_{t}\right)+o(1) .
\end{align*}
$$

where $s_{t}:=s+2 / \log h_{t}$. We now use (3.9) of Proposition 3.2 and get for any $\varepsilon>0$, for large $t$,

$$
\begin{equation*}
(5.4) \leq \int_{1-s-\varepsilon}^{1-r} e^{\kappa \phi(t)} \mathbb{P}(\mathbf{U} / t>1+a-x-\varepsilon) \mathrm{d} \mu_{t}(x)+o(1) . \tag{5.5}
\end{equation*}
$$

Let $0<r^{\prime}<s^{\prime}<1$ and $a^{\prime}>0$. Using first the uniform convergence of $u \mapsto e^{\kappa \phi(t)} \mathbb{P}(\mathbf{U} / t>u)$ on the compact $\left[a^{\prime}+r^{\prime}, a^{\prime}+s^{\prime}\right] \subset(0, \infty)$ and then the vague convergence of $\mu_{t}$ (see Lemma 5.1), we get

$$
\lim _{t \rightarrow+\infty} \int_{1-s^{\prime}}^{1-r^{\prime}} e^{\kappa \phi(t)} \mathbb{P}\left(\mathbf{U} / t>1+a^{\prime}-x\right) \mathrm{d} \mu_{t}(x)=\int_{1-s^{\prime}}^{1-r^{\prime}} \frac{x^{\kappa-1}\left(1+a^{\prime}-x\right)^{-\kappa}}{\Gamma(\kappa) \Gamma(1-\kappa)} \mathrm{d} x .
$$

Consequently, by letting $\varepsilon \rightarrow 0$, we obtain the first inequality of the following line

$$
\limsup _{t \rightarrow+\infty}(5.4) \leq \int_{1-s}^{1-r} \int_{a}^{\infty} \frac{\kappa}{\Gamma(1-\kappa) \Gamma(\kappa)}(1+y-x)^{-\kappa-1} x^{\kappa-1} \mathrm{~d} y \mathrm{~d} x \leq \liminf _{t \rightarrow+\infty}(5.4)
$$

We prove similarly the second inequality. Since we consider probability measures, the cases $r=0, s=1$ or $a=0$ follow, which concludes the proof of Proposition 1.4. Corollary 1.5 follows by straightforward computations.
Proof of Proposition 1.6: Let us denote by $\nu_{t}$ a positive measure on $\mathbb{R}_{+}$, such that for every $x \geq 0, \nu_{t}([0, x])=e^{-\kappa \phi(t)} \sum_{j=2}^{n_{t}-1} e^{-\frac{C_{\kappa} \lambda_{j}}{e^{\kappa \phi}(t)}} P\left(\sum_{27}^{j-1} U_{i} / t \leq x\right)$. We have, with the arguments
already used between (5.4) and (5.5), for any $a>0$ and $\varepsilon>0$,

$$
\left.\begin{array}{rl} 
& \mathbb{E}\left[e^{-\frac{C_{\kappa} \lambda^{\kappa} N_{t}}{e^{\phi \phi(t)}}}, H\left(m_{N_{t}+1}\right) / t \geq 1+a\right] \\
\leq & \sum_{j=1}^{n_{t}-1} e^{-\frac{C_{\kappa} \lambda^{\kappa}}{e^{\kappa \phi}(t)}} \mathbb{P}
\end{array} \sum_{i=1}^{j-1} U_{i} / t \leq 1,1+a-2 / \log h_{t} \leq \sum_{i=1}^{j} U_{i} / t\right]+o(1) .
$$

We prove similarly the lower bound. We now show that the Laplace transform of the measure $\nu_{t}$ converges when $t$ goes to infinity. We consider $\alpha$ such that $0<\lambda<\alpha$. We get,

$$
\int_{0}^{+\infty} e^{-\alpha u} \mathrm{~d} \nu_{t}(u)=\frac{\mathbb{E}\left(e^{-\frac{\alpha}{t} \mathbf{U}}\right)\left[1-\left[e^{-\frac{C_{\kappa} \lambda^{\kappa}}{e^{\kappa \phi}(t)}} \mathbb{E}\left(e^{-\frac{\alpha}{t} \mathbf{U}}\right)\right]^{n_{t}-2}\right]}{e^{\kappa \phi(t)+\frac{2 C_{\kappa} \lambda^{\kappa}}{e^{\kappa \phi}(t)}}}\left[1-e^{-\frac{C_{\kappa} \lambda^{\kappa}}{e^{\kappa \phi(t)}}} \mathbb{E}\left(e^{-\frac{\alpha}{t} \mathbf{U}}\right)\right] \quad o(1)=\frac{1}{C_{\kappa}\left(\alpha^{\kappa}+\lambda^{\kappa}\right)}+o(1)
$$

by Propositions 3.2 and then 4.7. We also notice that

$$
\frac{1}{\alpha^{\kappa}+\lambda^{\kappa}}=\sum_{j=0}^{+\infty} \frac{\left(-\lambda^{\kappa}\right)^{j}}{\alpha^{\kappa(1+j)}}=\sum_{j=0}^{+\infty} \frac{\left(-\lambda^{\kappa}\right)^{j}}{\Gamma[\kappa(1+j)]} \int_{0}^{+\infty} e^{-\alpha u} u^{\kappa(1+j)-1} \mathrm{~d} u
$$

So Fubini gives, since $\int_{0}^{\infty} \sum_{j=0}^{\infty}\left|\frac{e^{-\alpha u} u^{\kappa(1+j)-1}}{C_{\kappa}} \frac{\left(-\lambda^{\kappa}\right)^{j}}{\Gamma[\kappa(1+j)]}\right| \mathrm{d} u<\infty$,

$$
\forall \alpha>\lambda, \quad \lim _{t \rightarrow+\infty} \int_{0}^{+\infty} e^{-\alpha u} \mathrm{~d} \nu_{t}(u)=\int_{0}^{+\infty} e^{-\alpha u} \mathrm{~d} \nu(u)+o(1),
$$

where $\nu$ is the measure defined by $\mathrm{d} \nu(u)=\frac{1}{C_{\kappa}} \sum_{j=0}^{+\infty}\left(-\lambda^{\kappa}\right)^{j} \frac{u^{\kappa(1+j)-1}}{\Gamma[\kappa(1+j)]} \mathbb{1}_{\mathbb{R}_{+}}(u) \mathrm{d} u$. This pointwise convergence of the Laplace transform of $\nu_{t}$ leads to the vague convergence of $\nu_{t}$ to $\nu$. Using the uniform convergence of $x \mapsto e^{\kappa \phi(t)} \mathbb{P}(\mathbf{U} / t>1+a-x)$ on [0,1] provided by Lemma 5.1, we get

$$
\lim _{t \rightarrow+\infty} \mathbb{E}\left[e^{-\frac{C_{\kappa} \lambda^{\kappa} N_{t}}{e^{\kappa \phi}(t)}}, \frac{H\left(m_{N_{t}+1}\right)}{t} \geq 1+a\right]=\sum_{j=0}^{+\infty}\left(-\lambda^{\kappa}\right)^{j} \int_{0}^{1} \frac{(1+a-x)^{-\kappa} x^{\kappa(1+j)-1}}{\Gamma[\kappa(1+j)] \Gamma(1-\kappa)} \mathrm{d} x .
$$

and this remains true for $a=0$. Since $\int_{0}^{1} x^{a-1}(1-x)^{b-1} \mathrm{~d} x=\Gamma(a) \Gamma(b) / \Gamma(a+b)$ for every $a>0$ and $b>0$, changing $C_{\kappa} \lambda^{\kappa}$ into $u$ gives the pointwise convergence of $\mathbb{E}\left[\exp \left(-u N_{t} / e^{\kappa \phi(t)}\right)\right]$ to the right hand side of (1.4), which ends the proof of Proposition 1.6.
5.2. The localization : proof of Theorem 1.3. Let $\phi^{*}(t):=\phi(t) / \zeta$, where $0<\zeta<1$ will be chosen later. Let us define $H_{x \rightarrow y}:=H(y)-H(x)$ for $0<x<y, t^{*}:=t-e^{\phi^{*}(t)(1+2 \delta)}$,

$$
\begin{array}{ll}
\mathcal{A}_{0}:=\left\{1 \leq \tilde{N}_{t}<n_{t}\right\}, & \mathcal{A}_{1}:=\cap_{j=1}^{n_{t}-1}\left\{H_{\tilde{L}_{j} \rightarrow \tilde{m}_{j+1}} \leq 2 t / \log h_{t}\right\}, \\
\mathcal{A}_{2}:=\cap_{j=1}^{n_{t}-1}\left\{H_{\tilde{m}_{j} \rightarrow \tilde{m}_{j+1}}<H_{\tilde{m}_{j} \rightarrow \tilde{L}_{j}^{-}}\right\}, & \mathcal{A}_{3}:=\left\{H\left(\tilde{m}_{\tilde{N}_{t}}\right) \leq t^{*}\right\} .
\end{array}
$$

We also introduce $\mathbf{I}_{j}:=\left[\tilde{m}_{j}-\phi^{*}(t) / \zeta, \tilde{m}_{j}+\phi^{*}(t) / \zeta\right], j \in \mathbb{N}^{*}$. Let $\varepsilon>0$. We have:

$$
\begin{aligned}
\mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_{t}}\right) & \leq \mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_{t}}, \tilde{N}_{t}=\tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_{0}, \mathcal{A}_{1}, \mathcal{A}_{2}, \mathcal{A}_{3}\right)+\mathbb{P}\left(\tilde{N}_{t} \neq \tilde{N}_{t(1+\varepsilon)}\right) \\
& +\mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_{t}}, \tilde{N}_{t}=\tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_{0}, \mathcal{A}_{1}, \mathcal{A}_{2}, \overline{\mathcal{A}_{3}}\right)+\mathbb{P}\left(\overline{\mathcal{A}_{0}}\right)+\mathbb{P}\left(\overline{\mathcal{A}_{1}}\right)+\mathbb{P}\left(\overline{\mathcal{A}_{2}}\right) .
\end{aligned}
$$

We split the proof into three parts. We start with:

Part 1: we prove that there exists $c_{4}>0$ such that for large $t$,

$$
\begin{equation*}
\mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_{t}}, \tilde{N}_{t}=\tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_{0}, \mathcal{A}_{1}, \mathcal{A}_{2}, \mathcal{A}_{3}\right) \leq C_{+} n_{t}\left(h_{t} e^{-3 \phi^{*}(t)}+\phi^{*}(t) e^{-c_{4} \delta \phi^{*}(t)}\right) . \tag{5.6}
\end{equation*}
$$

Let $\mathcal{B}_{j}:=\left\{\tilde{N}_{t}=\tilde{N}_{t(1+\varepsilon)}=j\right\} \cap \mathcal{A}_{1} \cap \mathcal{A}_{2} \cap \mathcal{A}_{3}$. We have

$$
\begin{equation*}
\mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_{t}}, \tilde{N}_{t}=\tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_{0}, \mathcal{A}_{1}, \mathcal{A}_{2}, \mathcal{A}_{3}\right)=\sum_{j=1}^{n_{t}-1} \mathbb{P}\left(X(t) \notin \mathbf{I}_{j}, \mathcal{B}_{j}\right) . \tag{5.7}
\end{equation*}
$$

If $t$ is large enough, on $\mathcal{B}_{j}$, after first hitting $\tilde{m}_{j}, X$ stays in $\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right]$ at least until time $t(1+\varepsilon / 2)$. Therefore, conditioning on $H\left(\tilde{m}_{j}\right)$ and using the strong Markov property,

$$
\begin{align*}
\mathbb{P}\left(X(t) \notin \mathbf{I}_{j}, \mathcal{B}_{j}\right) & \leq E\left(\mathbb{P}^{W_{\kappa}}\left(X(t) \notin \mathbf{I}_{j}, H\left(\tilde{m}_{j}\right) \leq t^{*}, \forall u \in\left[H\left(\tilde{m}_{j}\right), t(1+\varepsilon / 2)\right], X(u) \in\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right]\right)\right) \\
& =E\left(\int_{0}^{t^{*}} \mathbb{P}^{W_{\kappa}}\left(X(t) \notin \mathbf{I}_{j}, H\left(\tilde{m}_{j}\right) \in d s, \forall u \in\left[s, t\left(1+\frac{\varepsilon}{2}\right)\right], X(u) \in\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right]\right)\right) \\
& \leq E\left(\sup _{0 \leq s \leq t^{*}} \mathbb{P}_{\tilde{m}_{j}}^{W_{j}}\left(X(t-s) \notin \mathbf{I}_{j}, \forall u \in[0, t(1+\varepsilon / 2)-s], X(u) \in\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right]\right)\right) . \tag{5.8}
\end{align*}
$$

So, as in ([5], proof of Prop. 4.1) we now introduce a coupling between $X$ (under $\mathbb{P}_{\tilde{m}_{j}^{\kappa}}^{W_{j}}$ ) and a reflected process $Y_{j}$ defined below. To this aim, let $\left(Y_{j}^{(x)}(u), u \geq 0\right)$ be the process defined by $Y_{j}^{(x)}(u):=A^{-1}\left(\widehat{B}_{j}^{(x)}\left(\widehat{T}_{x, j}^{-1}(u)\right)\right)$ for any $x \in\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right]$ and $u \geq 0$, where $\widehat{B}_{j}^{(x)}$ is a one-dimensional Brownian motion starting from $A(x)$ and reflected at $A\left(\tilde{L}_{j}^{-}\right)$and $A\left(\tilde{L}_{j}\right)$ and independent from $W_{\kappa}$, and $\widehat{T}_{x, j}$ is defined like $T$ replacing $B$ by $\hat{B}_{j}^{(x)}$. In words, $Y_{j}^{(x)}$ is a diffusion in the potential $W_{\kappa}$, starting from $x \in\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right]$ and reflected at $\tilde{L}_{j}^{-}$and $\tilde{L}_{j}$. We denote its law by $\widehat{P}_{j, x}^{W_{\kappa}}$.

This enables us to define $Y_{j}$ by $\widehat{P}_{j}^{W_{\kappa}}\left(Y_{j} \in.\right):=\int_{\tilde{L}_{j}^{-}}^{\tilde{L}_{j}} \widehat{P}_{j, x}^{W_{\kappa}}() .\mathrm{d} \tilde{\mu}_{j}(x)$, where

$$
\begin{equation*}
\mathrm{d} \tilde{\mu}_{j}(x):=\exp \left(-\tilde{V}^{(j)}(x)\right) \mathbb{1}_{\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right]}(x) \mathrm{d} x\left(\int_{\tilde{L}_{j}^{-}}^{\tilde{L}_{j}} \exp \left(-\tilde{V}^{(j)}(y)\right) d y\right)^{-1} \tag{5.9}
\end{equation*}
$$

As in ([5], proof of Prop. 4.1), $\tilde{\mu}_{j}$ is invariant for the semi-group of $Y_{j}$; in particular $\widehat{P}_{j}^{W_{\kappa}}\left(Y_{j}(s) \in\right.$ $U)=\tilde{\mu}_{j}(U)$ for every $s \geq 0$ and $U \subset\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right]$. We can now, as in [5], build a coupling $Q_{\tilde{m}_{j}}^{W_{\kappa}}$ of $X$ and $Y_{j}$, such that $Q_{\tilde{m}_{j}}^{W_{\kappa}}\left(Y_{j} \in.\right)=\widehat{P}_{j}^{W_{\kappa}}\left(Y_{j} \in.\right)$, and $Q_{\tilde{m}_{j}}^{W_{\kappa}}(X \in)=.\mathbb{P}_{\tilde{m}_{j}}^{W_{\kappa}}($.$) , these two Markov$ processes $Y_{j}$ and $X$ move independently until the first collision $\widehat{H}_{j}:=\inf \left\{u \geq 0, X(u)=Y_{j}(u)\right\}$, then $X(u)=Y_{j}(u)$ until $\widehat{H}_{j}^{e}:=\inf \left\{u>\widehat{H}_{j}, X(u) \notin\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right]\right\}$ and then $X$ and $Y_{j}$ move independently again.

We now prove that, with a large probability, under $Q_{\tilde{m}_{j}}^{W_{\kappa}}, X$ and $Y_{j}$ first meet before time $t_{1}:=t-t^{*}=e^{\phi^{*}(t)(1+2 \delta)}$. To this aim, we introduce $\widehat{L}_{j}^{+}:=\tilde{\tau}_{j}^{+}\left(\phi^{*}(t)\right)$ and $\widehat{L}_{j}^{-}:=\tilde{\tau}_{j}^{-}\left(\phi^{*}(t)\right)$, and we first prove a result concerning only $X$ :
Lemma 5.2. With a probability larger than $1-C_{+} e^{-c_{-} \delta \phi^{*}(t)}$, and with $x \vee y:=\max (x, y)$,

$$
\begin{equation*}
P_{\tilde{m}_{j}}^{W_{\kappa}}\left(H\left(\widehat{L}_{j}^{+}\right) \vee H\left(\widehat{L}_{j}^{-}\right) \leq e^{(1+\delta) \phi^{*}(t)}\right) \geq 1-C_{+} \phi^{*}(t) e^{-c_{-} \delta \phi^{*}(t)} \tag{5.10}
\end{equation*}
$$

Proof: We introduce $\widehat{L}_{j}^{++}:=\tilde{\tau}_{j}^{+}\left[\phi^{*}(t)(1+\delta / 2)\right]$ and $\widehat{L}_{j}^{--}:=\tilde{\tau}_{j}^{-}\left[\phi^{*}(t)(1+\delta / 2)\right]$. Then with a method similar as for (3.12), we get $P_{\tilde{m}_{j}}^{W_{k}}\left(H\left(\widehat{L}_{j}^{+}\right)>H\left(\widehat{L}_{j}^{--}\right)\right)=\widehat{Q}_{j} / \widehat{D}_{j}$, where

$$
\begin{aligned}
& \widehat{Q}_{j} \leq\left(\widehat{L}_{j}^{+}-\tilde{m}_{j}\right) \exp \left[\max _{\left[\tilde{m}_{j}, \hat{L}_{j}^{+}\right]} \tilde{V}^{(j)}(s)\right] \leq 8 \kappa^{-1} \phi^{*}(t) \exp \left(\phi^{*}(t)\right), \\
& \widehat{D}_{j} \geq\left[\tilde{\tau}_{j}^{-}\left((1+\delta / 4) \phi^{*}(t)\right)-\tilde{\tau}_{j}^{-}\left((1+\delta / 2) \phi^{*}(t)\right)\right] e^{(1+\delta / 8) \phi^{*}(t)} \geq e^{(1+\delta / 8) \phi^{*}(t)}
\end{aligned}
$$

with a probability larger than $1-e^{-c_{-} \delta \phi^{*}(t)}$, by (2.4) for the first line and by (3.2) and (3.3) for the second one this together with Lemma 2.3. Hence, with such a probability, $P_{\tilde{m}_{j}}^{W_{\kappa}}\left(H\left(\widehat{L}_{j}^{+}\right)<\right.$ $\left.H\left(\widehat{L}_{j}^{--}\right)\right) \geq 1-C_{+} \phi^{*}(t) e^{-\delta \phi^{*}(t) / 8}$.

Moreover, as in (3.13), we can write under $P_{\tilde{m}_{j}}^{W_{\kappa}}$ on the event $\left\{H\left(\widehat{L}_{j}^{+}\right)<H\left(\widehat{L}_{j}^{--}\right)\right\}$,

$$
H\left(\widehat{L}_{j}^{+}\right) \stackrel{\mathcal{L}}{=} \tilde{A}_{j}\left(\widehat{L}_{j}^{+}\right) \int_{\widehat{L}_{j}^{--}}^{\widehat{L}_{j}^{+}} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_{B}\left[\tau_{B}(1), \tilde{A}_{j}(u) / \tilde{A}_{j}\left(\widehat{L}_{j}^{+}\right)\right] \mathrm{d} u=: \mathcal{H}_{j} .
$$

Then using the work to get Lemma 4.6, we can prove using the same arguments that with a probability greater than $1-C_{+} e^{-c_{-} \delta \phi^{*}(t)}, P_{\tilde{m}_{j}}^{W_{k}}\left(\mathcal{H}_{j} \geq e^{(1+\delta) \phi^{*}(t)}\right) \leq C_{+} e^{-c_{-} \delta \phi^{*}(t)}$ and therefore $P_{\tilde{m}_{j}}^{W_{\kappa}}\left(H\left(\widehat{L}_{j}^{+}\right) \leq e^{(1+\delta) \phi^{*}(t)}\right) \geq 1-C_{+} \phi^{*}(t) e^{-c-\delta \phi^{*}(t)}$. We get the same result for $H\left(\widehat{L}_{j}^{-}\right)$, collecting both estimations we obtain (5.10).

We deduce from (5.10) that

$$
\begin{aligned}
Q_{\tilde{m}_{j}}^{W_{\kappa}}\left(\widehat{H}_{j}>t_{1}\right) \leq & Q_{\tilde{m}_{j}}^{W_{\kappa}}\left(\widehat{H}_{j}>H\left(\widehat{L}_{j}^{+}\right) \vee H\left(\widehat{L}_{j}^{-}\right)\right)+Q_{\tilde{m}_{j}}^{W_{\kappa}}\left(H\left(\widehat{L}_{j}^{+}\right) \vee H\left(\widehat{L}_{j}^{-}\right)>t_{1}\right) \\
\leq & Q_{\tilde{m}_{j}}^{W_{\kappa}}\left(\widehat{H}_{j}>H\left(\widehat{L}_{j}^{-}\right), Y_{j}(0)<\tilde{m}_{j}\right)+Q_{\tilde{m}_{j}}^{W_{\kappa}}\left(\widehat{H}_{j}>H\left(\widehat{L}_{j}^{+}\right), Y_{j}(0) \geq \tilde{m}_{j}\right) \\
& +C_{+} \phi^{*}(t) e^{-c_{-} \delta \phi^{*}(t)} .
\end{aligned}
$$

Hence, since $X$ and $Y$ are continuous,

$$
\begin{align*}
Q_{\tilde{m}_{j}}^{W_{\kappa}}\left(\widehat{H}_{j}>t_{1}\right) \leq & Q_{\tilde{m}_{j}}^{W_{\kappa}}\left(Y_{j}\left[H\left(\widehat{L}_{j}^{-}\right)\right] \in\left[\tilde{L}_{j}^{-}, \widehat{L}_{j}^{-}\right], \widehat{H}_{j}>H\left(\widehat{L}_{j}^{-}\right)\right) \\
& +Q_{\tilde{m}_{j}}^{W_{k}}\left(Y_{j}\left[H\left(\widehat{L}_{j}^{+}\right)\right] \in\left[\widehat{L}_{j}^{+}, \tilde{L}_{j}\right], \widehat{H}_{j}>H\left(\widehat{L}_{j}^{+}\right)\right)+C_{+} \phi^{*}(t) e^{-c_{-} \delta \phi^{*}(t)} \\
\leq & \tilde{\mu}_{j}\left(\left[\tilde{L}_{j}^{-}, \widehat{L}_{j}^{-}\right]\right)+\tilde{\mu}_{j}\left(\left[\left(\widehat{L}_{j}^{+}, \tilde{L}_{j}\right]\right)+C_{+} \phi^{*}(t) e^{-c_{-} \delta \phi^{*}(t)},\right. \tag{5.11}
\end{align*}
$$

where the last line comes from the independence of $X$ and $Y_{j}$ until $\widehat{H}_{j}$ and the fact that $\tilde{\mu}_{j}$ is the invariant probability measure for $Y_{j}$.

Let $s \in\left[0, t^{*}\right]$. Since $X(u)=Y_{j}(u)$ for every $\widehat{H}_{j} \leq u \leq \widehat{H}_{j}^{e}$ and $t_{1} \leq t-s \leq t(1+\varepsilon / 2)-s$,

$$
\begin{align*}
& Q_{\tilde{m}_{j}^{\kappa}}^{W_{\kappa}}\left(X(t-s) \notin \mathbf{I}_{j}, \forall u \in[0, t(1+\varepsilon / 2)-s], X(u) \in\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right]\right) \\
\leq & Q_{\tilde{m}_{j}^{\kappa}}^{W_{k}}\left(\widehat{H}_{j} \leq t_{1}, X(t-s) \notin \mathbf{I}_{j}, \widehat{H}_{j}^{e} \geq t(1+\varepsilon / 2)-s\right)+Q_{\tilde{m}_{j}}^{W_{\kappa}}\left(\widehat{H}_{j}>t_{1}\right) \\
\leq & \tilde{\mu}_{j}\left(\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right] \backslash \mathbf{I}_{j}\right)+Q_{\tilde{m}_{j}}^{W_{\kappa}}\left(\widehat{H}_{j}>t_{1}\right), \tag{5.12}
\end{align*}
$$

and notice that we can replace $Q_{\tilde{m}_{j}^{\kappa}}^{W_{\kappa}}$ by $\mathbb{P}_{\tilde{m}_{j}}^{W_{\kappa}}$ in the first line. We now prove a Lemma about $\tilde{\mu}_{j}$ :
Lemma 5.3. For all $j \leq n_{t}$, with a probability greater than $1-C_{+} e^{-c_{-} \delta \phi^{*}(t)}$,

$$
\begin{align*}
\tilde{\mu}_{j}\left(\left[\tilde{L}_{j}^{-}, \widehat{L}_{j}^{-}\right]\right)+\mu_{j}\left(\left[\hat{L}_{j}^{+}, \tilde{L}_{j}\right]\right) & \leq C_{+} h_{t} e^{-\kappa(1-\delta) \phi^{*}(t) / 16},  \tag{5.13}\\
\tilde{\mu}_{j}\left(\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right] \backslash \mathbf{I}_{j}\right) & \leq C_{+} h_{t} e^{-\kappa(1-\delta) \phi^{*}(t) / 16} . \tag{5.14}
\end{align*}
$$

Notice that for any $\zeta$, the right hand sides of (5.13) and (5.14) go to 0 as $t \rightarrow+\infty$.
Proof: Working on $\mathcal{V}_{t}$ (and then using Lemma 2.3, Fact 2.2 and Remark 2.4) we only have to prove (5.13) and (5.14) replacing $\tilde{\mu}_{j}$ by $\mu$ defined by

$$
\begin{aligned}
\mathrm{d} \mu(x):= & \frac{1}{\tilde{Z}}[ \\
& e^{-R^{(1)}(-x)} \mathbb{1}_{\left[-\tau^{R^{(1)}}\left(h_{t}^{+}\right), 0\right)}(x)+e^{-R^{(2)}(x)} \mathbb{1}_{\left[0, \tau^{R^{(2)}}\left(h_{t}\right)\right]}(x) \\
& +e^{-\tilde{W}_{\kappa}^{h_{t}}\left(x-\tau^{R^{(2)}}\left(h_{t}\right)\right)} \mathbb{1}_{\left[\tau^{R}(2)\right.}\left(h_{t}\right), \tau^{R^{(2)}}\left(h_{t}\right)+\tau^{\left.\tilde{W}_{\kappa}^{h_{t}}\left(h_{t} / 2\right)\right]} \text { (x)] } \mathrm{d} x,
\end{aligned}
$$

where $\tilde{Z}:=\int_{-\tau^{R}(1)}^{0}\left(h_{t}\right) e^{-R^{(1)}(-y)} \mathrm{d} y+\int_{0}^{\tau^{R^{(2)}}\left(h_{t}^{+}\right)} e^{-R^{(2)}(y)} \mathrm{d} y+\int_{0}^{\tau_{\kappa}^{\tilde{T}_{\kappa}^{h}}\left(h_{t} / 2\right)} e^{-\tilde{W}_{\kappa}^{h_{t}}(y)} \mathrm{d} y$. Also replace $\left(\tilde{L}_{j}^{-}, \widehat{L}_{j}^{-}, \widehat{L}_{j}^{-}, \tilde{L}_{j}\right)$ by $\left(-\tau^{R^{(1)}}\left(h_{t}^{+}\right),-\tau^{R^{(1)}}\left(\phi^{*}(t)\right), \tau^{R^{(2)}}\left(\phi^{*}(t)\right), \tau^{\tilde{W}_{\kappa}^{h t}}\left(h_{t} / 2\right)\right)$, and $\mathbf{I}_{j}$ by $\mathbf{I}:=$ $\left[-\phi^{*}(t) / \zeta, \phi^{*}(t) / \zeta\right]$.

We now prove simultaneously both inequalities. Let $\mathcal{E}_{1}^{5.3}:=\left\{\tau^{R^{(1)}}\left(\kappa \phi^{*}(t) / 8 \zeta\right) \leq \phi^{*}(t) / \zeta\right\}$ and $\mathcal{E}_{2}^{5.3}:=\left\{\tau^{R^{(2)}}\left(\kappa \phi^{*}(t) / 8 \zeta\right) \leq \phi^{*}(t) / \zeta\right\}$. Using formula (3.6) with $h=\kappa \phi^{*}(t) / 8 \zeta, P\left(\overline{\mathcal{E}_{1}^{5.3}}\right)=$ $P\left(\overline{\mathcal{E}_{2}^{5.3}}\right) \leq C_{+} e^{-\kappa^{2} \phi^{*}(t) /(16 \sqrt{2} \zeta)}$. Also we have

$$
\begin{equation*}
\mu\left(-\tau^{R^{(1)}}\left(h_{t}^{+}\right),-\tau^{R^{(1)}}\left(\phi^{*}(t)\right)\right)+\mu\left(\tau^{R^{(2)}}\left(\phi^{*}(t)\right), \tau^{\tilde{W}_{\kappa}^{h_{t}}}\left(h_{t} / 2\right)\right) \leq\left(\mathcal{J}_{3}(1)+\mathcal{J}_{4}(1)+\mathcal{J}_{5}\right) / \tilde{Z}, \tag{5.15}
\end{equation*}
$$

where for all $x>0$,

$$
\begin{gathered}
\mathcal{J}_{3}(x):=\int_{\tau^{R^{(1)}}\left(\kappa \phi^{*}(t) / 8 x\right)}^{\tau^{\left.R^{1}\right)}\left(h_{t}^{+}\right)} e^{-R^{(1)}(y)} \mathrm{d} y, \quad \mathcal{J}_{4}(x):=\int_{\tau^{R^{(2)}\left(\kappa \phi^{*}(t) / 8 x\right)}}^{\tau^{R^{(2)}}\left(h_{t}\right)} e^{-R^{(2)}(y)} \mathrm{d} y, \\
\mathcal{J}_{5}:=\int_{0}^{\tau \tilde{\tilde{W}}_{\kappa}^{h t}\left(h_{t} / 2\right)} e^{-\tilde{W}_{\kappa}^{h_{t}}(y)} \mathrm{d} y .
\end{gathered}
$$

Moreover on $\mathcal{E}_{1}^{5.3} \cap \mathcal{E}_{2}^{5.3}$,

$$
\begin{equation*}
\mu\left(\left[-\tau^{R^{(1)}}\left(h_{t}^{+}\right), \tau^{\tilde{W}_{\kappa}^{h_{t}}}\left(h_{t} / 2\right)\right] \backslash \mathbf{I}\right) \leq\left(\mathcal{J}_{3}(\zeta)+\mathcal{J}_{4}(\zeta)+\mathcal{J}_{5}\right) / \tilde{Z} \tag{5.16}
\end{equation*}
$$

We now estimate $\mathcal{J}_{3}(1)$ and $\mathcal{J}_{4}(1)$ similarly. We introduce $\mathcal{E}_{3}^{5.3}:=\left\{\inf \left\{R^{(1)}(s), \tau^{R^{(1)}}\left(\kappa \phi^{*}(t) / 8\right) \leq\right.\right.$ $\left.\left.s \leq \tau^{R^{(1)}}\left(h_{t}^{+}\right)\right\}>(1-\delta) \kappa \phi^{*}(t) / 8\right\}$ and $\mathcal{E}_{4}^{5.3}:=\left\{\tau^{R^{(1)}}\left(h_{t}^{+}\right) \leq 8 h_{t}^{+} / \kappa\right\}$. Using (3.7) with $h=$ $\kappa \phi^{*}(t) / 8, \gamma=1-\delta$ and $\alpha=1$ and $\omega=h_{t}^{+} / h$ for $t$ large enough we have $P\left(\overline{\mathcal{E}_{3}^{5.3}}\right) \leq 2 e^{-\delta \kappa^{2} \phi^{*}(t) / 8}$, and by (3.6), $P\left(\overline{\mathcal{E}_{4}^{5.3}}\right) \leq C_{+} e^{-\kappa h_{t}^{+} / 2 \sqrt{2}}$. On $\mathcal{E}_{3}^{5.3} \cap \mathcal{E}_{4}^{5.3}$, we have

$$
\mathcal{J}_{3}(1) \leq e^{-(1-\delta) \kappa \phi^{*}(t) / 8} \tau^{R^{(1)}}\left(h_{t}^{+}\right) \leq C_{+} h_{t}^{+} e^{-(1-\delta) \kappa \phi^{*}(t) / 8}=o(1),
$$

as $\log \log t=o\left(\phi^{*}(t)\right)$. With the same method, replacing $h_{t}^{+}$by $h_{t}$ and $R^{(1)}$ by $R^{(2)}$, we get the same inequality for $\mathcal{J}_{4}(1)$ on an event $\mathcal{E}_{5}^{5.3}$ satisfying $\mathbb{P}\left(\overline{\mathcal{E}_{5}^{5.3}}\right) \leq C_{+} e^{-c-\delta \phi^{*}(t)}$.

Let $\mathcal{E}_{6}^{5.3}:=\left\{\tau^{\tilde{N}_{\kappa}^{h_{t}}}\left(h_{t} / 2\right) \leq 2 h_{t} / \kappa\right\}$. Equation (2.7) gives $P\left(\overline{\mathcal{E}_{6}^{5.3}}\right)=P\left(\tau^{W_{\kappa}}\left(-h_{t} / 2\right) \geq 2 h_{t} / \kappa\right) \leq$ $e^{-\kappa h_{t} / 16}$. We have on $\mathcal{E}_{6}^{5.3}$,

$$
\begin{equation*}
\mathcal{J}_{5} \leq e^{-h_{t} / 2} \tau^{\tilde{W}_{\kappa}^{h_{t}}}\left(h_{t} / 2\right) \leq C_{+} h_{t} e^{-h_{t} / 2} \tag{5.17}
\end{equation*}
$$

Define $\mathcal{E}_{7}^{5.3}:=\left\{\tau^{R^{(1)}}\left[\alpha \phi^{*}(t)\right]-\tau^{R^{(1)}}\left[\alpha \phi^{*}(t) / 2\right] \geq 1\right\}$ with $\alpha:=(1-\delta) \kappa / 16$. By (3.3) $P\left(\overline{\mathcal{E}_{7}^{5.3}}\right) \leq$ $c_{+} e^{-c_{-} \alpha^{2} \phi^{*}(t)^{2} / 4}$. On $\mathcal{E}_{7}^{5.3}$,

$$
\begin{equation*}
\tilde{Z} \geq \int_{\tau^{R^{(1)}}\left[\alpha \phi^{*}(t) / 2\right]}^{\tau_{R^{(1)}}\left[\alpha \phi^{*}(t)\right]} e^{-R^{(1)}(y)} \mathrm{d} y \geq e^{-\alpha \phi^{*}(t)}\left(\tau^{R^{(1)}}\left[\alpha \phi^{*}(t)\right]-\tau^{R^{(1)}}\left[\alpha \phi^{*}(t) / 2\right]\right) \geq e^{-\alpha \phi^{*}(t)} . \tag{5.18}
\end{equation*}
$$

We get on $\cap_{i=1}^{7} \mathcal{E}_{i}^{5.3}$ for large $t$ since $\phi^{*}(t)=o\left(h_{t}\right)$,

$$
\begin{align*}
\left(\mathcal{J}_{3}(1)+\mathcal{J}_{4}(1)+\mathcal{J}_{5}\right) / \tilde{Z} & \leq C_{+}\left[2 h_{t}^{+} e^{-(1-\delta) \kappa \phi^{*}(t) / 8}+h_{t} e^{-h_{t} / 2}\right] e^{\alpha \phi^{*}(t)} \\
& \leq C_{+} h_{t} e^{-\alpha \phi^{*}(t) / 2} \tag{5.19}
\end{align*}
$$

Since $\mathbb{P}\left(\cap_{i=1}^{7} \mathcal{E}_{i}^{5.3}\right) \geq 1-C_{+} e^{-c_{-} \delta \phi^{*}(t)}$, this together with (5.15) and Lemma 2.3 proves (5.13). Since $\zeta<1$, replacing $\phi^{*}(t)$ by $\phi^{*}(t) / \zeta$ in this proof also gives (5.19) with $\mathcal{J}_{3}(\zeta)$ and $\mathcal{J}_{4}(\zeta)$ instead of $\mathcal{J}_{3}(1)$ and $\mathcal{J}_{4}(1)$, which together with (5.16) and Lemma 2.3 proves (5.14).

Finally collecting (5.7) to (5.14), we obtain (5.6), which ends the proof of this Part 1.

Part 2: we prove that there exists $c_{5}>0$ such that if $\zeta \leq \kappa /(16(1+2 \delta))$,

$$
\begin{equation*}
\mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_{t}}, \quad \tilde{N}_{t}=\tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_{0}, \mathcal{A}_{1}, \mathcal{A}_{2}, \overline{\mathcal{A}_{3}}\right) \leq C_{+} n_{t} e^{-c_{5} \phi^{*}(t) / \zeta} \tag{5.20}
\end{equation*}
$$

First, we prove similarly as in Part 1 that

$$
\begin{align*}
& \mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_{t}}, \tilde{N}_{t}=\tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_{0}, \mathcal{A}_{1}, \mathcal{A}_{2}, \overline{\mathcal{A}_{3}}\right)  \tag{5.21}\\
& \leq \sum_{j=1}^{n_{t}-1} E\left(\sup _{t^{*} \leq s \leq t} \mathbb{P}_{\tilde{m}_{j}}^{W_{\kappa}}\left(X(t-s) \notin \mathbf{I}_{j}, \forall u \in[0, t(1+\varepsilon / 2)-s], X(u) \in\left[\tilde{L}_{j}^{-}, \tilde{L}_{j}\right]\right)\right)+o(1) \\
& \leq \sum_{j=1}^{n_{t}-1} E\left(\sup _{0 \leq s \leq e^{\phi^{*}(t)(1+2 \delta)}} \mathbb{P}_{\tilde{m}_{j}}^{W_{\kappa}}\left(H\left(\tilde{m}_{j}-\phi^{*}(t) / \zeta\right) \wedge H\left(\tilde{m}_{j}+\phi^{*}(t) / \zeta\right) \leq s\right)\right)+o(1) \tag{5.22}
\end{align*}
$$

Notice that under $\mathbb{P}_{\tilde{m}_{j}}^{W_{\kappa}}$ on $\left\{\tilde{\tau}_{j}^{+}\left[\kappa \phi^{*}(t) /(8 \zeta)\right]-\tilde{m}_{j} \leq \phi^{*}(t) / \zeta\right\} \cap\left\{\tilde{m}_{j}-\tilde{\tau}_{j}^{-}\left[\kappa \phi^{*}(t) /(8 \zeta)\right] \leq \phi^{*}(t) / \zeta\right\}$, which has probability greater than $1-C_{+} e^{-\kappa^{2} \phi^{*}(t) /(16 \zeta \sqrt{2})}$ by (2.4) and Lemma 2.3, we have $H\left(\tilde{m}_{j}-\phi^{*}(t) / \zeta\right) \geq H\left(\tilde{\tau}_{j}^{-}\left[\kappa \phi^{*}(t) /(8 \zeta)\right]\right)$, and so there exists a Brownian motion $\tilde{B}$, independent of $W_{\kappa}$, such that

$$
\begin{align*}
& H\left(\tilde{m}_{j}-\frac{\phi^{*}(t)}{\zeta}\right) \geq \int_{\tilde{\tau}_{j}^{-}\left[\kappa \phi^{*}(t) /(8 \zeta)\right]}^{\tilde{m}_{j}} e^{-V^{(j)}(u)} \mathcal{L}_{\tilde{B}}\left[\tau^{\tilde{B}}\left(\tilde{A}_{j}\left(\tilde{\tau}_{j}^{-}\left[\frac{\kappa \phi^{*}(t)}{8 \zeta}\right]\right)\right), \tilde{A}_{j}(u)\right] \mathrm{d} u:=H_{j}^{-},  \tag{5.23}\\
& H\left(\tilde{m}_{j}+\frac{\phi^{*}(t)}{\zeta}\right) \geq \int_{\tilde{m}_{j}}^{\tilde{\tau}_{j}\left[\kappa \phi^{*}(t) /(8 \zeta)\right]} e^{-V^{(j)}(u)} \mathcal{L}_{\tilde{B}}\left[\tau^{\tilde{B}}\left(\tilde{A}_{j}\left(\tilde{\tau}_{j}\left[\frac{\kappa \phi^{*}(t)}{8 \zeta}\right]\right)\right), \tilde{A}_{j}(u)\right] \mathrm{d} u:=H_{j}^{+} \tag{5.24}
\end{align*}
$$

where for all $\tilde{L}_{j}^{-} \leq z \leq \tilde{L}_{j}, \tilde{A}_{j}(z):=\int_{\tilde{m}_{j}}^{z} e^{\tilde{V}^{(j)}(x)} \mathrm{d} x$. So we get, recalling that $t-t^{*}=e^{\phi^{*}(t)(1+2 \delta)}$,

$$
\begin{equation*}
(5.21) \leq n_{t} \max _{1 \leq j \leq n_{t}-1}\left[\mathbb{P}\left(H_{j}^{-} \leq e^{\phi^{*}(t)(1+2 \delta)}\right)+\mathbb{P}\left(H_{j}^{+} \leq e^{\phi^{*}(t)(1+2 \delta)}\right)+C_{+} e^{-\kappa^{2} \phi^{*}(t) /(16 \zeta \sqrt{2})}\right] \tag{5.25}
\end{equation*}
$$

Lemma 5.4. $\mathbb{P}\left(H_{j}^{ \pm} \leq e^{\phi^{*}(t)(1+2 \delta)}\right) \leq e^{-c_{-} \phi^{*}(t) / \zeta}$ for large $t$ if $\zeta \leq \kappa /(16(1+2 \delta))$.
Proof: First, let $\varepsilon>0$, define $\delta_{t}^{*}:=e^{-(1-3 \varepsilon) \phi^{*}(t) /(2 \zeta)}, x_{j}^{*}:=\tilde{A}_{j}^{-1}\left(\delta_{t}^{*} \tilde{\tau}_{j}^{+}\left[\kappa \phi^{*}(t) /(8 \zeta)\right]\right)-\tilde{m}_{j}$, and notice that as in (4.13),

$$
H_{j}^{+} \geq \tilde{A}_{j}\left(\tilde{\tau}_{j}^{+}\left[\kappa \phi^{*}(t) /(8 \zeta)\right]\right)\left(\int_{0}^{x_{j}^{*}} e^{-\tilde{V}^{(j)}\left(x+\tilde{m}_{j}\right)} \mathrm{d} x\right) \inf _{u \in\left[0, \delta_{t}^{*}\right]} \mathcal{L}_{\tilde{B}}(1, u)=: \mathcal{J}_{6} \mathcal{J}_{7} \mathcal{J}_{8}
$$

Now, for large $t$, we have as before by (4.6), $\mathcal{J}_{8} \geq\left[1-\left(\delta_{t}^{*}\right)^{1 / 3}\right] \mathbf{e}_{1} \geq \mathbf{e}_{1} / 2$ with probability $\geq 1-$ $\left(\delta_{t}^{*}\right)^{1 / 30}$, with $\mathbf{e}_{1}=\mathcal{L}_{\tilde{B}}(1,0) \geq e^{-6^{-1} \kappa \phi^{*}(t) /(8 \zeta)}$ with probability $\geq 1-e^{-6^{-1} \kappa \phi^{*}(t) /(8 \zeta)}$. Moreover, by Lemma 2.3, William's decomposition and (3.5), $\mathcal{J}_{6} \geq \exp \left[(1-\varepsilon) \kappa \phi^{*}(t) /(8 \zeta)\right]$ with probability $\geq 1-3 e^{-\kappa^{2} \varepsilon \phi^{*}(t) /(16 \zeta)}$. Finally, as in (4.15), $\tilde{A}_{j}\left(\tilde{\tau}_{j}^{+}\left[2^{-1} \kappa \phi^{*}(t) /(8 \zeta)\right]\right) \leq e^{2^{-1}(1+\varepsilon) \kappa \phi^{*}(t) /(8 \zeta)}$ with
probability $\geq 1-C_{+} e^{-\varepsilon \kappa \phi^{*}(t) /(16 \zeta)}$, which gives $x_{j}^{*} \geq \tilde{\tau}_{j}^{+}\left[2^{-1} \kappa \phi^{*}(t) /(8 \zeta)\right]$ as in (4.16), and then $\mathcal{J}_{7} \geq \tilde{\tau}_{j}^{+}\left(6^{-1} \kappa \phi^{*}(t) /(8 \zeta)\right) e^{-6^{-1} \kappa \phi^{*}(t) /(8 \zeta)} \geq e^{-6^{-1} \kappa \phi^{*}(t) /(8 \zeta)}$ with probability $\geq 1-e^{-\phi^{*}(t) / \zeta}$ by (3.3) and Lemma 2.3. Taking $\varepsilon=1 / 6$, this gives $\mathbb{P}\left(H_{j}^{+} \geq e^{\kappa \phi^{*}(t) /(16 \zeta)}\right) \geq 1-e^{-c_{-} \phi^{*}(t) / \zeta}$. This proves the lemma for $H_{j}^{+}$. The proof for $H_{j}^{-}$is similar.

This leads to (5.20), which ends this Part 2.
Part 3: We prove that $\mathbb{P}\left(\tilde{N}_{t} \neq \tilde{N}_{t(1+\varepsilon)}\right)+\mathbb{P}\left(\overline{\mathcal{A}_{0}}\right)+\mathbb{P}\left(\overline{\mathcal{A}_{1}}\right)+\mathbb{P}\left(\overline{\mathcal{A}_{2}}\right) \leq \varepsilon^{1-\kappa}+o(1)$. First, with Lemma 2.3 and then (1.3), $\mathbb{P}\left(\tilde{N}_{t} \neq \tilde{N}_{t(1+\varepsilon)}\right)=\mathbb{P}\left[H\left(m_{N_{t+1}}\right)<t(1+\varepsilon)\right]+o(1) \leq \varepsilon^{1-\kappa}+o(1)$. Moreover, we have $\mathbb{P}\left(\overline{\mathcal{A}_{0}}\right)=o(1)$ by (5.3) and Lemma 2.3. The fact that $\mathbb{P}\left(\overline{\mathcal{A}_{1}}\right)=o(1)$ comes from Lemma 3.5. Moreover, by Markov property,

$$
\begin{aligned}
\mathbb{P}_{\tilde{m}_{j}}^{W_{\kappa}}\left(H\left(\tilde{m}_{j+1}\right)<H\left(\tilde{L}_{j}^{-}\right)\right) & =\mathbb{P}_{\tilde{m}_{j}}^{W_{k}}\left(H\left(\tilde{L}_{j}\right)<H\left(\tilde{L}_{j}^{-}\right)\right) \times \mathbb{P}_{\tilde{L}_{j}}^{W_{k}}\left(H\left(\tilde{m}_{j+1}\right)<H\left(\tilde{L}_{j}^{-}\right)\right) \\
& \geq \mathbb{P}_{\tilde{m}_{j}}^{W_{k}}\left(H\left(\tilde{L}_{j}\right)<H\left(\tilde{L}_{j}^{-}\right)\right) \times \mathbb{P}_{\tilde{L}_{j}}^{W_{k}}\left(H\left(\tilde{m}_{j+1}\right)<H\left(\tilde{L}_{j}^{*}\right)\right) .
\end{aligned}
$$

Both probabilities of the last line have already been estimated, in (3.12) with probability $1-$ $o\left(1 / n_{t}\right)$ for the first one, and in Lemma 3.5 for the second one (since $Q_{i}^{*} / D_{i}^{*} \leq 2 e^{-h_{t} / 8}$ with probability $1-o\left(1 / n_{t}\right)$, as proved between (3.23) and (3.25)). This proves that $\mathbb{P}\left(\overline{\mathcal{A}_{2}}\right)=o(1)$.

Finally combining (5.6), (5.20) with Part 3, choosing $\zeta$ small enough, using Lemma 2.3 back again to transform the $\tilde{m}$. in $m$. letting $t$ goes to infinity and $\varepsilon$ to zero, we get Theorem 1.3.
5.3. The aging : Proof of Proposition 1.2. We fix $\alpha>1$. We recall that the r.v. $\left(m_{i}\right)_{i}$ depend on $t$. In what follows, we apply Theorem 1.3 first at time $t$ with function $\phi$, and second at time $\alpha t$ with a function $\phi_{\alpha}$ defined by $\log (\alpha t)-\phi_{\alpha}(\alpha t)=\log t-\phi(t)$, so that the r.v. $m_{i}$ are the same in both cases. Let $0<\zeta<1$. By Theorem 1.3 and since $\mathbb{P}\left(\mathcal{A}_{0}\right) \rightarrow_{t \rightarrow+\infty} 1$, we get

$$
\begin{aligned}
& \mathbb{P}\left(|X(\alpha t)-X(t)| \leq \mathcal{C}_{1} \phi(t) / \zeta, N_{t}<N_{\alpha t}\right) \\
\leq & \mathbb{P}\left(\left|m_{N_{\alpha t}}-m_{N_{t}}\right| \leq \mathcal{C}_{1}(1+1 / \zeta) \phi(t)+\mathcal{C}_{1} \phi_{\alpha}(\alpha t), 1 \leq N_{t}<N_{\alpha t} \leq n_{\alpha t}, N_{t} \leq n_{t}\right)+o(1) \\
\leq & \sum_{i=1}^{n_{t}} \sum_{j=i+1}^{n_{\alpha t}} \mathbb{P}\left(\left|m_{i}-m_{j}\right| \leq \mathcal{C}_{1}(1+1 / \zeta) \phi(t)+\mathcal{C}_{1} \phi_{\alpha}(\alpha t)\right)+o(1),
\end{aligned}
$$

as $t \rightarrow+\infty$. So, using Lemma 2.3 and (2.3), we get $\mathbb{P}\left(|X(\alpha t)-X(t)| \leq \mathcal{C}_{1} \phi(t), N_{t}<N_{\alpha t}\right)=$ $o(1)$. Consequently,

$$
\begin{align*}
\mathbb{P}\left(|X(\alpha t)-X(t)| \leq \mathcal{C}_{1} \phi(t) / \zeta\right) & =\mathbb{P}\left(|X(\alpha t)-X(t)| \leq \mathcal{C}_{1} \phi(t) / \zeta, N_{t}=N_{\alpha t}\right)+o(1) \\
& \leq \mathbb{P}\left(N_{t}=N_{\alpha t}\right)+o(1) \tag{5.26}
\end{align*}
$$

Moreover, by Theorem 1.3, we have for large $t$,

$$
\begin{aligned}
& \mathbb{P}\left(|X(\alpha t)-X(t)|>\mathcal{C}_{1} \phi(t) / \zeta, N_{t}=N_{\alpha t}\right) \\
\leq & \mathbb{P}\left(\left|X(\alpha t)-m_{N_{t}}\right|>\mathcal{C}_{1} \phi(t) / \zeta-\mathcal{C}_{1} \phi(t), N_{t}=N_{\alpha t}\right)+o(1) \\
\leq & \mathbb{P}\left(\left|X(\alpha t)-m_{N_{\alpha t}}\right|>\mathcal{C}_{1} \phi(t)(1 / \zeta-1)\right)+o(1) \\
\leq & \mathbb{P}\left(\left|X(\alpha t)-m_{N_{\alpha t}}\right|>\mathcal{C}_{1} \phi_{\alpha}(\alpha t)\right)+o(1) .
\end{aligned}
$$

for large $t$ if we choose $\zeta=1 / 3$. Hence $\mathbb{P}\left(|X(\alpha t)-X(t)|>\mathcal{C}_{2} \phi(t), N_{t}=N_{\alpha t}\right)=o(1)$ by Theorem 1.3 applied at time $\alpha t$ and $\mathcal{C}_{2}:=3 \mathcal{C}_{1}$. Therefore,
$\mathbb{P}\left(N_{t}=N_{\alpha t}\right)=\mathbb{P}\left(|X(\alpha t)-X(t)| \leq \mathcal{C}_{2} \phi(t), N_{t}=N_{\alpha 3}\right)+o(1) \leq \mathbb{P}\left(|X(\alpha t)-X(t)| \leq \mathcal{C}_{2} \phi(t)\right)+o(1)$.

This together with (5.26) gives $\mathbb{P}\left(|X(\alpha t)-X(t)| \leq \mathcal{C}_{2} \phi(t)\right)=\mathbb{P}\left(N_{t}=N_{\alpha t}\right)+o(1)=\mathbb{P}\left(H\left(m_{N_{t}+1}\right)>\right.$ $\alpha t)+o(1)$. This, combined with (1.3), the change of variables $u=1 /(1+x)$ proves Proposition 1.2 , since $\phi$ is choosen up to a multiplicative constant.

Acknowledgment : We would like to thank Dominique Lepingle for pointing out the reference [30].

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[^0]:    Date: 25/11/2013 à 10:00:01.
    2010 Mathematics Subject Classification. 60K37, 60J55.
    Key words and phrases. Diffusion in a random potential, Localization, Aging, Renewal theorem.
    This research was partially supported by the french ANR projects MEMEMO2 2010 BLAN 012502 and Randymeca.

