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► **To cite this version:**

Pierre Andreatti, Alexis Devulder. Localization and number of visited valleys for a transient diffusion in random environment. 2013. hal-00908626v1

HAL Id: hal-00908626

<https://hal.science/hal-00908626v1>

Preprint submitted on 25 Nov 2013 (v1), last revised 9 Mar 2015 (v3)

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LOCALIZATION AND NUMBER OF VISITED VALLEYS FOR A TRANSIENT DIFFUSION IN RANDOM ENVIRONMENT

PIERRE ANDREOLETTI AND ALEXIS DEVULDER

ABSTRACT. We consider a transient diffusion in a $(-\kappa/2)$ -drifted Brownian potential W_κ with $0 < \kappa < 1$. We prove its localization before time t in a neighborhood of some random points depending only on the environment, which are the positive h_t -minima of the environment, for h_t a bit smaller than $\log t$. We also prove an Aging phenomenon for the diffusion, and provide a central limit theorem for the number of valleys visited up to time t .

The proof relies on a Williams' decomposition of the trajectory of W_κ in the neighborhood of local minima, with the help of results of Faggionato [19], and on a precise analysis of exponential functionals of W_κ and of 3-dimensional $(-\kappa/2)$ -drifted Bessel processes.

1. INTRODUCTION AND NOTATION

1.1. Presentation of the model. We are interested in a diffusion $(X(t), t \geq 0)$ in a random càdlàg potential $(V(x), x \in \mathbb{R})$, defined informally by $X(0) = 0$ and

$$dX(t) = d\beta(t) - \frac{1}{2}V'(X(t))dt,$$

where β is a Brownian motion independent of V . More rigorously, X is defined by its conditional generator given V , which is

$$\frac{1}{2}e^{V(x)} \frac{d}{dx} \left(e^{-V(x)} \frac{d}{dx} \right).$$

These diffusions in random potentials are considered as a continuous time analogues of random walks in random environment (RWRE) (see e.g. P. Révész, [31], B.D. Hughes [22], Z. Shi [35] and O. Zeitouni [42] for reviews on RWRE).

The study of such a process starts with a choice for V . A classic one, originally introduced by S. Schumacher [33] and T. Brox [5], is to take for V a Lévy process. In fact only a few papers deal with the discontinuous case, see for example P. Carmona [6] or A. Singh [36, 37], and most of the results concern continuous V , that is to say

$$V(x) = W_\kappa(x) := W(x) - \frac{\kappa}{2}x, \quad x \in \mathbb{R},$$

with $\kappa \in \mathbb{R}$ and W a two sided Brownian motion. We denote by P the probability measure associated to $W_\kappa(\cdot)$. The probability conditionally on the potential W_κ is denoted by \mathbb{P}^{W_κ} and is called the quenched probability. We also define the annealed probability as

$$\mathbb{P}(\cdot) := \int \mathbb{P}^{W_\kappa}(\cdot) P(W_\kappa \in d\omega).$$

We denote respectively by \mathbb{E}^{W_κ} , \mathbb{E} , and E the expectancies with regard to \mathbb{P}^{W_κ} , \mathbb{P} and P .

Date: 25/11/2013 à 10:00:01.

2010 Mathematics Subject Classification. 60K37, 60J55.

Key words and phrases. Diffusion in a random potential, Localization, Aging, Renewal theorem.

This research was partially supported by the french ANR projects MEMEMO2 2010 BLAN 0125 02 and Randymeca.

In the case $\kappa = 0$, X is recurrent and [5] shows that it is sub-diffusive with asymptotic behavior in $(\log t)^2$, moreover X is localized, at time t , in the neighborhood of a random point $b_{\log t}$ depending only on t and W . This result can be written:

Theorem 1.1. (*Brox* [5]) *Assume $\kappa = 0$, then for all $\varepsilon > 0$*

$$\lim_{t \rightarrow +\infty} \mathbb{P} [X(t) \in [b_{\log t} - \varepsilon(\log t)^2, b_{\log t} + \varepsilon(\log t)^2]] = 1. \quad (1.1)$$

The limit law of $b_{\log t}/(\log t)^2$ and therefore of $X(t)/(\log t)^2$ were made explicit independently by H. Kesten [25] and A. O. Golosov [23]. For recent results for this case see for example [1] and [11].

In the case $\kappa > 0$, the diffusion X is a.s. transient, with a wide range of limiting behaviors, depending on the value of κ . It was first studied by K. Kawazu and H. Tanaka. Let us denote by $H(r)$ the hitting time of $r \in \mathbb{R}$ by X :

$$H(r) := \inf\{t > 0, X(t) = r\}.$$

Kawazu et al. [24] proved in particular that when $0 < \kappa < 1$, $H(r)/r^{1/\kappa}$ converges in law to a stable distribution (see also Y. Hu et al. [21], and H. Tanaka [40]). More recently we mention the results for large and moderate deviations, by M. Taleb ([38] and [39]), A. Devulder [8] and G. Faraud [18].

In this paper we follow a different approach from [21] and [24]. Indeed we focus on a quenched study, which has attracted much interest for transient RWRE in the last few years, see for example the works of N. Enriquez et al. [14], [15], [16], [17], D. Dolgopyat et al. [12], and J. Peterson et al. [27], [28], [29]. Heuristically, the diffusion goes to locations where the potential is low, hence it goes to $+\infty$, but it is slowed by "valleys" of the potential, which trap the diffusion for some time.

1.2. Main results. The goals of this paper are to localize the diffusion X , when $0 < \kappa < 1$, in some valleys of the potential W_κ , to understand the difference with Brox's result given by (1.1), and to prove an Aging phenomenon, as was done in [16] for transient zero-speed RWRE. We moreover obtain a central limit theorem for the number of valleys visited up to time t . We also prove some intermediate results, which we think will be useful for obtaining new results about the maximum local time of X .

Let $t \mapsto \phi(t)$ a positive increasing function of t , such that $\phi(t) = o(\log t)$ and $\log \log t = o(\phi(t))$, where $f(t) = o(g(t))$ means $\lim_{t \rightarrow +\infty} f(t)/g(t) = 0$. We prove the following aging phenomenon:

Proposition 1.2. *Assume $0 < \kappa < 1$. For all $\alpha > 1$, we have*

$$\lim_{t \rightarrow +\infty} \mathbb{P} (|X(\alpha t) - X(t)| \leq \phi(t)) = \frac{\sin(\kappa\pi)}{\pi} \int_0^{1/\alpha} u^{\kappa-1} (1-u)^{-\kappa} du.$$

This is actually a consequence of Theorem 1.3. Before stating it, we first introduce the notion of h -extrema, which were first introduced by Neveu et al. [26], and studied in the case of drifted Brownian motion by Faggionato [19]. For $h > 0$, we say that $x \in \mathbb{R}$ is an h -minimum for a given process V if there exist $u < x < v$ such that $V(y) \geq V(x)$ for all $y \in [u, v]$, $V(u) \geq V(x) + h$ and $V(v) \geq V(x) + h$. Moreover, x is an h -maximum for V if x is an h -minimum for $-V$, and x is an h -extrema for V iff it is an h -maximum or an h -minimum for V .

As we are studying the process X until time t , we are more especially interested in the h_t -extrema of W_κ where

$$h_t := \log t - \phi(t).$$

It is known (see [19]) that almost surely, the h_t -extrema of W_κ form a sequence indexed by \mathbb{Z} , unbounded from below and above, and that the h_t -minima and h_t -maxima alternate. We denote respectively by $(m_j, j \in \mathbb{Z})$ and $(M_j, j \in \mathbb{Z})$ the increasing sequences of h_t -minima and of h_t -maxima of W_κ , such that $m_0 \leq 0 < m_1$ and $m_j < M_j < m_{j+1}$ for every $j \in \mathbb{Z}$.

We also define

$$N_t := \max \left\{ k \in \mathbb{N}, \sup_{0 \leq s \leq t} X(s) \geq m_k \right\},$$

so that m_{N_t} is the largest h_t -minima visited by X until time t . The main result of this paper concerns the localization of the diffusion. It is stated as follows:

Theorem 1.3. *Assume $0 < \kappa < 1$. There exists a constant $C_1 > 0$, such that*

$$\lim_{t \rightarrow +\infty} \mathbb{P}(|X(t) - m_{N_t}| \leq C_1 \phi(t)) = 1.$$

We first recall that $X(t)$ is asymptotically of order t^κ (see e.g. [21]). So, the size $\phi(t)$ of the intervals in which X is localized, which can nearly be of the order of $\log \log t$, is small and is related to the minimum height h_t of our valleys. We could not say however if it the best that can be obtained. The main difference with the result of Brox (1.1) is the appearance of the (random) integer N_t , which is the number of typical valleys of height h_t visited before time t . In the recurrent case of Brox, the diffusion is, with a large probability, localized near the bottom of a unique valley of the potential, whereas in our transient case, the diffusion is localized near the bottom of one among several valleys of the potential.

We also prove a renewal theorem for the time to reach the last valley visited by X before t :

Proposition 1.4. *We have the following convergence in law under the annealed probability \mathbb{P} , as $t \rightarrow +\infty$,*

$$(H(m_{N_t})/t, H(m_{N_t+1})/t) \xrightarrow{\mathcal{L}} [\kappa \sin(\pi\kappa)/\pi] (y-x)^{-\kappa-1} x^{\kappa-1} \mathbf{1}_{[0,1]}(x) \mathbf{1}_{[1,\infty)}(y) dx dy.$$

As a consequence, we get the following results, which are useful for the proofs of Proposition 1.2 and Theorem 1.3:

Corollary 1.5. *Assume $0 < \kappa < 1$ and let $0 \leq r < s \leq 1$. Then,*

$$\lim_{t \rightarrow +\infty} \mathbb{P} \left(1-s \leq \frac{H(m_{N_t})}{t} \leq 1-r \right) = \frac{\sin(\pi\kappa)}{\pi} \int_{1-s}^{1-r} x^{\kappa-1} (1-x)^{-\kappa} dx, \quad (1.2)$$

$$\lim_{t \rightarrow +\infty} \mathbb{P}(H(m_{N_t+1})/t \geq 1+s) = \frac{\sin(\pi\kappa)}{\pi} \int_s^{+\infty} (1+x)^{-1} x^{-\kappa} dx. \quad (1.3)$$

Moreover, the total time spent in the last valley of height at least h_t visited before time t renormalized by t , that is $[H(m_{N_t+1}) - H(m_{N_t})]/t$ converges in law under \mathbb{P} to a r.v. with density $\sin(\pi\kappa)\pi^{-1}x^{-\kappa-1}[(1 - (1-x)^\kappa)\mathbf{1}_{[0,1]}(x) + \mathbf{1}_{(1,+\infty)}(x)]$.

We will see in Section 4 and 5, that this is due to the fact that for any integer $k \leq n_t$ with

$$n_t := \lfloor e^{\kappa\phi(t)(1+\delta)} \rfloor, \quad \delta > 0,$$

$H(m_k)$ can be approximated by a sum of i.i.d. random variables each of these random variables having the law of \mathbf{U} . We show that \mathbf{U} is the product of a random variable depending only on the environment and t , and an independent variable with exponential law of parameter 2. This first random variable can itself be approximated by a product of sums of functionals of 3-dimensional $(-\kappa/2)$ drifted Bessel processes and of W_κ (see Proposition 4.3).

These results are in accordance with those obtained by Enriquez et al. ([14], [15] and [16]) for transient RWRE. The work we present here is self contained, in particular we present in this same

paper the technical study of the Laplace transform of the first exit time \mathbf{U} . The study of the environment only requires continuous arguments of stochastic calculus, starting by a Williams decomposition of the trajectory of W_κ which mainly comes from the work of A. Faggionato [19].

The number N_t of valleys visited goes to $+\infty$ as $t \rightarrow +\infty$. More precisely, we prove the following central limit theorem for N_t , with renormalization $e^{\kappa\phi(t)}$:

Proposition 1.6. *Assume $0 < \kappa < 1$. Then $N_t e^{-\kappa\phi(t)} \xrightarrow{t \rightarrow +\infty} \mathcal{N}$ in law under the annealed law \mathbb{P} , where \mathcal{N} is a r.v. determined by its Laplace transform:*

$$\forall u > 0, \quad \mathbb{E}(e^{-u\mathcal{N}}) = \sum_{j=0}^{+\infty} \frac{1}{\Gamma(\kappa j + 1)} \left(\frac{-u}{C_\kappa}\right)^j \quad (1.4)$$

where $C_\kappa > 0$ is explicitly known (see Proposition 4.7).

Moreover we expect that the results of this paper will be useful to study other properties for the diffusion. In particular, let $(\mathcal{L}_X(t, x), t \geq 0, x \in \mathbb{R})$ be a bicontinuous version of the local time of X . It is known that the maximum local time of X at time t , that is $\mathcal{L}_X^*(t) := \max_{x \in \mathbb{R}} \mathcal{L}_X(t, x)$, satisfies $\limsup_{t \rightarrow +\infty} \mathcal{L}_X^*(t)/t = +\infty$ a.s. in the cases $\kappa = 0$ (see [34] and [11]) and even in the transient case $0 < \kappa < 1$ (see [10]). Hence the maximum local times of X exhibits very interesting properties, very different from those of the maximum local time of RWRE at time t , which is naturally bounded by $t/2$. We expect that the better understanding of the localization of X and some intermediate results provided in this paper will be useful to prove new results about \mathcal{L}_X^* (work in progress).

The rest of the paper is organized as follows. First, we give in Section 2 the main properties of the environment that will be useful for our study; in particular we present a Williams' decomposition of the trajectory of W_κ , close to the one detailed in [19]. Then, in Section 3, we approximate the trajectory of X by a sum of *i.i.d.* random variables, and we study one of these random variable in Section 4. We study the asymptotic behavior of the Laplace transform of the first exit time \mathbf{U} in Section 4.3. Finally, we prove the renewal results stated in Proposition 1.4, Theorem 1.3 and Proposition 1.2 in Section 5. Moreover, Sections 2, 3 and 4 start with basic facts on diffusion in random media and/or estimates on the drifted brownian motion and Bessel processes.

2. WILLIAMS' DECOMPOSITION AND STANDARD VALLEYS

We use a Williams' like decomposition ([41]), based on the results of A. Faggionato [19].

2.1. Williams' decomposition in the neighborhood of h_t -minima, (m_i, i) . We now recall Williams' decomposition of the trajectory in a neighborhood of the local minima $m_i, i \in \mathbb{N}^*$.

Let $a > 0$. For any process $(U(t), t \in \mathbb{R}^+)$ we denote by

$$\tau^U(a) := \inf\{t > 0, U(t) = a\},$$

the first time this process hit a , with the convention $\inf \emptyset = +\infty$. We denote by \mathcal{L}_U a bicontinuous version of the local time of U when it exists. We also denote by U^a the process U starting from a , and by P^a the law of U^a ; with the notation $U = U^0$.

Definition 2.1. *We recall the definition of a $(-\zeta/2)$ -drifted Brownian motion W_ζ Doob-conditioned to stay positive (see [3], Chapter VII.3), where $\zeta \neq 0$. For $z > 0$, $P_z^{-\zeta/2, \uparrow}$ is defined by $P_z^{-\zeta/2, \uparrow}(\Lambda) := \frac{1}{1 - e^{-\zeta z}} E^z[(1 - \exp(-\zeta W_\zeta^z(t))), \Lambda, t < \tau^{W_\zeta^z}(0)]$ for any $\Lambda \in \sigma(W_\zeta(u), 0 \leq u \leq t) =: \mathcal{F}_t, t \geq 0$, where E^z and P^z are the expectancy and probability related to W_ζ^z . This induces*

a unique probability measure $P_z^{-\zeta/2, \uparrow}$ on $\sigma(W_\zeta(u), u \geq 0)$. Moreover, $P_z^{-\zeta/2, \uparrow}$ converges weakly as $z \rightarrow 0^+$, in the space of Skorokhod $D(\mathbb{R}_+, \mathbb{R}_+)$ (see [3] Prop. 14) and in $C(\mathbb{R}_+, \mathbb{R}_+)$ (see [19]) to a probability measure denoted by $P_0^{-\zeta/2, \uparrow}$. The canonical process, which we denote by R , is a Feller process for the family $(P_z^{-\zeta/2, \uparrow}, z \geq 0)$; it takes values in \mathbb{R}_+ , and its infinitesimal generator is given for every $x > 0$ by (see [19] Lemma 6)

$$\frac{1}{2} \frac{d^2}{dx^2} + \frac{\zeta}{2} \coth\left(\frac{\zeta}{2}x\right) \frac{d}{dx}. \quad (2.1)$$

In the following, we call R under $P_z^{-\zeta/2, \uparrow}$ for $z \geq 0$ a 3 dimensional $(-\zeta/2)$ -drifted Bessel process starting from z . Of course this is a misuse of language as this process is not drifted directly but is obtained from a drifted process. We notice in particular that, by (2.1), the law of R is the same if ζ is replaced by $-\zeta$, that is, 3 dimensional $(-\kappa/2)$ -drifted Bessel processes have the same law as 3 dimensional $(\kappa/2)$ -drifted Bessel processes. Finally, when $\zeta < 0$, we have $P_z^{-\zeta/2, \uparrow}(\Lambda) = P^z(\Lambda | \tau^{W_\zeta}(0) = \infty)$ for every $\Lambda \in \mathcal{F}_t, t \geq 0$.

Let

$$V^{(i)}(\cdot) := W_\kappa(\cdot) - W_\kappa(m_i), \quad i \in \mathbb{N}^*,$$

which is the potential re-centered at the local minima m_i . We also define for $h > 0$

$$\tau_i^-(h) := \sup\{s < m_i, V^{(i)}(x) = h\}, \quad \tau_i(h) := \inf\{s > m_i, V^{(i)}(x) = h\}.$$

We have,

Fact 2.2. (*William's decomposition*)

For any $i \in \mathbb{N}^*$, let $P_1^{(i)}$ be the truncated process $(V^{(i)}(m_i - s), 0 \leq s \leq m_i - \tau_i^-(h_t))$, $P_2^{(i)}$ the truncated process $(V^{(i)}(m_i + s), 0 \leq s \leq \tau_i(h_t) - m_i)$ and $P_3^{(i)}$ the truncated process $(V^{(i)}(s + \tau_i(h_t)), 0 \leq s \leq \inf\{s > \tau_i(h_t), V^{(i)}(s) = a\} - \tau_i(h_t))$, for $a < h_t$. Then the truncated processes $((P_1^{(i)}, P_2^{(i)}), i \geq 1)$ are independent, and for a given i , $P_1^{(i)}, P_2^{(i)}$ and $P_3^{(i)}$ are independent as well.

Let us denote (R, h) the killed process $(R(s), 0 \leq s \leq \tau^R(h))$. Let $(R^{(1)}, h_t)$ and $(R^{(2)}, h_t)$ two independent copies of $(R, \tau(h_t))$ and $(\tilde{W}_\kappa^b, a) := (\tilde{W}_\kappa^b(s), 0 \leq s \leq \tau^{\tilde{W}_\kappa}(a))$ a $(-\kappa/2)$ -drifted Brownian motion starting from b and killed when it first hits $a < b$, and independent of $R^{(1)}$ and $R^{(2)}$. Then for all $i \geq 2$, $P_1^{(i)}$ is equal in law to $(R^{(1)}, h_t)$; for all $i \geq 1$, $P_2^{(i)}$ is equal in law to $(R^{(2)}, h_t)$, and $P_3^{(i)}$ is equal in law to $(\tilde{W}_\kappa^{h_t}, a)$, for $a < h_t$.

The results for the sequence $((P_1^{(i)}, P_2^{(i)}), i \geq 1)$, comes from Theorems 1 and 2 in [19]. The result for $P_3^{(i)}$ comes from the fact that $\tau_i(h_t)$ is a stopping time. We treat the *central slope* that is to say $P_1^{(1)}$ in Remark 2.4.

2.2. Standard h_t -minima (\tilde{m}_i, i) . Among the h_t -minima (m_i, i) only some of them are interesting for the analysis of the process X : the (\tilde{m}_i, i) h_t -minima. Recall that $\delta > 0$ is a positive real that can be chosen as small as needed (see the definition of n_t just after Corollary 1.5). Let

$$h_t^+ := (1 + \kappa + 2\delta)h_t.$$

We define $\tilde{L}_0^+ := 0$, $\tilde{m}_0 := 0$, and recursively for $i \geq 1$ (see Figure 1),

$$\begin{aligned}\tilde{L}_i^\# &:= \inf\{x > \tilde{L}_{i-1}^+, W_\kappa(x) = W_\kappa(\tilde{L}_{i-1}^+) - h_t^+\}, \\ \tilde{\tau}_i(h_t) &:= \inf\{x \geq \tilde{L}_i^\#, W_\kappa(x) - \inf_{[\tilde{L}_i^\#, x]} W_\kappa = h_t\}, \\ \tilde{m}_i &:= \inf\{x \geq \tilde{L}_i^\#, W_\kappa(x) = \inf_{[\tilde{L}_i^\#, \tilde{\tau}_i(h_t)]} W_\kappa\}, \\ \tilde{L}_i^+ &:= \inf\{x > \tilde{\tau}_i(h_t), W_\kappa(x) = W_\kappa(\tilde{\tau}_i(h_t)) - h_t^+ - h_t\}, \\ \tilde{\tau}_i^-(h) &:= \sup\{s < \tilde{m}_i, W_\kappa(x) - W_\kappa(\tilde{m}_i) = h\}, \quad h > 0.\end{aligned}$$

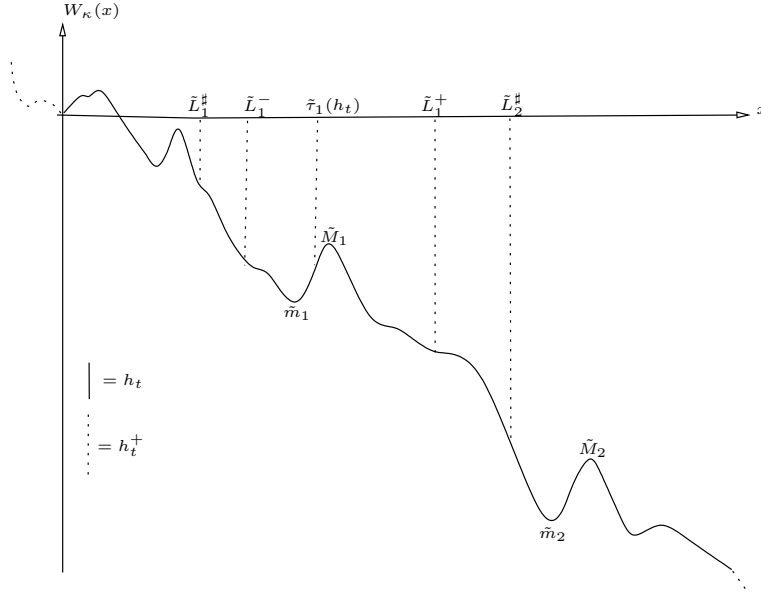


FIGURE 1. h_t standard valleys

We also introduce the equivalent of $V^{(i)}$, for the (\tilde{m}_i, i)

$$\tilde{V}^{(i)}(\cdot) := W_\kappa(\cdot) - W_\kappa(\tilde{m}_i).$$

We call i th valley, the re-centered truncated trajectory $(\tilde{V}^{(i)}(x), \tilde{L}_{i-1}^+ \leq x < \tilde{L}_i^+)$.

The next step is to show that with an overwhelming probability the first n_t positive h_t -minima $(m_i, 1 \leq i \leq n_t)$ coincide with $(\tilde{m}_i, 1 \leq i \leq n_t)$. We denote \mathcal{V}_t the event $\{(m_i, 1 \leq i \leq n_t) = (\tilde{m}_i, 1 \leq i \leq n_t)\}$.

In all the paper, C_+ and c_+ (resp. C_- and c_-) denote positive constants that may grow (resp. decrease) from line to line.

Lemma 2.3. *For any $0 < \delta < 1$, and any t large enough $P(\mathcal{V}_t) \geq 1 - C_1 w_t$, where $w_t := n_t e^{-\kappa h_t/2}$ and C_1 is a positive constant. Moreover, the sequence $((\tilde{V}^{(i)}(x + \tilde{L}_{i-1}^+), 0 \leq x < \tilde{L}_i^+ - \tilde{L}_{i-1}^+), i \geq 1)$ is i.i.d.*

Proof: We first notice that \tilde{m}_i is a h_t -minimum for W_κ for every $i \geq 1$, so $\{\tilde{m}_i, i \in \mathbb{N}^*\} \subset \{m_i, i \in \mathbb{N}^*\}$. Hence on the complementary of \mathcal{V}_t , there would exist $1 \leq i \leq n_t$ and $1 \leq j \leq n_t$ such that $\tilde{m}_{i-1} < m_j < \tilde{m}_i$. If for such i and j , $\tilde{L}_i^\# < m_j < \tilde{m}_i$, there would be a $v > m_j$ such that $W_\kappa(m_j) = \inf_{[m_j, v]} W_\kappa$ and $W_\kappa(v) \geq W_\kappa(m_j) + h_t$, so $m_j < \tilde{m}_i \leq \tilde{\tau}_i(h_t) \leq v$, then

$W_\kappa(m_j) \leq W_\kappa(\tilde{m}_i)$, which contradicts the definition of \tilde{m}_i . Hence, $\tilde{m}_{i-1} < m_j \leq \tilde{L}_i^\#$ and then $W_\kappa(m_j) \geq W_\kappa(\tilde{m}_{i-1}) - 2h_t^+$.

Consequently, $\bar{\mathcal{V}}_t \subset (\mathcal{E}_1^{2.3} \cap \mathcal{E}_2^{2.3}) \cup \overline{\mathcal{E}_1^{2.3}}$ where $\mathcal{E}_1^{2.3} := \{W_\kappa(m_1) < 0\} \cap \cap_{j=2}^{n_t} \{W_\kappa(m_j) < W_\kappa(m_{j-1})\}$ and $\mathcal{E}_2^{2.3} := \{W_\kappa(m_1) \geq -2h_t^+\} \cup \cup_{j=2}^{n_t} \{W_\kappa(m_j) \geq W_\kappa(m_{j-1}) - 2h_t^+\}$. For $j \geq 2$, $W_\kappa(m_j) - W_\kappa(m_{j-1})$ is by ([19] Prop. 1, Thm 1 and the remark before (2.26)) equal in law to $\zeta_+ - \zeta_-$, where ζ_+ and ζ_- are independent exponential r.v, such that the mean of ζ_\pm is $2\kappa^{-1} \sinh(\kappa h_t/2) e^{\mp \kappa h_t/2}$. So for $j \geq 2$,

$$\mathbb{P}[W_\kappa(m_j) \geq W_\kappa(m_{j-1}) - 2h_t^+] \leq \mathbb{P}(\zeta_+ > e^{\kappa h_t/2} - 2h_t^+) + \mathbb{P}(\zeta_- < e^{\kappa h_t/2}) \leq C_+ e^{-\kappa h_t/2}.$$

For $j = 1$, we notice that either there is an h_t -maximum between 0 and m_1 , with probability $\leq h_t e^{-\kappa h_t}$ by ([19], Thm. 1 and (2.25)), either $m_1 \leq \tau_1^*(h_t)$ with

$$\tau_1^*(h_t) := \inf\{u \geq 0, W_\kappa(u) - \inf_{[0,u]} W_\kappa \geq h_t\}. \quad (2.2)$$

In this case, $W_\kappa(m_1) = \inf_{[0, \tau_1^*(h_t)]} W_\kappa =: \beta^*$, where $-\beta^*$ is by ([19] Lemma 1) exponentially distributed with mean $2\kappa^{-1} \sinh(\kappa h_t/2) e^{\kappa h_t/2}$. Hence $\mathbb{P}[W_\kappa(m_1) \geq -2h_t^+] \leq h_t e^{-\kappa h_t} + \mathbb{P}[-\beta^* \leq 2h_t^+] \leq C_+ e^{-\kappa h_t/2}$.

Hence, $\mathbb{P}(\overline{\mathcal{E}_1^{2.3}}) \leq \mathbb{P}(\mathcal{E}_2^{2.3}) \leq C_+ n_t e^{-\kappa h_t/2}$, and then $P(\mathcal{V}_t) \geq 1 - C_1 n_t e^{-\kappa h_t/2}$.

Finally, the fact that the sequence $((\tilde{V}^{(i)}(x + \tilde{L}_{i-1}^+), 0 \leq x < \tilde{L}_i^+ - \tilde{L}_{i-1}^+), i \geq 1)$ is i.i.d. follows directly from the strong Markov property applied at times \tilde{L}_{i-1}^+ , which are stopping times. \square

The following Remark will be useful in the sequel

Remark 2.4. Lemma 2.3 implies that for all $1 \leq i \leq n_t$, $\tau_i^-(h_t^+) \geq 0$ with a probability larger than $1 - C_1 w_t$. Also by Fact 2.2, $(V^{(i)}(m_i - s), 0 \leq s \leq m_i - \tau_i^-(h_t))$ is equal in law to $(R^{(1)}, h_t)$, and by Corollary 1 in [30] this result can be extended until $\tau_i^-(h_t^+)$, as long as $\tau_i^-(h_t^+) \geq 0$ that is to say $(V^{(i)}(m_i - s), 0 \leq s \leq m_i - \tau_i^-(h_t^+))$ on $\tau_i^-(h_t^+) \geq 0$ is equal in law to $(R^{(1)}, h_t^+)$. So for every event \mathcal{A} which belongs to the σ -algebra generated by the truncated trajectory $(\tilde{V}^{(i)}(s), \tilde{\tau}_i^-(h_t^+) \leq s \leq \tilde{m}_i)$, $\mathcal{A} \equiv \mathcal{A}((\tilde{V}^{(i)}(s), \tilde{\tau}_i^-(h_t^+) \leq s \leq \tilde{m}_i))$ we can write $P(\mathcal{A}(R^{(1)}, h_t^+)) - C_1 w_t \leq P(\mathcal{A}) \leq P(\mathcal{A}(R^{(1)}, h_t^+)) + C_1 w_t$.

We also need the following intermediate random variables

$$\begin{aligned} \tilde{M}_i &:= \inf\{s > \tilde{m}_i, W_\kappa(s) = \max_{\tilde{m}_i \leq u \leq \tilde{L}_i^+} W_\kappa(u)\}, \\ \tilde{\tau}_i^+(h) &:= \inf\{s > \tilde{m}_i, W_\kappa(x) - W_\kappa(\tilde{m}_i) = h\}, \quad h > 0, \\ \tilde{L}_i^- &:= \tilde{\tau}_i^-(h_t^+). \end{aligned}$$

We have the following result for the distance between the points of a given valley:

Lemma 2.5. For all $0 \leq i \leq n_t$,

$$P(\tilde{m}_{i+1} - \tilde{M}_i \leq e^{\kappa h_t(1-\delta)}, \mathcal{V}_t) \leq C_+ e^{-\kappa \delta h_t}, \quad (2.3)$$

$$P(\tilde{\tau}_i^+(h) - \tilde{m}_i > 8h/\kappa, \mathcal{V}_t) \leq C_+ e^{-\kappa h/2\sqrt{2}}, \quad 0 \leq h \leq h_t, \quad (2.4)$$

$$P(\tilde{m}_{i+1} - \tilde{L}_{i+1}^- > 8h_t^+/\kappa, \mathcal{V}_t) \leq C_+ e^{-\kappa h_t^+/2\sqrt{2}}, \quad (2.5)$$

$$P(\tilde{L}_{i+1}^+ - \tilde{L}_{i+1}^- > 28h_t/\kappa, \mathcal{V}_t) \leq C_+ e^{-\kappa h_t/24}. \quad (2.6)$$

Equation (2.4) is also valid for $0 \leq h \leq h_t^+$ if $\tilde{\tau}_i^+(h) - \tilde{m}_i$ is replaced by $\tilde{m}_{i+1} - \tilde{\tau}_{i+1}^-(h)$.

Before giving the proof, we detail a basic result and its short proof:

Lemma 2.6. *Let $0 < \alpha < \omega$. For all h large enough, we have*

$$P(\tau^{W_\kappa}(-\alpha h) \geq 2\omega h/\kappa) \leq e^{-\kappa(\omega-\alpha)^2 h/4\omega}. \quad (2.7)$$

Proof: $P(\tau^{W_\kappa}(-\alpha h) \geq 2\omega h/\kappa) \leq P(W(2\omega h/\kappa) \geq (\omega - \alpha)h) = P(W(1) > \sqrt{\kappa h}(\omega - \alpha)/\sqrt{2\omega}) \leq e^{-\kappa(\omega-\alpha)^2 h/4\omega}$, where we have used $\mathbb{P}(W(1) \geq x) \leq e^{-x^2/2}$. \square

Proof of Lemma 2.5: Working on the event \mathcal{V}_t allows us to write that $\{\tilde{m}_{i+1} - \tilde{M}_i \leq e^{\kappa h_t(1-\delta)}, \mathcal{V}_t\} \subset \{m_{i+1} - M_i \leq e^{\kappa h_t(1-\delta)}\}$ and then use Faggionato's results. This idea is used several time in this proof and all along the paper. For $i \geq 1$, thanks to ([19] Thm 1), the law of $(W_\kappa(M_i + t) - W_\kappa(M_i), 0 \leq t \leq m_{i+1} - M_i)$ is $P_-^{\kappa/2}$, which is defined in [19] p. 1769.

Applying ([19] Proposition 1 p. 1769 and especially Formula 2.14), $m_{i+1} - M_i$ has the same law as a r.v. called ℓ_- , which Laplace transform is given by $E(e^{-\alpha \ell_-}) = \bar{\alpha} e^{-\kappa h_t/2} / [\bar{\alpha} \cosh(\bar{\alpha} h_t) - (\kappa/2) \sinh(\bar{\alpha} h_t)]$ for $\alpha > 0$, with $\bar{\alpha} := \sqrt{2\alpha + \kappa^2/4}$. In particular, with a Markov inequality with $\alpha = e^{-\kappa h_t(1-\delta)}$, we get $P(m_{i+1} - M_i \leq e^{\kappa h_t(1-\delta)}) \leq \exp(\alpha e^{\kappa h_t(1-\delta)}) E(e^{-\alpha \ell_-}) \leq C_+ e^{-\kappa \delta h_t}$.

Applying ([19] Thm 1 and formula 2.3) and taking $\alpha = -\kappa^2/16$, we get $E(e^{\kappa^2(\tau_i(h) - m_i)/16}) \sim_{h \rightarrow +\infty} e^{\kappa h(1-1/\sqrt{2})/2}/\sqrt{2}$. Then, (2.4) follows from Markov inequality. We get (2.5) by the same way.

To get the last inequality, we just need an upper bound for $\tilde{L}_i^+ - \tilde{\tau}_i(h_t)$. Since \tilde{L}_i^+ is a stopping time for W_κ (in particular we do not need to work on \mathcal{V}_t for this part), we have by using (2.7)

$$P(\tilde{L}_i^+ - \tilde{\tau}_i(h_t) \geq 12h_t/\kappa) = P(\tau^{W_\kappa}(-h_t^+ - h_t) \geq 12h_t/\kappa) \leq e^{-\kappa h_t/24}.$$

And we used the fact that $0 < \delta < 1$ hence $h_t^+ + h_t \leq 4h_t$. Combining these inequalities yields (2.6). \square

3. QUASI-INDEPENDENCE IN THE TRAJECTORIES OF X

In this section we show that the times to escape from the different valleys are asymptotically in t independent under the annealed measure. Then we prove that the time spent by X between the valleys is negligible.

We start with some basic facts about hitting times by X , R and W_κ .

3.1. About hitting times. We first introduce some notation. Let

$$A(r) := \int_0^r e^{W_\kappa(x)} dx, \quad r \in \mathbb{R},$$

and $A_\infty := \lim_{r \rightarrow +\infty} A(r) < \infty$ a.s. As in Brox [5], there exists a Brownian motion B such that $X(t) = A^{-1}[B(T^{-1}(t))]$, where

$$T(r) := \int_0^r \exp\{-2W_\kappa[A^{-1}(B(s))]\} ds, \quad 0 \leq r \leq \tau^B(A_\infty). \quad (3.1)$$

With these notations, we recall the following expression of $H(r)$, for all $r \geq 0$,

$$H(r) = T[\tau^B(A(r))] = \int_{-\infty}^r e^{-W_\kappa(u)} \mathcal{L}_B[\tau^B(A(r)), A(u)] du.$$

We also need some estimates on hitting times by W_κ and a $-(\kappa/2)$ -drifted Bessel process R :

Lemma 3.1. *Let $0 < \gamma < \alpha < \omega$. For all h large enough, we have*

$$P^{\alpha h}(\tau^R(\gamma h) < \tau^R(\omega h)) \leq 2e^{-\kappa(\alpha-\gamma)h}, \quad (3.2)$$

$$P(\tau^R(\omega h) - \tau^R(\alpha h) \leq 1) \leq C_+ e^{-c_+[(\omega-\alpha)h]^2}, \quad (3.3)$$

$$P\left(\tau^{W_\kappa^{(h)}}(\gamma h) \geq 1\right) \geq 1 - 2e^{-[(1-\gamma)h-\kappa/2]^2/2}, \quad \gamma \neq 1, \quad (3.4)$$

$$P\left(\int_0^{\tau^R(h)} e^{R(u)} du \geq e^{(1-\alpha)h}\right) \geq 1 - 3e^{-\kappa\alpha h/2}, \quad 0 < \alpha < 1 \quad (3.5)$$

$$P(\tau^R(h) > 8h/\kappa) \leq C_+ e^{-\kappa h/2\sqrt{2}}. \quad (3.6)$$

Proof: We recall that R has the same law as the $(\kappa/2)$ -drifted Brownian motion $W_{-\kappa}^0 = W_{-\kappa}$ Doob conditioned to stay positive, and more precisely that $P(R \in \Lambda) = \lim_{x \downarrow 0} P^x(\Lambda | \tau^{W_\kappa}(0) = \infty)$ for all $\Lambda \in \mathcal{G}_t$, where $(\mathcal{G}_t, t \geq 0)$ is the natural filtration (see [19] Lem 6. and the discussion before), and then for every $\Lambda \in \mathcal{G}_\tau$, where τ is an a.s. finite stopping time. Moreover, we know that a scale function of $W_{-\kappa}$ is given by $s_\kappa(u) := 2(1 - e^{-\kappa u})/\kappa = 4e^{-\kappa u/2} \sinh(\kappa u/2)/\kappa$, $u \in \mathbb{R}$ (see [19], (5.1)), that is, $P(\tau^{W_{-\kappa}}(y) < \tau^{W_{-\kappa}}(-x)) = s_\kappa(x)/s_\kappa(x+y)$ for $x > 0$ and $y > 0$. Hence, since $\tau_{\omega h}^{W_{-\kappa}} < \infty$ P^x -a.s.,

$$\begin{aligned} & \text{LHS of (3.2)} \\ &= P^{\alpha h}[\tau^{W_{-\kappa}}(\gamma h) < \tau^{W_{-\kappa}}(\omega h) \mid \tau^{W_{-\kappa}}(\infty) < \tau^{W_{-\kappa}}(0)] \\ &= P^{\alpha h}[\tau^{W_{-\kappa}}(\gamma h) < \tau^{W_{-\kappa}}(\omega h)] P^{\gamma h}[\tau^{W_{-\kappa}}(\infty) < \tau^{W_{-\kappa}}(0)] / P^{\alpha h}[\tau^{W_{-\kappa}}(\infty) < \tau^{W_{-\kappa}}(0)] \\ &= \left(1 - \frac{s_\kappa((\alpha - \gamma)h)}{s_\kappa((\omega - \gamma)h)}\right) \frac{s_\kappa(\gamma h)}{s_\kappa(\alpha h)} \\ &= \frac{\sinh(\kappa(\omega - \alpha)h/2) \sinh(\kappa\gamma h/2)}{\sinh(\kappa(\omega - \gamma)h/2) \sinh(\kappa\alpha h/2)}, \end{aligned} \quad (3.7)$$

where LHS means left hand side. This gives (3.2) for large h .

We now turn to the proof of (3.4). We have, if $0 < \gamma < 1$,

$$P\left(\tau^{W_\kappa^{(h)}}(\gamma h) \leq 1\right) = P\left(\inf_{[0,1]} (W(x) + h - \frac{\kappa}{2}x) \leq \gamma h\right) \leq P\left(\inf_{[0,1]} W \leq (\gamma - 1)h + \frac{\kappa}{2}\right).$$

This yields to (3.4), since $\inf_{[0,1]} W \stackrel{\mathcal{L}}{=} |W(1)|$, where $\stackrel{\mathcal{L}}{=}$ denotes equality in law, and $\mathbb{P}(|W(1)| \geq x) \leq 2e^{-x^2/2}$. The case $\gamma > 1$ is treated similarly.

Now, we notice that the left hand side of (3.3) is less than

$$\limsup_{x \downarrow 0} P^x[\tau^{W_{-\kappa}}(\alpha h) < \tau^{W_{-\kappa}}(0)] \times P^{\alpha h}[\tau^{W_{-\kappa}}(\omega h) - \tau^{W_{-\kappa}}(\alpha h) \leq 1] / P^x[\tau^{W_{-\kappa}}(0) = \infty].$$

Moreover

$$P^x[\tau^{W_{-\kappa}}(\omega h) - \tau^{W_{-\kappa}}(\alpha h) \leq 1] = P^{\alpha h}[\tau^{W_{-\kappa}}(\omega h) \leq 1] \leq 2 \exp(-[(\omega - \alpha)h - \kappa/2]^2/2),$$

where the inequality is proved in the same way as (3.4). Since $\lim_{x \rightarrow 0} P^x[\tau^{W_{-\kappa}}(\alpha h) < \tau^{W_{-\kappa}}(0)] / P^x[\tau^{W_{-\kappa}}(0) = \infty] = (1 - e^{-\kappa\alpha h})^{-1} \leq 2$ for large h , we get (3.3).

To prove (3.5), let $0 < \alpha < 1$. Notice that the probability of $\left\{\inf_{\tau^R[(1-\alpha/2)h] \leq u \leq \tau^R(h)} R(u) \geq (1-\alpha)h\right\} \cap \left\{\tau^R[(1-\alpha/2)h] - \tau^R(h) \geq 1\right\}$ is at least $1 - 3e^{-\kappa\alpha h/2}$ for large h by (3.2) and (3.3). Moreover, we have on this event,

$$\int_0^{\tau^R(h)} e^{R(u)} du \geq \int_{\tau^R[(1-\alpha/2)h]}^{\tau^R(h)} e^{R(u)} du. \geq [\tau^R[(1-\alpha/2)h] - \tau^R(h)] e^{(1-\alpha)h} \geq e^{(1-\alpha)h},$$

which proves (3.5). (3.6) is obtained similarly as (2.4). \square

3.2. Independence in a trajectory of X . We prove that the sequence $(U_i := H(\tilde{L}_i) - H(\tilde{m}_i), i \geq 1)$ is "nearly" i.i.d under \mathbb{P} for large t in the following sense: First we need some more points which belongs to the standard valleys, for all $i \geq 1$

$$\tilde{L}_i := \inf\{x > \tilde{\tau}_i(h_t), W_\kappa(x) - W_\kappa(\tilde{m}_i) = h_t/2\}.$$

Proposition 3.2. *There exists a constant $C_2 > 0$ such that for large t ,*

$$\forall \lambda > 0, 1 \leq n \leq n_t, \quad \left| \mathbb{E} \left(e^{-\lambda \sum_{i=1}^n U_i} \right) - \left[\mathbb{E} \left(e^{-\lambda \mathbf{U}} \right) \right]^n \right| \leq C_2 u(t, n), \quad (3.8)$$

where $u(t, n) := ne^{-\delta\kappa h_t}$ if δ is chosen small enough,

$$\mathbf{U} := \int_{\tilde{L}_2^-}^{\tilde{L}_2} e^{-\tilde{V}^{(2)}(u)} \mathcal{L}_B[\tau^B(\tilde{A}(\tilde{L}_2)), \tilde{A}(u)] du,$$

with $\tilde{A}(z) := \int_{\tilde{m}_2}^z e^{\tilde{V}^{(2)}(x)} dx$ for all $\tilde{L}_2^- \leq z \leq \tilde{L}_2$. Moreover for all $n \in \mathbb{N}^*$, $[a, b] \subset [0, 1]$ and $\alpha > 0$,

$$\left| \mathbb{P} \left(\sum_{i=1}^{n-1} \frac{U_i}{t} \in [a, b], \sum_{i=1}^n \frac{U_i}{t} \geq \alpha \right) - \int_a^b \mathbb{P} \left(\sum_{i=1}^{n-1} \frac{U_i}{t} \in dx \right) \mathbb{P}(\mathbf{U}/t \geq \alpha - x) \right| \leq C_+ e^{-\delta\kappa h_t}. \quad (3.9)$$

Proof: Let $\mathcal{E}_i := \{U_i < H'_i(\tilde{L}_i^-) - H(\tilde{m}_i)\}$, where $H'_i(\tilde{L}_i^-) := \inf\{s > H(\tilde{m}_i), X(s) = \tilde{L}_i^-\}$. We also introduce $\mathcal{F}_t := \sigma(X(u), 0 \leq u \leq t, W_\kappa(x), x \in \mathbb{R})$ for $t \geq 0$. We get

$$\begin{aligned} \mathbb{E} \left(e^{-\lambda \sum_{i=1}^n U_i} \prod_{i=1}^n \mathbf{1}_{\mathcal{E}_i} \right) &= E \left[\mathbb{E}^{W_\kappa} \left(\mathbb{E}^{W_\kappa} \left(e^{-\lambda U_n} \mathbf{1}_{\mathcal{E}_n} \mid \mathcal{F}_{H(\tilde{m}_n)} \right) e^{-\lambda \sum_{i=1}^{n-1} U_i} \prod_{i=1}^{n-1} \mathbf{1}_{\mathcal{E}_i} \right) \right] \\ &= E \left[\mathbb{E}^{W_\kappa} \left(\mathbb{E}_{\tilde{m}_n}^{W_\kappa} \left(e^{-\lambda H(\tilde{L}_n)} \mathbf{1}_{H(\tilde{L}_n) < H(\tilde{L}_n^-)} \right) e^{-\lambda \sum_{i=1}^{n-1} U_i} \prod_{i=1}^{n-1} \mathbf{1}_{\mathcal{E}_i} \right) \right], \end{aligned}$$

by the strong Markov property and since $\tilde{m}_i < \tilde{L}_i < \tilde{m}_n$ for $1 \leq i < n$. Hence we obtain by induction

$$\begin{aligned} \mathbb{E} \left(e^{-\lambda \sum_{i=1}^n U_i} \prod_{i=1}^n \mathbf{1}_{\mathcal{E}_i} \right) &= E \left(\prod_{i=1}^n \mathbb{E}_{\tilde{m}_i}^{W_\kappa} \left(e^{-\lambda H(\tilde{L}_i)} \mathbf{1}_{H(\tilde{L}_i) < H(\tilde{L}_i^-)} \right) \right) \\ &= E \left(\prod_{i=1}^n \left[L_\lambda(H(\tilde{L}_i)) \mathbb{P}^{W_\kappa}(\mathcal{E}_i) \right] \right), \end{aligned} \quad (3.10)$$

as $\mathbb{P}^{W_\kappa}(\mathcal{E}_i) = \mathbb{P}_{\tilde{m}_i}^{W_\kappa} \left(H(\tilde{L}_i) < H(\tilde{L}_i^-) \right)$ and with $L_\lambda[H(\tilde{L}_i)] := \mathbb{E}_{\tilde{m}_i}^{W_\kappa} \left(e^{-\lambda H(\tilde{L}_i)} \mid H(\tilde{L}_i) < H(\tilde{L}_i^-) \right)$.

We now need to prove that $\mathbb{P}^{W_\kappa}(\mathcal{E}_i)$ is closed to one with a large probability, so the next step is to get a lower bound for this probability. We now work under \mathcal{V}_t which allows us to use $V^{(i)}$ and its William's decomposition instead of $\tilde{V}^{(i)}$, we have

$$E \left(\prod_{i=1}^n \left[L_\lambda(H(\tilde{L}_i)) \mathbb{P}^{W_\kappa}(\mathcal{E}_i) \right] \right) \geq E \left(\prod_{i=1}^n \left[L_\lambda(H(\tilde{L}_i)) \mathbb{P}^{W_\kappa}(\mathcal{E}_i) \mathbf{1}_{\tilde{m}_i = m_i} \right] \right).$$

Let us give an upper bound for $\mathbb{P}^{W_\kappa}(\mathcal{E}_i)$ on $\{\tilde{m}_i = m_i\}$. Recall that A is a scale function of X under \mathbb{P}^{W_κ} (see e.g. [35] formula (2.2)), that is

$$\mathbb{P}^{W_\kappa}(\bar{\mathcal{E}}_i) = \mathbb{P}_{\tilde{m}_i}^{W_\kappa} \left(H(\tilde{L}_i) > H(\tilde{L}_i^-) \right) = \left(\int_{\tilde{m}_i}^{\tilde{L}_i} e^{\tilde{V}^{(i)}(x)} dx \right) \left(\int_{\tilde{L}_i^-}^{\tilde{L}_i} e^{\tilde{V}^{(i)}(u)} du \right)^{-1}, \quad (3.11)$$

so on $\{\tilde{m}_i = m_i\}$,

$$(3.11) \leq \left((\tilde{L}_i - \tilde{m}_i) e^{V^{(i)}(M_i)} \right) / \left(\int_{\tau_i^-(h_t^+)}^{\tau_i^-[h_t^+(1-\kappa/4)]} e^{V^{(i)}(x)} dx \right) =: Q_i/D_i,$$

recalling that $\tau_i^-(z) = \sup\{s < m_i, V^{(i)}(s) \geq z\}$ for $z > 0$.

We start with the denominator D_i . First on $\{\tilde{m}_i = m_i\}$, $\tau_i^-(h_t^+) \geq 0$ so by Remark 2.4 ($V^{(i)}(m_i - x)$, $m_i - \tau_i^-[h_t^+] \leq x \leq m_i - \tau_i^-[h_t^+(1-\kappa/4)]$) is equal in law to $(R^{(2)}(x), \tau^{R^{(2)}}[h_t^+(1-\kappa/4)] \leq x \leq \tau^{R^{(2)}}(h_t^+))$. Then, formula (3.2) applied with $h = h_t^+$, $\gamma = 1 - \kappa/2$, $\alpha = 1 - \kappa/4$ and $\omega = 1$ gives for t large enough, with a probability larger than $1 - C_+ e^{-\kappa^2 h_t/4}$,

$$D_i \mathbf{1}_{\tilde{m}_i = m_i} \geq e^{h_t^+(1-\kappa/2)} [\tau_i^-(h_t^+) - \tau_i^-(h_t^+(1-\kappa/4))] \mathbf{1}_{\tilde{m}_i = m_i}.$$

Moreover by (3.3), we have

$$P[\tau_1^{R^{(2)}}(h_t^+) - \tau_1^{R^{(2)}}(h_t^+(1-\kappa/4)) \geq 1] \geq 1 - C_+ e^{-c-h_t^2}.$$

So finally, we get $D_i \mathbf{1}_{\tilde{m}_i = m_i} \geq e^{h_t^+(1-\kappa/2)} \mathbf{1}_{\tilde{m}_i = m_i}$ with a probability larger than $1 - C_+ e^{-\kappa^2 h_t}$.

For the numerator Q_i , first by (2.6) and Lemma 2.3 $P(\tilde{L}_i - \tilde{L}_i^- > 28h_t/\kappa) \leq 2e^{-\kappa h_t/24}$. Also thanks to Williams' decomposition (Fact 2.2),

$$P[V^{(i)}(M_i) > h_t(1+\delta)] \leq P\left(\sup_{s \geq 0} W_\kappa(s) > \delta h_t\right) \leq e^{-\delta \kappa h_t},$$

where the last inequality comes from formula 1.1.4 (1) page 251 of [4]. Finally for δ small enough and t large enough, with probability greater than $1 - 2e^{-\delta \kappa h_t}$, $Q_i \mathbf{1}_{\tilde{m}_i = m_i} \leq 28h_t \kappa^{-1} e^{h_t(1+\delta)} \mathbf{1}_{\tilde{m}_i = m_i}$ and $D_i \mathbf{1}_{\tilde{m}_i = m_i} \geq e^{h_t^+(1-\kappa/2)} \mathbf{1}_{\tilde{m}_i = m_i}$, so

$$\mathbb{P}^{W_\kappa}(\mathcal{E}_i) \mathbf{1}_{\tilde{m}_i = m_i} \geq (1 - C_+ h_t e^{-(\kappa+2(1-\kappa)\delta)h_t/2}) \mathbf{1}_{\tilde{m}_i = m_i}. \quad (3.12)$$

Collecting what we did above, we get

$$\begin{aligned} & E \left[\prod_{i=1}^n \left(L_\lambda(H(\tilde{L}_i)) \mathbb{P}^{W_\kappa}(\mathcal{E}_i) \mathbf{1}_{\tilde{m}_i = m_i} \right) \right] \\ & \geq (1 - C_+ e^{-\delta \kappa h_t})^n E \left(\mathbf{1}_{\mathcal{V}_t} \prod_{i=1}^n \mathbb{E}_{\tilde{m}_i}^{W_\kappa} \left(e^{-\lambda H(\tilde{L}_i)} \mid H(\tilde{L}_i) < H(\tilde{L}_i^-) \right) \right). \end{aligned}$$

Using Lemma 2.3 and considering (3.10),

$$\mathbb{E} \left(e^{-\lambda \sum_{i=1}^n U_i} \prod_{i=1}^n \mathbf{1}_{\mathcal{E}_i} \right) - E \left(\prod_{i=1}^n \mathbb{E}_{\tilde{m}_i}^{W_\kappa} \left(e^{-\lambda H(\tilde{L}_i)} \mid H(\tilde{L}_i) < H(\tilde{L}_i^-) \right) \right) \geq -C_+ n e^{-\delta \kappa h_t}.$$

With similar ideas for the upper bound, we finally get

$$\left| \mathbb{E} \left(e^{-\lambda \sum_{i=1}^n U_i} \right) - E \left(\prod_{i=1}^n \mathbb{E}_{\tilde{m}_i}^{W_\kappa} \left(e^{-\lambda H(\tilde{L}_i)} \mid H(\tilde{L}_i) < H(\tilde{L}_i^-) \right) \right) \right| \leq C_+ n e^{-\delta \kappa h_t}.$$

For every fixed W_κ , we have under $\mathbb{P}_{\tilde{m}_i}^{W_\kappa}$ on $\{H(\tilde{L}_i) < H(\tilde{L}_i^-)\}$,

$$H(\tilde{L}_i) = \int_{\tilde{L}_i^-}^{\tilde{L}_i} e^{-\tilde{V}^{(i)}(u)} \mathcal{L}_{\tilde{B}}[\tau^{\tilde{B}}(\tilde{A}_i(\tilde{L}_i)), \tilde{A}_i(u)] du, \quad (3.13)$$

where for all $\tilde{L}_i^- \leq z \leq \tilde{L}_j$, $\tilde{A}_i(z) := \int_{\tilde{m}_j}^z e^{\tilde{V}^{(i)}(x)} dx$ and \tilde{B} is a standard Brownian motion. This and (3.11) show that the left hand side of (3.11) and $\mathbb{E}_{\tilde{m}_i}^{W_\kappa}(e^{-\lambda H(\tilde{L}_i)} | H(\tilde{L}_i) < H(\tilde{L}_i^-))$ are measurable with respect to the σ -algebra generated by $(\tilde{V}^{(i)}(x), \tilde{L}_i^- \leq x \leq \tilde{L}_i)$.

Using (3.13), and the fact that $\{(\tilde{V}^{(i)}(x), \tilde{L}_i^- \leq x \leq \tilde{L}_i), i \leq n\}$ is i.i.d. by Lemma 2.3 give the upper bound of (3.8). Notice that we choose the second valley in the definition of \mathbf{U} in order to avoid the central slope (see Fact 2.2) when working under \mathcal{V}_t .

For (3.9), we obtain

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{n-1} \frac{U_i}{t} \in [a, b], \sum_{i=1}^n \frac{U_i}{t} \geq \alpha, \mathcal{E}_n\right) &= \mathbb{E}\left[\mathbb{1}_{\sum_{i=1}^{n-1} \frac{U_i}{t} \in [a, b]} \mathbb{E}^{W_\kappa}\left(\mathbb{1}_{U_n \geq \alpha - \sum_{i=1}^{n-1} \frac{U_i}{t}} \mathbb{1}_{\mathcal{E}_n} | \mathcal{F}_{H(\tilde{m}_n)}\right)\right] \\ &= \int_a^b \mathbb{P}\left(\sum_{i=1}^{n-1} \frac{U_i}{t} \in dx\right) \mathbb{P}(U_n/t \geq \alpha - x, \mathcal{E}_n), \end{aligned}$$

since U_i is for $i \leq n-1$ measurable with respect to $\sigma(W_\kappa(x), x \leq \tilde{L}_{n-1}, X(u), u \leq H(\tilde{L}_{n-1}))$, whereas $\mathbb{E}^{W_\kappa}\left(\mathbb{1}_{U_n \geq \alpha - x} \mathbb{1}_{\mathcal{E}_n} | \mathcal{F}_{H(\tilde{m}_n)}\right)$ is for every $x \in \mathbb{R}$ measurable with respect to $\sigma(W_\kappa(x + \tilde{m}_n) - W_\kappa(\tilde{m}_n), x \geq \tilde{L}_n^- - \tilde{m}_n, X(u + H(\tilde{m}_n)) - \tilde{m}_n, 0 \leq u \leq H(\tilde{L}_n) - H(\tilde{m}_n))$. Since U_n is equal in law to \mathbf{U} and $\mathbb{P}(\mathcal{E}_n) \leq C_+ e^{-\delta h t}$, we get (3.9). \square

3.3. Negligible parts in the trajectory of X . We now prove that the total time spent between the first n_t large valleys is negligible compared to t .

We first need to give estimates concerning the hitting times of \tilde{m}_1 and $\tau_1(h_t)$. To this aim, notice that $H(r) = H_-(r) + H_+(r)$, where $H_-(r) := \int_{-\infty}^0 e^{-W_\kappa(u)} \mathcal{L}_B[\tau^B(A(r)), A(u)] du$ and $H_+(r) := \int_0^r e^{-W_\kappa(u)} \mathcal{L}_B[\tau^B(A(r)), A(u)] du$. Actually, $H_-(r)$ (resp. $H_+(r)$) is the time spent by X in \mathbb{R}_- (resp. in \mathbb{R}_+) before it hits r for the first time. We start with the following lemma about H_- ; it comes from [10] and the proof is given for the sake of completeness:

Lemma 3.3. *Let $\kappa > 0$ and $H_-(+\infty) := \lim_{r \rightarrow +\infty} H_-(r)$. For z large enough,*

$$\mathbb{P}(H_-(+\infty) > z) \leq C_+ [(\log z)/z]^{\kappa/(\kappa+2)}. \quad (3.14)$$

Proof: For $a > 0$, $\alpha > 0$ and $b > 0$, let

$$\mathcal{E}_1^{3.3} := \left\{ \sup_{x < 0} e^{-W_\kappa(x)} \leq a \right\}, \quad \mathcal{E}_2^{3.3} := \{A_\infty \leq \alpha\}, \quad \mathcal{E}_3^{3.3} := \left\{ \sup_{y < 0} \mathcal{L}_B[\tau^B(\alpha), y] \leq b \right\}.$$

$$\mathcal{L}_X^{*-}(+\infty) := \sup_{r \geq 0} \sup_{x < 0} \mathcal{L}_X(H(r), x) = \sup_{t \geq 0} \sup_{x < 0} \mathcal{L}_X(t, x).$$

We first prove an inequality with regards to $\mathcal{L}_X^{*-}(+\infty)$. We notice that

$$\mathcal{L}_X^{*-}(+\infty) = \sup_{r \geq 0} \sup_{x < 0} \left\{ e^{-W_\kappa(x)} \mathcal{L}_B[\tau^B(A(r)), A(x)] \right\} \leq \left(\sup_{x < 0} e^{-W_\kappa(x)} \right) \sup_{y < 0} \mathcal{L}_B[\tau^B(A_\infty), y].$$

By the first Ray–Knight theorem, there exist two Bessel processes R_2 and R_0 , of dimensions 2 and 0 respectively, starting from 0 and $R_2(\alpha)$, such that $\mathcal{L}_B(\tau^B(\alpha), x)$ is equal to $R_2^2(\alpha - x)$ for $x \in [0, \alpha]$ and to $R_0^2(-x)$ for $x < 0$. Hence, for $\alpha \leq b$,

$$\mathbb{P}(\overline{\mathcal{E}_3^{3.3}}) = \mathbb{P}(R_2^2(\alpha) > b) + \int_0^b \mathbb{P}_x \left(\sup_{y > 0} R_0^2(y) > b \right) \mathbb{P}(R_2^2(\alpha) \in [x, x + dx]).$$

Consequently,

$$\mathbb{P}(\overline{\mathcal{E}_3^{3.3}}) = \exp\left(-\frac{b}{2\alpha}\right) + \mathbb{E}\left(\frac{R_2^2(\alpha)}{b} \mathbf{1}_{\{R_2^2(\alpha) \leq b\}}\right) \leq [2 + \mathbb{E}(R_2^2(1))] \frac{\alpha}{b} =: c_0 \frac{\alpha}{b},$$

with $c_0 > 0$. Now, let $a := z^{\frac{1}{\kappa+2}}$, $\alpha := z^{\frac{1}{\kappa+2}}$ and $b := z^{\frac{\kappa+1}{\kappa+2}}$. Notice that on $\mathcal{E}_1^{3.3} \cap \mathcal{E}_2^{3.3} \cap \mathcal{E}_3^{3.3}$, $\mathcal{L}_X^*(+\infty) \leq z$ and recall $A_\infty \stackrel{\mathcal{L}}{=} 2/\gamma_\kappa$ (see Dufresne, [13]), where γ_κ is a gamma variable of parameter $(\kappa, 1)$. We have for z large enough,

$$\mathbb{P}(\mathcal{L}_X^*(+\infty) > z) \leq \mathbb{P}(\overline{\mathcal{E}_1^{3.3}}) + \mathbb{P}(\overline{\mathcal{E}_2^{3.3}}) + \mathbb{P}(\overline{\mathcal{E}_3^{3.3}}) \leq a^{-\kappa} + [2/\alpha]^\kappa / (\kappa\Gamma(\kappa)) + c_0\alpha/b \leq c_1 z^{-\frac{\kappa}{\kappa+2}}, \quad (3.15)$$

with $c_1 > 0$. We now turn back to $H_-(+\infty)$. Define for $c > 0$,

$$\mathcal{E}_4^{3.3} := \left\{ \min_{0 \leq s \leq \tau^{B(\alpha)}} B(s) > -\alpha z^{\frac{\kappa+1}{\kappa+2}} \right\}, \quad \mathcal{E}_5^{3.3} := \{|A^{-1}(-z)| \leq c \log z\}.$$

On $\mathcal{E}_1^{3.3} \cap \mathcal{E}_2^{3.3} \cap \mathcal{E}_3^{3.3} \cap \mathcal{E}_4^{3.3} \cap \mathcal{E}_5^{3.3}$, we have for $r \geq 0$,

$$\begin{aligned} H_-(+\infty) &\leq \lim_{r \rightarrow +\infty} \int_{A^{-1}(\min_{0 \leq s \leq \tau^{B(A(r))}} B(s))}^0 \mathcal{L}_X^*(+\infty) dx \\ &\leq \left| A^{-1}(\min_{0 \leq s \leq \tau^{B(A_\infty)}} B(s)) \right| \mathcal{L}_X^*(+\infty) \\ &\leq |A^{-1}(-z)| \mathcal{L}_X^*(+\infty) \leq cz \log z. \end{aligned} \quad (3.16)$$

Moreover, for $c > 2/\kappa$, and $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(\overline{\mathcal{E}_5^{3.3}}) &= \mathbb{P}\left(z > \int_0^{c \log z} e^{W(u) + \kappa u/2} du\right) \leq \mathbb{P}\left[z > \exp\left(\inf_{0 \leq u \leq c \log z} W(u)\right) \frac{2}{\kappa} (z^{\kappa c/2} - 1)\right] \\ &\leq 2z^{-\frac{1}{2c}(\frac{\kappa c}{2} - 1 - \varepsilon)^2} \end{aligned} \quad (3.17)$$

for all large z . Moreover, $\mathbb{P}(\overline{\mathcal{E}_4^{3.3}}) = \alpha/[\alpha + \alpha z^{\frac{\kappa+1}{\kappa+2}}] \leq z^{-\frac{\kappa+1}{\kappa+2}}$. Choosing c large enough, this, together with (3.15), (3.16) and (3.17) gives (3.14). \square

Lemma 3.4. *There exists a constant $C_5 > 0$ such that for every $h > 0$,*

$$\mathbb{E}[H_+(\tau_1^*(h))] \leq C_5 e^h \quad (3.18)$$

where $\tau_1^*(h)$ is define in (2.2). Moreover,

$$\mathbb{P}[H_-(\tilde{m}_1) \geq t/\log h_t] \leq C_+ [(\log t)^2/t]^{\kappa/(\kappa+2)}. \quad (3.19)$$

Proof: We first remind that $\mathbb{E}[\mathcal{L}_B(\tau^B(1), y)] = 2(1-y)$ for $0 \leq y \leq 1$; which is a consequence of the first Ray–Knight theorem (see e.g. [32]). We notice that by the scale property of B , recalling that $A(u) \geq 0$ for all $u \geq 0$ and A is independent of B , we have for every $r \geq 0$, which can depend on the environment W_κ ,

$$\begin{aligned} \mathbb{E}^{W_\kappa}(H_+(r)) &= \mathbb{E}^{W_\kappa}\left(\int_0^r e^{-W_\kappa(u)} A(r) \mathcal{L}_B(\tau^B(1), A(u)/A(r)) du\right) \\ &= \int_0^r e^{-W_\kappa(u)} 2(A(r) - A(u)) du \\ &= 2 \int_0^r \int_u^r e^{W_\kappa(v) - W_\kappa(u)} dv du, \end{aligned} \quad (3.20)$$

by Fubini. Hence, applying this to $r = \tau_1^*(h)$, we get

$$\begin{aligned} E[H_+(\tau_1^*(h))] &= 2E\left(\int_0^{\tau_1^*(h)} \int_u^{\tau_1^*(h)} e^{W_\kappa(v)-W_\kappa(u)} dv du\right) \\ &\leq 2E\left(\int_0^\infty \mathbb{1}_{u \leq \tau_1^*(h)} \int_u^{\tau_1^*(u,h)} e^{W_\kappa(v)-W_\kappa(u)} dv du\right) \end{aligned}$$

where $\tau_1^*(u, h) := \inf\{t > u, W_\kappa(t) - \inf_{[u,t]} W_\kappa \geq h\}$. Applying Fubini followed by the Markov property at time u , we get

$$\begin{aligned} \mathbb{E}[H_+(\tau_1^*(h))] &\leq 2 \int_0^\infty E\left(\mathbb{1}_{u \leq \tau_1^*(h)} \int_0^{\tau_1^*(u,h)-u} e^{W_\kappa(\alpha+u)-W_\kappa(u)} d\alpha\right) du \\ &= 2 \int_0^\infty E(\mathbb{1}_{u \leq \tau_1^*(h)}) E\left(\int_0^{\tau_1^*(h)} e^{W_\kappa(\alpha)} d\alpha\right) du \\ &= 2\beta_1(h)\beta_2(h), \end{aligned} \tag{3.21}$$

where, similarly as in Enriquez et al. ([15], Lem. 4.9),

$$\beta_1(h) := E(\tau_1^*(h)), \quad \beta_2(h) := E\left(\int_0^{\tau_1^*(h)} e^{W_\kappa(u)} du\right).$$

Using [19] (formula (2.3) and (2.7)) we have $\beta_1(h) \leq c_1 e^{\kappa h}$, with $c_1 > 0$.

We now cut the integral which appears in the definition of $\beta_2(h)$ into several parts. to show that $\beta_2(h) \leq C_+ e^{(1-\kappa)h}$ for h large enough. To this aim, we introduce $e_0 := 0$ and

$$\begin{aligned} e_j &:= \inf\{s > e_{j-1}, W_\kappa(s) = -j\}, \quad j \geq 1, \\ H_j &:= \sup_{e_j \leq u \leq v \leq e_{j+1}} (W_\kappa(v) - W_\kappa(u)), \quad j \in \mathbb{N}, \\ \Xi_k &:= \{e_k \leq \tau_1^*(h) \leq e_{k+1}\}, \quad k \in \mathbb{N}. \end{aligned}$$

We have,

$$\begin{aligned} \beta_2(h) &= E\left(\sum_{k=0}^\infty \mathbb{1}_{\Xi_k} \int_0^{\tau_1^*(h)} e^{W_\kappa(u)} du\right) \\ &= \sum_{k=0}^\infty E\left[\mathbb{1}_{\{H_0 < h\}} \times \cdots \times \mathbb{1}_{\{H_{k-1} < h\}} \mathbb{1}_{\{H_k \geq h\}} \left(\sum_{i=0}^{k-1} e^{W_\kappa(e_i)} J_i + e^{W_\kappa(e_k)} \bar{J}_k\right)\right] \end{aligned}$$

where $J_i := \int_{e_i}^{e_{i+1}} e^{W_\kappa(u)-W_\kappa(e_i)} du$ and $\bar{J}_k := \int_{e_k}^{\tau_1^*(h)} e^{W_\kappa(u)-W_\kappa(e_k)} du$. Hence, applying Markov at times e_i and since $W_\kappa(e_i) = -i$, we get

$$\begin{aligned} \beta_2(h) &= \sum_{k=0}^\infty \left[\sum_{i=0}^{k-1} e^{-i} (1-q)^k q E(J_0 | H_0 < h) + (1-q)^k q e^{-k} E(\bar{J}_0 | H_0 \geq h) \right] \\ &\leq \sum_{k=0}^\infty \left[(1-e^{-1})^{-1} (1-q)^k q E(J_0 | H_0 < h) \right] + \frac{q}{1-(1-q)e^{-1}} E(\bar{J}_0 | H_0 \geq h) \end{aligned}$$

with $q := P(H_0 \geq h)$. Hence $\beta_2(h) \leq C[E(J_0 | H_0 < h) + E(\bar{J}_0 \mathbb{1}_{\{H_0 \geq h\}})]$. To finish first by 3.10.7 (a) page 317 in [4], taking $x = 0$, $a = -1$, $\alpha = 0$ and $b = h$, we easily get $E(J_0 \mathbb{1}_{\{H_0 < h\}}) \leq E(J_0 \mathbb{1}_{\{\tau^{W_\kappa(-1)} < \tau^{W_\kappa(h)}\}}) \leq C e^{(1-\kappa)h}$. In the same way using formula 3.10.7 (b) of the same reference with the same parameters except $b = h - 1$, we also get $E(\bar{J}_0 \mathbb{1}_{\{H_0 \geq h\}}) \leq$

$E(\overline{J_0} \mathbf{1}_{\{\tau^{W_\kappa}(-1) > \tau^{W_\kappa}(h-1)\}}) \leq Ce^{h(1-\kappa)}$. This, combined with $q \sim_{h \rightarrow +\infty} Ce^{-\kappa h}$ gives $\beta_2(h) \leq Ce^{(1-\kappa)h}$ for large h , which together with (3.21) and $\beta_1(h) \leq c_1 e^{\kappa h}$ gives (3.18).

The second inequality, that is (3.19), comes directly from (3.14). \square

We now have all the tools needed to bound the time spent between the deep valleys:

Lemma 3.5. *For any δ small enough ($\delta < 2^{-3/2}$ and $\kappa(1 + 2\delta) < 1$) and t large enough*

$$\mathbb{P}(H(\tilde{m}_1) < \tilde{v}_t) \geq \mathbb{P}\left(\bigcap_{k=1}^{n_t} \left\{0 \leq H(\tilde{m}_k) - \sum_{i=1}^{k-1} U_i < \tilde{v}_t\right\}\right) \geq 1 - C_3 v_t,$$

where $\tilde{v}_t := 2t/\log h_t$, $v_t := n_t(\log h_t)e^{-\phi(t)} = o(1)$, $C_3 > 0$ and $\sum_{i=1}^0 \dots = 0$ by convention.

Proof : We have, for every $1 \leq k \leq n_t$,

$$H(\tilde{m}_k) = H(\tilde{m}_1) + \sum_{i=1}^{k-1} U_i + \sum_{i=1}^{k-1} (H(\tilde{m}_{i+1}) - H(\tilde{L}_i)), \quad (3.22)$$

so we just have to prove that $H(\tilde{m}_1) + \sum_{i=1}^{n_t-1} (H(\tilde{m}_{i+1}) - H(\tilde{L}_i))$ is less than or equal to \tilde{v}_t with large probability. We consider

$$\tilde{L}_i^* := \inf\{x \geq \tilde{\tau}_i(h_t), W_\kappa(x) - W_\kappa(\tilde{m}_i) = 3h_t/4\}.$$

We define $X_i(t) := X(t + H(\tilde{L}_i))$ (resp. $X_i^*(t) := X(t + H(\tilde{L}_i^*))$), $t \geq 0$, which is a diffusion in the environment W_κ , starting from \tilde{L}_i (resp. \tilde{L}_i^*). We also denote by $H_{X_i}(r)$ the hitting time of r by X_i , for $r \geq \tilde{L}_i$ and $A_\infty^x := \int_x^\infty e^{W_\kappa(u) - W_\kappa(x)} du$. We introduce the following events:

$$\begin{aligned} \mathcal{E}_1^{3.5} &:= \bigcap_{i=1}^{n_t-1} \{H_{X_i}(\tilde{m}_{i+1}) < H_{X_i}(\tilde{L}_i^*)\}, \\ \mathcal{E}_2^{3.5} &:= \bigcap_{i=1}^{n_t} \{A_\infty^{\tilde{L}_i} \leq e^{h_t/16}, A_\infty^{\tilde{\tau}_{i+1}(h_t)} \leq e^{h_t/16}, A_\infty^{\tilde{L}_i^*} \geq e^{-h_t/16}\}. \end{aligned}$$

We recall that $A_\infty := A_\infty^0$ has the same law as $2/\gamma_\kappa$, where γ_κ is a gamma variable of parameter $(\kappa, 1)$, with density $e^{-x} x^{\kappa-1} \mathbf{1}_{\mathbb{R}_+}(x)/\Gamma(\kappa)$ (see [13], or [4] IV.48). Hence, $P(A_\infty \geq y) \leq Cy^{-\kappa}$ for $y > 0$ and $C > 0$, and $P(A_\infty \leq y) \leq e^{-1/y}$ for small $y > 0$. Moreover, since \tilde{L}_i , $\tilde{\tau}_{i+1}(h_t)$ and \tilde{L}_i^* are stopping times for the natural filtration of W_κ , $A_\infty^{\tilde{L}_i}$, $A_\infty^{\tilde{\tau}_{i+1}(h_t)}$ and $A_\infty^{\tilde{L}_i^*}$ have the same law as A_∞ under P by the strong Markov property. Consequently,

$$P(\overline{\mathcal{E}_2^{3.5}}) \leq n_t(Ce^{-\kappa h_t/16} + e^{-e^{h_t/16}}). \quad (3.23)$$

Moreover, we have for $1 \leq i \leq n_t - 1$ on $\mathcal{E}_2^{3.5}$, $\mathbb{P}^{W_\kappa}[H_{X_i}(\tilde{m}_{i+1}) > H_{X_i}(\tilde{L}_i^*)] = Q_i^*/D_i^*$, where

$$\begin{aligned} Q_i^* &:= \int_{\tilde{L}_i}^{\tilde{m}_{i+1}} e^{W_\kappa(x)} dx \leq e^{W_\kappa(\tilde{L}_i)} A_\infty^{\tilde{L}_i} \leq \exp(W_\kappa(\tilde{m}_i) + h_t/2 + h_t/16), \\ D_i^* &:= \int_{\tilde{L}_i^*}^{\tilde{m}_{i+1}} e^{W_\kappa(x)} dx = e^{W_\kappa(\tilde{L}_i^*)} A_\infty^{\tilde{L}_i^*} - \int_{\tilde{m}_{i+1}}^\infty e^{W_\kappa(x)} dx. \end{aligned} \quad (3.24)$$

Moreover, we have on $\mathcal{E}_2^{3.5} \cap \mathcal{E}_3^{3.5}$, where $\mathcal{E}_3^{3.5} := \bigcap_{j=1}^{n_t-1} \{\tilde{\tau}_{j+1}(h_t) - \tilde{m}_{j+1} \leq 8h_t/\kappa\}$,

$$\int_{\tilde{m}_{i+1}}^\infty e^{W_\kappa(x)} dx \leq [\tilde{\tau}_{i+1}(h_t) - \tilde{m}_{i+1} + A_\infty^{\tilde{\tau}_{i+1}(h_t)}] e^{W_\kappa(\tilde{m}_{i+1}) + h_t} \leq (8h_t/\kappa + e^{h_t/16}) e^{W_\kappa(\tilde{m}_i)}$$

for large t since $W_\kappa(\tilde{m}_{i+1}) \leq W_\kappa(\tilde{L}_i^+) \leq W_\kappa(\tilde{m}_i) - h_t$. This yields to $D_i^* \geq e^{W_\kappa(\tilde{m}_i) + 11h_t/16}/2$ for large t . Hence,

$$\mathbb{P}(\overline{\mathcal{E}_1^{3.5}}) \leq \mathbb{P}(\overline{\mathcal{E}_1^{3.5}} \cap \mathcal{E}_2^{3.5} \cap \mathcal{E}_3^{3.5}) + P(\overline{\mathcal{E}_2^{3.5}}) + P(\overline{\mathcal{E}_3^{3.5}}) \leq 2n_t e^{-h_t/8} + P(\overline{\mathcal{E}_2^{3.5}}) + P(\overline{\mathcal{E}_3^{3.5}}), \quad (3.25)$$

where $P(\overline{\mathcal{E}_3^{3.5}}) \leq C + n_t e^{-\kappa h_t/2^{3/2}}$ by (2.4) and Lemma 2.3.

Let $\tilde{\tau}_{i+1}^*(h_t) := \inf\{u \geq \tilde{L}_i^*, W_\kappa(u) - \inf_{[\tilde{L}_i^*, u]} W_\kappa \geq h_t\}$ for $i \geq 1$, and $\mathcal{E}_4^{3.5} := \cap_{i=1}^{n_t-1} \{\tilde{\tau}_{i+1}^*(h_t) = \tilde{\tau}_{i+1}(h_t)\}$. By definition of $\tilde{\tau}_i(h_t)$, $\{\tilde{\tau}_{i+1}^*(h_t) \neq \tilde{\tau}_{i+1}(h_t)\} = \{\tilde{\tau}_{i+1}^*(h_t) \leq \tilde{L}_{i+1}^\# \} = \{\inf_{[\tilde{L}_i^*, \tilde{\tau}_{i+1}^*(h_t)]} W_\kappa - W_\kappa(\tilde{L}_i^*) \geq -2h_t^+ - 3h_t/4\}$. Since \tilde{L}_i^* is a stopping time, using the strong Markov property, we obtain $P(\tilde{\tau}_{i+1}^*(h_t) = \tilde{\tau}_{i+1}(h_t)) \leq \mathbb{P}(\inf_{[0, \tau_1^*(h_t)]} W_\kappa \geq -2h_t^+ - 3h_t/4) \leq C_+ e^{-\kappa h_t/2}$ as in the proof of Lemma 2.3. Then $P(\overline{\mathcal{E}_4^{3.5}}) \leq C_+ n_t e^{-\kappa h_t/2}$.

On $\mathcal{E}_4^{3.5}$, $H(\tilde{m}_{i+1}) - H(\tilde{L}_i) = H_{X_i}(\tilde{m}_{i+1}) \leq H_{X_i}(\tilde{\tau}_{i+1}(h_t)) = H_{X_i}(\tilde{\tau}_{i+1}^*(h_t))$, which is, on $\mathcal{E}_1^{3.5}$, the total time spent by X_i in $[\tilde{L}_i^*, +\infty)$ before hitting $\tilde{\tau}_{i+1}(h_t)$. This last quantity is less than or equal to the total time spent in $[\tilde{L}_i^*, +\infty)$ by X_i^* before hitting $\tilde{\tau}_{i+1}^*(h_t)$, which has the same law as $H_+(\tau_1^*(h_t))$ under the annealed probability \mathbb{P} , since \tilde{L}_i^* is a stopping time for W_κ and then $(W_\kappa(\tilde{L}_i^* + x) - W_\kappa(\tilde{L}_i^*), x \geq 0)$ is a standard Brownian motion. Consequently, $\mathbb{E}[(H(\tilde{m}_{i+1}) - H(\tilde{L}_i))1_{\mathcal{E}_4^{3.5}}] \leq \mathbb{E}[H_+(\tau_1^*(h_t))]$ for $1 \leq i \leq n_t - 1$. This, together with Lemma 3.4 and a Markov inequality lead to

$$\mathbb{P}\left(H_+(\tilde{m}_1) + \sum_{i=1}^{n_t-1} (H(\tilde{m}_{i+1}) - H(\tilde{L}_i)) \geq t/(\log h_t), \mathcal{E}_1^{3.5}, \mathcal{E}_4^{3.5}, \mathcal{E}_5^{3.5}, \mathcal{V}_t\right) \leq C_+ n_t e^{h_t} \frac{\log h_t}{t},$$

where $\mathcal{E}_5^{3.5} := \{m_1 \leq \tau_1^*(h_t)\}$. Recall that $\mathbb{P}(\overline{\mathcal{E}_5^{3.5}}) \leq \mathbb{P}(0 \leq M_0 < m_1) \leq h_t e^{-\kappa h_t}$ as seen in the proof of Lemma 2.3, case $j = 1$. This, combined with (3.19), (3.23), (3.25) and Lemma 2.3, concludes the proof. \square

4. TIME SPENT IN A STANDARD VALLEY

The aim of this section is to prove Proposition 4.6. First we need additional estimates given below.

4.1. Some technical estimates. Recall that $(R(s), s \geq 0)$ is a $(-\kappa/2)$ drifted 3-dimensional Bessel process, and let $(\tilde{W}_\kappa^b(s), 0 \leq s \leq \tau^{\tilde{W}_\kappa^b}(a))$ a $(-\kappa/2)$ -drifted Brownian motion starting from b and killed when it first hits $a < b$. We now introduce

$$F^\pm(a) := \int_0^{\tau^R(a)} e^{\pm R(s)} ds, \quad G^\pm(a, b) := \int_0^{\tau^{\tilde{W}_\kappa^b}(a)} e^{\pm \tilde{W}_\kappa^b(s)}(s) ds,$$

Lemma 4.1. *There exists $C_4 > 0$, $M > 0$ and $\eta_1 \in (0, 1)$ such that $\forall y > M, \forall \gamma \in (0, \eta_1]$,*

$$\left| E\left(e^{-\gamma F^-(y)}\right) - [1 + 2\gamma/(\kappa + 1)]^{-1} \right| \leq C_4 \max(e^{-\kappa y}, \gamma^{3/2}), \quad (4.1)$$

$$\left| E\left(e^{-\gamma F^+(y)/e^y}\right) - [1 - 2\gamma/(\kappa + 1)] \right| \leq C_4 \max(e^{-\kappa y}, \gamma^{3/2}), \quad (4.2)$$

$$\left| E\left(e^{-\gamma G^+(y/2, y)/e^y}\right) - [1 - \Gamma(1 - \kappa)(2\gamma)^\kappa / \Gamma(1 + \kappa)] \right| \leq C_4 \max(\gamma^\kappa e^{-\kappa y/2}, \gamma). \quad (4.3)$$

Moreover, there exists $c_1 > 0$, such that for all $y > 0$, $E(F^+(y)/e^y) \leq c_1$.

Proof: We denote by I_κ and K_κ the modified Bessel functions, respectively of the first and second kind (see e.g. [4] p. 638). First, ([4], 2.10.3 page 302) with $\alpha = 0$, $x = y$, $z = \omega y < y$, $\beta = 1/2$, and $\mu = -\kappa/2$ gives

$$E\left(e^{-\gamma G^+(\omega y, y)}\right) = \frac{e^{\kappa(1-\omega)y/2} K_\kappa(2\sqrt{2\gamma}e^{y/2})}{K_\kappa(2\sqrt{2\gamma}e^{\omega y/2})}, \quad \gamma > 0.$$

Note that this expression can be deduced from the fact that $G^+(\omega y, y) \stackrel{\mathcal{L}}{=} \int_0^{\tau^{\tilde{W}_\kappa(\omega y)}} e^{\tilde{W}_\kappa(x)} dx$, $A_\infty \stackrel{\mathcal{L}}{=} \int_0^{\tau^{\tilde{W}_\kappa(\omega y)}} e^{\tilde{W}_\kappa(x)} dx + e^{\omega y} A'_\infty$, with $A'_\infty \stackrel{\mathcal{L}}{=} A_\infty$ independent of $\int_0^{\tau^{\tilde{W}_\kappa(\omega y)}} e^{\tilde{W}_\kappa(x)} dx$ and [13].

Recall that $\mathbb{P}(R \in \Lambda) = \lim_{x \downarrow 0} P^x(\Lambda | \tau^{W_\kappa}(0) = \infty)$ for every $\Lambda \in \mathcal{G}_{\tau^{W_\kappa}(a)}$. Hence, we have for $\gamma > 0$, by the strong Markov property,

$$\begin{aligned} E\left(e^{-\gamma F^\pm(y)}\right) &= \lim_{x \downarrow 0} E^x \left[\exp\left(-\gamma \int_0^{\tau^{W_\kappa}(y)} e^{\pm W_\kappa(s)} ds\right) \mathbf{1}_{\{T_y < T_0\}} \right] \frac{P^y(T_0 = \infty)}{P^x(T_0 = \infty)} \\ &= \frac{\pm \kappa^{-1} \sinh(\kappa y/2)}{I_\kappa(2\sqrt{2\gamma}e^{\pm y/2})K_\kappa(2\sqrt{2\gamma}) - K_\kappa(2\sqrt{2\gamma}e^{\pm y/2})I_\kappa(2\sqrt{2\gamma})}, \end{aligned} \quad (4.4)$$

where the last equality comes from ([4], 3.10.7(b) page 317) with $\alpha = 0$, $a = 0$, $b = a$, $\beta = \pm 1/2$, and $x > 0$, since $P^x(T_0 = \infty) = 1 - e^{-\kappa x}$ and $I'_\kappa(u)K_\kappa(u) - I_\kappa(u)K'_\kappa(u) = 1/u$ for $u > 0$. Now, notice that when $\max(v^{2\kappa}, u^3) \rightarrow 0$,

$$f(u, v) := \frac{-\kappa^{-1}(v^{-\kappa} - v^\kappa)}{I_\kappa(uv)K_\kappa(u) - K_\kappa(uv)I_\kappa(u)} = [1 + u^2/(4(\kappa + 1))]^{-1} + O(\max(v^{2\kappa}, u^3)). \quad (4.5)$$

Hence, there exist $C'_4 > 0$ and $\eta'_1 > 0$ such that

$$\max(v^{2\kappa}, u^3) < \eta'_1 \implies |f(u, v) - [1 + u^2/(4(\kappa + 1))]^{-1}| \leq C'_4 \max(v^{2\kappa}, u^3).$$

Consequently, taking $u = 2\sqrt{2\gamma}$ and $v = e^{-y/2}$, there exist $M > 0$ and $\eta_1 > 0$ satisfying (4.1). Similarly, we get as $u \rightarrow 0$ and $y \rightarrow +\infty$,

$$\begin{aligned} E(e^{-uF^+(y)/e^y}) &= 1 - 2u/(\kappa + 1) + o(\max(u^{3/2}, e^{-\kappa y})), \\ E(e^{-uG^+(y/2, y)/e^y}) &= 1 - \Gamma(1 - \kappa)(2u)^\kappa/\Gamma(1 + \kappa) + o(\max(u, u^\kappa e^{-\kappa y/2})), \end{aligned}$$

which yield respectively to (4.2) and (4.3). Finally we deduce the result for the mean of $F^+(y)/e^y$ computing the first derivative of (4.4). \square

Before proving Proposition 4.6, we still need to introduce the following technical lemma:

Lemma 4.2. *B being a standard two-sided Brownian motion, there exists a constant c_3 such that for every $0 < \varepsilon < 1$, $0 < \delta < 1$ and $x > 0$,*

$$\mathbb{P}\left(\sup_{u \in [-\delta, \delta]} |\mathcal{L}_B(\tau^B(1), u) - \mathcal{L}_B(\tau^B(1), 0)| > \varepsilon \mathcal{L}_B(\tau^B(1), 0)\right) \leq c_3 \frac{\delta^{1/6}}{\varepsilon^{2/5}}, \quad (4.6)$$

$$\mathbb{P}\left(\sup_{u \in [0, 1]} \mathcal{L}_B(\tau^B(1), u) \geq x\right) \leq 4e^{-x/2}, \quad (4.7)$$

$$\mathbb{P}\left(\sup_{u \leq 0} \mathcal{L}_B(\tau^B(1), u) \geq x\right) \leq 1/x. \quad (4.8)$$

Proof: First, (4.7) and (4.8) follow respectively by [11] Lemma 2.3 and [39] Lemma 3.1 together with the first Ray-Knight theorem.

Let $0 < \varepsilon < 1$, $0 < \delta < 1$ and

$$\mathcal{E}^{4.2} := \left\{ \sup_{u \in [-\delta, \delta]} |\mathcal{L}_B(\tau^B(1), u) - \mathcal{L}_B(\tau^B(1), 0)| > \varepsilon \mathcal{L}_B(\tau^B(1), 0) \right\}.$$

We have, for $\alpha > 0$ and $\beta > 0$,

$$\begin{aligned} \mathbb{P}(\mathcal{E}^{4.2}) &= \mathbb{P}(\mathcal{E}^{4.2} \cap \{\mathcal{L}_B(\tau^B(1), 0) \geq \alpha\}) + \mathbb{P}(\mathcal{E}^{4.2} \cap \{\mathcal{L}_B(\tau^B(1), 0) < \alpha\}) \\ &\leq \mathbb{P}\left(\sup_{u \in [-\delta, \delta]} |\mathcal{L}_B(\tau^B(1), u) - \mathcal{L}_B(\tau^B(1), 0)| > \varepsilon \alpha\right) + \mathbb{P}[\mathcal{L}_B(\tau^B(1), 0) < \alpha] \\ &\leq \mathbb{P}[\tau^B(1) \geq \beta] + \mathbb{P}\left(\sup_{u \in [-\delta, \delta], 0 \leq s \leq \beta} |\mathcal{L}_B(s, u) - \mathcal{L}_B(s, 0)| > \varepsilon \alpha\right) + \frac{\alpha}{2} \end{aligned} \quad (4.9)$$

since $\mathcal{L}_B(\tau^B(1), 0)$ is an exponential variable with mean 2. Now notice that

$$\mathbb{P}[\tau^B(1) \geq \beta] = \mathbb{P}\left(\sup_{0 \leq u \leq \beta} B(u) < 1\right) = \mathbb{P}(\sqrt{\beta}|B(1)| < 1) \leq 2/\sqrt{2\pi\beta}.$$

Let $0 < \varepsilon_0 < 1/2$. The second term of (4.9) is less than or equal to

$$\begin{aligned} & \mathbb{P}\left(\sup_{u \in [-\delta, \delta] - \{0\}, 0 \leq s \leq \beta} \frac{|\mathcal{L}_B(s, u) - \mathcal{L}_B(s, 0)|}{|u|^{1/2-\varepsilon_0}} > \frac{\varepsilon\alpha}{\delta^{1/2-\varepsilon_0}}\right) \\ & \leq \frac{\delta^{1/2-\varepsilon_0}}{\varepsilon\alpha} \mathbb{E}\left(\sup_{a \neq b, 0 \leq s \leq \beta} \frac{|\mathcal{L}_B(s, b) - \mathcal{L}_B(s, a)|}{|a-b|^{1/2-\varepsilon_0}}\right), \end{aligned} \quad (4.10)$$

the last inequality being a consequence of Markov inequality. Now, applying Barlow and Yor ([2], (ii) p. 199 with $\gamma = 1$) to the continuous martingale $B(\cdot \wedge \beta)$, we can say that the expectancy in (4.10) is less than or equal to $C_{1,\varepsilon_0}(\sqrt{\beta})^{1/2+\varepsilon_0}$ where $C_{1,\varepsilon_0} > 0$. Consequently,

$$\mathbb{P}(\mathcal{E}^{4.2}) \leq c_3/\sqrt{\beta} + C_{1,\varepsilon_0}(\sqrt{\beta})^{1/2+\varepsilon_0}\delta^{1/2-\varepsilon_0}(\varepsilon\alpha)^{-1} + \alpha/2.$$

Now, we choose $\alpha = \varepsilon^{-2/5}\delta^{1/5}$, $\beta = \varepsilon^{4/5}\delta^{-2/5}$ and $\varepsilon_0 < 1/36$; we get $\mathbb{P}(\mathcal{E}^{4.2}) \leq c_3\delta^{1/6}\varepsilon^{-2/5}$, which concludes the proof. \square

4.2. Approximation of the time to escape from a typical valley.

We now prove that a standard exit time can be approximated by product and sums of independent well known random variables. We recall that \mathbf{U} is defined in Proposition 3.2.

Proposition 4.3. *Let $\varepsilon_t := 3e^{-(1-3\delta)h_t/6}$. There exist independent random variables $\mathcal{I}_1^+, \mathcal{I}_2^+, \mathcal{I}_1^-$ and \mathcal{I}_2^- , depending on t and independent of \mathbf{e}_1 a random variable with exponential law with mean 2, such that \mathcal{I}_1^+ is distributed as $F^+(h_t)$, \mathcal{I}_2^+ as $G^+(h_t/2, h_t)$, and \mathcal{I}_1^- and \mathcal{I}_2^- as $F^-(h_t/2)$, such that for t large enough, $\mathbb{P}(\mathcal{A}_t) \geq 1 - C_+e^{-\delta^2 c_- h_t}$, where*

$$\mathcal{A}_t := \left\{ |\mathbf{U} - (\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)\mathbf{e}_1| \leq (\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)\mathbf{e}_1\varepsilon_t \right\}.$$

The proof of this Proposition involves 3 Lemmata, the first two are straightforward consequence of what we have already discussed or proved the last one is more technical.

First applying Williams decomposition (see Fact 2.2), we have the following

Lemma 4.4. *We have*

$$\begin{aligned} A_1^+ &:= \int_{m_2}^{\tau_2(h_t)} e^{V^{(2)}(x)} dx \stackrel{\mathcal{L}}{=} F^+(h_t), & A_1^- &:= \int_{m_i}^{\tau_2(h_t/2)} e^{-V^{(2)}(x)} dx \stackrel{\mathcal{L}}{=} F^-(h_t/2) \\ A_2^\pm &:= \int_{\tau_2(h_t)}^{L_2} e^{\pm V^{(2)}(x)} dx \stackrel{\mathcal{L}}{=} G^\pm(h_t/2, h_t), & B_1^\pm &:= \int_{m_2}^{\tau_2^-(h_t/2)} e^{\pm V^{(2)}(x)} dx \stackrel{\mathcal{L}}{=} F^\pm(h_t/2), \end{aligned} \quad (4.11)$$

with $L_2 := \inf\{x > \tau_2(h_t), W_\kappa(x) - W_\kappa(m_2) = h_t/2\}$,

Recall that \tilde{A} is defined just before (3.9). We have,

Lemma 4.5. *For all $0 < \zeta \leq 1$, $0 < \varepsilon < 1/2$ and t large enough, we have*

$$P[e^{\zeta h_t(1-\varepsilon)} \leq \tilde{A}(\tilde{\tau}_1(\zeta h_t)) \leq e^{\zeta h_t(1+\varepsilon)}] \geq 1 - C_+e^{-\varepsilon\kappa\zeta h_t}. \quad (4.12)$$

Proof: Working on \mathcal{V}_t , and considering the first expression in (4.11) (replacing h_t by ζh_t), we know that the probability in 4.12 is larger than $P[e^{\zeta h_t(1-\varepsilon)} \leq F^+(\zeta h_t) \leq e^{\zeta h_t(1+\varepsilon)}] - P(\tilde{\mathcal{V}}_t)$. Then by Markov inequality and Lemma 4.1 $P[F^+(\zeta h_t) > e^{\zeta h_t(1+\varepsilon)}] \leq c_1e^{-\varepsilon\zeta h_t}$. For the lower

bound, taking $\gamma = \zeta(1 - \varepsilon/2)$, $\alpha = \zeta(1 - \varepsilon)$ and $\omega = \zeta$ in (3.2) we get with a probability larger than $1 - C_+e^{-\varepsilon\kappa h_t}$,

$$F^+(\zeta h_t) \geq e^{\zeta h_t(1-\varepsilon)}[\tau^R(\zeta h_t) - \tau^R(\zeta h_t(1 - \varepsilon/2))].$$

Finally using (3.3) we get the result. \square

We first prove the following intermediate result

Lemma 4.6. *For all $0 < \varepsilon < \inf(2/27, \kappa^2/2)$, and t large enough,*

$$\mathbb{P}\left(\left|\mathbf{U} - \tilde{A}(\tilde{L}_2)\mathcal{I}^- \mathbf{e}_1\right| \leq 3e^{-(1-3\varepsilon)h_t/6} \tilde{A}(\tilde{L}_2)\mathcal{I}^- \mathbf{e}_1\right) \geq 1 - C_+e^{-c-\varepsilon h_t},$$

where $\mathcal{I}^- := \int_{\tilde{\tau}_2^-(h_t/2)}^{\tilde{\tau}_2^+(h_t/2)} e^{-\tilde{V}^{(2)}(u)} du$. Also \mathbf{e}_1 is independent of W_κ .

Proof Let $\delta_t := e^{-h_t(1-3\varepsilon)/2}$. We introduce $x_0^+ := \tilde{A}^{-1}(\delta_t \tilde{A}(\tilde{L}_2))$ and $x_0^- := \tilde{A}^{-1}(-\delta_t \tilde{A}(\tilde{L}_2))$. We cut the integral defining \mathbf{U} into three parts:

$$\mathbf{U} = \int_{\tilde{L}_1^-}^{x_0^-} + \int_{x_0^-}^{x_0^+} + \int_{x_0^+}^{\tilde{L}_1} e^{-\tilde{V}^{(2)}(u)} \tilde{A}(\tilde{L}_2) \mathcal{L}_{B'}(\tau_{B'}(1), \tilde{A}(u)/\tilde{A}(\tilde{L}_2)) du =: \mathcal{J}_0 + \mathcal{J}_1 + \mathcal{J}_2, \quad (4.13)$$

where $B'(\cdot) = B([\tilde{A}(\tilde{L}_2)]^2)/\tilde{A}(\tilde{L}_2)$, and $(B'_u)_{u \geq 0}$ is a standard Brownian motion independent of W_κ , which we still denote by B in the rest of this proof. In what follows, we show that the main contribution comes from \mathcal{J}_1 .

For \mathcal{J}_2 , we prove that for t large enough

$$\mathbb{P}(\mathcal{J}_2 \geq \tilde{A}(\tilde{L}_2)c_+h_t^2e^{-(1-\varepsilon)h_t/2}) \leq C_+e^{-\varepsilon c_-h_t}. \quad (4.14)$$

To this aim, we first notice that $x_0^+ \in (\tilde{m}_2, \tilde{L}_2)$ since $0 \leq \delta_t \leq 1$. Moreover, $\tilde{A}(x) \leq \delta_t \tilde{A}(\tilde{L}_2)$ iff $x \leq x_0^+$. We introduce $\mathcal{E}_1^{4.6} := \{x_0^+ \geq \tilde{\tau}_2^+(h_t/2)\}$, and prove that $P(\overline{\mathcal{E}_1^{4.6}}) \leq C_+e^{-c-\varepsilon h_t}$. First by (4.12)

$$P(\tilde{A}(\tilde{\tau}_2^+(h_t/2))) \leq e^{h_t(1+\varepsilon)/2} \geq 1 - C_+e^{-\varepsilon h_t/2}. \quad (4.15)$$

Working on \mathcal{V}_t , $\tilde{A}(\tilde{L}_2) = A_1^+ + A_2^+$, so by Lemma 4.4 $P(\tilde{A}(\tilde{L}_2) \geq e^{h_t(1-\varepsilon)}) \geq P(F^+(h_t) \geq e^{h_t(1-\varepsilon)}) - P(\overline{\mathcal{V}_t})$. Hence, we get $P(\tilde{A}(\tilde{L}_2) \geq e^{h_t(1-\varepsilon)}) \geq 1 - 4e^{-\kappa\varepsilon h_t/2}$ by (3.5). This, together with (4.15) gives

$$P(\mathcal{E}_1^{4.6}) = P[\delta_t \geq \tilde{A}(\tilde{\tau}_2^+(h_t/2))/\tilde{A}(\tilde{L}_2)] \geq 1 - C_+e^{-c-\varepsilon h_t}. \quad (4.16)$$

We now turn back to \mathcal{J}_2 . We have on $\mathcal{E}_1^{4.6}$,

$$\mathcal{J}_2 \leq \sup_{u \in [0,1]} \mathcal{L}_B(\tau^B(1), u) \tilde{A}(\tilde{L}_2) \int_{\tilde{\tau}_2^+(h_t/2)}^{\tilde{L}_2} e^{-\tilde{V}^{(2)}(u)} du. \quad (4.17)$$

Let $\mathcal{E}_2^{4.6} := \{\inf_{\tilde{\tau}_2^+(h_t/2) \leq u \leq \tilde{\tau}_2(h_t)} [\tilde{V}^{(2)}(u)] > (1-\varepsilon)h_t/2\}$ and $\mathcal{E}_3^{4.6} := \{\tilde{L}_2^+ - \tilde{L}_2^- \leq 28h_t/\kappa\}$. Working on \mathcal{V}_t , $\mathcal{E}_2^{4.6} = \{\inf_{\tau_2(h_t/2) \leq u \leq \tau_2(h_t)} [V^{(2)}(u)] > (1-\varepsilon)h_t/2\}$, then Williams' decomposition, equation (3.2) with $\alpha = 1/2$, $\gamma = 1/2(1-\varepsilon)$ and $\omega = 1$ give $P(\overline{\mathcal{E}_2^{4.6}}) \leq P(\overline{\mathcal{E}_2^{4.6}}, \mathcal{V}_t) + P(\overline{\mathcal{V}_t}) \leq 2e^{-\kappa\varepsilon h_t/2} + P(\overline{\mathcal{V}_t})$. Moreover, $P(\overline{\mathcal{E}_3^{4.6}}) \leq C_+e^{-\kappa h_t/24}$ by (2.6) and Lemma 2.3. We have on $\mathcal{E}_2^{4.6} \cap \mathcal{E}_3^{4.6}$,

$$\int_{\tilde{\tau}_2^+(h_t/2)}^{\tilde{L}_2} e^{-\tilde{V}^{(2)}(u)} du \leq e^{-(1-\varepsilon)h_t/2} (\tilde{L}_2 - \tilde{\tau}_2^+(h_t/2)) \leq h_t \frac{28}{\kappa} e^{-(1-\varepsilon)h_t/2}.$$

This, together with (4.17) and (4.7) with $M = h_t$ proves (4.14).

For \mathcal{J}_0 , we prove that for t large enough

$$\mathbb{P}(\mathcal{J}_0 \geq C_+\tilde{A}(\tilde{L}_2)h_t e^{-(1/2-2\varepsilon)h_t}) \leq C_+e^{-\varepsilon\delta c_-h_t}. \quad (4.18)$$

With the same arguments as those used for (4.16), we get $P(\mathcal{E}_4^{4.6}) \geq 1 - C_+ e^{-c-\varepsilon h_t}$, where $\mathcal{E}_4^{4.6} := \{x_0^- \leq \tilde{\tau}_2^-(h_t/2)\}$.

We now define $\mathcal{E}_5^{4.6} := \{x_0^- \geq \tilde{L}_2^-\}$ and prove it has a large probability. In the one hand we have

$$P(\tilde{A}(\tilde{L}_2) \geq e^{h_t(1+\varepsilon/2)}) \leq C_+ e^{-c-\varepsilon h_t/2}. \quad (4.19)$$

Indeed $\tilde{A}(\tilde{L}_2) = \tilde{A}(\tilde{\tau}_2(h_t)) + \tilde{A}(\tilde{L}_2) - \tilde{A}(\tilde{\tau}_2(h_t))$. Thanks to Lemma 4.5 $P(\tilde{A}(\tilde{\tau}_2(h_t)) > 2e^{h_t(1+\varepsilon/2)}) \leq C_+ e^{-\varepsilon h_t/2}/2$. Also the strong Markov property gives $\tilde{A}(\tilde{L}_2) - \tilde{A}(\tilde{\tau}_2(h_t)) \stackrel{\mathcal{L}}{=} G^+(h_t, h_t/2)$, then we notice that $G^+(h_t, h_t/2) \stackrel{\mathcal{L}}{=} e^{h_t} G^+(0, -h_t/2) \leq e^{h_t} A_\infty$, where A_∞ is defined just above (3.1). So $P(\tilde{A}(\tilde{L}_2) - \tilde{A}(\tilde{\tau}_2(h_t)) > e^{h_t(1+\varepsilon/2)}/2) \leq P(A_\infty > e^{\varepsilon h_t/2}/2) \leq C_+ e^{-\kappa \varepsilon h_t/2}/2 \leq C_+ e^{-c-\varepsilon h_t/2}/2$, as $A_\infty \stackrel{\mathcal{L}}{=} 2/\gamma_\kappa$.

On the other hand $P(-\tilde{A}(\tilde{L}_2^-) \geq e^{h_t(1+\kappa/2)}) \geq P(F^+(h_t^+) \geq e^{h_t(1+\kappa/2)}) - P(\tilde{\mathcal{V}}_t)$ (see Remark 2.4). So by (3.5) $P(-\tilde{A}(\tilde{L}_2^-) \geq e^{h_t(1+\kappa/2)}) \geq 1 - 4e^{-\kappa^2 h_t/4}$ for large t . This, together with (4.19), gives

$$P(\mathcal{E}_5^{4.6}) = P(-\delta_t \geq \tilde{A}(\tilde{L}_2^-)/\tilde{A}(\tilde{L}_2)) \geq 1 - C_+ e^{-c-\varepsilon h_t/4}.$$

Let $\mathcal{E}_6^{4.6} := \{\inf_{\tilde{L}_2^- \leq u \leq \tilde{\tau}_2^-(h_t/2)} (\tilde{V}^{(2)}(u)) \geq e^{(1/2-\varepsilon)h_t}\}$ and $\mathcal{E}_7^{4.6} := \{\sup_{s \leq 0} \mathcal{L}_B(\tau^B(1), s) \leq e^{\varepsilon h_t}\}$. Also by Remark 2.4, on $\{\tau_2^-(h_t^+) \geq 0\}$, $(V^{(2)}(u), \tau_2^-(h_t^+) \leq u \leq \tau_2^-(h_t/2))$ has the same law as $(R^{(2)}(s), \tau^{R^{(2)}}(h_t/2) \leq s \leq \tau^{R^{(2)}}(h_t^+))$. Hence (3.2) with $\gamma = 1/2 - \varepsilon$, $\alpha = 1/2$ and $\omega = 1 + \kappa + 2\delta$ gives $P(\mathcal{E}_6^{4.6}) \leq P(\mathcal{E}_6^{4.6}, \mathcal{V}_t) + P(\tilde{\mathcal{V}}_t) \leq 2e^{-\kappa \varepsilon h_t}$. Moreover, by (4.8), $\mathbb{P}(\mathcal{E}_7^{4.6}) \leq e^{-\varepsilon h_t}$.

Therefore, on $\cap_{i=3}^7 \mathcal{E}_i^{4.6} \cap \mathcal{V}_t$, i.e with a probability larger than $1 - C_+ e^{-c-\varepsilon h_t}$, we obtain

$$\mathcal{J}_0 \leq \tilde{A}(\tilde{L}_2) \sup_{s \leq 0} \mathcal{L}_B(\tau^B(1), s) \int_{\tilde{L}_1^-}^{\tilde{\tau}_1^-(h_t/2)} e^{-\tilde{V}^{(1)}(u)} du \leq C_+ h_t \tilde{A}(\tilde{L}_2) e^{-(1/2-2\varepsilon)h_t}, \quad (4.20)$$

which yields (4.18).

For \mathcal{J}_1 , we prove that for t large enough,

$$\mathbb{P}[\mathcal{J}_1 \leq \mathbf{e}_1 \tilde{A}(\tilde{L}_2) e^{-\varepsilon h_t}/2] \leq C_+ e^{-c-\varepsilon h_t}, \quad (4.21)$$

where $\mathbf{e}_1 := \mathcal{L}_B(\tau^B(1), 0)$ is an exponential r.v. with mean 2 by the first Ray-Knight theorem, and is independent of W_κ . First, let $\hat{\mathcal{J}}_1 := \int_{x_0^-}^{\tilde{m}_1} + \int_{\tilde{m}_1}^{x_0^+} e^{-(W_\kappa(u) - W_\kappa(\tilde{m}_1))} du =: \hat{\mathcal{J}}_1^- + \hat{\mathcal{J}}_1^+$ and

$$\mathcal{E}_8^{4.6} := \left\{ \sup_{u \in [-\delta_t, \delta_t]} |\mathcal{L}_B(\tau^B(1), u) - \mathcal{L}_B(\tau^B(1), 0)| \leq \delta_t^{1/3} \mathcal{L}_B(\tau^B(1), 0) \right\}.$$

We know that $\mathbb{P}(\overline{\mathcal{E}_8^{4.6}}) \leq C_+ \delta_t^{1/30}$ by (4.6). We have on $\mathcal{E}_8^{4.6}$,

$$\left| \mathcal{J}_1 - \tilde{A}(\tilde{L}_2) \hat{\mathcal{J}}_1 \mathbf{e}_1 \right| \leq \delta_t^{1/3} \tilde{A}(\tilde{L}_2) \hat{\mathcal{J}}_1 \mathbf{e}_1. \quad (4.22)$$

We finally need that $\hat{\mathcal{J}}_1$ is not too small, which we prove with a similar argument as before : first we have $\hat{\mathcal{J}}_1 \geq \hat{\mathcal{J}}_1^+ \geq e^{-\varepsilon h_t} (\tilde{\tau}_2^+(\varepsilon h_t) - \tilde{m}_2) \geq e^{-\varepsilon h_t}$ on $\mathcal{E}_1^{4.6} \cap \mathcal{E}_9^{4.6}$, where $\mathcal{E}_9^{4.6} := \{\tilde{\tau}_2^+(\varepsilon h_t) - \tilde{m}_2 \geq 1\}$ has a probability $P(\mathcal{E}_9^{4.6}) \geq P(\mathcal{E}_9^{4.6}, \mathcal{V}_t) - P(\tilde{\mathcal{V}}_t) \geq 1 - C_+ e^{-c-\varepsilon^2 h_t^2/4} - C_1 n_t e^{-\kappa h_t/2}$ by Lemma 2.3, Williams decomposition and (3.3). Hence $\mathcal{J}_1 \geq (1 - \delta_t^{1/3}) \tilde{A}(\tilde{L}_2) e^{-\varepsilon h_t} \mathbf{e}_1$ on $\mathcal{E}_1^{4.6} \cap \mathcal{E}_8^{4.6} \cap \mathcal{E}_9^{4.6} \cap \mathcal{V}_t$, which gives (4.21) for large t .

End of the proof. To finish we have to prove that $\hat{\mathcal{J}}_1$ is nearly equal to \mathcal{I}^- .

On $\mathcal{E}_1^{4.6} \cap \mathcal{E}_4^{4.6} \cap \mathcal{E}_5^{4.6}$, we have $\tilde{L}_2^- \leq x_0^- \leq \tilde{\tau}_2^-(h_t/2) \leq \tilde{\tau}_2^+(h_t/2) \leq x_0^+ \leq \tilde{L}_2$, and $\tilde{V}^{(2)}(x) \geq (1-\varepsilon)h_t/2$ for every $x \in [\tilde{L}_2^-, \tilde{\tau}_2^-(h_t/2)] \cap [\tilde{\tau}_2^+(h_t/2), \tilde{L}_2]$ on $\mathcal{E}_2^{4.6} \cap \mathcal{E}_6^{4.6}$, hence on $\cap_{i=4}^9 \mathcal{E}_i^{4.6}$,

$$0 \leq \hat{\mathcal{J}}_1 - \mathcal{I}^- \leq \int_{\tilde{L}_2^-}^{\tilde{\tau}_2^-(h_t/2)} + \int_{\tilde{\tau}_2^+(h_t/2)}^{\tilde{L}_2} e^{-\tilde{V}^{(2)}(x)} dx \leq e^{-(1-\varepsilon)h_t/2} (\tilde{L}_2 - \tilde{L}_2^-) \leq C_+ h_t e^{-(1-\varepsilon)h_t/2}.$$

Let $\mathcal{E}_{10}^{4.6} := \{\mathbf{e}_1 \geq e^{-\varepsilon h_t/2}\}$, and observe that $\mathbb{P}(\mathcal{E}_{10}^{4.6}) \geq 1 - e^{-\varepsilon h_t/2}$. We now combine all the previous intermediate results. On $\cap_{i=1}^{10} \mathcal{E}_i^{4.6}$, for t large enough, $\hat{\mathcal{J}}_1 \geq \tilde{A}(\tilde{L}_2) \mathbf{e}_1 e^{-\varepsilon h_t/2} \geq \tilde{A}(\tilde{L}_2) e^{-2\varepsilon h_t/2}$ and $\mathcal{J}_0 + \mathcal{J}_2 \leq \tilde{A}(\tilde{L}_2) e^{(-1/2+3\varepsilon)h_t} \leq 2\mathcal{J}_1 e^{(-1/2+5\varepsilon)h_t}$. This, together with (4.22), yields

$$(1 - e^{-h_t(1-3\varepsilon)/6}) \tilde{A}(\tilde{L}_2) \hat{\mathcal{J}}_1 \mathbf{e}_1 \leq \mathcal{J}_1 \leq \mathbf{U} \leq (1 + 2e^{(-1/2+5\varepsilon)h_t}) \mathcal{J}_1 \leq (1 + 2e^{-h_t(1-3\varepsilon)/6}) \tilde{A}(\tilde{L}_2) \hat{\mathcal{J}}_1 \mathbf{e}_1 \quad (4.23)$$

Moreover, $\hat{\mathcal{J}}_1 \geq \mathcal{I}^- \geq e^{-\varepsilon h_t} (\tilde{\tau}_2^+(\varepsilon h_t) - \tilde{m}_1) \geq e^{-\varepsilon h_t}$, and $\hat{\mathcal{J}}_1 = (\hat{\mathcal{J}}_1 - \mathcal{I}^-) + \mathcal{I}^- \leq C_+ h_t e^{-(1-\varepsilon)h_t/2} + \mathcal{I}^- \leq (1 + e^{(-1/2+2\varepsilon)h_t}) \mathcal{I}^-$ for large t . This and (4.23) give $(1 - e^{-h_t(1-3\varepsilon)/6}) \tilde{A}(\tilde{L}_2) \mathcal{I}^- \mathbf{e}_1 \leq \mathbf{U} \leq (1 + 3e^{-h_t(1-3\varepsilon)/6}) \tilde{A}(\tilde{L}_2) \hat{\mathcal{J}}_1 \mathbf{e}_1$ which proves the lemma. \square

We are now ready to prove the proposition :

Proof of Proposition 4.3: We define $\mathcal{I}_1^- := \int_{m_2}^{\tau_2(h_t/2)} e^{-V^{(2)}(s)} ds = A_1^-$,

$$\mathcal{I}_2^- := \int_{\tau_2^-(h_t/2)}^{m_2} e^{-V^{(2)}(s)} ds = B_1^-, \mathcal{I}_3^+ := \int_{\tau_2(h_t/2)}^{\tau_2(h_t)} e^{V^{(2)}(s)} ds, \text{ and } \mathcal{I}_2^+ := \int_{\tau_2(h_t)}^{L_2} e^{V^{(2)}(s)} ds = A_2^+.$$

By Williams' decomposition and the strong Markov property, \mathcal{I}_1^- , \mathcal{I}_2^- and \mathcal{I}_2^+ are independent, $\mathcal{I}_1^- \stackrel{\mathcal{L}}{=} F^-(h_t/2)$, $\mathcal{I}_2^- \stackrel{\mathcal{L}}{=} F^-(h_t/2)$ and $\mathcal{I}_2^+ \stackrel{\mathcal{L}}{=} G^+(h_t/2, h_t)$, and they are independent of $(W_\kappa(s + \tau_2(h_t/2)) - W_\kappa(m_2), 0 \leq s \leq \tau_2(h_t) - \tau_2(h_t/2))$ which has the same law as a $(\kappa/2)$ -drifted 3 dimensional Bessel process started from $h_t/2$ and killed when it first hits h_t .

We now consider, possibly on an enlarged probability space, a process $(R^{(1)}(s), 0 \leq s \leq \tau^{R^{(1)}}(h_t/2))$, independent of W_κ and \mathbf{e}_1 and then independent of \mathcal{I}_1^- , \mathcal{I}_2^- and \mathcal{I}_2^+ , and distributed as $(R(s), 0 \leq s \leq \tau^R(h_t/2))$. We now extend this process by setting $R^{(1)}(s) := W_\kappa[s - \tau^{R^{(1)}}(h_t/2) + \tau_2(h_t/2)] - W_\kappa(m_2)$ for $\tau^{R^{(1)}}(h_t/2) \leq s \leq \tau^{R^{(1)}}(h_t/2) + \tau_2(h_t) - \tau_2(h_t/2)$. We also introduce $\mathcal{I}_1^+ := \int_0^{\tau^{R^{(1)}}(h_t)} e^{R^{(1)}(s)} ds$. By the strong Markov property, $R^{(1)}$ (and hence also \mathcal{I}_1^+) is independent of \mathcal{I}_2^+ , \mathcal{I}_1^- , \mathcal{I}_2^- and \mathbf{e}_1 .

Moreover, with the same notation as in Lemma 4.6, we have on \mathcal{V}_t , $\mathcal{I}^- = \mathcal{I}_1^- + \mathcal{I}_2^-$ and $\tilde{A}(\tilde{L}_2) = A_1^+ + \mathcal{I}_2^+$ where \mathcal{I}^- is defined in Lemma 4.6.

We now prove that A_1^+ can be approximated by \mathcal{I}_1^+ . Since $A_1^+ - \mathcal{I}_3^+ = \int_{m_2}^{\tau_2(h_t/2)} e^{V^{(2)}(s)} ds \stackrel{\mathcal{L}}{=} F^+(h_t/2)$, and by Lemma 4.1

$$P\left(\mathcal{I}_1^+ - \mathcal{I}_3^+ > e^{(1+\delta)h_t/2}\right) = P\left(A_1^+ - \mathcal{I}_3^+ > e^{(1+\delta)h_t/2}\right) = P\left(\frac{F^+(h_t/2)}{e^{h_t/2}} > e^{\delta h_t/2}\right) \leq \frac{C_+}{e^{\delta h_t/2}}.$$

Moreover by (3.5), with a probability larger than $1 - 3e^{-\kappa\delta h_t/2}$ for large t , $\mathcal{I}_1^+ \geq e^{h_t(1-\delta)}$. Therefore, with a probability greater than $1 - 4e^{-\kappa\delta h_t/2}$ for large t , $A_1^+ \leq \mathcal{I}_3^+ + e^{-h_t(1-3\delta)/2} \mathcal{I}_1^+ \leq \mathcal{I}_1^+ (1 + e^{-h_t(1-3\delta)/2})$ and $A_1^+ \geq \mathcal{I}_1^+ - (\mathcal{I}_1^+ - \mathcal{I}_3^+) \geq (1 - e^{-h_t(1-3\delta)/2}) \mathcal{I}_1^+$, and then

$$(1 - e^{-h_t(1-3\delta)/2}) \mathcal{I}_1^+ \leq A_1^+ \leq (1 + e^{-h_t(1-3\delta)/2}) \mathcal{I}_1^+,$$

which gives $(1 - e^{-h_t(1-3\delta)/2}) \leq \tilde{A}(\tilde{L}_2) \mathcal{I}^- / [(\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)] \leq (1 + e^{-h_t(1-3\delta)/2})$. To finish we use Lemma 4.6 with $\varepsilon = \delta$. \square

4.3. Second order Laplace transform of a standard exit time.

We are now ready to prove a key point of this paper which is the second order of the Laplace transform of \mathbf{U}/t :

Proposition 4.7. *Let $\lambda > 0$. For t large enough,*

$$e^{\kappa\phi(t)} \left(1 - E \left(e^{-\frac{\lambda}{t}\mathbf{U}} \right)\right) = C_\kappa \lambda^\kappa + o(1)$$

with $C_\kappa := 8^\kappa(C_0 + |\Upsilon_0|)$, with $C_0 := \Gamma(1 - \kappa)\Gamma(\kappa + 2)/(1 + \kappa)^\kappa$ and

$$\Upsilon_0 := \int_0^\infty \int_0^\infty \frac{u^\kappa y^{-\kappa-1} e^{-u}}{\Gamma(\kappa)} \left[\frac{y^\kappa}{[\Gamma(\kappa + 1)I_\kappa(\sqrt{y})]^2} - \left[1 + \frac{y}{(\kappa + 1)} \right]^{-2} \right] dy du.$$

Proof: We have,

$$E \left(e^{-\frac{\lambda}{t}\mathbf{U}} \right) = E \left(e^{-\frac{\lambda}{t}\mathbf{U}} \mathbf{1}_{\mathcal{A}_t} \right) + E \left(e^{-\frac{\lambda}{t}\mathbf{U}} \mathbf{1}_{\overline{\mathcal{A}_t}} \right) \leq E \left(e^{-\frac{\lambda}{t}\mathbf{U}} \mathbf{1}_{\mathcal{A}_t} \right) + P(\overline{\mathcal{A}_t}).$$

Hence by definition of \mathcal{A}_t and Proposition 4.3, we get with $\lambda_t^\pm := 2\lambda(1 \pm \varepsilon_t)/t$,

$$S_0^+ - C_+ e^{-\delta^2 c - h_t} \leq E \left(e^{-\frac{\lambda}{t}\mathbf{U}} \right) \leq S_0^- + C_+ e^{-\delta^2 c - h_t},$$

where $S_0^\pm := E \left(e^{-(\lambda_t^\pm/2)(\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)} \mathbf{e}_1 \right)$. Let $\varepsilon_1 > 0$, $\theta := 3\kappa/4 + \varepsilon_1$, $\mathcal{B}_1 := \{(\mathcal{I}_1^+ + \mathcal{I}_2^+) > e^{\log t - \theta\phi(t)}\}$, and $\mathcal{B}_2 := \{(\mathcal{I}_1^+ + \mathcal{I}_2^+) \leq e^{\log t - \theta\phi(t)}\}$. Since \mathbf{e}_1 is an exponential r.v. with mean 2, we get $S_0^\pm = S_1^\pm + S_2^\pm$, where for $1 \leq i \leq 2$,

$$\begin{aligned} S_2^\pm &:= \int_0^\infty dz e^{-z} E \left(\mathbf{1}_{\mathcal{B}_i} e^{-z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+)(\mathcal{I}_1^- + \mathcal{I}_2^-)} \right) \\ &= \int_0^{+\infty} dz e^{-z} E \left(\mathbf{1}_{\mathcal{B}_i} \left[E \left(e^{-z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+)\mathcal{I}_1^-} \mid \mathcal{I}_1^+, \mathcal{I}_2^+ \right) \right]^2 \right), \end{aligned} \quad (4.24)$$

since \mathcal{I}_1^- and \mathcal{I}_2^- are independent and independent of \mathcal{I}_1^+ and \mathcal{I}_2^+ and have the same law.

We start with S_2^\pm . Let $a(t) := e^{-(3/4)\kappa\phi(t)}$. Now, consider $0 \leq z \leq \eta_1 a(t) e^{\theta\phi(t) - \log t} / \lambda_t^\pm = \eta_1 a(t) e^{\theta\phi(t)} / (2\lambda(1 \pm \varepsilon_t))$. On \mathcal{B}_2 , we have $z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+) \leq \eta_1 a(t)$, which gives by (4.1) applied to $\mathcal{I}_1^- \stackrel{\mathcal{L}}{=} F^-(h_t/2)$ for t so large that $h_t/2 \geq M$,

$$E \left(e^{-z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+)\mathcal{I}_1^-} \mid \mathcal{I}_1^+, \mathcal{I}_2^+ \right) \leq Z(z) + C_4 \max(e^{-\kappa h_t/2}, [z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+)]^{\frac{3}{2}}).$$

with $Z(x) := (1 + 2\lambda_t^\pm x(\mathcal{I}_1^+ + \mathcal{I}_2^+) / (\kappa + 1))^{-1}$ for any $x > 0$. Therefore, for such z and t , by (4.24),

$$\begin{aligned} S_2^\pm &\leq \int_0^{\frac{\eta_1 a(t) e^{\theta\phi(t)}}{t\lambda_t^\pm}} \frac{dz}{e^z} E \left(\mathbf{1}_{\mathcal{B}_2} Z^2(z) + \mathbf{1}_{\mathcal{B}_2} 3C_4 ([z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+)]^{\frac{3}{2}} + e^{-\kappa h_t/2}) \right) \\ &\quad + \int_{\frac{\eta_1 a(t) e^{\theta\phi(t)}}{t\lambda_t^\pm}}^{+\infty} e^{-z} dz. \end{aligned}$$

We notice that $\int_{\frac{\eta_1 a(t) e^{\theta\phi(t)}}{t\lambda_t^\pm}}^{+\infty} e^{-z} dz \leq e^{-\phi(t)}$ for large t since $a(t) e^{\theta\phi(t)} \geq \phi(t)$. Moreover, $\int_0^{\frac{\eta_1 a(t) e^{\theta\phi(t)}}{t\lambda_t^\pm}} dz e^{-z} E \left(\mathbf{1}_{\mathcal{B}_2} [z\lambda_t^\pm(\mathcal{I}_1^+ + \mathcal{I}_2^+)]^{3/2} \right) \leq [\eta_1 a(t)]^{3/2}$. So for t large,

$$S_2^\pm \leq \int_0^\infty dz e^{-z} E \left(\mathbf{1}_{\mathcal{B}_2} Z^2(z) \right) + o(e^{-\kappa\phi(t)}).$$

We also define,

$$S_3^\pm := \int_0^{+\infty} dz e^{-z} E \left(\mathbf{1}_{\mathcal{B}_1} Z^2(z) \right).$$

Recall that for any $\lambda > 0$ and any random variable $X \geq 0$, we have by Fubini,

$$E\left((1 + \lambda X)^{-2}\right) = \int_0^{+\infty} du \int_u^{+\infty} dx E\left(e^{-x(1+\lambda X)}\right) dx.$$

So, by independence of \mathcal{I}_1^+ and \mathcal{I}_2^+ , and defining $\rho_\kappa^\pm := \frac{4(1 \pm \varepsilon t)}{\kappa + 1}$

$$E(Z^2(z)) = \int_0^{+\infty} du \int_u^{+\infty} dx e^{-x} E\left(e^{-\frac{\rho_\kappa^\pm zx \lambda \mathcal{I}_1^+}{t}}\right) E\left(e^{-\frac{\rho_\kappa^\pm zx \lambda \mathcal{I}_2^+}{t}}\right). \quad (4.25)$$

Recall that $t = e^{ht} e^{\phi(t)}$, so using (4.2), (4.3) and the laws of \mathcal{I}_1^+ and \mathcal{I}_2^+ , we notice that whenever $\rho_\kappa^\pm zx \lambda / e^{\phi(t)} \leq \eta_1$ and $y \geq M$,

$$\begin{aligned} E\left(e^{-\frac{\rho_\kappa^\pm zx \lambda \mathcal{I}_1^+}{t}}\right) &\leq 1 - \frac{2\rho_\kappa^\pm zx \lambda}{(\kappa + 1)e^{\phi(t)}} + C_4 \max\left(e^{-\kappa ht}, \left(\rho_\kappa^\pm zx \lambda / e^{\phi(t)}\right)^{3/2}\right), \\ E\left(e^{-\frac{\rho_\kappa^\pm zx \lambda \mathcal{I}_2^+}{t}}\right) &\leq 1 - \frac{\Gamma(1 - \kappa)}{\Gamma(1 + \kappa)} \frac{(2\rho_\kappa^\pm zx \lambda)^\kappa}{e^{\kappa\phi(t)}} + C_4 \max\left(\left(\rho_\kappa^\pm zx \lambda / e^{\phi(t)}\right)^\kappa e^{-\kappa ht/2}, \frac{\rho_\kappa^\pm zx \lambda}{e^{\phi(t)}}\right). \end{aligned}$$

So, we get, C being a positive constant,

$$\begin{aligned} (4.25) &\leq \int_0^{+\infty} du \int_u^{+\infty} dx e^{-x} \left(1 - \frac{\Gamma(1 - \kappa)}{\Gamma(1 + \kappa)} \frac{(2\rho_\kappa^\pm zx \lambda)^\kappa}{e^{\kappa\phi(t)}}\right) \mathbf{1}_{\rho_\kappa^\pm zx \lambda / e^{\phi(t)} \leq \eta_1} \\ &\quad + C \int_0^{+\infty} du \int_u^{+\infty} dx e^{-x} \left(\left(\frac{\rho_\kappa^\pm zx \lambda}{e^{\phi(t)}}\right)^\kappa e^{-\kappa ht/2} + \frac{\rho_\kappa^\pm zx \lambda}{e^{\phi(t)}} + e^{-\kappa ht/2}\right) \mathbf{1}_{\rho_\kappa^\pm zx \lambda / e^{\phi(t)} \leq \eta_1} \\ &\quad + \int_0^{+\infty} du \int_u^{+\infty} e^{-x} dx \mathbf{1}_{\rho_\kappa^\pm zx \lambda / e^{\phi(t)} > \eta_1}. \end{aligned}$$

Finally, notice that $\int_0^{+\infty} e^{-z} dz \int_0^{+\infty} du \int_u^{+\infty} dx e^{-x} \rho_\kappa^\pm zx \lambda / e^{\phi(t)} = 2\rho_\kappa^\pm \lambda / e^{\phi(t)}$. Moreover, we have $\int_0^{+\infty} e^{-z} dz \int_0^{+\infty} du \int_u^{+\infty} dx e^{-x} [\rho_\kappa^\pm zx \lambda / e^{\phi(t)}]^\kappa = O(e^{-\kappa\phi(t)})$, and furthermore $\int_0^{+\infty} dz e^{-z} \int_0^{+\infty} du \int_u^{+\infty} e^{-x} dx \mathbf{1}_{\rho_\kappa^\pm zx \lambda / e^{\phi(t)} > \eta_1} = O(e^{-\phi(t)})$. Hence, we get

$$\begin{aligned} &\int_0^{+\infty} dz e^{-z} E(Z^2(z)) \\ &\leq \int_0^{+\infty} dz e^{-z} \int_0^{+\infty} du \int_u^{+\infty} dx e^{-x} \left(1 - \frac{\Gamma(1 - \kappa)}{\Gamma(1 + \kappa)} \frac{(2\rho_\kappa^\pm zx \lambda)^\kappa}{e^{\kappa\phi(t)}}\right) \mathbf{1}_{\rho_\kappa^\pm zx \lambda / e^{\phi(t)} \leq \eta_1} + O(e^{-\phi(t)}) \\ &= 1 - \frac{\Gamma(1 - \kappa)}{\Gamma(1 + \kappa)} \frac{8^\kappa \lambda^\kappa}{(1 + \kappa)^\kappa e^{\kappa\phi(t)}} (1 \pm \varepsilon t)^\kappa \int_0^{+\infty} dz e^{-z} z^\kappa \int_0^{+\infty} dx e^{-x} x^{\kappa+1} \\ &\quad + \frac{\Gamma(1 - \kappa)(2\lambda\rho_\kappa^\pm)^\kappa}{\Gamma(1 + \kappa)e^{\kappa\phi(t)}} \int_0^{+\infty} dz e^{-z} \int_0^{+\infty} du \int_u^{+\infty} dx e^{-x} (zx)^\kappa \mathbf{1}_{\rho_\kappa^\pm zx \lambda / e^{\phi(t)} > \eta_1} + O(e^{-\phi(t)}) \quad (4.26) \\ &= 1 - \frac{\Gamma(1 - \kappa)\Gamma(\kappa + 2)8^\kappa \lambda^\kappa}{(1 + \kappa)^\kappa e^{\kappa\phi(t)}} (1 \pm \varepsilon t)^\kappa + o(e^{-\kappa\phi(t)}). \quad (4.27) \end{aligned}$$

since, by the dominated convergence theorem, the integral in (4.26) goes to 0 as $t \rightarrow +\infty$ and then Line (4.26) = $o(e^{-\kappa\phi(t)})$. Combining equations (4.24) to (4.27) we get $S_2^\pm + S_3^\pm \leq 1 - C_0 8^\kappa \lambda^\kappa e^{-\kappa\phi(t)} + o(e^{-\kappa\phi(t)})$. We prove similarly that $S_2^\pm + S_3^\pm \geq 1 - C_0 8^\kappa \lambda^\kappa e^{-\kappa\phi(t)} + o(e^{-\kappa\phi(t)})$.

We now turn to the estimation of $S_1^\pm - S_3^\pm$. We assume that $\phi(t) \geq 4 \log \log t$, $0 < \varepsilon < 1/2$, $b(t) = -\theta\phi(t)$.

Step 1: Approximation of $\mathcal{I}_1^+ + \mathcal{I}_2^+$. Since $\varepsilon e^{(1-3\kappa/4-\varepsilon_1)\phi(t)} \geq 8h_t/\kappa$ for large t , we have

$$\mathbb{P}(\mathcal{I}_1^+ \geq \varepsilon t e^{-(3\kappa/4+\varepsilon_1)\phi(t)}) \leq \mathbb{P}(\tau^R(h_t) \geq \varepsilon e^{(1-3\kappa/4-\varepsilon_1)\phi(t)}) \leq P(\tau^R(h_t) \geq 8h_t/\kappa) \leq C_+ e^{-\kappa h_t/2\sqrt{2}},$$

since $F_+(h_t) \leq e^{h_t} \tau^R(h_t)$, $\tau^R(h_t) \stackrel{\mathcal{L}}{=} \tau_1(h_t) - m_1$ by Fact 2.2, and thanks to (2.4).

Moreover, we can write $\mathcal{I}_2^+ = e^{ht} \widehat{A}_\infty - e^{ht/2} \widetilde{A}_\infty$, where $\widehat{A}_\infty \stackrel{\mathcal{L}}{=} \widetilde{A}_\infty \stackrel{\mathcal{L}}{=} A_\infty$. We have $\mathbb{P}(e^{ht/2} \widetilde{A}_\infty \geq \varepsilon t e^{-(3\kappa/4+\varepsilon_1)\phi(t)}) \leq \mathbb{P}(\widetilde{A}_\infty \geq t^{1/3}) \leq C t^{-\kappa/3}$ since $P(A_\infty \geq y) \leq C y^{-\kappa}$ (see Dufresne). Now, we have on $\mathcal{B}_1 \cap \{\mathcal{I}_1^+ < \varepsilon t e^{-(3\kappa/4+\varepsilon_1)\phi(t)}\} \cap \{e^{ht/2} \widetilde{A}_\infty < \varepsilon t e^{-(3\kappa/4+\varepsilon_1)\phi(t)}\}$,

$$t e^{-(3\kappa/4+\varepsilon_1)\phi(t)} < \mathcal{I}_1^+ + \mathcal{I}_2^+ = \mathcal{I}_1^+ + e^{ht} \widehat{A}_\infty - e^{ht/2} \widetilde{A}_\infty < e^{ht} \widehat{A}_\infty + \varepsilon t e^{-(3\kappa/4+\varepsilon_1)\phi(t)}.$$

This yields $e^{ht} \widehat{A}_\infty \geq (1-\varepsilon) t e^{-(3\kappa/4+\varepsilon_1)\phi(t)}$, and then $\mathcal{I}_1^+ + \mathcal{I}_2^+ \leq (1 + \frac{\varepsilon}{1-\varepsilon}) e^{ht} \widehat{A}_\infty \leq (1+2\varepsilon) e^{ht} \widehat{A}_\infty$. Similarly, $\mathcal{I}_1^+ + \mathcal{I}_2^+ \geq e^{ht} \widehat{A}_\infty - \varepsilon t e^{-(3\kappa/4+\varepsilon_1)\phi(t)} \geq (1 - \frac{\varepsilon}{1-\varepsilon}) e^{ht} \widehat{A}_\infty \geq (1-2\varepsilon) e^{ht} \widehat{A}_\infty$. Consequently, replacing ε by $\varepsilon/2$,

$$\mathbb{P}(\mathcal{B}_1 \cap \{(1-\varepsilon) e^{ht} A_\infty \leq \mathcal{I}_1^+ + \mathcal{I}_2^+ \leq (1+\varepsilon) e^{ht} A_\infty\}^c) = O(e^{-\phi(t)}). \quad (4.28)$$

Step 2: Simplification. Thanks to (4.28), we get

$$\begin{aligned} S_1^\pm - S_3^\pm &\leq \int_0^\infty E \left[\left(e^{-\lambda_t^\pm u (\mathcal{I}_1^+ + \mathcal{I}_2^+) (\mathcal{I}_1^- + \mathcal{I}_2^-)} - Z^2(u) \right) \mathbf{1}_{\{\mathcal{I}_1^+ + \mathcal{I}_2^+ \geq t e^{b(t)}\}} \right. \\ &\quad \left. \mathbf{1}_{\{(1-\varepsilon) e^{ht} \widehat{A}_\infty \leq \mathcal{I}_1^+ + \mathcal{I}_2^+ \leq (1+\varepsilon) e^{ht} \widehat{A}_\infty\}} \right] e^{-u} du + e^{-\phi(t)} \\ &\leq S_4^\pm + S_5^\pm + e^{-\phi(t)}, \end{aligned} \quad (4.29)$$

where

$$\begin{aligned} S_4^\pm &:= \int_0^\infty E \left[\left(e^{-\lambda_t^\pm u (1-\varepsilon) e^{ht} \widehat{A}_\infty (\mathcal{I}_1^- + \mathcal{I}_2^-)} - \right) \mathbf{1}_{(1+\varepsilon) e^{ht} \widehat{A}_\infty \geq t e^{b(t)}} \right. \\ &\quad \left. \mathbf{1}_{\{(1-\varepsilon) e^{ht} \widehat{A}_\infty \leq \mathcal{I}_1^+ + \mathcal{I}_2^+ \leq (1+\varepsilon) e^{ht} \widehat{A}_\infty\}} \right] e^{-u} du, \\ S_5^\pm &:= \int_0^\infty E \left[\left(e^{-\lambda_t^\pm u (1-\varepsilon) e^{ht} \widehat{A}_\infty (\mathcal{I}_1^- + \mathcal{I}_2^-)} - Z_\infty^2(u) \right) \right. \\ &\quad \left. [\mathbf{1}_{\mathcal{I}_1^+ + \mathcal{I}_2^+ \geq t e^{b(t)}} - \mathbf{1}_{(1+\varepsilon) e^{ht} \widehat{A}_\infty \geq t e^{b(t)}}] \mathbf{1}_{\{(1-\varepsilon) e^{ht} \widehat{A}_\infty \leq \mathcal{I}_1^+ + \mathcal{I}_2^+ \leq (1+\varepsilon) e^{ht} \widehat{A}_\infty\}} \right] e^{-u} du, \\ Z_\infty(u) &:= \left[1 + 2\lambda_t^\pm u (1+\varepsilon) e^{ht} \widehat{A}_\infty / (\kappa + 1) \right]^{-1}. \end{aligned}$$

Using once more (4.28) leads to

$$\begin{aligned} S_4^\pm &\leq \int_0^\infty E \left[\left(e^{-\lambda_t^\pm u (1-\varepsilon) e^{ht} \widehat{A}_\infty (\mathcal{I}_1^- + \mathcal{I}_2^-)} - Z_\infty^2(u) \right) \mathbf{1}_{(1+\varepsilon) e^{ht} \widehat{A}_\infty \geq t e^{b(t)}} \right] e^{-u} du \\ &\quad + 2e^{-\phi(t)}. \end{aligned} \quad (4.30)$$

Now, for t large enough so that $0 \leq \varepsilon_t \leq \varepsilon$, the integral in (4.30) can be written as

$$\begin{aligned} &\int_0^\infty e^{-u} du E \left[\left(E \left(e^{-\lambda_t^\pm u (1-\varepsilon) e^{ht} \widehat{A}_\infty \mathcal{I}_1^-} \mid \widehat{A}_\infty \right)^2 - Z_\infty^2(u) \right) \mathbf{1}_{\widehat{A}_\infty \geq \frac{e^{b(t)+\phi(t)}}{1+\varepsilon}} \right] \\ &\leq \int_0^\infty e^{-u} du E \left[\left(E \left(e^{-4\lambda(1-\varepsilon)^2 u e^{-\phi(t)} \gamma_\kappa^{-1} \mathcal{I}_1^-} \mid \gamma_\kappa \right)^2 - Z_\infty^2(u) \right) \mathbf{1}_{\gamma_\kappa \leq \frac{2(1+\varepsilon)}{e^{b(t)+\phi(t)}}} \right]. \end{aligned} \quad (4.31)$$

By Dufresne [13], since γ_κ has density $x^{\kappa-1} e^{-x} / \Gamma(\kappa) \mathbf{1}_{\mathbb{R}_+}(x)$ and is, as \widehat{A}_∞ , independent of \mathcal{I}_1^- then the RHS of (4.31) is equal to

$$\int_0^\infty e^{-u} du \int_{\frac{2(1+\varepsilon)}{e^{b(t)+\phi(t)}}}^\infty \left[E \left(e^{-4\lambda(1-\varepsilon)^2 u e^{-\phi(t)} x^{-1} \mathcal{I}_1^-} \right)^2 - \left(1 + \frac{8\lambda(1+\varepsilon)^2 u}{(\kappa+1)e^{\phi(t)} x} \right)^{-2} \right] \frac{x^{\kappa-1} e^{-x}}{\Gamma(\kappa)} dx. \quad (4.32)$$

With the change of variables $y = 8u\lambda e^{-\phi(t)}x^{-1}$, this is equal to $(8\lambda)^\kappa e^{-\kappa\phi(t)}\Upsilon_\epsilon(t)$, with $\Upsilon_\epsilon(t) := \int_0^\infty \int_0^\infty f_t(u, y) dy du$ and

$$f_t(u, y) := \mathbb{1}_{\{y \geq \frac{4\lambda u e^{-\theta\phi(t)}}{(1+\epsilon)}\}} u^\kappa e^{-u} \left[E \left(e^{-(1-\epsilon)^2 y \mathcal{I}_1^- / 2} \right)^2 - \left(1 + \frac{(1+\epsilon)^2 y}{(\kappa+1)} \right)^{-2} \right] \frac{e^{-8\lambda u y^{-1} e^{-\phi(t)}}}{\Gamma(\kappa) y^{\kappa+1}}.$$

Step 3: pointwise convergence.

Notice that thanks to (4.4), $\lim_{y \rightarrow +\infty} E \left(e^{-\gamma F^-(y)} \right) = \frac{(2\gamma)^{\kappa/2}}{\kappa \Gamma(\kappa) I_\kappa(2\sqrt{2\gamma})}$ for $\gamma > 0$. Hence, for every $u > 0$ and $y > 0$,

$$f_t(u, y) \rightarrow_{t \rightarrow +\infty} f(u, y) := u^\kappa e^{-u} \left[\frac{(1-\epsilon)^{2\kappa} y^\kappa}{[\Gamma(\kappa+1) I_\kappa(2(1-\epsilon)\sqrt{y})]^2} - \left(1 + \frac{(1+\epsilon)^2 y}{(\kappa+1)} \right)^{-2} \right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)}.$$

Step 4: dominated convergence. We notice that $f_t(u, y) = a_t(u, y) + b_t(u, y)$ and $f(u, y) = a(u, y) + b(u, y)$, where

$$a_t(u, y) := \mathbb{1}_{\{y \geq \frac{4\lambda u e^{-\theta\phi(t)}}{(1+\epsilon)}\}} u^\kappa e^{-u} \left[E \left(e^{-(1-\epsilon)^2 y \mathcal{I}_1^- / 2} \right)^2 - \left(1 + \frac{(1-\epsilon)^2 y}{(\kappa+1)} \right)^{-2} \right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)} e^{-8\lambda u y^{-1} e^{-\phi(t)}}$$

$$b_t(u, y) := \mathbb{1}_{\{y \geq \frac{4\lambda u e^{-\theta\phi(t)}}{(1+\epsilon)}\}} u^\kappa e^{-u} \left[\left(1 + \frac{(1-\epsilon)^2 y}{(\kappa+1)} \right)^{-2} - \left(1 + \frac{(1+\epsilon)^2 y}{(\kappa+1)} \right)^{-2} \right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)} e^{-8\lambda u y^{-1} e^{-\phi(t)}}$$

and their pointwise limits on $(\mathbb{R}_+^*)^2$ as $t \rightarrow +\infty$ are respectively

$$a(u, y) := u^\kappa e^{-u} \left[\frac{(1-\epsilon)^{2\kappa} y^\kappa}{[\Gamma(\kappa+1) I_\kappa(2(1-\epsilon)\sqrt{y})]^2} - \left(1 + \frac{(1-\epsilon)^2 y}{(\kappa+1)} \right)^{-2} \right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)},$$

$$b(u, y) := u^\kappa e^{-u} \left[\left(1 + \frac{(1-\epsilon)^2 y}{(\kappa+1)} \right)^{-2} - \left(1 + \frac{(1+\epsilon)^2 y}{(\kappa+1)} \right)^{-2} \right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)}.$$

Since $\forall x > 0$, $I_\kappa(x) > \frac{(x/2)^\kappa}{\Gamma(\kappa+1)} + \frac{(x/2)^{\kappa+2}}{\Gamma(\kappa+2)}$, we get $\frac{\gamma^\kappa}{[\Gamma(\kappa+1) I_\kappa(2\sqrt{\gamma})]^2} - \left(1 + \frac{\gamma}{(\kappa+1)} \right)^{-2} < 0$ for every $\gamma > 0$. Consequently, $\forall (u, t) \in \mathbb{R}_+^2$, $a(u, t) < 0$. This, and the fact that $t \mapsto E \left(e^{-\gamma F^-(h_t/2)} \right)$ is nonincreasing for $\gamma \geq 0$, lead to

$$\forall (u, y) \in (\mathbb{R}_+)^2, \quad a(u, y) \leq a_t(u, y) \leq u^\kappa e^{-u} \left[1 - \left(1 + \frac{(1-\epsilon)^2 y}{(\kappa+1)} \right)^{-2} \right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)} =: h(u, y). \quad (4.33)$$

Hence, $|a_t(u, y)| \leq |a(u, y)| + |h(u, y)|$ and $|b_t(u, y)| \leq |b(u, y)|$. for every $(u, y) \in \mathbb{R}_+^2$. Moreover, since $0 < \kappa < 1$, $|h|$, $|b|$ and $|a|$ have finite integrals over \mathbb{R}_+^2 (notice for example that $e^u u^{-\kappa} a(u, y) = O(y^{-\kappa})$ as $y \rightarrow 0$). Thus, by the dominated convergence theorem,

$$\lim_{t \rightarrow +\infty} \Upsilon_\epsilon(t) = \int_0^\infty \int_0^\infty f(u, y) dy du =: \Upsilon_\epsilon.$$

Hence for small ϵ , $\limsup_{t \rightarrow +\infty} ((S_4^\pm)^\kappa e^{\kappa\phi(t)} \lambda^{-\kappa}) \leq (1-\epsilon) 8^\kappa \Upsilon_\epsilon$, since $\lim_{x \rightarrow 0} J(x) = J(0) < 0$.

Step 5: S_5^\pm is negligible. First, we have

$$\begin{aligned} |S_5^\pm| &\leq \int_0^\infty E \left[\left| e^{-\lambda_t^\pm u (1-\epsilon) e^{ht} A_\infty(\mathcal{I}_1^- + \mathcal{I}_2^-)} - Z_\infty^2(u) \right| \right. \\ &\quad \left. \mathbb{1}_{\{\mathcal{I}_1^+ + \mathcal{I}_2^+ < t e^{b(t)}\}} \mathbb{1}_{\{(1+\epsilon) e^{ht} A_\infty \geq t e^{b(t)}\}} \mathbb{1}_{\{(1-\epsilon) e^{ht} A_\infty \leq \mathcal{I}_1^+ + \mathcal{I}_2^+ \leq (1+\epsilon) e^{ht} A_\infty\}} \right] e^{-u} du \\ &\leq \int_0^\infty E \left[\left| e^{-\lambda_t^\pm u (1-\epsilon) e^{ht} A_\infty(\mathcal{I}_1^- + \mathcal{I}_2^-)} - Z_\infty^2(u) \right| \right. \\ &\quad \left. \mathbb{1}_{\{(1-\epsilon) e^{ht} A_\infty \leq t e^{b(t)} \leq (1+\epsilon) e^{ht} A_\infty\}} \right] e^{-u} du, \end{aligned} \quad (4.34)$$

where we used $|(\mathbb{1}_{E_1} - \mathbb{1}_{E_2})\mathbb{1}_{E_3}| = \mathbb{1}_{\overline{E_1}}\mathbb{1}_{E_2}\mathbb{1}_{E_3} + \mathbb{1}_{E_1}\mathbb{1}_{\overline{E_2}}\mathbb{1}_{E_3}$, for events E_i with $\mathbb{1}_{E_1}\mathbb{1}_{\overline{E_2}}\mathbb{1}_{E_3} = 0$ in our case. The same calculations as for S_4^\pm give $|S_5^\pm| \leq \frac{(8\lambda)^\kappa}{e^{\kappa\phi(t)}} \int_0^\infty \int_0^\infty (|F_t(u, y, \varepsilon)| + |F_t(u, y, -\varepsilon)|) dy du$ for large t , with

$$F_t(u, y, z) := \mathbb{1}_{\left\{\frac{4\lambda ue^{-\theta\phi(t)}}{(1+\varepsilon)} \leq y \leq \frac{4\lambda ue^{-\theta\phi(t)}}{(1-\varepsilon)}\right\}} u^\kappa e^{-u} \left[E \left(e^{-(1-z)^2 y \mathcal{I}_1^- / 2} \right)^2 - \left(1 + \frac{(1+z)^2 y}{(\kappa+1)} \right)^{-2} \right] \frac{y^{-\kappa-1}}{\Gamma(\kappa)}.$$

Using the same domination as before, the dominated convergence theorem gives $S_5^\pm = o(e^{-\kappa\phi(t)})$.

Conclusion: Consequently, $\limsup_{t \rightarrow +\infty} [(S_4^\pm + S_5^\pm) e^{\kappa\phi(t)} \lambda^{-\kappa}] \leq (1 - 2\varepsilon) 8^\kappa \Upsilon_\varepsilon$, then

$$\limsup_{t \rightarrow +\infty} \left[(S_1^\pm - S_3^\pm) e^{\kappa\phi(t)} \lambda^{-\kappa} \right] \leq (1 - 2\varepsilon) 8^\kappa \Upsilon_\varepsilon$$

by (4.29). We prove with the same method that $\liminf_{t \rightarrow +\infty} [(S_1^\pm - S_3^\pm) e^{\kappa\phi(t)} \lambda^{-\kappa}] \geq (1 + 2\varepsilon) 8^\kappa \Upsilon_{-\varepsilon}$. Moreover $\lim_{x \rightarrow 0} \Upsilon_x = \Upsilon_0$ which yields $(S_1^\pm - S_3^\pm) \sim_{t \rightarrow +\infty} 8^\kappa \Upsilon_0 \lambda^\kappa e^{-\kappa\phi(t)}$. Since we already proved that $S_2^\pm + S_3^\pm = 1 - C_0 8^\kappa \lambda^\kappa e^{-\kappa\phi(t)} + o(e^{-\kappa\phi(t)})$, we get

$$1 - E \left(e^{-\frac{\lambda}{t} \mathbf{U}} \right) \sim_{t \rightarrow +\infty} 8^\kappa (C_0 - \Upsilon_0) \lambda^\kappa e^{-\kappa\phi(t)},$$

which proves the proposition. \square

5. PROOF OF THE MAIN RESULTS

5.1. The renewal results : In this section we prove Propositions 1.4, 1.6 and Corollary 1.5. We start with the following important intermediate result on the first exit time \mathbf{U} .

In what follows, we mainly use the same ideas as in [16] pages 18 to 22, inspired from the book [20] pages 470-472. We first need a lemma and give its proof for the sake of completeness. Recall that for all $i \geq 1$, $U_i = H(\tilde{L}_i) - H(\tilde{m}_i)$.

Lemma 5.1. *For $t > 0$, let μ_t be the positive measure on \mathbb{R}_+ such that for all $x \geq 0$, $\mu_t([0, x]) = e^{-\kappa\phi(t)} \sum_{j=1}^{n_t-2} \mathbb{P}(\sum_{i=1}^j U_i / t \leq x)$. Then, $(\mu_t)_t$ converges vaguely, as $t \rightarrow +\infty$, to the measure μ defined by $d\mu(x) := (C_\kappa \Gamma(\kappa))^{-1} x^{\kappa-1} \mathbb{1}_{(0, +\infty)}(x) dx$. Moreover, on every compact subset in $(0, +\infty)$, $x \mapsto e^{\kappa\phi(t)} \mathbb{P}(\mathbf{U}/t \geq x)$ converges uniformly when $t \rightarrow +\infty$ to $x \mapsto C_\kappa x^{-\kappa} / \Gamma(1 - \kappa)$.*

Proof: First, let us prove that for all $\lambda > 0$, we have as $t \rightarrow +\infty$,

$$\int_0^{+\infty} e^{-\lambda x} d\mu_t(x) = \int_0^{+\infty} \frac{e^{-\lambda x} x^{\kappa-1}}{C_\kappa \Gamma(\kappa)} dx + o(1), \quad (5.1)$$

$$\int_0^{+\infty} e^{-\lambda x} e^{\kappa\phi(t)} \mathbb{P}(\mathbf{U}/t \geq x) dx = \int_0^{+\infty} e^{-\lambda x} \frac{C_\kappa}{\Gamma(1 - \kappa) x^\kappa} dx + o(1). \quad (5.2)$$

Let $\lambda > 0$. First, we have, by Proposition 3.2,

$$\int_0^{+\infty} e^{-\lambda x} d\mu_t(x) = \frac{1}{e^{\kappa\phi(t)}} \sum_{j=1}^{n_t-2} \mathbb{E} \left(e^{-\lambda \sum_{i=1}^j \frac{U_i}{t}} \right) = \sum_{j=1}^{n_t-2} \left(\frac{1}{e^{\kappa\phi(t)}} \left(\mathbb{E} \left(e^{-\lambda \frac{U}{t}} \right) \right)^j \right) + O \left(\frac{n_t u(t, n_t)}{e^{\kappa\phi(t)}} \right).$$

We notice that, by Proposition 4.7, $[\mathbb{E} (e^{-\lambda \mathbf{U}/t})]^{n_t-1} = o(1)$, since $n_t e^{-\kappa\phi(t)} \rightarrow_{t \rightarrow +\infty} +\infty$ and $C_\kappa > 0$. Hence, again by Proposition 4.7, we get since $n_t u(t, n_t) e^{-\kappa\phi(t)} \rightarrow 0$ as $t \rightarrow +\infty$,

$$\int_0^{+\infty} e^{-\lambda x} d\mu_t(x) = \frac{e^{-\kappa\phi(t)} (1 + o(1))}{1 - \mathbb{E} (e^{-\lambda \mathbf{U}/t})} + o(1) = \frac{1}{C_\kappa \lambda^\kappa} + o(1) = \int_0^{+\infty} \frac{e^{-\lambda x} x^{\kappa-1}}{C_\kappa \Gamma(\kappa)} dx + o(1),$$

which gives (5.1). This implies the pointwise convergence of the Laplace transforms of $(\mu_t)_t$ to that of μ , and then the vague convergence of $(\mu_t)_t$ to μ . Now, we have by Fubini,

$$\mathbb{E}\left(1 - e^{-\frac{\lambda}{t}\mathbf{U}}\right) = \int_0^\infty \int_0^u \lambda e^{-\lambda x} dx \mathbb{P}(\mathbf{U}/t \in du) = \lambda \int_0^{+\infty} e^{-\lambda x} \mathbb{P}(\mathbf{U}/t \geq x) dx.$$

Since $\lambda^\kappa = \lambda \int_0^{+\infty} e^{-\lambda x} x^{-\kappa} dx / \Gamma(1 - \kappa)$ and using Proposition 4.7, we get (5.2). From this pointwise convergence of the Laplace transform of $e^{\kappa\phi(t)} \mathbb{P}(\mathbf{U}/t \geq x)$, we get the uniform convergence of $e^{\kappa\phi(t)} P(\mathbf{U}/t \geq x)$ when t goes to infinity as in [16]. \square

Proof of Proposition 1.4 and Corollary 1.5: Let \tilde{N}_t be the unique integer such that $H(\tilde{m}_{\tilde{N}_t}) \leq t < H(\tilde{m}_{\tilde{N}_t+1})$.

First, by Lemma 2.3 and (3.22), $\mathbb{P}(N_t \geq n_t) \leq \mathbb{P}(\sum_{j=1}^{n_t-1} U_j \leq H(\tilde{m}_{n_t}) \leq t) + C_1 w_t$. By the exponential Markov inequality, this is $\leq e \mathbb{E}(e^{-\sum_{i=1}^{n_t-1} U_i/t}) + C_1 w_t$. Finally, Propositions 3.2 and 4.7 give since $\phi(t) = o(\log t)$,

$$\mathbb{P}(N_t \geq n_t) \leq e \left(\mathbb{E}(e^{-\mathbf{U}/t}) \right)^{n_t} + C_1 w_t + C_2 u(t, n_t) \leq C_+ e^{-c - n_t e^{-\kappa\phi(t)}} + 2C_2 u(t, n_t) = o(1). \quad (5.3)$$

Assume first that $0 < r < s < 1$, and $a > 0$. Then, Lemmata 2.3, 3.5 and (5.3) yield to

$$\begin{aligned} & \mathbb{P}[1 - s \leq H(m_{N_t})/t \leq 1 - r, H(m_{N_t+1})/t \geq 1 + a] \\ & \leq \sum_{j=1}^{n_t-1} \mathbb{P}\left(1 - s \leq \frac{H(\tilde{m}_{\tilde{N}_t})}{t} \leq 1 - r, H(\tilde{m}_{\tilde{N}_t+1})/t \geq 1 + a, \tilde{N}_t = j, \mathcal{V}_t\right) \\ & \quad + \mathbb{P}(N_t \notin [1, n_t]) + P(\bar{\mathcal{V}}_t) \\ & \leq \sum_{j=1}^{n_t-1} \mathbb{P}\left(1 - s_t \leq \sum_{l=1}^{j-1} U_l/t \leq 1 - r, \sum_{l=1}^j U_l/t > 1 + a - 2/\log h_t\right) + o(1). \end{aligned} \quad (5.4)$$

where $s_t := s + 2/\log h_t$. We now use (3.9) of Proposition 3.2 and get for any $\varepsilon > 0$, for large t ,

$$(5.4) \leq \int_{1-s-\varepsilon}^{1-r} e^{\kappa\phi(t)} \mathbb{P}(\mathbf{U}/t > 1 + a - x - \varepsilon) d\mu_t(x) + o(1). \quad (5.5)$$

Let $0 < r' < s' < 1$ and $a' > 0$. Using first the uniform convergence of $u \mapsto e^{\kappa\phi(t)} \mathbb{P}(\mathbf{U}/t > u)$ on the compact $[a' + r', a' + s'] \subset (0, \infty)$ and then the vague convergence of μ_t (see Lemma 5.1), we get

$$\lim_{t \rightarrow +\infty} \int_{1-s'}^{1-r'} e^{\kappa\phi(t)} \mathbb{P}(\mathbf{U}/t > 1 + a' - x) d\mu_t(x) = \int_{1-s'}^{1-r'} \frac{x^{\kappa-1} (1 + a' - x)^{-\kappa}}{\Gamma(\kappa)\Gamma(1 - \kappa)} dx.$$

Consequently, by letting $\varepsilon \rightarrow 0$, we obtain the first inequality of the following line

$$\limsup_{t \rightarrow +\infty} (5.4) \leq \int_{1-s}^{1-r} \int_a^\infty \frac{\kappa}{\Gamma(1 - \kappa)\Gamma(\kappa)} (1 + y - x)^{-\kappa-1} x^{\kappa-1} dy dx \leq \liminf_{t \rightarrow +\infty} (5.4).$$

We prove similarly the second inequality. Since we consider probability measures, the cases $r = 0$, $s = 1$ or $a = 0$ follow, which concludes the proof of Proposition 1.4. Corollary 1.5 follows by straightforward computations. \square

Proof of Proposition 1.6: Let us denote by ν_t a positive measure on \mathbb{R}_+ , such that for every $x \geq 0$, $\nu_t([0, x]) = e^{-\kappa\phi(t)} \sum_{j=2}^{n_t-1} e^{-\frac{C_\kappa \lambda^\kappa j}{e^{\kappa\phi(t)}}} P(\sum_{i=1}^{j-1} U_i/t \leq x)$. We have, with the arguments

already used between (5.4) and (5.5), for any $a > 0$ and $\varepsilon > 0$,

$$\begin{aligned} & \mathbb{E} \left[e^{-\frac{C_\kappa \lambda^\kappa N_t}{e^{\kappa \phi(t)}}}, H(m_{N_t+1})/t \geq 1 + a \right] \\ & \leq \sum_{j=1}^{n_t-1} e^{-\frac{C_\kappa \lambda^\kappa j}{e^{\kappa \phi(t)}}} \mathbb{P} \left[\sum_{i=1}^{j-1} U_i/t \leq 1, 1 + a - 2/\log h_t \leq \sum_{i=1}^j U_i/t \right] + o(1) \\ & \leq \int_0^1 d\nu_t(x) e^{\kappa \phi(t)} \mathbb{P} \left(\frac{\mathbf{U}}{t} > 1 + a - \varepsilon - x \right) + o(1). \end{aligned}$$

We prove similarly the lower bound. We now show that the Laplace transform of the measure ν_t converges when t goes to infinity. We consider α such that $0 < \lambda < \alpha$. We get,

$$\int_0^{+\infty} e^{-\alpha u} d\nu_t(u) = \frac{\mathbb{E} \left(e^{-\frac{\alpha}{t} \mathbf{U}} \right) \left[1 - \left[e^{-\frac{C_\kappa \lambda^\kappa}{e^{\kappa \phi(t)}}} \mathbb{E} \left(e^{-\frac{\alpha}{t} \mathbf{U}} \right) \right]^{n_t-2} \right]}{e^{\kappa \phi(t) + \frac{2C_\kappa \lambda^\kappa}{e^{\kappa \phi(t)}}} \left[1 - e^{-\frac{C_\kappa \lambda^\kappa}{e^{\kappa \phi(t)}}} \mathbb{E} \left(e^{-\frac{\alpha}{t} \mathbf{U}} \right) \right]} + o(1) = \frac{1}{C_\kappa (\alpha^\kappa + \lambda^\kappa)} + o(1),$$

by Propositions 3.2 and then 4.7. We also notice that

$$\frac{1}{\alpha^\kappa + \lambda^\kappa} = \sum_{j=0}^{+\infty} \frac{(-\lambda^\kappa)^j}{\alpha^{\kappa(1+j)}} = \sum_{j=0}^{+\infty} \frac{(-\lambda^\kappa)^j}{\Gamma[\kappa(1+j)]} \int_0^{+\infty} e^{-\alpha u} u^{\kappa(1+j)-1} du.$$

So Fubini gives, since $\int_0^\infty \sum_{j=0}^\infty \left| \frac{e^{-\alpha u} u^{\kappa(1+j)-1}}{C_\kappa} \frac{(-\lambda^\kappa)^j}{\Gamma[\kappa(1+j)]} \right| du < \infty$,

$$\forall \alpha > \lambda, \quad \lim_{t \rightarrow +\infty} \int_0^{+\infty} e^{-\alpha u} d\nu_t(u) = \int_0^{+\infty} e^{-\alpha u} d\nu(u) + o(1),$$

where ν is the measure defined by $d\nu(u) = \frac{1}{C_\kappa} \sum_{j=0}^{+\infty} (-\lambda^\kappa)^j \frac{u^{\kappa(1+j)-1}}{\Gamma[\kappa(1+j)]} \mathbf{1}_{\mathbb{R}_+}(u) du$. This pointwise convergence of the Laplace transform of ν_t leads to the vague convergence of ν_t to ν . Using the uniform convergence of $x \mapsto e^{\kappa \phi(t)} \mathbb{P}(\mathbf{U}/t > 1 + a - x)$ on $[0, 1]$ provided by Lemma 5.1, we get

$$\lim_{t \rightarrow +\infty} \mathbb{E} \left[e^{-\frac{C_\kappa \lambda^\kappa N_t}{e^{\kappa \phi(t)}}}, \frac{H(m_{N_t+1})}{t} \geq 1 + a \right] = \sum_{j=0}^{+\infty} (-\lambda^\kappa)^j \int_0^1 \frac{(1+a-x)^{-\kappa} x^{\kappa(1+j)-1}}{\Gamma[\kappa(1+j)] \Gamma(1-\kappa)} dx.$$

and this remains true for $a = 0$. Since $\int_0^1 x^{a-1} (1-x)^{b-1} dx = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ for every $a > 0$ and $b > 0$, changing $C_\kappa \lambda^\kappa$ into u gives the pointwise convergence of $\mathbb{E}[\exp(-uN_t/e^{\kappa \phi(t)})]$ to the right hand side of (1.4), which ends the proof of Proposition 1.6. \square

5.2. The localization : proof of Theorem 1.3. Let $\phi^*(t) := \phi(t)/\zeta$, where $0 < \zeta < 1$ will be chosen later. Let us define $H_{x \rightarrow y} := H(y) - H(x)$ for $0 < x < y$, $t^* := t - e^{\phi^*(t)(1+2\delta)}$,

$$\begin{aligned} \mathcal{A}_0 &:= \{1 \leq \tilde{N}_t < n_t\}, & \mathcal{A}_1 &:= \cap_{j=1}^{n_t-1} \{H_{\tilde{L}_j \rightarrow \tilde{m}_{j+1}} \leq 2t/\log h_t\}, \\ \mathcal{A}_2 &:= \cap_{j=1}^{n_t-1} \{H_{\tilde{m}_j \rightarrow \tilde{m}_{j+1}} < H_{\tilde{m}_j \rightarrow \tilde{L}_j}\}, & \mathcal{A}_3 &:= \{H(\tilde{m}_{\tilde{N}_t}) \leq t^*\}. \end{aligned}$$

We also introduce $\mathbf{I}_j := [\tilde{m}_j - \phi^*(t)/\zeta, \tilde{m}_j + \phi^*(t)/\zeta]$, $j \in \mathbb{N}^*$. Let $\varepsilon > 0$. We have:

$$\begin{aligned} \mathbb{P} \left(X(t) \notin \mathbf{I}_{\tilde{N}_t} \right) & \leq \mathbb{P} \left(X(t) \notin \mathbf{I}_{\tilde{N}_t}, \tilde{N}_t = \tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \right) + \mathbb{P}(\tilde{N}_t \neq \tilde{N}_{t(1+\varepsilon)}) \\ & \quad + \mathbb{P} \left(X(t) \notin \mathbf{I}_{\tilde{N}_t}, \tilde{N}_t = \tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \overline{\mathcal{A}_3} \right) + \mathbb{P}(\overline{\mathcal{A}_0}) + \mathbb{P}(\overline{\mathcal{A}_1}) + \mathbb{P}(\overline{\mathcal{A}_2}). \end{aligned}$$

We split the proof into three parts. We start with:

Part 1: we prove that there exists $c_4 > 0$ such that for large t ,

$$\mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_t}, \tilde{N}_t = \tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\right) \leq C_+ n_t \left(h_t e^{-3\phi^*(t)} + \phi^*(t) e^{-c_4 \delta \phi^*(t)}\right). \quad (5.6)$$

Let $\mathcal{B}_j := \{\tilde{N}_t = \tilde{N}_{t(1+\varepsilon)} = j\} \cap \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$. We have

$$\mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_t}, \tilde{N}_t = \tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\right) = \sum_{j=1}^{n_t-1} \mathbb{P}\left(X(t) \notin \mathbf{I}_j, \mathcal{B}_j\right). \quad (5.7)$$

If t is large enough, on \mathcal{B}_j , after first hitting \tilde{m}_j , X stays in $[\tilde{L}_j^-, \tilde{L}_j]$ at least until time $t(1+\varepsilon/2)$. Therefore, conditioning on $H(\tilde{m}_j)$ and using the strong Markov property,

$$\begin{aligned} \mathbb{P}\left(X(t) \notin \mathbf{I}_j, \mathcal{B}_j\right) &\leq E\left(\mathbb{P}^{W_\kappa}(X(t) \notin \mathbf{I}_j, H(\tilde{m}_j) \leq t^*, \forall u \in [H(\tilde{m}_j), t(1+\varepsilon/2)], X(u) \in [\tilde{L}_j^-, \tilde{L}_j])\right) \\ &= E\left(\int_0^{t^*} \mathbb{P}^{W_\kappa}\left(X(t) \notin \mathbf{I}_j, H(\tilde{m}_j) \in ds, \forall u \in [s, t(1+\frac{\varepsilon}{2})], X(u) \in [\tilde{L}_j^-, \tilde{L}_j]\right)\right) \\ &\leq E\left(\sup_{0 \leq s \leq t^*} \mathbb{P}_{\tilde{m}_j}^{W_\kappa}\left(X(t-s) \notin \mathbf{I}_j, \forall u \in [0, t(1+\varepsilon/2)-s], X(u) \in [\tilde{L}_j^-, \tilde{L}_j]\right)\right). \end{aligned} \quad (5.8)$$

So, as in ([5], proof of Prop. 4.1) we now introduce a coupling between X (under $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$) and a reflected process Y_j defined below. To this aim, let $(Y_j^{(x)}(u), u \geq 0)$ be the process defined by $Y_j^{(x)}(u) := A^{-1}\left(\hat{B}_j^{(x)}(\hat{T}_{x,j}^{-1}(u))\right)$ for any $x \in [\tilde{L}_j^-, \tilde{L}_j]$ and $u \geq 0$, where $\hat{B}_j^{(x)}$ is a one-dimensional Brownian motion starting from $A(x)$ and reflected at $A(\tilde{L}_j^-)$ and $A(\tilde{L}_j)$ and independent from W_κ , and $\hat{T}_{x,j}$ is defined like T replacing B by $\hat{B}_j^{(x)}$. In words, $Y_j^{(x)}$ is a diffusion in the potential W_κ , starting from $x \in [\tilde{L}_j^-, \tilde{L}_j]$ and reflected at \tilde{L}_j^- and \tilde{L}_j . We denote its law by $\hat{P}_{j,x}^{W_\kappa}$.

This enables us to define Y_j by $\hat{P}_j^{W_\kappa}(Y_j \in \cdot) := \int_{\tilde{L}_j^-}^{\tilde{L}_j} \hat{P}_{j,x}^{W_\kappa}(\cdot) d\tilde{\mu}_j(x)$, where

$$d\tilde{\mu}_j(x) := \exp(-\tilde{V}^{(j)}(x)) \mathbf{1}_{[\tilde{L}_j^-, \tilde{L}_j]}(x) dx \left(\int_{\tilde{L}_j^-}^{\tilde{L}_j} \exp(-\tilde{V}^{(j)}(y)) dy \right)^{-1}. \quad (5.9)$$

As in ([5], proof of Prop. 4.1), $\tilde{\mu}_j$ is invariant for the semi-group of Y_j ; in particular $\hat{P}_j^{W_\kappa}(Y_j(s) \in U) = \tilde{\mu}_j(U)$ for every $s \geq 0$ and $U \subset [\tilde{L}_j^-, \tilde{L}_j]$. We can now, as in [5], build a coupling $Q_{\tilde{m}_j}^{W_\kappa}$ of X and Y_j , such that $Q_{\tilde{m}_j}^{W_\kappa}(Y_j \in \cdot) = \hat{P}_j^{W_\kappa}(Y_j \in \cdot)$, and $Q_{\tilde{m}_j}^{W_\kappa}(X \in \cdot) = \mathbb{P}_{\tilde{m}_j}^{W_\kappa}(\cdot)$, these two Markov processes Y_j and X move independently until the first collision $\hat{H}_j := \inf\{u \geq 0, X(u) = Y_j(u)\}$, then $X(u) = Y_j(u)$ until $\hat{H}_j^e := \inf\{u > \hat{H}_j, X(u) \notin [\tilde{L}_j^-, \tilde{L}_j]\}$ and then X and Y_j move independently again.

We now prove that, with a large probability, under $Q_{\tilde{m}_j}^{W_\kappa}$, X and Y_j first meet before time $t_1 := t - t^* = e^{\phi^*(t)(1+2\delta)}$. To this aim, we introduce $\hat{L}_j^+ := \tilde{\tau}_j^+(\phi^*(t))$ and $\hat{L}_j^- := \tilde{\tau}_j^-(\phi^*(t))$, and we first prove a result concerning only X :

Lemma 5.2. *With a probability larger than $1 - C_+ e^{-c-\delta\phi^*(t)}$, and with $x \vee y := \max(x, y)$,*

$$P_{\tilde{m}_j}^{W_\kappa}\left(H(\hat{L}_j^+) \vee H(\hat{L}_j^-) \leq e^{(1+\delta)\phi^*(t)}\right) \geq 1 - C_+ \phi^*(t) e^{-c-\delta\phi^*(t)}. \quad (5.10)$$

Proof: We introduce $\widehat{L}_j^{++} := \tilde{\tau}_j^+[\phi^*(t)(1 + \delta/2)]$ and $\widehat{L}_j^{--} := \tilde{\tau}_j^-[\phi^*(t)(1 + \delta/2)]$. Then with a method similar as for (3.12), we get $P_{\tilde{m}_j}^{W_\kappa} \left(H(\widehat{L}_j^+) > H(\widehat{L}_j^-) \right) = \widehat{Q}_j / \widehat{D}_j$, where

$$\begin{aligned} \widehat{Q}_j &\leq (\widehat{L}_j^+ - \tilde{m}_j) \exp\left[\max_{[\tilde{m}_j, \widehat{L}_j^+]} \tilde{V}^{(j)}(s) \right] \leq 8\kappa^{-1} \phi^*(t) \exp(\phi^*(t)), \\ \widehat{D}_j &\geq [\tilde{\tau}_j^-((1 + \delta/4)\phi^*(t)) - \tilde{\tau}_j^-((1 + \delta/2)\phi^*(t))] e^{(1+\delta/8)\phi^*(t)} \geq e^{(1+\delta/8)\phi^*(t)} \end{aligned}$$

with a probability larger than $1 - e^{-c-\delta\phi^*(t)}$, by (2.4) for the first line and by (3.2) and (3.3) for the second one this together with Lemma 2.3. Hence, with such a probability, $P_{\tilde{m}_j}^{W_\kappa} (H(\widehat{L}_j^+) < H(\widehat{L}_j^-)) \geq 1 - C_+ \phi^*(t) e^{-\delta\phi^*(t)/8}$.

Moreover, as in (3.13), we can write under $P_{\tilde{m}_j}^{W_\kappa}$ on the event $\{H(\widehat{L}_j^+) < H(\widehat{L}_j^-)\}$,

$$H(\widehat{L}_j^+) \stackrel{\mathcal{L}}{=} \tilde{A}_j(\widehat{L}_j^+) \int_{\widehat{L}_j^-}^{\widehat{L}_j^+} e^{-\tilde{V}^{(j)}(u)} \mathcal{L}_B[\tau_B(1), \tilde{A}_j(u)/\tilde{A}_j(\widehat{L}_j^+)] du =: \mathcal{H}_j.$$

Then using the work to get Lemma 4.6, we can prove using the same arguments that with a probability greater than $1 - C_+ e^{-c-\delta\phi^*(t)}$, $P_{\tilde{m}_j}^{W_\kappa} (\mathcal{H}_j \geq e^{(1+\delta)\phi^*(t)}) \leq C_+ e^{-c-\delta\phi^*(t)}$ and therefore $P_{\tilde{m}_j}^{W_\kappa} \left(H(\widehat{L}_j^+) \leq e^{(1+\delta)\phi^*(t)} \right) \geq 1 - C_+ \phi^*(t) e^{-c-\delta\phi^*(t)}$. We get the same result for $H(\widehat{L}_j^-)$, collecting both estimations we obtain (5.10). \square

We deduce from (5.10) that

$$\begin{aligned} Q_{\tilde{m}_j}^{W_\kappa} \left(\widehat{H}_j > t_1 \right) &\leq Q_{\tilde{m}_j}^{W_\kappa} \left(\widehat{H}_j > H(\widehat{L}_j^+) \vee H(\widehat{L}_j^-) \right) + Q_{\tilde{m}_j}^{W_\kappa} \left(H(\widehat{L}_j^+) \vee H(\widehat{L}_j^-) > t_1 \right) \\ &\leq Q_{\tilde{m}_j}^{W_\kappa} \left(\widehat{H}_j > H(\widehat{L}_j^-), Y_j(0) < \tilde{m}_j \right) + Q_{\tilde{m}_j}^{W_\kappa} \left(\widehat{H}_j > H(\widehat{L}_j^+), Y_j(0) \geq \tilde{m}_j \right) \\ &\quad + C_+ \phi^*(t) e^{-c-\delta\phi^*(t)}. \end{aligned}$$

Hence, since X and Y are continuous,

$$\begin{aligned} Q_{\tilde{m}_j}^{W_\kappa} \left(\widehat{H}_j > t_1 \right) &\leq Q_{\tilde{m}_j}^{W_\kappa} \left(Y_j[H(\widehat{L}_j^-)] \in [\tilde{L}_j^-, \widehat{L}_j^-], \widehat{H}_j > H(\widehat{L}_j^-) \right) \\ &\quad + Q_{\tilde{m}_j}^{W_\kappa} \left(Y_j[H(\widehat{L}_j^+)] \in [\widehat{L}_j^+, \tilde{L}_j], \widehat{H}_j > H(\widehat{L}_j^+) \right) + C_+ \phi^*(t) e^{-c-\delta\phi^*(t)} \\ &\leq \tilde{\mu}_j([\tilde{L}_j^-, \widehat{L}_j^-]) + \tilde{\mu}_j([\widehat{L}_j^+, \tilde{L}_j]) + C_+ \phi^*(t) e^{-c-\delta\phi^*(t)}, \end{aligned} \quad (5.11)$$

where the last line comes from the independence of X and Y_j until \widehat{H}_j and the fact that $\tilde{\mu}_j$ is the invariant probability measure for Y_j .

Let $s \in [0, t^*]$. Since $X(u) = Y_j(u)$ for every $\widehat{H}_j \leq u \leq \widehat{H}_j^e$ and $t_1 \leq t - s \leq t(1 + \varepsilon/2) - s$,

$$\begin{aligned} &Q_{\tilde{m}_j}^{W_\kappa} \left(X(t-s) \notin \mathbf{I}_j, \forall u \in [0, t(1 + \varepsilon/2) - s], X(u) \in [\tilde{L}_j^-, \tilde{L}_j] \right) \\ &\leq Q_{\tilde{m}_j}^{W_\kappa} \left(\widehat{H}_j \leq t_1, X(t-s) \notin \mathbf{I}_j, \widehat{H}_j^e \geq t(1 + \varepsilon/2) - s \right) + Q_{\tilde{m}_j}^{W_\kappa} (\widehat{H}_j > t_1) \\ &\leq \tilde{\mu}_j([\tilde{L}_j^-, \tilde{L}_j] \setminus \mathbf{I}_j) + Q_{\tilde{m}_j}^{W_\kappa} (\widehat{H}_j > t_1), \end{aligned} \quad (5.12)$$

and notice that we can replace $Q_{\tilde{m}_j}^{W_\kappa}$ by $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$ in the first line. We now prove a Lemma about $\tilde{\mu}_j$:

Lemma 5.3. *For all $j \leq n_t$, with a probability greater than $1 - C_+ e^{-c-\delta\phi^*(t)}$,*

$$\tilde{\mu}_j([\tilde{L}_j^-, \widehat{L}_j^-]) + \mu_j([\widehat{L}_j^+, \tilde{L}_j]) \leq C_+ h_t e^{-\kappa(1-\delta)\phi^*(t)/16}, \quad (5.13)$$

$$\tilde{\mu}_j([\tilde{L}_j^-, \tilde{L}_j] \setminus \mathbf{I}_j) \leq C_+ h_t e^{-\kappa(1-\delta)\phi^*(t)/16}. \quad (5.14)$$

Notice that for any ζ , the right hand sides of (5.13) and (5.14) go to 0 as $t \rightarrow +\infty$.

Proof: Working on \mathcal{V}_t (and then using Lemma 2.3, Fact 2.2 and Remark 2.4) we only have to prove (5.13) and (5.14) replacing $\tilde{\mu}_j$ by μ defined by

$$\begin{aligned} d\mu(x) &:= \frac{1}{\tilde{Z}} [e^{-R^{(1)}(-x)} \mathbf{1}_{[-\tau R^{(1)}(h_t^+), 0)}(x) + e^{-R^{(2)}(x)} \mathbf{1}_{[0, \tau R^{(2)}(h_t)]}(x) \\ &\quad + e^{-\tilde{W}_\kappa^{h_t}(x - \tau R^{(2)}(h_t))} \mathbf{1}_{[\tau R^{(2)}(h_t), \tau R^{(2)}(h_t) + \tau \tilde{W}_\kappa^{h_t}(h_t/2)]}(x)] dx, \end{aligned}$$

where $\tilde{Z} := \int_{-\tau R^{(1)}(h_t)}^0 e^{-R^{(1)}(-y)} dy + \int_0^{\tau R^{(2)}(h_t^+)} e^{-R^{(2)}(y)} dy + \int_0^{\tau \tilde{W}_\kappa^{h_t}(h_t/2)} e^{-\tilde{W}_\kappa^{h_t}(y)} dy$. Also replace $(\tilde{L}_j^-, \hat{L}_j^-, \tilde{L}_j^-, \tilde{L}_j)$ by $(-\tau R^{(1)}(h_t^+), -\tau R^{(1)}(\phi^*(t)), \tau R^{(2)}(\phi^*(t)), \tau \tilde{W}_\kappa^{h_t}(h_t/2))$, and \mathbf{I}_j by $\mathbf{I} := [-\phi^*(t)/\zeta, \phi^*(t)/\zeta]$.

We now prove simultaneously both inequalities. Let $\mathcal{E}_1^{5.3} := \{\tau R^{(1)}(\kappa\phi^*(t)/8\zeta) \leq \phi^*(t)/\zeta\}$ and $\mathcal{E}_2^{5.3} := \{\tau R^{(2)}(\kappa\phi^*(t)/8\zeta) \leq \phi^*(t)/\zeta\}$. Using formula (3.6) with $h = \kappa\phi^*(t)/8\zeta$, $P(\overline{\mathcal{E}_1^{5.3}}) = P(\overline{\mathcal{E}_2^{5.3}}) \leq C_+ e^{-\kappa^2\phi^*(t)/(16\sqrt{2}\zeta)}$. Also we have

$$\mu(-\tau R^{(1)}(h_t^+), -\tau R^{(1)}(\phi^*(t))) + \mu(\tau R^{(2)}(\phi^*(t)), \tau \tilde{W}_\kappa^{h_t}(h_t/2)) \leq (\mathcal{J}_3(1) + \mathcal{J}_4(1) + \mathcal{J}_5)/\tilde{Z}, \quad (5.15)$$

where for all $x > 0$,

$$\mathcal{J}_3(x) := \int_{\tau R^{(1)}(\kappa\phi^*(t)/8x)}^{\tau R^{(1)}(h_t^+)} e^{-R^{(1)}(y)} dy, \quad \mathcal{J}_4(x) := \int_{\tau R^{(2)}(\kappa\phi^*(t)/8x)}^{\tau R^{(2)}(h_t)} e^{-R^{(2)}(y)} dy,$$

$$\mathcal{J}_5 := \int_0^{\tau \tilde{W}_\kappa^{h_t}(h_t/2)} e^{-\tilde{W}_\kappa^{h_t}(y)} dy.$$

Moreover on $\mathcal{E}_1^{5.3} \cap \mathcal{E}_2^{5.3}$,

$$\mu([-\tau R^{(1)}(h_t^+), \tau \tilde{W}_\kappa^{h_t}(h_t/2)] \setminus \mathbf{I}) \leq (\mathcal{J}_3(\zeta) + \mathcal{J}_4(\zeta) + \mathcal{J}_5)/\tilde{Z}. \quad (5.16)$$

We now estimate $\mathcal{J}_3(1)$ and $\mathcal{J}_4(1)$ similarly. We introduce $\mathcal{E}_3^{5.3} := \{\inf\{R^{(1)}(s), \tau R^{(1)}(\kappa\phi^*(t)/8) \leq s \leq \tau R^{(1)}(h_t^+)\} > (1 - \delta)\kappa\phi^*(t)/8\}$ and $\mathcal{E}_4^{5.3} := \{\tau R^{(1)}(h_t^+) \leq 8h_t^+/\kappa\}$. Using (3.7) with $h = \kappa\phi^*(t)/8$, $\gamma = 1 - \delta$ and $\alpha = 1$ and $\omega = h_t^+/h$ for t large enough we have $P(\overline{\mathcal{E}_3^{5.3}}) \leq 2e^{-\delta\kappa^2\phi^*(t)/8}$, and by (3.6), $P(\overline{\mathcal{E}_4^{5.3}}) \leq C_+ e^{-\kappa h_t^+/2\sqrt{2}}$. On $\mathcal{E}_3^{5.3} \cap \mathcal{E}_4^{5.3}$, we have

$$\mathcal{J}_3(1) \leq e^{-(1-\delta)\kappa\phi^*(t)/8} \tau R^{(1)}(h_t^+) \leq C_+ h_t^+ e^{-(1-\delta)\kappa\phi^*(t)/8} = o(1),$$

as $\log \log t = o(\phi^*(t))$. With the same method, replacing h_t^+ by h_t and $R^{(1)}$ by $R^{(2)}$, we get the same inequality for $\mathcal{J}_4(1)$ on an event $\mathcal{E}_5^{5.3}$ satisfying $\mathbb{P}(\overline{\mathcal{E}_5^{5.3}}) \leq C_+ e^{-c-\delta\phi^*(t)}$.

Let $\mathcal{E}_6^{5.3} := \{\tau \tilde{W}_\kappa^{h_t}(h_t/2) \leq 2h_t/\kappa\}$. Equation (2.7) gives $P(\overline{\mathcal{E}_6^{5.3}}) = P(\tau W_\kappa(-h_t/2) \geq 2h_t/\kappa) \leq e^{-\kappa h_t/16}$. We have on $\mathcal{E}_6^{5.3}$,

$$\mathcal{J}_5 \leq e^{-h_t/2} \tau \tilde{W}_\kappa^{h_t}(h_t/2) \leq C_+ h_t e^{-h_t/2}. \quad (5.17)$$

Define $\mathcal{E}_7^{5.3} := \{\tau R^{(1)}[\alpha\phi^*(t)] - \tau R^{(1)}[\alpha\phi^*(t)/2] \geq 1\}$ with $\alpha := (1 - \delta)\kappa/16$. By (3.3) $P(\overline{\mathcal{E}_7^{5.3}}) \leq C_+ e^{-c-\alpha^2\phi^*(t)^2/4}$. On $\mathcal{E}_7^{5.3}$,

$$\tilde{Z} \geq \int_{\tau R^{(1)}[\alpha\phi^*(t)/2]}^{\tau R^{(1)}[\alpha\phi^*(t)]} e^{-R^{(1)}(y)} dy \geq e^{-\alpha\phi^*(t)} (\tau R^{(1)}[\alpha\phi^*(t)] - \tau R^{(1)}[\alpha\phi^*(t)/2]) \geq e^{-\alpha\phi^*(t)}. \quad (5.18)$$

We get on $\cap_{i=1}^7 \mathcal{E}_i^{5.3}$ for large t since $\phi^*(t) = o(h_t)$,

$$\begin{aligned} (\mathcal{J}_3(1) + \mathcal{J}_4(1) + \mathcal{J}_5)/\tilde{Z} &\leq C_+[2h_t^+ e^{-(1-\delta)\kappa\phi^*(t)/8} + h_t e^{-h_t/2}]e^{\alpha\phi^*(t)} \\ &\leq C_+ h_t e^{-\alpha\phi^*(t)/2}. \end{aligned} \quad (5.19)$$

Since $\mathbb{P}(\cap_{i=1}^7 \mathcal{E}_i^{5.3}) \geq 1 - C_+ e^{-c-\delta\phi^*(t)}$, this together with (5.15) and Lemma 2.3 proves (5.13). Since $\zeta < 1$, replacing $\phi^*(t)$ by $\phi^*(t)/\zeta$ in this proof also gives (5.19) with $\mathcal{J}_3(\zeta)$ and $\mathcal{J}_4(\zeta)$ instead of $\mathcal{J}_3(1)$ and $\mathcal{J}_4(1)$, which together with (5.16) and Lemma 2.3 proves (5.14). \square

Finally collecting (5.7) to (5.14), we obtain (5.6), which ends the proof of this Part 1.

Part 2 : we prove that there exists $c_5 > 0$ such that if $\zeta \leq \kappa/(16(1+2\delta))$,

$$\mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_t}, \tilde{N}_t = \tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \overline{\mathcal{A}_3}\right) \leq C_+ n_t e^{-c_5\phi^*(t)/\zeta}. \quad (5.20)$$

First, we prove similarly as in Part 1 that

$$\begin{aligned} &\mathbb{P}\left(X(t) \notin \mathbf{I}_{\tilde{N}_t}, \tilde{N}_t = \tilde{N}_{t(1+\varepsilon)}, \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \overline{\mathcal{A}_3}\right) \\ &\leq \sum_{j=1}^{n_t-1} E\left(\sup_{t^* \leq s \leq t} \mathbb{P}_{\tilde{m}_j}^{W_\kappa}\left(X(t-s) \notin \mathbf{I}_j, \forall u \in [0, t(1+\varepsilon/2) - s], X(u) \in [\tilde{L}_j^-, \tilde{L}_j]\right)\right) + o(1) \\ &\leq \sum_{j=1}^{n_t-1} E\left(\sup_{0 \leq s \leq e^{\phi^*(t)(1+2\delta)}} \mathbb{P}_{\tilde{m}_j}^{W_\kappa}(H(\tilde{m}_j - \phi^*(t)/\zeta) \wedge H(\tilde{m}_j + \phi^*(t)/\zeta) \leq s)\right) + o(1). \end{aligned} \quad (5.21)$$

Notice that under $\mathbb{P}_{\tilde{m}_j}^{W_\kappa}$ on $\{\tilde{\tau}_j^+[\kappa\phi^*(t)/(8\zeta)] - \tilde{m}_j \leq \phi^*(t)/\zeta\} \cap \{\tilde{m}_j - \tilde{\tau}_j^-[\kappa\phi^*(t)/(8\zeta)] \leq \phi^*(t)/\zeta\}$, which has probability greater than $1 - C_+ e^{-\kappa^2\phi^*(t)/(16\zeta\sqrt{2})}$ by (2.4) and Lemma 2.3, we have $H(\tilde{m}_j - \phi^*(t)/\zeta) \geq H(\tilde{\tau}_j^-[\kappa\phi^*(t)/(8\zeta)])$, and so there exists a Brownian motion \tilde{B} , independent of W_κ , such that

$$H\left(\tilde{m}_j - \frac{\phi^*(t)}{\zeta}\right) \geq \int_{\tilde{\tau}_j^-[\kappa\phi^*(t)/(8\zeta)]}^{\tilde{m}_j} e^{-V^{(j)}(u)} \mathcal{L}_{\tilde{B}}\left[\tau^{\tilde{B}}\left(\tilde{A}_j\left(\tilde{\tau}_j^- \left[\frac{\kappa\phi^*(t)}{8\zeta}\right]\right)\right), \tilde{A}_j(u)\right] du := H_j^-, \quad (5.23)$$

$$H\left(\tilde{m}_j + \frac{\phi^*(t)}{\zeta}\right) \geq \int_{\tilde{m}_j}^{\tilde{\tau}_j^+[\kappa\phi^*(t)/(8\zeta)]} e^{-V^{(j)}(u)} \mathcal{L}_{\tilde{B}}\left[\tau^{\tilde{B}}\left(\tilde{A}_j\left(\tilde{\tau}_j^+ \left[\frac{\kappa\phi^*(t)}{8\zeta}\right]\right)\right), \tilde{A}_j(u)\right] du := H_j^+, \quad (5.24)$$

where for all $\tilde{L}_j^- \leq z \leq \tilde{L}_j$, $\tilde{A}_j(z) := \int_{\tilde{m}_j}^z e^{\tilde{V}^{(j)}(x)} dx$. So we get, recalling that $t - t^* = e^{\phi^*(t)(1+2\delta)}$,

$$(5.21) \leq n_t \max_{1 \leq j \leq n_t-1} \left[\mathbb{P}\left(H_j^- \leq e^{\phi^*(t)(1+2\delta)}\right) + \mathbb{P}\left(H_j^+ \leq e^{\phi^*(t)(1+2\delta)}\right) + C_+ e^{-\kappa^2\phi^*(t)/(16\zeta\sqrt{2})} \right]. \quad (5.25)$$

Lemma 5.4. $\mathbb{P}\left(H_j^\pm \leq e^{\phi^*(t)(1+2\delta)}\right) \leq e^{-c-\phi^*(t)/\zeta}$ for large t if $\zeta \leq \kappa/(16(1+2\delta))$.

Proof: First, let $\varepsilon > 0$, define $\delta_t^* := e^{-(1-3\varepsilon)\phi^*(t)/(2\zeta)}$, $x_j^* := \tilde{A}_j^{-1}(\delta_t^* \tilde{\tau}_j^+[\kappa\phi^*(t)/(8\zeta)]) - \tilde{m}_j$, and notice that as in (4.13),

$$H_j^+ \geq \tilde{A}_j(\tilde{\tau}_j^+[\kappa\phi^*(t)/(8\zeta)]) \left(\int_0^{x_j^*} e^{-\tilde{V}^{(j)}(x+\tilde{m}_j)} dx \right) \inf_{u \in [0, \delta_t^*]} \mathcal{L}_{\tilde{B}}(1, u) =: \mathcal{J}_6 \mathcal{J}_7 \mathcal{J}_8.$$

Now, for large t , we have as before by (4.6), $\mathcal{J}_8 \geq [1 - (\delta_t^*)^{1/3}] \mathbf{e}_1 \geq \mathbf{e}_1/2$ with probability $\geq 1 - (\delta_t^*)^{1/30}$, with $\mathbf{e}_1 = \mathcal{L}_{\tilde{B}}(1, 0) \geq e^{-6^{-1}\kappa\phi^*(t)/(8\zeta)}$ with probability $\geq 1 - e^{-6^{-1}\kappa\phi^*(t)/(8\zeta)}$. Moreover, by Lemma 2.3, William's decomposition and (3.5), $\mathcal{J}_6 \geq \exp[(1-\varepsilon)\kappa\phi^*(t)/(8\zeta)]$ with probability $\geq 1 - 3e^{-\kappa^2\varepsilon\phi^*(t)/(16\zeta)}$. Finally, as in (4.15), $\tilde{A}_j(\tilde{\tau}_j^+[2^{-1}\kappa\phi^*(t)/(8\zeta)]) \leq e^{2^{-1}(1+\varepsilon)\kappa\phi^*(t)/(8\zeta)}$ with

probability $\geq 1 - C_+ e^{-\varepsilon \kappa \phi^*(t)/(16\zeta)}$, which gives $x_j^* \geq \tilde{\tau}_j^+ [2^{-1} \kappa \phi^*(t)/(8\zeta)]$ as in (4.16), and then $\mathcal{J}_7 \geq \tilde{\tau}_j^+ (6^{-1} \kappa \phi^*(t)/(8\zeta)) e^{-6^{-1} \kappa \phi^*(t)/(8\zeta)} \geq e^{-6^{-1} \kappa \phi^*(t)/(8\zeta)}$ with probability $\geq 1 - e^{-\phi^*(t)/\zeta}$ by (3.3) and Lemma 2.3. Taking $\varepsilon = 1/6$, this gives $\mathbb{P}\left(H_j^+ \geq e^{\kappa \phi^*(t)/(16\zeta)}\right) \geq 1 - e^{-\phi^*(t)/\zeta}$. This proves the lemma for H_j^+ . The proof for H_j^- is similar. \square

This leads to (5.20), which ends this Part 2.

Part 3: We prove that $\mathbb{P}\left(\tilde{N}_t \neq \tilde{N}_{t(1+\varepsilon)}\right) + \mathbb{P}(\overline{\mathcal{A}}_0) + \mathbb{P}(\overline{\mathcal{A}}_1) + \mathbb{P}(\overline{\mathcal{A}}_2) \leq \varepsilon^{1-\kappa} + o(1)$. First, with Lemma 2.3 and then (1.3), $\mathbb{P}\left(\tilde{N}_t \neq \tilde{N}_{t(1+\varepsilon)}\right) = \mathbb{P}[H(m_{N_{t+1}}) < t(1+\varepsilon)] + o(1) \leq \varepsilon^{1-\kappa} + o(1)$. Moreover, we have $\mathbb{P}(\overline{\mathcal{A}}_0) = o(1)$ by (5.3) and Lemma 2.3. The fact that $\mathbb{P}(\overline{\mathcal{A}}_1) = o(1)$ comes from Lemma 3.5. Moreover, by Markov property,

$$\begin{aligned} \mathbb{P}_{\tilde{m}_j}^{W_\kappa}\left(H(\tilde{m}_{j+1}) < H(\tilde{L}_j^-)\right) &= \mathbb{P}_{\tilde{m}_j}^{W_\kappa}\left(H(\tilde{L}_j) < H(\tilde{L}_j^-)\right) \times \mathbb{P}_{\tilde{L}_j}^{W_\kappa}\left(H(\tilde{m}_{j+1}) < H(\tilde{L}_j^-)\right) \\ &\geq \mathbb{P}_{\tilde{m}_j}^{W_\kappa}\left(H(\tilde{L}_j) < H(\tilde{L}_j^-)\right) \times \mathbb{P}_{\tilde{L}_j}^{W_\kappa}\left(H(\tilde{m}_{j+1}) < H(\tilde{L}_j^*)\right). \end{aligned}$$

Both probabilities of the last line have already been estimated, in (3.12) with probability $1 - o(1/n_t)$ for the first one, and in Lemma 3.5 for the second one (since $Q_i^*/D_i^* \leq 2e^{-h_t/8}$ with probability $1 - o(1/n_t)$, as proved between (3.23) and (3.25)). This proves that $\mathbb{P}(\overline{\mathcal{A}}_2) = o(1)$.

Finally combining (5.6), (5.20) with Part 3, choosing ζ small enough, using Lemma 2.3 back again to transform the \tilde{m} in m , letting t goes to infinity and ε to zero, we get Theorem 1.3.

5.3. The aging : Proof of Proposition 1.2. We fix $\alpha > 1$. We recall that the r.v. $(m_i)_i$ depend on t . In what follows, we apply Theorem 1.3 first at time t with function ϕ , and second at time αt with a function ϕ_α defined by $\log(\alpha t) - \phi_\alpha(\alpha t) = \log t - \phi(t)$, so that the r.v. m_i are the same in both cases. Let $0 < \zeta < 1$. By Theorem 1.3 and since $\mathbb{P}(\mathcal{A}_0) \rightarrow_{t \rightarrow +\infty} 1$, we get

$$\begin{aligned} &\mathbb{P}(|X(\alpha t) - X(t)| \leq \mathcal{C}_1 \phi(t)/\zeta, N_t < N_{\alpha t}) \\ &\leq \mathbb{P}(|m_{N_{\alpha t}} - m_{N_t}| \leq \mathcal{C}_1(1+1/\zeta)\phi(t) + \mathcal{C}_1 \phi_\alpha(\alpha t), 1 \leq N_t < N_{\alpha t} \leq n_{\alpha t}, N_t \leq n_t) + o(1) \\ &\leq \sum_{i=1}^{n_t} \sum_{j=i+1}^{n_{\alpha t}} \mathbb{P}(|m_i - m_j| \leq \mathcal{C}_1(1+1/\zeta)\phi(t) + \mathcal{C}_1 \phi_\alpha(\alpha t)) + o(1), \end{aligned}$$

as $t \rightarrow +\infty$. So, using Lemma 2.3 and (2.3), we get $\mathbb{P}(|X(\alpha t) - X(t)| \leq \mathcal{C}_1 \phi(t), N_t < N_{\alpha t}) = o(1)$. Consequently,

$$\begin{aligned} \mathbb{P}(|X(\alpha t) - X(t)| \leq \mathcal{C}_1 \phi(t)/\zeta) &= \mathbb{P}(|X(\alpha t) - X(t)| \leq \mathcal{C}_1 \phi(t)/\zeta, N_t = N_{\alpha t}) + o(1) \\ &\leq \mathbb{P}(N_t = N_{\alpha t}) + o(1). \end{aligned} \tag{5.26}$$

Moreover, by Theorem 1.3, we have for large t ,

$$\begin{aligned} &\mathbb{P}(|X(\alpha t) - X(t)| > \mathcal{C}_1 \phi(t)/\zeta, N_t = N_{\alpha t}) \\ &\leq \mathbb{P}(|X(\alpha t) - m_{N_t}| > \mathcal{C}_1 \phi(t)/\zeta - \mathcal{C}_1 \phi(t), N_t = N_{\alpha t}) + o(1) \\ &\leq \mathbb{P}(|X(\alpha t) - m_{N_{\alpha t}}| > \mathcal{C}_1 \phi(t)(1/\zeta - 1)) + o(1) \\ &\leq \mathbb{P}(|X(\alpha t) - m_{N_{\alpha t}}| > \mathcal{C}_1 \phi_\alpha(\alpha t)) + o(1). \end{aligned}$$

for large t if we choose $\zeta = 1/3$. Hence $\mathbb{P}(|X(\alpha t) - X(t)| > \mathcal{C}_2 \phi(t), N_t = N_{\alpha t}) = o(1)$ by Theorem 1.3 applied at time αt and $\mathcal{C}_2 := 3\mathcal{C}_1$. Therefore,

$$\mathbb{P}(N_t = N_{\alpha t}) = \mathbb{P}(|X(\alpha t) - X(t)| \leq \mathcal{C}_2 \phi(t), N_t = N_{\alpha t}) + o(1) \leq \mathbb{P}(|X(\alpha t) - X(t)| \leq \mathcal{C}_2 \phi(t)) + o(1).$$

This together with (5.26) gives $\mathbb{P}(|X(\alpha t) - X(t)| \leq C_2\phi(t)) = \mathbb{P}(N_t = N_{\alpha t}) + o(1) = \mathbb{P}(H(m_{N_{t+1}}) > \alpha t) + o(1)$. This, combined with (1.3), the change of variables $u = 1/(1+x)$ proves Proposition 1.2, since ϕ is chosen up to a multiplicative constant. \square

Acknowledgment : We would like to thank Dominique Lepingle for pointing out the reference [30].

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