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Claude Daviau, Jacques Bertrand

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New insights in the standard model of quantum physics in Clifford Algebra

Claude Daviau
Le Moulin de la Lande
44522 Pouillé-les-coteaux
France
email: claudio.daviau@nordnet.fr

Jacques Bertrand
15 avenue Danielle Casanova
95210 Saint-Gratien
France
email: bertrandjacques-m@orange.fr

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Introduction

To see from where comes the standard model that rules today quantum physics, we first return to its beginning. When the idea of a wave associated to the move of a particle was found, Louis de Broglie was following consequences of the restricted relativity [25]. The first wave equation found by Schrödinger [50] was not relativistic, and could not be the true wave equation. In the same time the spin of the electron was discovered. This remains the main change from pre-quantum physics, since the spin 1/2 has none classical equivalent. Pauli gave a wave equation for a non-relativistic equation with spin. This equation was the starting point of the attempt made by Dirac [30] to get a relativistic wave equation for the electron. This Dirac equation was a very great success. Until now it is still considered as the wave equation for each particle with spin 1/2, electrons but also positrons, muons and anti-muons, neutrinos, quarks.

This wave equation was intensively studied by Louis de Broglie and his students. He published a first book in 1934 [26] explaining how this equation gives in the case of the hydrogen atom the quantification of energy levels, awaited quantum numbers, the true number of quantum states, the true energy levels and the Landé factors. The main novelty in physics coming with the Dirac theory is the fact that the wave has not vector or tensor properties under a Lorentz rotation, the wave is a spinor and transforms very differently. It results from this transformation that the Dirac equation is form invariant under Lorentz rotations. This form invariance is the departure of our study and is the central thread of this book.

The Dirac equation was built from the Pauli equation and is based on 4×4 complex matrices, which were constructed from the Pauli matrices. Many years after this first construction, D. Hestenes [32] used the Clifford algebra of space-time to get a different form of the same wave equation. Tensors which are constructed from the Dirac spinors appear differently and the relations between these tensors are more easily obtained.

One of the parameters of the Dirac wave, the Yvon-Takabayasi angle [51], was completely different from all classical physics. G. Lochak understood that this angle allows a second gauge invariance and he found a wave equation for a magnetic monopole from this second gauge invariance [40]. He showed that a wave equation with a nonlinear mass term was possible for his magnetic monopole. When this mass term is null, the wave is made of two independent Weyl spinors.

This mass term is compatible with the electric gauge ruling the Dirac equation. So it can replace the linear term of the Dirac equation of the electron [7]. A nonlinear wave equation for the electron was awaited by de Broglie, because it was necessary to link the particle to the wave. But this does not explain how to choose the nonlinearity. And the nonlinearity is a formidable problem in quantum physics: quantum theory is a linear theory, it is by solving the linear wave equation that the quantification of energy levels and quantum numbers are obtained in the hydrogen atom. If you start from a nonlinear wave equation, usually you will not be even able to get quantification and quantum numbers.

Nevertheless the study of this nonlinear wave equation began in the case where the Dirac equation is its linear approximation. In this case the wave equation is homogeneous. It is obtained from a Lagrangian density which differs from the Lagrangian of the linear theory only by the mass term. Therefore many results are similar. For instance the dynamics of the electron are the same, and the electron follows the Lorentz force.

Two formalisms were available, the Dirac formalism with 4×4 complex matrices, and the real Clifford algebra of space-time. A matrix representation links these formalisms. Since the hydrogen case gave the main result, a first attempt was made to solve the nonlinear equation in this case. Heinz Krüger gave a precious tool [37] by finding a way to separate the spherical coordinates. Moreover the beginning of this resolution by separation of variables was the same in the case of the linear Dirac equation and in the case of the nonlinear homogeneous equation. But then there was a great difficulty: The Yvon-Takabayasi angle is null in the $x^3 = 0$ plane; This angle is a complicated function of an angular variable and of the radial variable; Moreover for any solution with a not constant radial polynomial, circles exist where the Yvon-Takabayasi angle is not defined; In the vicinity of these circles this angle is not small and the solutions of the Dirac equation have no reason to be linear approximations of solutions of the nonlinear homogeneous equation. Finally it was possible to compute [8] another orthonormal set of solutions of the Dirac equation, which have everywhere a well defined and small Yvon-Takabayasi angle. These solutions are linear approximations of the solutions of the nonlinear equation. The existence of this set of orthonormalized solutions is a powerful argument for our nonlinear wave equation.

When you have two formalisms for the same theory the question necessarily comes: which one is the best formalism? Comparing advantages of these formalisms, the possibility of a third formalism which could be the true one aroused. A third formalism is really available [10] to read the Dirac theory: it is the Clifford algebra of the physical space used by W.E. Baylis [2]. This Clifford algebra is isomorphic, as real algebra, to the matrix algebra generated by Pauli matrices. Quantum physics knew very early this formalism, since these Pauli matrices were invented to get the first wave equation with spin 1/2. Until now this formalism is also used to get the form invariance of the Dirac equation. Having then three formalisms for the same theory, the question was, once more: which is the true one?

The criterion of the best choice was necessarily the Lorentz invariance of

the wave equation. Therefore a complete study, from the start, of this form invariance of the Dirac theory was made [12]. This problem was a classical one, treated by many books, but always with mathematical flaws. The reason is that two different Lie groups may have the same Lie algebra. The Lie algebra of a Lie group is the algebra generated by infinitesimal operators of the Lie group. Quantum mechanics uses only these infinitesimal operators and it is then very difficult to avoid ambiguities. But it is possible to avoid any infinitesimal operator. And when you work without them you can easily see that the fundamental invariance group is larger than expected.

The first consequence of this larger group is the possibility to define, from the Dirac wave, Lorentz dilations from an intrinsic space-time manifold to the usual relativistic space-time. So the space-time is double and the Dirac wave is the link between these two manifolds. They are very different, the intrinsic manifold is not isotropic and has a torsion.

In several articles and in two previous books [16][17], were presented several consequences of this larger invariance group. This invariance group governs not only the Dirac theory, but also all the electromagnetism, with or without magnetic monopoles, with or without photons. But it is not all of the thing, since this form invariance is also the rule for electro-weak and strong interactions [18].

Because it is impossible to read this article without knowledge of the Cl_3 algebra a first section presents Clifford algebras at an elementary level.

Section 2 reviews the Dirac equation, firstly with Dirac matrices, where we get a mathematical correct form of the relativistic invariance of the theory. This necessitates the use of the space algebra Cl_3 . Next we explain the form of the Dirac equation in this simple frame, we review the relativistic form invariance of the Dirac wave. We explain with the tensors without derivative how the classical matrix formalism is deficient. We review plane waves. We present the invariant form of the wave equation. Its scalar part is the Lagrangian density, another true novelty allowed by Clifford algebra. Finally we present the charge conjugation in this frame.

Section 3 introduces our homogeneous nonlinear equation and explains why this equation is better than the Dirac equation which is its linear approximation. We review its two gauge invariances. We explain why plane waves have only positive energy. The form of the spinorial wave and the form of its relativistic invariance introduce the dilation generated by the wave from an intrinsic space-time manifold onto the usual relative space-time manifold, main geometric novelty of quantum physics. We explain the physical reason to normalize the wave. The link between the wave of the particle and the wave of the antiparticle coming from relativistic quantum mechanics gives a charge conjugation where only the differential term of the wave equation changes sign. This makes the CPT theorem trivial and it is also a powerful argument for this wave equation. We get the quantification of the energy in the case of the hydrogen atom and all results of the linear theory with this homogeneous nonlinear wave equation.

Section 4 presents the invariance of electromagnetism under Cl_3^* , the group of the invertible elements in Cl_3 , for the Maxwell-de Broglie electromagnetism

with massive photons, for the electromagnetism with magnetic monopoles, for the four photons of de Broglie-Lochak.

Section 5 presents some first consequences of these novelties. The anisotropy of the intrinsic space-time explains why we see muons and tauons beside electrons, their similarities and differences. The intrinsic manifold has a torsion whose components were calculated for plane waves. The mass term is linked to this torsion. Next we present the building of the de Broglie's wave of a system of electrons as a wave in the ordinary space-time, and not in a configuration space where space and time do not have the same status, with value onto the space algebra. We present also as a counter-example a wave equation [20] without Lagrangian formalism, we solve this wave equation in the hydrogen case.

Section 6 is devoted to our main progress since [17]. We present the electro-weak gauge theory in the frame of the space-time algebra, first for the lepton case, secondly for the quark case. Next we use the $Cl_{5,1}$ Clifford algebra to extend the gauge to strong interactions. Even if our aim is the same as in [18], we use here a different Clifford algebra, because we need the link between the wave of the particle and the wave of the antiparticle that is used in the standard model of electro-weak and strong interactions. We get a $U(1) \times SU(2) \times SU(3)$ gauge group in this frame. The addition from the standard model is the comprehension of the insensitivity of leptons to strong interactions. We extend to the $Cl_{5,1}$ frame the form invariance of the gauge interactions. This induces the use of a complex 6-dimensional space-time into which the usual 4-dimensional space-time is well separated from supplementary dimensions. Finally the nullity of right waves of the neutrino and of quarks induces two remarkable identities. They imply that waves of leptons and the full wave of the lepton and of three colored quarks have an invertible value. These identities allow a wave equation with a mass term. We propose a wave equation with a mass term for a pair electron-neutrino. This wave equation gives both the homogeneous non-linear equation studied in section 3 and the electro-weak gauge invariance studied in section 6. It is well-known that the standard model has a great difficulty with the mass of the electron: the mass term of the Dirac equation links left wave to right wave. In the electro-weak interactions these left and right waves act differently. Therefore the standard model firstly cancels the mass of the electron, finally it must put back this mass from a very complicated mechanism of spontaneously broken symmetry. We get here a wave equation with a mass term, both form invariant (and consequently relativistic invariant) and gauge invariant under the $U(1) \times SU(2)$ gauge group of electro-weak interactions. This makes the standard model much stronger.

Section 7 is devoted to magnetic monopoles. We explain Russian experiments and our french experiments. We precise their results, particularly the wavelength. We use our study of electro-weak interactions in the case of the magnetic monopole.

Section 8 presents our conclusions about the major change explained here in our way to see the standard model of our physical universe.

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1 Clifford algebras

This section presents what is a Clifford algebra, then we study the algebra of an Euclidean plane and the algebra of the three-dimensional physical space which is also the algebra of the Pauli matrices. We put there the space-time and the relativistic invariance. Then we present the space-time algebra and the Dirac matrices. We finally present the Clifford algebra of a 6-dimensional space-time, needed by electro-weak and strong interactions.

It is quite usual in a physics article to put into appendices mathematics even if they are necessary to understand the main part of the article. As it is impossible to expose the part containing physics without the Clifford algebras, we make here again a complete presentation of this necessary tool.¹

We shall only speak here about Clifford algebras on the real field. Algebras on the complex field also exist and it could be expected to complex algebras to be key point for quantum physics. The main algebra used here is also an algebra on the complex field, but it is its structure of real algebra which is useful in this frame.²

Our aim is not to say everything about any Clifford algebra but simply to give to our lecturer tools to understand the next sections of this article. Another main book on this subject was written by Doran and Lasenby [39]. It is more devoted to space-time algebra. It is probably less easy for physicists, because it does not use complex matrices.

1.1 What is a Clifford algebra?

1 - It is an algebra [5][16], there are two operations, noted $A + B$ and AB , such as, for any A, B, C :

$$\begin{aligned} A + (B + C) &= (A + B) + C \quad ; \quad A + B = B + A \\ A + 0 &= A \quad ; \quad A + (-A) = 0 \\ A(B + C) &= AB + AC \quad ; \quad (A + B)C = AC + BC \\ A(BC) &= (AB)C. \end{aligned} \tag{1.1}$$

2 - The algebra contains a set of vectors, noted with arrows, in which a scalar product exists and the intern Clifford multiplication $\vec{u}\vec{v}$ is supposed to satisfy for any vector \vec{u} :

$$\vec{u}\vec{u} = \vec{u} \cdot \vec{u}. \tag{1.2}$$

¹Readers being in the know may do a quick review. On the contrary a complete lecture is strongly advised for each reader who really wants to understand physics contained in the following sections.

²A Clifford algebra on the real field has components of vectors which are real numbers and which cannot be multiplied by i . A Clifford algebra on the complex field has components of vectors which are complex numbers and which can be multiplied by i .

where $\vec{u} \cdot \vec{v}$ is the usual notation for the scalar product of two vectors.³ This implies, since $\vec{u} \cdot \vec{u}$ is a real number, that the algebra contains vectors but also real numbers.

3 - Real numbers are commuting with any member of the algebra: if a is a real number and if A is any element in the algebra :

$$aA = Aa \quad (1.3)$$

$$1A = A. \quad (1.4)$$

Such an algebra exists for any finite-dimensional linear space which are the ones that we need in this article.

The smaller one is single, to within an isomorphism.

Remark 1: (1.1) and (1.4) imply that the algebra is itself a linear space, not to be confused with the first one. If the initial linear space is n -dimensional, we get a Clifford algebra which is 2^n -dimensional. We shall see for instance in 1.3 that the Clifford algebra of the 3-dimensional physical space is a 8-dimensional linear space on the real field. We do not need here to distinguish the left or right linear space, since real numbers commute with each element of the algebra. We also do not need to consider the multiplication by a real number as a third operation, because it is a particular case of the multiplication.

Remark 2: If \vec{u} and \vec{v} are two orthogonal vectors, ($\vec{u} \cdot \vec{v} = 0$), the equality $(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = (\vec{u} + \vec{v})(\vec{u} + \vec{v})$ implies $\vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} = \vec{u}\vec{u} + \vec{u}\vec{v} + \vec{v}\vec{u} + \vec{v}\vec{v}$, so we get :

$$0 = \vec{u}\vec{v} + \vec{v}\vec{u} ; \quad \vec{v}\vec{u} = -\vec{u}\vec{v} \quad (1.5)$$

It's **the** change to usual rules on numbers, the multiplication is not commutative, we must be as careful as with matrix calculations.

Remark 3: The addition is defined in the whole algebra, which contains numbers and vectors. So we shall get sums of numbers and vectors: $3 + 5\vec{i}$ is **authorized**. It is perhaps strange or disturbing, but we shall see next it is not different from $3 + 5i$. And everyone using complex numbers finally gets used to.

Even sub-algebra: It's the sub-algebra generated by the products of an even number of vectors: $\vec{u}\vec{v}$, $\vec{e}_1\vec{e}_2\vec{e}_3\vec{e}_4, \dots$

Reversion: The reversion $A \mapsto \widetilde{A}$ changes orders of products. Reversion does not change numbers a nor vectors $\widetilde{a} = a$, $\widetilde{\vec{u}} = \vec{u}$, and we get, for any \vec{u} and \vec{v} , A and B :

$$\widetilde{\vec{u}\vec{v}} = \vec{v}\vec{u} ; \quad \widetilde{AB} = \widetilde{B}\widetilde{A} ; \quad \widetilde{A+B} = \widetilde{A} + \widetilde{B}. \quad (1.6)$$

1.2 Clifford algebra of an Euclidean plan: Cl_2

Cl_2 contains the real numbers and the vectors of an Euclidean plan, which read $\vec{u} = x\vec{e}_1 + y\vec{e}_2$, where \vec{e}_1 and \vec{e}_2 form an orthonormal basis of the plan: $\vec{e}_1^2 = \vec{e}_2^2 = 1$, $\vec{e}_1 \cdot \vec{e}_2 = 0$. Usually we set: $\vec{e}_1\vec{e}_2 = i$. The general element of the algebra is :

$$A = a + x\vec{e}_1 + y\vec{e}_2 + ib \quad (1.7)$$

³This equality seems strange, but gives nice properties. We need these properties in the next sections.

where a , x , y and b are real numbers. This is enough because :

$$\begin{aligned}\vec{e}_1 i &= \vec{e}_1 (\vec{e}_1 \vec{e}_2) = (\vec{e}_1 \vec{e}_1) \vec{e}_2 = 1 \vec{e}_2 = \vec{e}_2 \\ \vec{e}_2 i &= -\vec{e}_1 \quad ; \quad i \vec{e}_2 = \vec{e}_1 \quad ; \quad i \vec{e}_1 = -\vec{e}_2 \\ i^2 &= i i = i (\vec{e}_1 \vec{e}_2) = (i \vec{e}_1) \vec{e}_2 = -\vec{e}_2 \vec{e}_2 = -1\end{aligned}\tag{1.8}$$

Two remarks must be made:

1- The even sub-algebra Cl_2^+ is the set formed by all $a + ib$, so it is the complex field. We may say that complex numbers are underlying as soon as the dimension of the linear space is greater than one. This even sub-algebra is commutative.

2 - The reversion is here the usual conjugation: $\tilde{i} = \widetilde{\vec{e}_1 \vec{e}_2} = \vec{e}_2 \vec{e}_1 = -i$

We get then, for any \vec{u} and any \vec{v} in the plane: $\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + i \det(\vec{u}, \vec{v})$ where $\det(\vec{u}, \vec{v})$ is the determinant.

To establish that $(\vec{u} \cdot \vec{v})^2 + [\det(\vec{u}, \vec{v})]^2 = \vec{u}^2 \vec{v}^2$, it is possible to use $\vec{u}\vec{v}\vec{v}\vec{u}$ which can be calculated by two ways, and we can use $\vec{v}\vec{v}$ which is a real number and commutes with anything in the algebra.

1.3 Clifford algebra of the physical space: Cl_3

Cl_3 contains [2] the real numbers and the vectors of the physical space which read : $\vec{u} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$, where x , y and z are real numbers, \vec{e}_1 , \vec{e}_2 and \vec{e}_3 form an orthonormal basis:

$$\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_2 \cdot \vec{e}_3 = \vec{e}_3 \cdot \vec{e}_1 = 0 \quad ; \quad \vec{e}_1^2 = \vec{e}_2^2 = \vec{e}_3^2 = 1.\tag{1.9}$$

We let:

$$i_1 = \vec{e}_2 \vec{e}_3 \quad ; \quad i_2 = \vec{e}_3 \vec{e}_1 \quad ; \quad i_3 = \vec{e}_1 \vec{e}_2 \quad ; \quad i = \vec{e}_1 \vec{e}_2 \vec{e}_3.\tag{1.10}$$

Then we get:

$$i_1^2 = i_2^2 = i_3^2 = i^2 = -1\tag{1.11}$$

$$i\vec{u} = \vec{u}i \quad ; \quad i\vec{e}_j = i_j \quad , \quad j = 1, 2, 3.\tag{1.12}$$

To satisfy (1.11) we can use the same way we used to get (1.8). To satisfy (1.12) we may firstly justify that i commutes with each \vec{e}_j .

The general element of Cl_3 reads: $A = a + \vec{u} + i\vec{v} + ib$. This gives $1+3+3+1 = 8 = 2^3$ dimensions for Cl_3 .

Several remarks:

1 - The center of Cl_3 is the set of the $a + ib$, only elements which commute with every other ones in the algebra. It is isomorphic to the complex field.

2 - The even sub-algebra Cl_3^+ is the set of the $a + i\vec{v}$, isomorphic to the quaternion field. Therefore quaternions are implicitly present into calculations as soon as the dimension of the linear space is greater or equal to three. This even sub-algebra is not commutative.

3 - $\tilde{A} = a + \vec{u} - i\vec{v} - ib$; The reversion is the conjugation, for complex numbers but also for the quaternions contained into Cl_3 .

4 - $i\vec{v}$ is what is usually called "axial vector" or "pseudo-vector", whilst \vec{u} is usually called vector. It is well known that it is specific to dimension 3.

5 - There are now four different terms with square -1, four ways to get complex numbers. Quantum theory is used to only one term with square -1. When complex numbers are used in quantum mechanics, it is necessary to ask the question of which i is used: i or i_3 ?

1.3.1 Cross-product, orientation

$\vec{u} \times \vec{v}$ is the cross-product of \vec{u} and \vec{v} . Using coordinates in the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, we can easily establish for any \vec{u} and \vec{v} :

$$\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + i \vec{u} \times \vec{v} \quad (1.13)$$

$$(\vec{u} \cdot \vec{v})^2 + (\vec{u} \times \vec{v})^2 = \vec{u}^2 \vec{v}^2 \quad (1.14)$$

$\det(\vec{u}, \vec{v}, \vec{w})$ is the determinant whose columns contain the components of vectors \vec{u} , \vec{v} , \vec{w} , in the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. Again using coordinates, it is possible to establish, for any \vec{u} , \vec{v} , \vec{w} :

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det(\vec{u}, \vec{v}, \vec{w}) \quad (1.15)$$

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{w} \cdot \vec{u})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w} \quad (1.16)$$

$$\vec{u}\vec{v}\vec{w} = i \det(\vec{u}, \vec{v}, \vec{w}) + (\vec{v} \cdot \vec{w})\vec{u} - (\vec{w} \cdot \vec{u})\vec{v} + (\vec{u} \cdot \vec{v})\vec{w} \quad (1.17)$$

From (1.15) comes that $\vec{u} \times \vec{v}$ is orthogonal to \vec{u} and \vec{v} . (1.14) allows to calculate the length of $\vec{u} \times \vec{v}$, and (1.15) gives its orientation. We recall that a basis $(\vec{u}, \vec{v}, \vec{w})$ is said to be direct, or to have the same orientation as $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ if $\det(\vec{u}, \vec{v}, \vec{w}) > 0$ and to be inverse, or to have other orientation if $\det(\vec{u}, \vec{v}, \vec{w}) < 0$. (1.17) allows to establish that, if $B = (\vec{u}, \vec{v}, \vec{w})$ is any orthonormal basis, then $\vec{u}\vec{v}\vec{w} = i$ if and only if B is direct, and $\vec{u}\vec{v}\vec{w} = -i$ if and only if B is inverse. So i is strictly linked to the orientation of the physical space. To change i to $-i$ is equivalent to change the space orientation (it is the same for a plan). The fact that i rules the orientation of the physical space will play an important role in the next sections.

1.3.2 Pauli algebra

The Pauli algebra, introduced in physics as soon as 1926 to account for the spin of the electron, is the algebra of 2×2 complex matrices. It is identical (isomorphic) to Cl_3 , but only as algebras on the real field. ⁴ Identifying complex numbers to scalar matrices and the e_j to the Pauli matrices σ_j is enough. ⁵ So,

⁴The dimension of the Pauli algebra is 8 on the real field, but only 4 on the complex field.

⁵This identifying process may be considered a lack of rigor, but in fact it is frequent in mathematics. The same process allows to include integer numbers into relative numbers, or real numbers into complex numbers. To go without implies very complicated notations. This identifying process considers the three σ_j as forming a direct basis of the physical space.

z being any complex number, we have

$$z = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \quad (1.18)$$

$$\vec{e}_1 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \vec{e}_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \vec{e}_3 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1.19)$$

This is fully compatible with all preceding calculations, because:

$$\sigma_1 \sigma_2 \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = i \quad (1.20)$$

$$\sigma_1 \sigma_2 = i \sigma_3 \quad ; \quad \sigma_2 \sigma_3 = i \sigma_1 \quad ; \quad \sigma_3 \sigma_1 = i \sigma_2 \quad (1.21)$$

And the reverse is then identical to the adjoint matrix:

$$\tilde{A} = A^\dagger = (A^*)^t \quad (1.22)$$

Consequently we shall say now equally Pauli algebra or space algebra. This is not liked by pure Clifford algebraist. Most physicists are used to Pauli algebra and its old and cumbersome notations. They do not often use conveniences of the Clifford algebra Cl_3 of the physical space.

1.3.3 Three conjugations are used:

$A = a + \vec{u} + i\vec{v} + ib$ is the sum of the even part $A_1 = a + i\vec{v}$ (quaternion) and the odd part $A_2 = \vec{u} + ib$. From this we define the conjugation (involutive automorphism) $A \mapsto \hat{A}$ such as

$$\hat{A} = A_1 - A_2 = a - \vec{u} + i\vec{v} - ib \quad (1.23)$$

This conjugation satisfies, for any element A and any B in Cl_3 :

$$\widehat{A+B} = \hat{A} + \hat{B} \quad ; \quad \widehat{AB} = \hat{A}\hat{B}. \quad (1.24)$$

It is the main automorphism of this algebra, and each Clifford algebra owns such an automorphism.

From this conjugation and from the reversion we may form the third conjugation:

$$\bar{A} = \hat{A}^\dagger = a - \vec{u} - i\vec{v} + ib \quad ; \quad \overline{A+B} = \bar{A} + \bar{B} \quad ; \quad \overline{AB} = \bar{B} \bar{A} \quad (1.25)$$

Composing, in any order, two of these three conjugations gives the third one. Only $A \mapsto \hat{A}$ preserves the order of products, $A \mapsto \bar{A}$ and $A \mapsto A^\dagger$ inverse the order of products.

Now a, b, c, d are any complex numbers and a^* is the complex conjugate of a . We can satisfy that for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have:

$$\tilde{A} = A^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \quad ; \quad \hat{A} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix} \quad ; \quad \bar{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (1.26)$$

$$A\bar{A} = \bar{A}A = \det(A) = ad - bc \quad ; \quad \widehat{A}A^\dagger = A^\dagger\hat{A} = [\det(A)]^* \quad (1.27)$$

1.3.4 Gradient, divergence and curl

In Cl_3 there exists one important differential operator, because all may be made with it:

$$\vec{\partial} = \vec{e}_1\partial_1 + \vec{e}_2\partial_2 + \vec{e}_3\partial_3 = \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix} \quad (1.28)$$

with ⁶

$$\vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 \quad ; \quad \partial_j = \frac{\partial}{\partial x^j} \quad (1.29)$$

The Laplacian is simply the square of $\vec{\partial}$:

$$\Delta = (\partial_1)^2 + (\partial_2)^2 + (\partial_3)^2 = \vec{\partial}\vec{\partial} \quad (1.30)$$

Applied to a scalar a , $\vec{\partial}$ gives the gradient and applied to a vector \vec{u} it gives both the divergence and the curl:

$$\vec{\partial}a = \vec{\text{grad}} a \quad (1.31)$$

$$\vec{\partial}\vec{u} = \vec{\partial} \cdot \vec{u} + i \vec{\partial} \times \vec{u} \quad ; \quad \vec{\partial} \cdot \vec{u} = \text{div}\vec{u} \quad ; \quad \vec{\partial} \times \vec{u} = \text{curl} \vec{u}. \quad (1.32)$$

1.3.5 Space-time in space algebra:

With

$$x^0 = ct \quad ; \quad \vec{x} = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 \quad ; \quad \partial_\mu = \frac{\partial}{\partial x^\mu} \quad (1.33)$$

we let [2][47]

$$x = x^0 + \vec{x} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \quad (1.34)$$

Then the space-time is made of the auto-adjoint part of the Pauli algebra ($x^\dagger = x$) and we get

$$\hat{x} = \bar{x} = x^0 - \vec{x} \quad (1.35)$$

$$\det(x) = x\hat{x} = x \cdot x = (x^0)^2 - \vec{x}^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \quad (1.36)$$

The square of the pseudo-norm of any space-time vector is then simply the determinant. ⁷ Any element M of the Pauli algebra is the sum of a space-time vector v and of the product by i of another space-time vector w :

$$M = v + iw \quad (1.37)$$

$$v = \frac{1}{2}(M + M^\dagger) \quad ; \quad v^\dagger = v \quad (1.38)$$

$$iw = \frac{1}{2}(M - M^\dagger) \quad ; \quad w^\dagger = w. \quad (1.39)$$

⁶This operator $\vec{\partial}$ is usually notated, in quantum mechanics, as a scalar product, for instance $\vec{\sigma} \cdot \vec{\nabla}$. From this come many convoluted complications. To use simple notations fully simplifies calculations.

⁷We must notice that the pseudo-norm of the space-time metric comes not from a scalar product, a symmetric bilinear form, but from a determinant, an antisymmetric bilinear form. We are here very far from Riemannian spaces.

Space-time vectors v and w are uniquely defined. We need two linked differential operators:

$$\nabla = \partial_0 - \vec{\partial} \ ; \ \widehat{\nabla} = \partial_0 + \vec{\partial} \quad (1.40)$$

They allow to calculate the D'alembertian:

$$\nabla \widehat{\nabla} = \widehat{\nabla} \nabla = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2 = \square \quad (1.41)$$

1.3.6 Relativistic invariance:

If M is any not null element in Cl_3 and if R is the transformation from the space-time into itself, which to any x associates x' such as

$$x' = x'^0 + \vec{x}' = R(x) = MxM^\dagger \quad (1.42)$$

we note, if $\det(M) \neq 0$:

$$\det(M) = re^{i\theta} \ , \ r = |\det(M)|. \quad (1.43)$$

We get then:

$$\begin{aligned} (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 &= \det(x') = \det(MxM^\dagger) \\ &= re^{i\theta} \det(x)re^{-i\theta} = r^2[(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2] \end{aligned} \quad (1.44)$$

R multiplies then by r any space-time distance and is called Lorentz dilation with ratio r . If we let, with the usual convention summing the up and down indices:

$$x'^\mu = R_\nu^\mu x^\nu \quad (1.45)$$

we get for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$:

$$2R_0^0 = |a|^2 + |b|^2 + |c|^2 + |d|^2 > 0 \quad (1.46)$$

x'^0 has then the same sign as x^0 at the origin: R conserves the arrow of the time. Even more we get (the calculation is in [14] A.2.4):

$$\det(R_\nu^\mu) = r^4 \quad (1.47)$$

R conserves therefore the orientation of space-time and as it conserves the time orientation it conserves also the space orientation.

Let f be the application which to M associates R . Let M' be another matrix, with:

$$\det(M') = r'e^{i\theta'} \ ; \ R' = f(M') \ ; \ x'' = M'x'M'^\dagger. \quad (1.48)$$

We get

$$\begin{aligned} x'' &= M'x'M'^\dagger = M'(MxM^\dagger)M'^\dagger = (M'M)x(M'M)^\dagger \\ f(M') \circ f(M) &= f(M'M) \end{aligned} \quad (1.49)$$

f is then an homomorphism. If we restrict us to $r \neq 0$, f is a homomorphism from the group (Cl_3^*, \times) into the group (D^*, \circ) , where D^* is the set of dilations with not null ratio. These two groups are Lie groups. (Cl_3^*, \times) is a 8-dimensional Lie group, because Cl_3 is 8-dimensional. On the contrary (D^*, \circ) is only a 7-dimensional Lie group, one dimension is lost because the kernel of f is not reduced to the neutral element: let θ be any real number and let M be

$$M = e^{i\frac{\theta}{2}} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} ; \quad \det(M) = e^{i\theta} \quad (1.50)$$

we get then:

$$x' = MxM^\dagger = e^{i\frac{\theta}{2}}xe^{-i\frac{\theta}{2}} = x. \quad (1.51)$$

$f(M)$ is therefore the identity and M belongs to the kernel of f , which is a group with only one parameter: θ , and we get only 7 parameters in D^* , 6 angles defining a Lorentz rotation, plus the ratio of the dilation. For instance, if

$$M = e^{a+b\sigma_1} = e^a(\cosh(b) + \sinh(b)\sigma_1) \quad (1.52)$$

then the R transformation defined in (1.42) satisfies

$$\begin{aligned} x' &= MxM^\dagger = e^{a+b\sigma_1}(x^0 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3)e^{a+b\sigma_1} \\ &= e^{2a}[e^{2b\sigma_1}(x^0 + x^1\sigma_1) + x^2\sigma_2 + x^3\sigma_3] \end{aligned} \quad (1.53)$$

We then get

$$\begin{aligned} x'^0 + x'^1\sigma_1 &= e^{2a}(\cosh(2b) + \sinh(2b)\sigma_1)(x^0 + x^1\sigma_1) \\ x'^0 &= e^{2a}(\cosh(2b)x^0 + \sinh(2b)x^1) \end{aligned} \quad (1.54)$$

$$x'^1 = e^{2a}(\sinh(2b)x^0 + \cosh(2b)x^1) \quad (1.55)$$

$$x'^2 = e^{2a}x^2 \quad (1.56)$$

$$x'^3 = e^{2a}x^3 \quad (1.57)$$

We can recognize R as the product of a Lorentz boost mixing the temporal component x^0 and the spacial component x^1 and of an homothety with ratio e^{2a} . Now if

$$M = e^{a+bi\sigma_1} = e^a(\cos(b) + \sin(b)i\sigma_1) \quad (1.58)$$

then the R transformation defined in (1.42) satisfies

$$\begin{aligned} x' &= MxM^\dagger = e^{a+bi\sigma_1}(x^0 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3)e^{a-bi\sigma_1} \\ &= e^{2a}[x^0 + x^1\sigma_1 + e^{2bi\sigma_1}(x^2\sigma_2 + x^3\sigma_3)] \end{aligned} \quad (1.59)$$

We then get

$$x'^2\sigma_2 + x'^3\sigma_3 = e^{2a}(\cos(2b) + \sin(2b)i\sigma_1)(x^2\sigma_2 + x^3\sigma_3)$$

$$x'^0 = e^{2a}x^0 \quad (1.60)$$

$$x'^1 = e^{2a}x^1 \quad (1.61)$$

$$x'^2 = e^{2a}(\cos(2b)x^2 + \sin(2b)x^3) \quad (1.62)$$

$$x'^3 = e^{2a}(-\sin(2b)x^2 + \cos(2b)x^3) \quad (1.63)$$

We can recognize R as the product of a rotation with axis Ox_1 and angle $2b$ and a homothety with ratio e^{2a} .

1.3.7 Restricted Lorentz group:

If we impose now the condition $\det(M) = 1$, the set of the M is called $SL(2, \mathbb{C})$, (1.44) becomes:

$$(x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \quad (1.64)$$

R is then a Lorentz rotation and the set of the R is the Lorentz restricted group \mathcal{L}_+^\uparrow (conserving space and time orientation). With (1.50) we get:

$$1 = e^{i\theta} \ ; \ \theta = k2\pi \ ; \ \frac{\theta}{2} = k\pi \ ; \ M = \pm 1 \quad (1.65)$$

Remark 1: Quantum mechanics do not distinguish M from R , they confuse $SL(2, \mathbb{C})$ and \mathcal{L}_+^\uparrow and they name bi-valued representations of \mathcal{L}_+^\uparrow the representations of $SL(2, \mathbb{C})$. This comes mainly from the use in quantum theory of infinitesimal rotations, so they work only in the vicinity of the origin of the group, and they work then not in the group but in the Lie algebra of the group. And it happens that the Lie algebras of $SL(2, \mathbb{C})$ and of \mathcal{L}_+^\uparrow are identical. $SL(2, \mathbb{C})$ is the covering group of \mathcal{L}_+^\uparrow . Globally $SL(2, \mathbb{C})$ and \mathcal{L}_+^\uparrow are quite different, for instance any element reads e^A in \mathcal{L}_+^\uparrow and this is false in $SL(2, \mathbb{C})$. It is therefore perfectly intolerable to have so long neglected the fact that when an angle b is present in M , it is an angle $2b$ which is present in R .

Remark 2: We are forced to distinguish the group of the M from the group of the R , as soon as θ is not null, because these two groups do not have the same dimension and are not similar even in the vicinity of the origin.

Remark 3: $SL(2, \mathbb{C})$ contains as subgroup the $SU(2)$ group of the unitary 2×2 complex matrices with determinant 1. The restriction of f to this subgroup is an homomorphism from $SU(2)$ into the $SO(3)$ group of rotations. The kernel of this homomorphism is also ± 1 .

Remark 4: There are two non-equivalent homomorphisms from (Cl_3^*, \times) into the group (D^*, \circ) . the second homomorphism \widehat{f} is defined by

$$x' = \widehat{R}(x) = \widehat{M}x\overline{M} \ ; \ \widehat{R} = \widehat{f}(M). \quad (1.66)$$

1.4 Clifford algebra of the space-time: $Cl_{1,3}$

$Cl_{1,3}$ contains real numbers and space-time vectors \mathbf{x} such as

$$\mathbf{x} = x^0\gamma_0 + x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3 = x^\mu\gamma_\mu. \quad (1.67)$$

The four γ_μ form an orthonormal basis of space-time: ⁸

$$(\gamma_0)^2 = 1 \quad ; \quad (\gamma_1)^2 = (\gamma_2)^2 = (\gamma_3)^2 = -1 \quad ; \quad \gamma_\mu \cdot \gamma_\nu = 0, \quad \mu \neq \nu. \quad (1.68)$$

The general term in $Cl_{1,3}$ is a sum:

$$N = s + v + b + pv + ps \quad (1.69)$$

where s is a real number, v is a space-time vector, b is a bivector, pv is a pseudo-vector and ps is a pseudo-scalar. There are $1+4+6+4+1 = 16 = 2^4$ dimensions on the real field because: There are 6 independent bivectors: $\gamma_{01} = \gamma_0\gamma_1$, γ_{02} , γ_{03} , γ_{12} , γ_{23} , γ_{31} , and $\gamma_{ji} = -\gamma_{ij}$, $j \neq i$, 4 independent pseudo-vectors: γ_{012} , γ_{023} , γ_{031} , γ_{123} and one pseudoscalar:

$$ps = p\gamma_{0123} \quad ; \quad \gamma_{0123} = \gamma_0\gamma_1\gamma_2\gamma_3 \quad (1.70)$$

where p is a real number.

The even part of N is $s + b + ps$, the odd part is $v + pv$. The main automorphism is $N \mapsto \tilde{N} = s - v + b - pv + ps$

The reverse of N is

$$\tilde{N} = s + v - b - pv + ps \quad (1.71)$$

Amongst the 16 generators of $Cl_{1,3}$, 10 have for square -1 and 6 have for square 1:

$$\begin{aligned} 1^2 &= \gamma_{01}^2 = \gamma_{02}^2 = \gamma_{03}^2 = \gamma_0^2 = \gamma_{123}^2 = 1 \\ \gamma_1^2 &= \gamma_2^2 = \gamma_3^2 = \gamma_{12}^2 = \gamma_{23}^2 = \gamma_{31}^2 \\ &= \gamma_{012}^2 = \gamma_{023}^2 = \gamma_{031}^2 = \gamma_{0123}^2 = -1 \end{aligned} \quad (1.72)$$

Remark 1: If we use the + sign for the space, then we get 10 generators with square 1 and 6 with square -1. The two Clifford algebras are not identical. And yet there is no known physical reason to prefer one to the other algebra.

Remark 2: The even sub-algebra $Cl_{1,3}^+$, formed by all the even elements $N = s + b + ps$ is 8-dimensional and is isomorphic to Cl_3 . We shall see this in the next page, by using the Dirac matrices. The even sub-algebra of $Cl_{3,1}$ is also isomorphic to Cl_3 .

The privileged differential operator, in $Cl_{1,3}$, is:

$$\partial = \gamma^\mu \partial_\mu \quad ; \quad \gamma^0 = \gamma_0 \quad ; \quad \gamma^j = -\gamma_j, \quad j = 1, 2, 3. \quad (1.73)$$

It satisfies:

$$\partial\partial = \square = (\partial_0)^2 - (\partial_1)^2 - (\partial_2)^2 - (\partial_3)^2 \quad (1.74)$$

⁸Users of Clifford algebras are nearly equally parted between users of a + sign for the time (Hestenes [32][34]), and users of a - sign for the time (Dehevels [29]). It seems that no physical property of the space-time can allow to prefer one to the other. We use here a + sign for the time, which was the Hestenes's choice. It is another main difference with general relativity where the metric tensor has physical properties linked to gravitation.

1.4.1 Dirac matrices:

Most physicists do not use directly the Clifford algebra of space-time but they use a matrix algebra⁹, generated by the Dirac matrices. These matrices are not uniquely defined. The easier way, to link $Cl_{1,3}$ to Cl_3 , is to let¹⁰ :

$$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} ; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \quad \gamma^j = -\gamma_j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad (1.75)$$

Then we get:

$$\boldsymbol{\partial} = \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix} \quad (1.76)$$

It is easy to satisfy:

$$\gamma_{0j} = \begin{pmatrix} -\sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} ; \quad \gamma_{23} = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix} ; \quad \gamma_{0123} = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} \quad (1.77)$$

Isomorphism between $Cl_{1,3}^+$ and Cl_3 : Let N be any even element. With:

$$\begin{aligned} N &= a + Bi + ps \quad ; \quad Bi = u_1\gamma_{10} + u_2\gamma_{20} + u_3\gamma_{30} + v_1\gamma_{32} + v_2\gamma_{13} + v_3\gamma_{21} \\ ps &= b\gamma_{0123} \end{aligned} \quad (1.78)$$

$$\begin{aligned} M &= a + \vec{u} + i\vec{v} + ib \quad ; \quad \vec{u} = u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3 \\ \vec{v} &= v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3 \end{aligned} \quad (1.79)$$

Bi is a bivector and ps a pseudo-vector in space-time. We get, with the choice (1.75) made for the Dirac matrices:

$$N = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} ; \quad \widetilde{N} = \begin{pmatrix} \overline{M} & 0 \\ 0 & M^\dagger \end{pmatrix}. \quad (1.80)$$

Since the conjugation $M \mapsto \widehat{M}$ is compatible with the addition and with the multiplication, the algebra of the M is exactly isomorphic to the algebra of the N . As N contains both M and \widehat{M} , the Dirac matrices combine both inequivalent representations of Cl_3^* .

1.5 Clifford Algebra $Cl_{5,1} = Cl_{1,5}$

We shall use also later in this article the Clifford algebra of a larger space-time with a 5-dimensional space. The general element \underline{x} of this larger space-time

⁹Generally the matrix algebra used is $M_4(\mathbb{C})$, an algebra on the complex field. This algebra is 16-dimensional on the complex field and therefore it is also an algebra on the real field, 32-dimensional on the real field. This is enough to prove that $M_4(\mathbb{C}) \neq Cl_{1,3}$.

¹⁰This choice of the Dirac matrices is not the choice used in the Dirac theory to calculate the solutions in the hydrogen atom case, but the choice used when high velocities and restricted relativity are required. It is also the usual choice in the electro-weak theory. We shall see that it is also a convenient choice to solve the wave equation for the hydrogen case.

reads

$$\underline{x} = x^a \Lambda_a = x^\mu \Lambda_\mu + x^4 \Lambda_4 + x^5 \Lambda_5 \quad (1.81)$$

$$x^a \Lambda_a = \sum_{n=0}^{n=5} x^n \Lambda_n ; \quad x^\mu \Lambda_\mu = \sum_{n=0}^{n=3} x^n \Lambda_n \quad (1.82)$$

$$(\underline{x})^2 = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2. \quad (1.83)$$

We link the preceding $Cl_{1,3}$ space-time algebra to this greater algebra by using the following matrix representation, $\mu = 0, 1, 2, 3$:

$$\Lambda_\mu = \begin{pmatrix} 0 & -\gamma^\mu \\ \gamma_\mu & 0 \end{pmatrix} ; \quad \Lambda_4 = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} ; \quad \Lambda_5 = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \quad (1.84)$$

where I_4 is the unity for 4×4 matrices and $\mathbf{i} = \gamma_{0123}$ (See (1.77)). We always use the matrix representation (1.75). We get

$$\Lambda_{\mu\nu} = \Lambda_\mu \Lambda_\nu = \begin{pmatrix} -\gamma^{\mu\nu} & 0 \\ 0 & -\gamma_{\mu\nu} \end{pmatrix} \quad (1.85)$$

$$\Lambda_{\mu\nu\rho} = \Lambda_{\mu\nu} \Lambda_\rho = \begin{pmatrix} 0 & \gamma^{\mu\nu\rho} \\ -\gamma_{\mu\nu\rho} & 0 \end{pmatrix} \quad (1.86)$$

$$\Lambda_{0123} = \Lambda_{01} \Lambda_{23} = \begin{pmatrix} \gamma_{0123} & 0 \\ 0 & \gamma_{0123} \end{pmatrix} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} \quad (1.87)$$

$$\Lambda_{45} = \Lambda_4 \Lambda_5 = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \quad (1.88)$$

$$\Lambda_{012345} = \Lambda_{0123} \Lambda_{45} = \begin{pmatrix} -I_4 & 0 \\ 0 & I_4 \end{pmatrix} \quad (1.89)$$

The general term of this algebra reads

$$\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 + \Psi_6 ; \quad \Psi_0 = sI_8, s \in \mathbb{R}, \quad (1.90)$$

$$\begin{aligned} \Psi_1 &= \sum_{a=0}^{a=5} N^a \Lambda_a, \quad \Psi_2 = \sum_{0 \leq a < b \leq 5} N^{ab} \Lambda_{ab}, \quad \Psi_3 = \sum_{0 \leq a < b < c \leq 5} N^{abc} \Lambda_{abc} \\ \Psi_4 &= \sum_{0 \leq a < b < c < d \leq 5} N^{abcd} \Lambda_{abcd}, \quad \Psi_5 = \sum_{0 \leq a < b < c < d < e \leq 5} N^{abcde} \Lambda_{abcde} \end{aligned} \quad (1.91)$$

$$\Psi_6 = p \Lambda_{012345}, \quad p \in \mathbb{R}. \quad (1.92)$$

where N^{ind} are real numbers. We shall need in section 6:

$$\begin{aligned} P^+ &= \frac{1}{2}(I_8 - \Lambda_{012345}) = \begin{pmatrix} I_4 & 0 \\ 0 & 0 \end{pmatrix} \\ P^- &= \frac{1}{2}(I_8 + \Lambda_{012345}) = \begin{pmatrix} 0 & 0 \\ 0 & I_4 \end{pmatrix}. \end{aligned} \quad (1.93)$$

Now we consider six elements of this algebra:

$$L_a = \Lambda_{012345} \Lambda_a ; \quad a = 0, 1, 2, 3, 4, 5. \quad (1.94)$$

We then get

$$L_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}; \quad L_4 = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}; \quad L_5 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} \quad (1.95)$$

$$L_{ab} = -\Lambda_{ab}, \quad 0 \leq a \leq b \leq 5 \quad (1.96)$$

$$L_{abcd} = \Lambda_{abcd}, \quad 0 \leq a < b < c < d \leq 5 \quad (1.97)$$

$$L_{012345} = -\Lambda_{012345} = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} \quad (1.98)$$

$$\Lambda_a = L_a L_{012345}. \quad (1.99)$$

These six L_a are the generators of the $L_{1,5}$ algebra since (1.96) implies

$$(L_0)^2 = 1; \quad (L_a)^2 = -1, \quad a = 1, 2, 3, 4, 5. \quad (1.100)$$

And the algebra generated by the L_a is a sub-algebra of the algebra generated by the Λ_a . Inversely (1.99) implies that we should may start with the L_a and get the Λ_a from them. This explains how $Cl_{5,1} = Cl_{1,5}$. Vectors in $Cl_{1,5}$ are pseudo-vectors of $Cl_{5,1}$ and vice-versa. Then the sums of vectors and pseudo-vectors that we shall need in section 6 are independent on the choice of the signature.

2 Dirac equation

We present here in two different frames the Dirac wave equation for the electron. We study it firstly with the Dirac matrices: second order equation, conservative current, tensors without derivative, gauge invariance. Relativistic invariance requires to use the space algebra, into which we rewrite all the Dirac theory. We get new tensors. We study the link between the invariant form of the Dirac equation and the Lagrangian density. We review the charge conjugation.

An important part of the standard model of quantum physics is the Dirac wave equation. This comes from the fact that electrons, neutrinos, quarks, are quantum objects with spin 1/2. And the standard model explains the spin 1/2 as a relativistic consequence of the Dirac wave equation.

This equation is detailed in the present section, slightly changed in section 3, and the modified wave equation will be generalized in section 6 as a wave equation for the electron-neutrino.

2.1 With the Dirac matrices

The starting point of the Dirac's work was the Pauli wave equation for the electron, which used a wave with two complex components mixed by Pauli matrices.

The Schrödinger and Pauli wave equations include a first order time derivative, and second order derivatives for the space coordinates. This is inappropriate for a relativistic wave equation. So Dirac sought a wave equation with only first order derivatives, giving at the second order the equation for material waves. This necessitated to use matrices as Pauli had done. Dirac [30] understood that more components were necessary. His wave equation proves that four components are enough. With the notations (1.75) this equation reads ¹¹

$$0 = [\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi ; \quad q = \frac{e}{\hbar c} ; \quad m = \frac{m_0 c}{\hbar} \quad (2.1)$$

with the usual convention summing up and down indexes. The A_μ are components of the space-time vector which is the exterior electromagnetic potential, e is the charge of the electron, m_0 is its proper mass. Even with a well defined signature for the space-time, the matrices of the theory are not uniquely defined. The choice we have made in (1.75) allows to use the Weyl spinors ξ and η which play a fundamental role, firstly for the relativistic invariance of the theory, secondly for the Lochak's theory for a magnetic monopole [42] or the electro-weak interactions in section 6. With them the wave ψ is the matrix-column:

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} ; \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} ; \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \quad (2.2)$$

The Dirac equation is, as the Schrödinger equation, a linear wave equation, it contains only partial derivatives and products by matrices, so linear combinations of solutions are also solutions of the wave equation.

2.1.1 Second order equation

Without exterior electromagnetic field and with (1.76) the Dirac equation reads

$$\boldsymbol{\partial}\psi = -im\psi \quad (2.3)$$

and gives at the second order:

$$\square\psi = \boldsymbol{\partial}\boldsymbol{\partial}\psi = \boldsymbol{\partial}(-im\psi) = -im\boldsymbol{\partial}\psi = (-im)(-im)\psi = -m^2\psi \quad (2.4)$$

we get then the awaited equation at the second order:

$$(\square + m^2)\psi = 0 \quad (2.5)$$

¹¹First works about Dirac equation [30] [26] use an imaginary temporal variable which allows to use a + + + + signature for space-time and avoids to distinguish covariant and contravariant indexes. This brings also difficulties, the tensor components are either real or pure imaginary. It also hides that matrices of the relativistic theory cannot be all hermitian. The algebra on the complex field generated by the Dirac matrices is the $M_4(\mathbb{C})$ algebra which is 16-dimensional on the complex field and 32-dimensional on the real field. Therefore this algebra cannot be isomorphic to the Clifford algebra of space-time, 16-dimensional on the real field, even if we have used in 1.4.1 a sub-algebra of $M_4(\mathbb{C})$ to represent $Cl_{1,3}$ and to link $Cl_{1,3}$ to Cl_3 .

2.1.2 Conservative current

Taking the adjoint matrices into the Dirac equation we get

$$0 = \psi^\dagger[(\partial_\mu - iqA_\mu)\gamma^{\mu\dagger} - im] \quad (2.6)$$

We multiply now on the right by γ_0 , we let¹²

$$\bar{\psi} = \psi^\dagger\gamma_0 \quad (2.7)$$

and we get the wave equation on the equivalent form:

$$0 = \bar{\psi}[(\partial_\mu - iqA_\mu)\gamma^\mu - im] \quad (2.8)$$

Multiplying (2.1) on the left by $\bar{\psi}$, (2.8) on the right by ψ and adding, we get:

$$0 = \partial_\mu(\bar{\psi}\gamma^\mu\psi) \quad (2.9)$$

A conservative current therefore exists: $(\partial_\mu J^\mu = 0)$, the J current, whose contravariant components are defined as

$$J^\mu = \bar{\psi}\gamma^\mu\psi \quad (2.10)$$

The temporal component of this space-time vector satisfies

$$J^0 = |\xi_1|^2 + |\xi_2|^2 + |\eta_1|^2 + |\eta_2|^2 \quad (2.11)$$

and so may be interpreted, as with the Schrödinger equation, as giving the probability density of the electron's presence.

2.1.3 Tensors

The J current is one of the tensors of the Dirac theory whose definition, from the wave, is made without partial derivative. Other same quantities were remarked, firstly a scalar

$$\Omega_1 = \bar{\psi}\psi ; \quad \bar{\psi} = \psi^\dagger\gamma_0 = (\eta^\dagger \xi^\dagger). \quad (2.12)$$

Then the six

$$S^{\mu\nu} = i\bar{\psi}\gamma^\mu\gamma^\nu\psi \quad (2.13)$$

are the components of an antisymmetric two-ranked tensor. The four K^μ

$$K^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi ; \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (2.14)$$

are components of a pseudo-vector in space-time, dual of a three-ranked anti-symmetric tensor. Finally

$$\Omega_2 = -i\bar{\psi}\gamma_5\psi \quad (2.15)$$

¹²The $\bar{\psi}$ in the Dirac formalism has nothing to do with the \bar{M} in the Pauli algebra.

is a pseudo-scalar and allows to define the invariant ρ and the angle β of Yvon-Takabayasi :

$$\Omega_1 = \rho \cos \beta ; \quad \Omega_2 = \rho \sin \beta ; \quad \Omega_1 + i\Omega_2 = \rho e^{i\beta}. \quad (2.16)$$

We get with the Weyl spinors:

$$\begin{aligned} \Omega_1 &= \xi^\dagger \eta + \eta^\dagger \xi ; \quad \Omega_2 = i(\xi^\dagger \eta - \eta^\dagger \xi) \\ \rho e^{i\beta} &= \Omega_1 + i\Omega_2 = 2\eta^\dagger \xi = 2(\eta_1^* \xi_1 + \eta_2^* \xi_2) \\ \rho e^{-i\beta} &= \Omega_1 - i\Omega_2 = 2\xi^\dagger \eta = 2(\eta_1 \xi_1^* + \eta_2 \xi_2^*) \end{aligned} \quad (2.17)$$

These tensors were so more intensively studied that physicists hoped to link the theory to pre-quantum physics, with quantities which were vectors and tensors. In fact these 16 tensorial densities that we have just encountered know nothing about the phase of the wave because they are gauge invariant. We see this now.

2.1.4 Gauge invariances

The Dirac equation (2.1) is invariant under the gauge transformation:

$$\psi \mapsto \psi' = e^{ia} \psi ; \quad A_\mu \mapsto A_\mu' = A_\mu - \frac{1}{q} \partial_\mu a \quad (2.18)$$

because we get:

$$\begin{aligned} &[\gamma^\mu (\partial_\mu + iqA_\mu') + im] \psi' \\ &= \gamma^\mu [i\partial_\mu a e^{ia} \psi + e^{ia} \partial_\mu \psi + iq(A_\mu - \frac{1}{q} \partial_\mu a) e^{ia} \psi] + im e^{ia} \psi \\ &= e^{ia} [\gamma^\mu (\partial_\mu + iqA_\mu) + im] \psi \end{aligned} \quad (2.19)$$

The first consequence of this gauge invariance is that the wave may be multiplied by i and this is the reason to use complex linear spaces in the Dirac theory. Next the Noether's theorem allows to link this gauge invariance to the existence of a conservative current (see [16] B.1.2). In the case of the Dirac equation the variational calculus allows to get the linear wave equation (2.1) from a Lagrangian density:

$$-\mathcal{L} = \frac{1}{2} [(\bar{\psi} \gamma^\mu (-i\partial_\mu + qA_\mu) \psi) + (\bar{\psi} \gamma^\mu (-i\partial_\mu + qA_\mu) \psi)^\dagger] + m \bar{\psi} \psi \quad (2.20)$$

And the Noether's theorem links the invariance of this Lagrangian density under the gauge transformation (2.18) to the conservation of the J current. Tensorial densities of the preceding page contains all a ψ factor and a $\bar{\psi}$ factor, the first one is multiplied by e^{ia} and the second by e^{-ia} , and so these tensorial densities are invariant under gauge transformations.

The starting point of the G. Lochak's monopole theory [42] is the existence in the case of a null proper mass of another gauge invariance which is possible because γ_{0123} anti-commutes with each of the four γ^μ matrices. He established

that the Dirac equation could be invariant under this gauge invariance if the mass term was replaced by a not linear mass term. And the gauge may be local if an adequate potential term is added:

$$[\gamma^\mu(\partial_\mu - \frac{ig}{\hbar c}B_\mu\gamma_5) + \frac{1}{2}\frac{m(\rho^2)c}{\hbar}(\Omega_1 - i\Omega_2\gamma_5)]\psi = 0 \quad (2.21)$$

where g is the charge of the monopole, the B_μ are the pseudo-potentials of Cabibbo and Ferrari. We will use later this mass term in a particular case.

2.1.5 Relativistic invariance

With the notations and results of paragraph 1.3.6 the transformation R defined by (1.42) is a Lorentz dilation. With the N matrix in (1.80) the reverse is

$$\tilde{N} = \begin{pmatrix} \overline{M} & 0 \\ 0 & M^\dagger \end{pmatrix} \quad (2.22)$$

And we get with the R in (1.42) and (1.45), with the N in (1.80), for any M and $\nu = 0, 1, 2, 3$ (a proof is in [14] A.2.2):

$$R_\mu^\nu \gamma^\mu = \tilde{N} \gamma^\nu N \quad (2.23)$$

We get also

$$\partial'_\nu = \frac{\partial}{\partial x'^\nu} ; \quad \partial_\mu = R_\mu^\nu \partial'_\nu ; \quad A_\mu = R_\mu^\nu A'_\nu \quad (2.24)$$

and so we get:

$$\begin{aligned} 0 &= [\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi \\ &= [\gamma^\mu R_\mu^\nu(\partial'_\nu + iqA'_\nu) + im]\psi \\ &= [\tilde{N}\gamma^\nu N(\partial'_\nu + iqA'_\nu) + im]\psi. \end{aligned}$$

And if we restrict M to $SL(2, \mathbb{C})$, we get $M\overline{M} = \det(M) = 1$, so $\overline{M} = M^{-1}$ and $\tilde{N} = N^{-1}$ which allows to write

$$[\tilde{N}\gamma^\nu N(\partial'_\nu + iqA'_\nu) + im]\psi = N^{-1}[\gamma^\nu(\partial'_\nu + iqA'_\nu) + im]N\psi \quad (2.25)$$

So the Dirac theory supposes:

$$\psi' = N\psi \quad (2.26)$$

and it gets

$$0 = [\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi = N^{-1}[\gamma^\mu(\partial'_\mu + iqA'_\mu) + im]\psi' \quad (2.27)$$

This is why the Dirac equation is said form invariant under the Lorentz group. We must remark:

- 1 - Only transformations of the restricted Lorentz group \mathcal{L}_+^\uparrow are obtained.
- 2 - Same γ^μ matrices appear in the two systems of coordinates, the x^μ system and the x'^μ system. Dirac matrices are independent on the used system, they do

not depend on the moving observer seeing the wave. This is quite different from the Hestenes' study [34] where the γ_μ form a basis of space-time and change from an observer to another one.

3 - ξ and η change differently:

$$\psi' = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} ; \quad \xi' = M\xi ; \quad \eta' = \widehat{M}\eta \quad (2.28)$$

Left and right Weyl spinors are linked to one of two non-equivalent representations of $SL(2, \mathbb{C})$.

4 - Only one factor M or \widehat{M} appears into these last relations, whilst two M factors are present in $x' = MxM^\dagger$. In case of a rotation the wave turns only by θ when we rotate by 2θ .

5 - It is somewhat incorrect to say that the Dirac equation is relativistic invariant, while the equation is in fact form invariant under another group, $SL(2, \mathbb{C})$, which is not isomorphic to the Lorentz group.

Nevertheless for any relativistic object with a Dirac wave it is not possible to avoid the $SL(2, \mathbb{C})$ group, therefore it is impossible to avoid the algebra Cl_3 which contains this group. Now we shall explain how we are able to write all the Dirac theory in the Cl_3 frame.

2.2 The wave with the space algebra

To read the Dirac equation, using only the Clifford algebra of the physical space which is the Pauli algebra, we start again from (2.1) using the Weyl spinors ξ and η . With:

$$\vec{A} = A^1\sigma_1 + A^2\sigma_2 + A^3\sigma_3 ; \quad A = A^0 + \vec{A} \quad (2.29)$$

and with (1.75) the Dirac equation reads

$$\begin{pmatrix} 0 & \nabla + iqA \\ \widehat{\nabla} + iq\widehat{A} & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + im \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0. \quad (2.30)$$

This gives the following system, equivalent to the Dirac equation:

$$(\nabla + iqA)\eta + im\xi = 0 \quad (2.31)$$

$$(\widehat{\nabla} + iq\widehat{A})\xi + im\eta = 0. \quad (2.32)$$

We use the complex conjugation on (2.32), then we multiply on the left by $-i\sigma_2$:¹³

$$(-i\sigma_2)(\widehat{\nabla}^* - iq\widehat{A}^*)\xi^* - im(-i\sigma_2)\eta^* = 0. \quad (2.33)$$

But we have:

$$(-i\sigma_2)(\widehat{\nabla}^* - iq\widehat{A}^*) = (\nabla - iqA)(-i\sigma_2). \quad (2.34)$$

¹³Whichever formalism is used to read the Dirac wave equation we can see that the third direction is privileged, and we shall explain this farther. The 12 or 21 planes are also privileged, but indexes 1 and 2 play the same role. When a i is added, this is the case with the electric interaction and the electric gauge invariance, then indexes 1 and 2 do not play the same role because σ_1 is real while σ_2 is pure imaginary. Therefore the use here of σ_2 is necessary.

So (2.32) is equivalent to:

$$\nabla(-i\sigma_2\xi^*) + iqA(i\sigma_2\xi^*) + im(i\sigma_2\eta^*) = 0. \quad (2.35)$$

The system composed of (2.31)-(2.32) is consequently equivalent to one matrix equation ¹⁴:

$$\nabla(\eta \quad -i\sigma_2\xi^*) + iqA(\eta \quad i\sigma_2\xi^*) + im(\xi \quad i\sigma_2\eta^*) = 0 \quad (2.36)$$

Now we let

$$\phi = \sqrt{2}(\xi \quad -i\sigma_2\eta^*) = \sqrt{2} \begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix}. \quad (2.37)$$

This gives

$$\hat{\phi} = \sqrt{2}(\eta \quad -i\sigma_2\xi^*) = \sqrt{2} \begin{pmatrix} \eta_1 & -\xi_2^* \\ \eta_2 & \xi_1^* \end{pmatrix} \quad (2.38)$$

and also

$$\phi\sigma_3 = \sqrt{2}(\xi \quad i\sigma_2\eta^*); \quad \hat{\phi}\sigma_3 = \sqrt{2}(\eta \quad i\sigma_2\xi^*). \quad (2.39)$$

So (2.36) which is equivalent to the Dirac equation (2.1) reads

$$\nabla\hat{\phi} + iqA\hat{\phi}\sigma_3 + im\phi\sigma_3 = 0 \quad (2.40)$$

which we shall write with

$$\sigma_{12} = \sigma_1\sigma_2 = i\sigma_3; \quad \sigma_{21} = \sigma_2\sigma_1 = -i\sigma_3 \quad (2.41)$$

$$\begin{aligned} 0 &= \nabla\hat{\phi} + qA\hat{\phi}\sigma_{12} + m\phi\sigma_{12} \\ 0 &= \nabla\hat{\phi}\sigma_{21} + qA\hat{\phi} + m\phi. \end{aligned} \quad (2.42)$$

Even if this equation seems very different from the well known form (2.1), it is necessary to insist on the fact that this wave equation is strictly the Dirac equation.¹⁵

2.2.1 Relativistic invariance

Under a dilation R defined by any M matrix satisfying (1.39) and (1.40), we got in (2.28) $\xi' = M\xi$, $\eta' = \widehat{M}\eta$, and these relations are not reserved to the

¹⁴This is possible because when we compute the product of two matrices we multiply each column of the right matrix by the left matrix. Terms between brackets in (2.36) are the column-matrices that we got separately in (2.31) and (2.35)

¹⁵The indistinct i in quantum theory also generator of the gauge invariance (2.18) is changed here into the multiplication on the right by $\sigma_{12} = i_3$. This is so more interesting than i_3 is not the alone element with square -1 . In space algebra there are four independent terms with square -1 . These terms generate a Lie algebra which is exactly the Lie algebra of the $SU(2) \times U(1)$ Lie group. Hestenes [33] was the first to remark this Lie algebra and to compare with the Lie group of the electro-weak theory.

particular case $r = 1$ and $\theta = 0$. More, we get

$$\begin{aligned} -i\sigma_2\eta'^* &= -i\sigma_2\widehat{M}^*\eta^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \eta^* = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \eta^* \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \eta^* = M(-i\sigma_2\eta^*). \end{aligned} \quad (2.43)$$

So with

$$\phi' = \sqrt{2}(\xi' \quad -i\sigma_2\eta'^*) = \sqrt{2} \begin{pmatrix} \xi_1' & -\eta_2'^* \\ \xi_2' & \eta_1'^* \end{pmatrix} \quad (2.44)$$

formulas (2.28) are equivalent to

$$\phi' = M\phi \quad (2.45)$$

This signifies that the link between the Weyl spinors ξ , η and ϕ is not only relativistic invariant, it is also invariant under the greater group Cl_3^* of the invertible elements in Cl_3 . In addition, with

$$\nabla' = \sigma^\mu \partial'_\mu ; \quad \partial'_\mu = \frac{\partial}{\partial x'^\mu} ; \quad \sigma_0 = \sigma^0 = 1 ; \quad \sigma^j = -\sigma_j, \quad j = 1, 2, 3. \quad (2.46)$$

we get (see [14] A.2.1), for any M :

$$\nabla = \overline{M}\nabla'\widehat{M} \quad (2.47)$$

and the electric gauge invariance impose then

$$qA = \overline{M}q'A'\widehat{M} \quad (2.48)$$

which gives

$$\begin{aligned} 0 &= \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + m\phi \\ &= \overline{M}\nabla'\widehat{M}\widehat{\phi}\sigma_{21} + q'\overline{M}A'\widehat{M}\widehat{\phi} + m\phi \\ &= \overline{M}(\nabla'\widehat{\phi}'\sigma_{21} + q'A'\widehat{\phi}') + m\phi \end{aligned} \quad (2.49)$$

Form invariance under Cl_3^* of the Dirac equation signifies that we have

$$\begin{aligned} 0 &= \nabla'\widehat{\phi}'\sigma_{21} + q'A'\widehat{\phi}' + m'\phi' ; \quad \nabla'\widehat{\phi}'\sigma_{21} + q'A'\widehat{\phi}' = -m'\phi' \\ 0 &= \overline{M}(-m'\phi') + m\phi = -m'\overline{M}M\phi + m\phi \\ &= (-m're^{i\theta} + m)\phi \end{aligned} \quad (2.50)$$

We get then the invariance of the wave equation under the Cl_3^* group if and only if

$$m = m're^{i\theta}. \quad (2.51)$$

Evidently in the case where we restrict to $r = 1$ and $\theta = 0$ we get $m' = m$.

2.2.2 More tensors

Tensorial densities of the Dirac theory appear very different when the Pauli algebra is used. We shall see this here only for tensors without derivative. The 16 tensorial densities of 2.1.3 are usually presented as the only possible ones, as a result of the 16 dimensions of the algebra generated by the Dirac matrices¹⁶. But it is completely false ! (Detailed calculations are in [16] appendix A). Invariants Ω_1 and Ω_2 satisfy

$$\det(\phi) = \Omega_1 + i\Omega_2 = \rho e^{i\beta}. \quad (2.52)$$

So ρ is the modulus and the Yvon-Takabayasi angle β is the argument of the determinant of ϕ .¹⁷ And ϕ is invertible if and only if $\rho \neq 0$. The calculation of components, using ξ and η , gives:

$$J = J^\mu \sigma_\mu = \phi \sigma_0 \phi^\dagger ; \quad K = K^\mu \sigma_\mu = \phi \sigma_3 \phi^\dagger. \quad (2.53)$$

But now we may see immediately that these two space-time vectors which were known orthogonal and with opposite squares are part of a list (D_0, D_1, D_2, D_3) containing four space-time vectors:

$$D_0 = J ; \quad D_1 = \phi \sigma_1 \phi^\dagger ; \quad D_2 = \phi \sigma_2 \phi^\dagger ; \quad D_3 = K. \quad (2.54)$$

The components of D_1 and D_2 are not combinations of the 16 quantities known by the complex formalism. For a Lorentz dilation R defined by a M matrix, the four D_μ vectors transform in the same way:

$$D'_\mu = \phi' \sigma_\mu \phi'^\dagger = (M\phi) \sigma_\mu (M\phi)^\dagger = M \phi \sigma_\mu \phi^\dagger M^\dagger = M D_\mu M^\dagger \quad (2.55)$$

The D_μ behave then as the space-time vectors x . We shall say that they are **contravariant**. They are also vectors with the same length. More, they are orthogonal and form a mobile basis of space-time:

$$\begin{aligned} 2D_\mu \cdot D_\nu &= D_\mu \widehat{D}_\nu + D_\nu \widehat{D}_\mu \\ &= \phi \sigma_\mu \phi^\dagger \widehat{\phi} \widehat{\sigma}_\nu \overline{\phi} + \phi \sigma_\nu \phi^\dagger \widehat{\phi} \widehat{\sigma}_\mu \overline{\phi} \\ &= \phi \sigma_\mu \rho e^{-i\beta} \widehat{\sigma}_\nu \overline{\phi} + \phi \sigma_\nu \rho e^{-i\beta} \widehat{\sigma}_\mu \overline{\phi} \\ &= \rho e^{-i\beta} \phi (\sigma_\mu \widehat{\sigma}_\nu + \sigma_\nu \widehat{\sigma}_\mu) \overline{\phi} = \rho e^{-i\beta} \phi 2\delta_{\mu\nu} \overline{\phi} \\ &= 2\delta_{\mu\nu} \rho e^{-i\beta} \phi \overline{\phi} = 2\delta_{\mu\nu} \rho e^{-i\beta} \rho e^{i\beta} \\ D_\mu \cdot D_\nu &= \delta_{\mu\nu} \rho^2. \end{aligned} \quad (2.56)$$

Of course, as we use the space-time of the restricted relativity, with the choice of a + sign for the time, we get

$$\delta_{00} = 1 ; \quad \delta_{11} = \delta_{22} = \delta_{33} = -1 ; \quad \delta_{\mu\nu} = 0 , \quad \mu \neq \nu. \quad (2.57)$$

¹⁶This false idea is one of many consequences of the confusion between real algebras and complex algebras. The tensorial densities are real quantities, not complex quantities, the dimension of the algebra on the real field is 32, not 16.

¹⁷This explains the $\sqrt{2}$ factor that we put in (2.37) and (2.38).

Among these ten relations (2.56), only three were known and computed with difficulty by the formalism of Dirac matrices:

$$J^2 = \rho^2 ; \quad K^2 = -\rho^2 ; \quad J \cdot K = 0 \quad (2.58)$$

For the $S^{\mu\nu}$ tensor, we let:

$$S_3 = S^{23}\sigma_1 + S^{31}\sigma_2 + S^{12}\sigma_3 + S^{10}i\sigma_1 + S^{20}i\sigma_2 + S^{30}i\sigma_3. \quad (2.59)$$

And we get:

$$S_3 = \phi\sigma_3\bar{\phi} \quad (2.60)$$

We see immediately that S_3 is one of four analog terms

$$S_\mu = \phi\sigma_\mu\bar{\phi}. \quad (2.61)$$

We have already encountered S_0 , because:

$$S_0 = \phi\sigma_0\bar{\phi} = \phi\bar{\phi} = \rho e^{i\beta} = \det(\phi). \quad (2.62)$$

With the 4 D_μ that have each 4 components, S_0 which has 2 components and the 3 S_j that have each 6 components, we get 36 components of tensors without derivative, instead of only 16 from the complex formalism.¹⁸

Under a dilation R defined by a M matrix, the S_μ are transformed into

$$S'_\mu = \phi'\sigma_\mu\bar{\phi}' = M\phi\sigma_\mu\bar{M}\bar{\phi} = M\phi\sigma_\mu\bar{\phi}\bar{M} = MS_\mu\bar{M}. \quad (2.63)$$

We get as a particular case:

$$\begin{aligned} \rho' e^{i\beta'} &= S'_0 = MS_0\bar{M} = M\rho e^{i\beta}\bar{M} = \rho e^{i\beta}M\bar{M} = \rho e^{i\beta}r e^{i\theta} \\ \rho' &= r\rho ; \quad \beta' = \beta + \theta. \end{aligned} \quad (2.64)$$

Between these 36 components of tensors without derivative, which depend only on the 8 real parameters of the ϕ wave, exist many relations, rather difficult to get with the formalism of Dirac matrices, but straightforward (see [14] A.2.5) with Cl_3 , for instance, with $j = 1, 2, 3$:

$$\begin{aligned} S_j^2 &= (\Omega_1 + i\Omega_2)^2 ; \quad D_0\hat{S}_j = (-\Omega_1 + i\Omega_2)D_j \\ D_j\hat{S}_j &= (-\Omega_1 + i\Omega_2)D_0. \end{aligned} \quad (2.65)$$

¹⁸Using the linear space of the linear applications from Cl_3 into Cl_3 , which is 64-dimensional, we can establish [11] that 64 terms of a particular basis can be split into

$$28 = \frac{8 \times 7}{2}$$

terms forming a basis of the Lie algebra of the $O(8)$ Lie group, and

$$36 = \frac{9 \times 8}{2}$$

terms which gives the 36 components of tensors without derivative. The number 36 is not at random. And the 16 tensorial components previously known are the invariant ones under the electric gauge (2.18).

Formulas (2.63) are completely different from formulas giving the transformation of two-ranked anti-symmetric tensors: $S^{\rho\sigma} = R_\mu^\rho R_\nu^\sigma S^{\mu\nu}$. As R_μ^ν is quadratic in M and multiplies each space-time length by r , the presence of two R factors signifies a multiplication by r^2 while (2.63) is quadratic in M and multiplies the length only by r . We can consider the two formalism as equivalent only if we restrict the invariance to the \mathcal{L}_+^\uparrow and $SL(2, \mathbb{C})$ groups, where $r^2 = r = 1$. To consider truly the invariance under the greater, consequently more restrictive, Cl_3^* group implies to abandon the formalism of Dirac matrices. The Pauli algebra is not only simpler than the algebra of Dirac matrices, it is the only formalism allowing to think the larger invariance group.

2.2.3 Plane waves

We study the simpler case, where the interaction with exterior fields is negligible, we can then take $A = 0$. The Dirac equation, in the Pauli algebra, is reduced to

$$\nabla \widehat{\phi} \sigma_{21} + m\phi = 0 \quad (2.66)$$

We consider a plane wave with a phasis φ such as:

$$\phi = \phi_0 e^{-\varphi \sigma_{12}} ; \quad \varphi = mv_\mu x^\mu. \quad (2.67)$$

We shall use the reduced speed space-time vector:

$$v = \sigma^\mu v_\mu \quad (2.68)$$

and ϕ_0 is a fixed term, which gives

$$\nabla \widehat{\phi} \sigma_{21} = \sigma^\mu \partial_\mu (\widehat{\phi}_0 e^{-\varphi \sigma_{12}}) \sigma_{21} = -mv \widehat{\phi}. \quad (2.69)$$

Consequently the wave equation (2.66) is equivalent to

$$\phi = v \widehat{\phi}. \quad (2.70)$$

Conjugating, this is equivalent to

$$\widehat{\phi} = \widehat{v} \phi. \quad (2.71)$$

Combining now the two preceding equalities, we get

$$\phi = v(\widehat{v}\phi) = (v\widehat{v})\phi = (v \cdot v)\phi. \quad (2.72)$$

So we must have

$$1 = v \cdot v = v_0^2 - \vec{v}^2 \quad (2.73)$$

$$v_0^2 = 1 + \vec{v}^2 ; \quad v_0 = \pm \sqrt{1 + \vec{v}^2} \quad (2.74)$$

a priori with two possibilities for the sign. The minus sign implies a negative energy for the particle, this has been at the beginning a disappointment for

Dirac. It is impossible to suppress these troublesome negative energies. For instance they are necessary if we want to write any wave as a sum of plane waves by using the Fourier's transformation. After the discovery of the positron, a particle with the same mass and a charge opposite to the charge of the electron, these plane waves with negative energy were associated to the positron, even though positrons seem to have a proper energy equal to, not opposite to the energy of the electron.

2.3 The Dirac equation in space-time algebra

(2.42) and its conjugation give:

$$\nabla\widehat{\phi} = qA\widehat{\phi}\sigma_{21} + m\phi\sigma_{21} ; \quad \widehat{\nabla}\phi = q\widehat{A}\phi\sigma_{21} + m\widehat{\phi}\sigma_{21} \quad (2.75)$$

We let now

$$\Psi = \begin{pmatrix} \phi & 0 \\ 0 & \widehat{\phi} \end{pmatrix} ; \quad \mathbf{A} = \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix} \quad (2.76)$$

and we get

$$\begin{aligned} \partial\Psi &= \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix} \begin{pmatrix} \phi & 0 \\ 0 & \widehat{\phi} \end{pmatrix} = \begin{pmatrix} 0 & \nabla\widehat{\phi} \\ \widehat{\nabla}\phi & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & qA\widehat{\phi}\sigma_{21} + m\phi\sigma_{21} \\ q\widehat{A}\phi\sigma_{21} + m\widehat{\phi}\sigma_{21} & 0 \end{pmatrix} \\ &= q \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix} \begin{pmatrix} \phi & 0 \\ 0 & \widehat{\phi} \end{pmatrix} \begin{pmatrix} \sigma_{21} & 0 \\ 0 & \sigma_{21} \end{pmatrix} + m \begin{pmatrix} 0 & \phi \\ \widehat{\phi} & 0 \end{pmatrix} \begin{pmatrix} \sigma_{21} & 0 \\ 0 & \sigma_{21} \end{pmatrix} \end{aligned} \quad (2.77)$$

And (1.75) gives:

$$\begin{aligned} \gamma_{12} = \gamma_1\gamma_2 &= \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_1\sigma_2 & 0 \\ 0 & -\sigma_1\sigma_2 \end{pmatrix} = \begin{pmatrix} \sigma_{21} & 0 \\ 0 & \sigma_{21} \end{pmatrix} \\ \Psi\gamma_0 &= \begin{pmatrix} \phi & 0 \\ 0 & \widehat{\phi} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & \phi \\ \widehat{\phi} & 0 \end{pmatrix} \end{aligned} \quad (2.78)$$

then (2.77) reads

$$\partial\Psi = q\mathbf{A}\Psi\gamma_{12} + m\Psi\gamma_{012} \quad (2.79)$$

which is the Hestenes' form of the Dirac equation [34]. But the interpretation of Hestenes considers the four γ_μ as a basis of space-time, while the Dirac theory considers them as fixed matrices. Since the relativistic form invariance of the Dirac wave comes from the implicit use of the Cl_3 algebra, we get with (1.76), (2.45) and (2.76)

$$\Psi' = \begin{pmatrix} \phi' & 0 \\ 0 & \widehat{\phi}' \end{pmatrix} = \begin{pmatrix} M\phi & 0 \\ 0 & \widehat{M}\widehat{\phi} \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} \begin{pmatrix} \phi & 0 \\ 0 & \widehat{\phi} \end{pmatrix} = N\Psi. \quad (2.80)$$

2.4 Invariant Dirac equation

The form invariance of the Dirac theory uses $\nabla = \overline{M}\nabla'\widehat{M}$. Since $\phi' = M\phi$ implies $\overline{\phi}' = \overline{\phi}\overline{M}$ the factor \overline{M} on the left induces to consider a multiplication by $\overline{\phi}$ on the left side of the wave equation. When and where $\rho \neq 0$, ϕ is invertible and if we multiply by $\overline{\phi}$ the Dirac equation is equivalent to

$$\overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\overline{\phi}\phi = 0. \quad (2.81)$$

Under the Lorentz dilation R defined by an element M in Cl_3 by (1.42) we get (2.45), (2.47) and (2.48) which imply

$$\overline{\phi}(\nabla\widehat{\phi})\sigma_{21} = \overline{\phi}(\overline{M}\nabla'\widehat{M}\widehat{\phi})\sigma_{21} = \overline{\phi}'(\nabla'\widehat{\phi}')\sigma_{21} \quad (2.82)$$

$$\overline{\phi}qA\widehat{\phi} = \overline{\phi}\overline{M}q'A'\widehat{M}\widehat{\phi} = \overline{\phi}'q'A'\widehat{\phi}' \quad (2.83)$$

The two first terms of (2.81) are then form invariant and the mass term is also form invariant if we have

$$m\overline{\phi}\phi = m'\overline{\phi}'\phi' = m'\overline{\phi}'\overline{M}M\phi = re^{i\theta}m'\overline{\phi}\phi \quad (2.84)$$

which is equivalent to (2.51). This mass term reads

$$m\overline{\phi}\phi = m\Omega_1 + im\Omega_2 \quad (2.85)$$

it is then the sum of a scalar and a pseudo-scalar term. The second term of the invariant Dirac equation (2.81) has another peculiarity: it is a space-time vector because it is self-adjoint:

$$(\overline{\phi}qA\widehat{\phi})^\dagger = \widehat{\phi}^\dagger qA^\dagger \overline{\phi}^\dagger = \overline{\phi}qA\widehat{\phi}. \quad (2.86)$$

We can then let with (1.38)

$$\overline{\phi}qA\widehat{\phi} = V^0 + \vec{V} = V^\mu\sigma_\mu. \quad (2.87)$$

where V is a space-time vector. Only the first term of (2.81) is general, but we can also find its peculiarities with

$$\overline{\phi}(\nabla\widehat{\phi}) = \frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) + (\overline{\phi}\nabla)\widehat{\phi}] + \frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) - (\overline{\phi}\nabla)\widehat{\phi}] \quad (2.88)$$

$$\frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) + (\overline{\phi}\nabla)\widehat{\phi}] = \frac{1}{2}\partial_\mu(\overline{\phi}\sigma^\mu\widehat{\phi}) = v = v^\mu\sigma_\mu \quad (2.89)$$

$$\frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) - (\overline{\phi}\nabla)\widehat{\phi}] = iw = iw^\mu\sigma_\mu \quad (2.90)$$

where v and w are two space-time vectors, because $v^\dagger = v$ and $(iw)^\dagger = -iw$. This gives

$$\begin{aligned} \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} &= (v + iw)\sigma_{21} \\ &= (v^0 + v^1\sigma_1 + v^2\sigma_2 + v^3\sigma_3 + iw^0 + w^1i\sigma_1 + w^2i\sigma_2 + w^3i\sigma_3)(-i\sigma_3) \\ &= w^3 + v^2\sigma_1 - v^1\sigma_2 + w^0\sigma_3 + i(-v^3 + w^2\sigma_1 - w^1\sigma_2 - v^0\sigma_3). \end{aligned} \quad (2.91)$$

Therefore the Dirac equation is equivalent to the system

$$0 = w^3 + V^0 + m\Omega_1 \quad (2.92)$$

$$0 = v^2 + V^1 \quad (2.93)$$

$$0 = -v^1 + V^2 \quad (2.94)$$

$$0 = w^0 + V^3 \quad (2.95)$$

$$0 = -v^3 + m\Omega_2 \quad (2.96)$$

$$0 = w^2 \quad (2.97)$$

$$0 = -w^1 \quad (2.98)$$

$$0 = -v^0 \quad (2.99)$$

First evidence, the gauge invariance concerns only four of the eight equations, these containing the V^μ . This is a consequence of the fact that classical electromagnetism is based on the absence of magnetic monopoles, as we will explain farther. Less evident and of great importance in the Dirac theory, the first equation is exactly $\mathcal{L} = 0$ because (a detailed calculation is in appendix A.1):

$$\begin{aligned} -\mathcal{L} &= \frac{1}{2}[(\bar{\psi}\gamma^\mu(-i\partial_\mu + qA_\mu)\psi) + (\bar{\psi}\gamma^\mu(-i\partial_\mu + qA_\mu)\psi)^\dagger] + m\bar{\psi}\psi \\ &= w^3 + V^0 + m\Omega_1 \end{aligned} \quad (2.100)$$

It is well known that varying the \mathcal{L} Lagrangian density we get the Dirac wave equation. Moreover the fact that the Dirac equation is homogeneous implies that $\mathcal{L} = 0$ when the Dirac equation is verified. Here we get the inverse situation, the equation $\mathcal{L} = 0$ is one of the wave equations and the Lagrangian formalism is a consequence of the wave equation.¹⁹ And the four equations containing the symmetric part v of $\widehat{\phi}(\nabla\widehat{\phi})$ are respectively, for indexes 0, 3, 2, 1 and the D_μ of (2.54) (see A.1):

$$0 = \nabla \cdot D_0 \quad (2.101)$$

$$0 = \nabla \cdot D_3 + 2m\Omega_2 \quad (2.102)$$

$$0 = \nabla \cdot D_1 - 2qA \cdot D_2 \quad (2.103)$$

$$0 = \nabla \cdot D_2 + 2qA \cdot D_1 \quad (2.104)$$

Equation (2.101) which is also (2.99) or (2.9) and which is known as the equation of the conservation of the probability, is now exactly one of the eight equations equivalent to the Dirac wave equation. (2.102) is known as the relation of Uhlenbeck and Laporte. (2.103) and (2.104) indicate that the D_1 and D_2 space-time vectors are not gauge invariant, the electric gauge transformation induces

¹⁹Each law of movement, in classical mechanics and in electromagnetism, may be obtained from a Lagrangian mechanism. We know nowadays this comes from the Lagrangian form and from the universality of quantum mechanics. But from where comes the Lagrangian form of quantum mechanics ? Here we see this as totally determined since the Lagrangian density is the scalar part of the wave equation and since the Lagrangian formalism implies the wave equation.

a rotation in the $D_1 - D_2$ plane. In spite of the peculiar aspect of (2.81), this invariant equation appears as the true wave equation since it is form invariant and it has so many interesting aspects.

2.4.1 Charge conjugation

Many years after the discovery of electrons the positrons were also discovered. The only difference between these particles is the sign of the charge, negative for the electron, positive for the positron. This charge is linked to the fundamental quantities \hbar , c and α , fine structure constant, by $\alpha = \frac{e^2}{\hbar c}$ This must then be seen as

$$e^2 = \alpha \hbar c ; \quad e = \pm \sqrt{\alpha \hbar c} \quad (2.105)$$

$$q = \frac{e}{\hbar c} = \pm \sqrt{\frac{\alpha}{\hbar c}} \quad (2.106)$$

The minus sign is the sign for the electron:

$$q = -\sqrt{\frac{\alpha}{\hbar c}} \quad (2.107)$$

The other sign is the sign for the positron. This gives

$$0 = \nabla \hat{\phi} \sigma_{21} - q A \hat{\phi} + m \phi \quad (2.108)$$

$$0 = \bar{\phi} (\nabla \hat{\phi}) \sigma_{21} - \bar{\phi} q A \hat{\phi} + m \bar{\phi} \phi \quad (2.109)$$

Other properties of the positron are similar to properties of the electron. Instead of the system (2.92)...(2.99) we get

$$0 = w^3 - V^0 + m \Omega_1 \quad (2.110)$$

$$0 = v^2 - V^1 \quad (2.111)$$

$$0 = -v^1 - V^2 \quad (2.112)$$

$$0 = w^0 - V^3 \quad (2.113)$$

$$0 = -v^3 + m \Omega_2 \quad (2.114)$$

$$0 = w^2 \quad (2.115)$$

$$0 = -w^1 \quad (2.116)$$

$$0 = -v^0 \quad (2.117)$$

There is also a Lagrangian formalism, whose density is the scalar part of the wave equation. There is also a conservation of the probability, which is the pseudo-scalar part of the wave equation and so on.

2.4.2 Link between the wave of the particle and the wave of the antiparticle

From the Dirac equation of a particle (2.1), quantum theory gets the wave equation of the antiparticle as follows. The wave of the electron is noted as ψ_e

and the wave of the positron is noted as ψ_p . We take the complex conjugate of (2.1):

$$0 = [\gamma^{\mu*}(\partial_\mu - iqA_\mu) - im]\psi_e^*. \quad (2.118)$$

Since (1.75) gives $\gamma_2\gamma^{\mu*} = -\gamma^\mu\gamma_2$, $\mu = 0, 1, 2, 3$, multiplying (2.118) by $i\gamma_2$ on the left we get

$$0 = -[\gamma^\mu(\partial_\mu - iqA_\mu) + im]i\gamma_2\psi_e^*. \quad (2.119)$$

Therefore, up to an arbitrary phase, quantum theory supposes ²⁰

$$\psi_p = i\gamma_2\psi_e^*. \quad (2.120)$$

which gives (2.1) with the change of q into $-q$:

$$0 = [\gamma^\mu(\partial_\mu - iqA_\mu) + im]\psi_p. \quad (2.121)$$

Using (2.2) and indexes e for the electron and p for the positron (2.120) reads

$$\begin{pmatrix} \xi_{1p} \\ \xi_{2p} \\ \eta_{1p} \\ \eta_{2p} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_{1e}^* \\ \xi_{2e}^* \\ \eta_{1e}^* \\ \eta_{2e}^* \end{pmatrix} \quad (2.122)$$

which gives

$$\xi_{1p} = \eta_{2e}^*, \quad \xi_{2p} = -\eta_{1e}^*; \quad \eta_{1p} = -\xi_{2e}^*; \quad \eta_{2p} = \xi_{1e}^* \quad (2.123)$$

Now with (2.38) and indexes e for the electron and p for the positron we get

$$\widehat{\phi}_e = \sqrt{2} \begin{pmatrix} \eta_{1e} & -\xi_{2e}^* \\ \eta_{2e} & \xi_{1e}^* \end{pmatrix}; \quad \widehat{\phi}_p = \sqrt{2} \begin{pmatrix} \eta_{1p} & -\xi_{2p}^* \\ \eta_{2p} & \xi_{1p}^* \end{pmatrix}. \quad (2.124)$$

Then (2.120), which is equivalent to (2.123), is also equivalent to

$$\widehat{\phi}_p = \widehat{\phi}_e \sigma_1 \quad (2.125)$$

or to

$$\phi_p = -\phi_e \sigma_1. \quad (2.126)$$

We shall use this equality in section 6.

2.5 About the Pauli algebra

It is in fact very strange that the Pauli algebra is fully able to read the Dirac theory. As it is a relativistic wave equation, space-time algebra should be better. But the Dirac wave has value not in the space-time algebra, only in its even sub-algebra, isomorphic to the Pauli algebra. This signifies that in space-time algebra there is too much symmetry between space and time. In the Pauli

²⁰Quantum theory uses γ_2 because only this matrix contains imaginary terms, other γ_μ matrices with the choice (1.75) are real.

algebra, even if space and time may be added inside this algebra, space and time are always distinct, do not have the same geometrical status, and each keeps its own **orientation**. Physically there is no way to change the orientation of the time, no way to change the orientation of the space. P and T transformations are purely theoretical.

There is another reason, a mathematical one, to prefer the space algebra to the space-time algebra. All the well known results about limits, continuous or differentiable functions, were obtained for **separated** topological spaces, that is to say topological spaces where two distinct points have separated vicinities. Metric spaces are separated, because two distinct points are at finite and not null distance, therefore are center of separated balls.²¹ The Cl_3 algebra is a 8-dimensional linear space, so it is a separated metric space. All the usual theorems about limits and continuous functions apply here without any difficulty. The space-time of the restricted relativity is not equipped of an Euclidean scalar product, but of a pseudo-Euclidean scalar product, as a result of the signature of space-time. Two distinct points may be at a null distance in space-time, with its pseudo-metric **the space-time is not a separated topological space**. On the mathematical point of view, we get the separation only by distinguishing truly the time and the space, and this is equivalent to work with Cl_3 . This separation is a necessary underground to define continuity or partial derivatives of functions which are used by wave equations.

On the physical point of view, if you consider two photons with a null mass produced at the same point E of the space-time, and absorbed at points A_1 and A_2 , A_1 and A_2 are separated points, but E and A_1 are not separated, E and A_2 are not separated points in space-time. Therefore you cannot define properly wave equations in space-time algebra.

The space algebra is not only the necessary frame to describe particles with spin 1/2, we shall see that it is the basic frame allowing to construct a greater space-time able to describe all relativistic quantum waves.

3 The homogeneous nonlinear wave equation.

We discuss the question of negative energies. We prove that with our wave equation all usual plane waves have a positive energy. We study the relativistic invariance, which introduces a greater invariance group, and a second space-time manifold. We discuss the wave normalization and the charge conjugation. We explain how this nonlinear wave equation gets the quantification and the true results in the case of the H atom.

When Dirac was searching his wave equation he was hoping to get a wave equation without the negative energies which came from the relativistic Klein-Gordon equation. But his wave equation also had solutions with negative energies since two signs are possible in (2.74). When positrons were discovered

²¹If two points P_1 and P_2 are separated by a distance d then the ball B_1 centered in P_1 with radius $d/3$ and the ball B_2 centered in P_2 with radius $d/3$ have none common point.

six years later, solutions with negative energy were associated to positrons and considered as a success of the new wave equation. But the creation of a pair electron-positron necessitates an amount of energy: $+ 2 \times 511$ keV. The link of the positron to negative energy implies an interpretation in another theoretical frame and is now understood only with the second quantification and a complicated reasoning.

But it is possible to solve the problem of negative energy with a simple modification of the Dirac equation. With a plane wave defined by (2.67), (2.68) we get (2.70) which gives

$$\det(\phi) = \det(v) \det(\widehat{\phi}) \quad (3.1)$$

And we have with (2.72) and (2.73)

$$1 = v \cdot v = v\widehat{v} = v\bar{v} = \det(v) \quad (3.2)$$

Therefore (3.1) reads

$$\rho e^{i\beta} = 1 \times \rho e^{-i\beta} ; \quad e^{i\beta} = \pm 1 \quad (3.3)$$

Now (2.70) gives also

$$\phi^\dagger = (v\widehat{\phi})^\dagger = \bar{\phi}v \quad (3.4)$$

$$D_0 = \phi\phi^\dagger = \phi\bar{\phi}v = \rho e^{i\beta}v = \pm\rho v \quad (3.5)$$

$$D_0^0 = \pm\rho v^0 \quad (3.6)$$

Since $D_0^0 > 0$ and since $\rho > 0$ we have only two possibilities:

$$v^0 = \sqrt{1 + \bar{v}^2} ; \quad e^{i\beta} = 1 ; \quad \phi\bar{\phi} = \rho e^{i\beta} = \rho \quad (3.7)$$

$$v^0 = -\sqrt{1 + \bar{v}^2} ; \quad e^{i\beta} = -1 ; \quad \phi\bar{\phi} = \rho e^{i\beta} = -\rho \quad (3.8)$$

The $\phi\bar{\phi}$ term is equal to its modulus ρ in the case (3.7) of a positive energy, and this is false in the case (3.8) of a negative energy.

Now we generalize and we get a new wave equation, by replacing the $\phi\bar{\phi} = \bar{\phi}\phi$ term in the invariant Dirac equation (2.81) by the modulus ρ of this term:

$$\bar{\phi}(\nabla\widehat{\phi})\sigma_{21} + \bar{\phi}qA\widehat{\phi} + m\rho = 0. \quad (3.9)$$

Multiplying by the left by $\bar{\phi}^{-1}$ we get with $\rho = e^{-i\beta}\bar{\phi}\phi$ the equivalent equation

$$\nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + me^{-i\beta}\phi = 0. \quad (3.10)$$

Equations (3.9) and (3.10) are the two main forms of the wave equation that we study in this section. We firstly obtained this wave equation from a different way. We started from the wave equation for a magnetic monopole of G. Lochak (2.21), suppressing the potential term:

$$[\gamma^\mu\partial_\mu + \frac{1}{2}\frac{m(\rho^2)c}{\hbar}(\Omega_1 - i\Omega_2\gamma_5)]\psi = 0. \quad (3.11)$$

When we choose

$$\frac{1}{2} \frac{m(\rho^2)c}{\hbar} = \frac{im}{\rho}$$

this equation becomes:

$$[\gamma^\mu \partial_\mu + ime^{-i\beta\gamma_5}]\psi = 0. \quad (3.12)$$

Since the Yvon-Takabayasi β angle is electric gauge invariant, it is perfectly possible to add an electric potential term, this gives [7]:

$$[\gamma^\mu(\partial_\mu + iqA_\mu) + ime^{-i\beta\gamma_5}]\psi = 0. \quad (3.13)$$

This wave equation is nonlinear, because β depends on the value of ψ . It is homogeneous, because if we multiply a solution ψ by a fixed real number k , β does not change, so $k\psi$ is also a solution of the equation. Our equation has many common properties with the Dirac equation. We must immediately say that, if β is null or negligible $ime^{-i\beta\gamma_5} \approx im$ and (3.13) has the Dirac equation as linear approximation.

To write this equation in the Pauli algebra, we proceed as with the Dirac equation, and in the place of (2.30) we get

$$\begin{pmatrix} 0 & \nabla + iqA \\ \widehat{\nabla} + iq\widehat{A} & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + im \begin{pmatrix} e^{-i\beta}I & 0 \\ 0 & e^{i\beta}I \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = 0 \quad (3.14)$$

this gives the following system, equivalent to (3.13):

$$(\nabla + iqA)\eta + ime^{-i\beta}\xi = 0 \quad (3.15)$$

$$(\widehat{\nabla} + iq\widehat{A})\xi + ime^{i\beta}\eta = 0 \quad (3.16)$$

Using complex conjugation on the last equation, next multiplying on the left by $-i\sigma_2$ we get:

$$(-i\sigma_2)(\widehat{\nabla}^* - iq\widehat{A}^*)\xi^* - ime^{-i\beta}(-i\sigma_2)\eta^* = 0. \quad (3.17)$$

Using again (2.34), we get

$$\nabla(-i\sigma_2\xi^*) + iqA(i\sigma_2\xi^*) + ime^{-i\beta}(i\sigma_2\eta^*) = 0. \quad (3.18)$$

The system made of (3.15) and (3.16) is then equivalent to the matrix equation :

$$\nabla \begin{pmatrix} \eta & -i\sigma_2\xi^* \end{pmatrix} + iqA \begin{pmatrix} \eta & -i\sigma_2\xi^* \end{pmatrix} + ime^{-i\beta} \begin{pmatrix} \xi & i\sigma_2\eta^* \end{pmatrix} = 0. \quad (3.19)$$

With (2.37), with (2.38) and (2.39), the homogeneous nonlinear equation (3.13) becomes [9]

$$\nabla\widehat{\phi} + qA\widehat{\phi}\sigma_{12} + me^{-i\beta}\phi\sigma_{12} = 0. \quad (3.20)$$

which is equivalent to (3.10) or to the invariant equation (3.9). The differential term $\overline{\phi}(\nabla\widehat{\phi})\sigma_{21}$ and the gauge term $\overline{\phi}qA\widehat{\phi}$ are those of the linear Dirac equation

and the only change is in the mass term where $\bar{\phi}\phi = \Omega_1 + i\Omega_2$ is replaced by $\rho = \sqrt{\Omega_1^2 + \Omega_2^2}$. We therefore get instead of (2.92) to (2.99) and with notations of section 2 the system:

$$0 = w^3 + V^0 + m\rho \quad (3.21)$$

$$0 = v^2 + V^1 \quad (3.22)$$

$$0 = -v^1 + V^2 \quad (3.23)$$

$$0 = w^0 + V^3 \quad (3.24)$$

$$0 = -v^3 \quad (3.25)$$

$$0 = w^2 \quad (3.26)$$

$$0 = -w^1 \quad (3.27)$$

$$0 = -v^0. \quad (3.28)$$

3.1 Gauge invariances

Since the differential term and the gauge term are the same and since the mass term is gauge invariant, the homogeneous nonlinear wave equation is also invariant under the electric gauge which reads in the Pauli algebra

$$\phi \mapsto \phi' = \phi e^{ia\sigma_3} ; \quad A_\mu \mapsto A'_\mu = A_\mu - \frac{1}{q}\partial_\mu a \quad (3.29)$$

As with the Dirac equation, the scalar equation (3.21) gives the Lagrangian density:

$$\begin{aligned} -\mathcal{L} &= \frac{1}{2}[(\bar{\psi}\gamma^\mu(-i\partial_\mu + qA_\mu)\psi) + (\bar{\psi}\gamma^\mu(-i\partial_\mu + qA_\mu)\psi)^\dagger] + m\rho \\ &= w^3 + V^0 + m\rho \end{aligned} \quad (3.30)$$

The conservative current linked to the gauge invariance (3.29) by Noether's theorem is here also the probability current $J = D_0$ and (2.99), which is (3.28) is exactly the conservation law (2.101).

But the homogeneous nonlinear equation allows a second gauge invariance, only a global one:

$$\phi \mapsto \phi' = e^{ia}\phi ; \quad \bar{\phi} \mapsto \bar{\phi}' = e^{ia}\bar{\phi} ; \quad \partial_\mu a = 0, \quad (3.31)$$

which gives

$$\begin{aligned} \rho e^{i\beta} = \bar{\phi}\phi \mapsto \rho' e^{i\beta'} = \bar{\phi}'\phi' = e^{2ia}\bar{\phi}\phi = \rho e^{i(\beta+2a)} \\ \rho \mapsto \rho' = \rho ; \quad \beta \mapsto \beta' = \beta + 2a. \end{aligned} \quad (3.32)$$

We shall see in section 6 that this second gauge is part of the electro-weak gauge group. Since (3.31) is also the chiral gauge²² of G. Lochak [40] we get his

²²The electric gauge multiply ξ and η by the same factor e^{ia} while the chiral gauge multiply ξ by e^{ia} and η by e^{-ia} . This gauge is a local one in the Lochak's theory or in the electro-weak theory.

result on the current: the Noether's theorem implies the existence of another conservative current, $K = D_3$ (see [16] B.1.3), and this replaces the Uhlenbeck and Laporte relation (2.102) by the conservative law:

$$0 = \nabla \cdot D_3. \quad (3.33)$$

This is with the change in the Lagrangian density and the scalar equation, the only changes, the 6 other equations are unchanged, for instance we have again (2.103) and (2.104). Since the chiral gauge multiplies ϕ by e^{ia} , $\hat{\phi}$ is multiplied by e^{-ia} , the Weyl spinor ξ is multiplied by e^{ia} and η is multiplied by e^{-ia} [42]. The generator i of the chiral gauge is exactly the i of the Pauli algebra which rules the orientation of the physical space (see 1.3.1).

Since we have lost linearity the sum $\phi_1 + \phi_2$ of two solutions of (3.20) is not necessarily a solution of (3.20). But since the equation is homogeneous and invariant under the chiral gauge, if ϕ is a solution and z is any complex number then $z\phi$ is also a solution of (3.20). This property, usual with the Schrödinger equation, is not true with the Dirac equation in Cl_3 .

3.2 Plane waves

We repeat what has been made in 2.2.3 for the linear equation. Our equation is now reduced, for $A = 0$, to:

$$\nabla \hat{\phi} + me^{-i\beta} \phi \sigma_{12} = 0. \quad (3.34)$$

If we consider a plane wave with a phase φ satisfying

$$\phi = \phi_0 e^{-\varphi \sigma_{12}}; \quad \varphi = mv_\mu x^\mu; \quad v = \sigma^\mu v_\mu. \quad (3.35)$$

where v is a fixed reduced speed and ϕ_0 is also a fixed term, we get:

$$\nabla \hat{\phi} = \sigma^\mu \partial_\mu (\hat{\phi}_0 e^{-\varphi \sigma_{12}}) = -mv \hat{\phi} \sigma_{12}. \quad (3.36)$$

(3.15) is then equivalent to

$$\phi = e^{i\beta} v \hat{\phi} \quad (3.37)$$

or to

$$\hat{\phi} = e^{-i\beta} \hat{v} \phi \quad (3.38)$$

which implies

$$\phi = e^{i\beta} v (e^{-i\beta} \hat{v} \phi) = v \hat{v} \phi = (v \cdot v) \phi. \quad (3.39)$$

So, if ϕ_0 is invertible, we must take

$$1 = v \cdot v = v_0^2 - \vec{v}^2 \quad (3.40)$$

$$v_0^2 = 1 + \vec{v}^2; \quad v_0 = \pm \sqrt{1 + \vec{v}^2}. \quad (3.41)$$

which is the expected relation for the reduced speed of the particle. More, with the nonlinear equation, we have:

$$D_0 = \phi \phi^\dagger = (e^{i\beta} v \hat{\phi}) \phi^\dagger = e^{i\beta} v (\hat{\phi} \phi^\dagger) = e^{i\beta} v \rho e^{-i\beta} = v \rho. \quad (3.42)$$

So we get

$$D_0^0 = \rho v^0 \quad (3.43)$$

and since D_0^0 and ρ are always positive, (3.40) is obtained only if

$$v_0 = \sqrt{1 + \bar{v}^2} \quad (3.44)$$

This proves that the replacement of $\bar{\phi}\phi$ by ρ in the mass term of the invariant equation is enough to rid the Dirac theory of unphysical negative energies in the electron case.

3.3 Relativistic invariance

We start from the invariant form (3.9). With a Lorentz dilation R with ratio $r = |\det(M)|$ satisfying

$$\begin{aligned} x' &= R(x) = MxM^\dagger, \quad \det(M) = re^{i\theta}, \quad \phi' = M\phi \\ \nabla &= \bar{M}\nabla'\widehat{M}; \quad qA = \bar{M}q'A'\widehat{M} \end{aligned} \quad (3.45)$$

we have also

$$\begin{aligned} \rho' e^{i\beta'} &= \det(\phi') = \phi'\bar{\phi}' = M\phi\bar{\phi}\bar{M} = M\rho e^{i\beta}\bar{M} \\ &= M\bar{M}\rho e^{i\beta} = re^{i\theta}\rho e^{i\beta} = r\rho e^{i(\beta+\theta)} \end{aligned} \quad (3.46)$$

$$\rho' = r\rho \quad (3.47)$$

$$\beta' = \beta + \theta. \quad (3.48)$$

And so we get:

$$\begin{aligned} 0 &= \bar{\phi}(\nabla\widehat{\phi})\sigma_{21} + \bar{\phi}qA\widehat{\phi} + m\rho \\ &= \bar{\phi}\bar{M}\nabla'\widehat{M}\widehat{\phi}\sigma_{21} + \bar{\phi}\bar{M}q'A'\widehat{M}\widehat{\phi} + m\rho \\ &= \bar{\phi}'(\nabla'\widehat{\phi}')\sigma_{21} + \bar{\phi}'q'A'\widehat{\phi}' + m\rho \end{aligned} \quad (3.49)$$

The homogeneous nonlinear equation is form invariant under Cl_3^* , group of invertible elements in Cl_3 , if and only if

$$m\rho = m'\rho' \quad (3.50)$$

$$m\rho = m'r\rho \quad (3.51)$$

We get then the form invariance of the wave equation under $Cl_3^* = GL(2, \mathbb{C})$ if and only if ²³

$$m = m'r \quad (3.52)$$

²³The simplification that we see here, from (2.51), is a powerful argument for the homogeneous nonlinear equation. A factor $e^{i\theta}$ in the mass term is not annoying because $m\widehat{m} = |m|^2$. But it indicates a lack of symmetry, and it explains by itself why the greater group of invariance Cl_3^* was not previously seen. The form invariance of the electromagnetism, that we shall study in the next section, and the form invariance of the Dirac theory, are fully compatible only with the homogeneous nonlinear equation and this transformation of masses.

What is the signification of this equality for physics ? If the true invariance group for the electromagnetism is not only the Lorentz group, and not even its covering group, but the greater group Cl_3^* , it must happen things similar to what comes when we go from Galilean physics to relativistic physics: there are less invariant quantities. The proper mass m_0 and ρ are both invariant under Lorentz rotations. Under Lorentz dilations induced by all M matrices, m and ρ are no more separately invariant, it is the product $m\rho$ alone which is invariant

$$m\rho = m'r\rho = m'\rho'. \quad (3.53)$$

What says to us the invariance of $m\rho$? It is the product of a reduced mass and a dilation ratio which is invariant. A reduced mass $m = \frac{m_0c}{\hbar}$ is proportional to the inverse of a space-time length, which is a frequency. This is exactly what says $E = h\nu$. Otherwise, the existence of the Planck's constant is linked to the fact that m and ρ are not separately invariant, but only their product. Or again, the existence of the Planck's constant is linked to the invariance under the Cl_3^* group, greater than the invariance group of the restricted relativity. Somewhere we can say: the existence of the Planck's constant was not fully understood on the physical point of view. To consider this greater invariance group will enable us to see things otherwise and to understand **why there is a Planck constant**.

The invariance of the $m\rho$ product has also another consequence. If we restrict the invariance to the subgroup of Lorentz rotations, m is there invariant. Since the $m\rho$ product is a constant, this implies that ρ has a physically determined value. But if we multiply ψ or ϕ by a real constant k , ρ is multiplied by k^2 . To say that ρ has a physically determined value is equivalent to say that the wave is normalized, or that, in a way or another, there is a physical condition which fixes the amplitude of the wave. We shall see this at the next page.

With the wave equation (3.10) we get the invariance of the wave equation under Cl_3^* with the condition

$$m = m'r. \quad (3.54)$$

We have implicitly considered before r and ρ on the same foot, this is natural since $\rho' = r\rho$. More generally: **There is no difference of structure between the M matrix defining the dilation R and the ϕ wave, which are both complex 2×2 matrices, elements of the space algebra Cl_3 . More precisely ϕ is a function from space-time with value into Cl_3 . Consequently ϕ , as M , can define a Lorentz dilation D , with ratio ρ , by:**

$$D : y \mapsto x = \phi y \phi^\dagger. \quad (3.55)$$

And the components D_μ^ν of the four D_μ are justly the 16 terms of the matrix of the dilation because

$$x = x^\mu \sigma_\mu = \phi y^\nu \sigma_\nu \phi^\dagger = y^\nu \phi \sigma_\nu \phi^\dagger = y^\nu D_\nu = y^\nu D_\nu^\mu \sigma_\mu; \quad x^\mu = D_\nu^\mu y^\nu. \quad (3.56)$$

There is no difference between the matrix product $M'M$ which gives the composition of dilations $R' \circ R$ and the product $M\phi$ which gives the

transformation of the wave under a dilation, and this induces then a composition of dilations $R \circ D$:

$$x' = MxM^\dagger = M\phi y\phi^\dagger M^\dagger = (M\phi)y(M\phi)^\dagger = \phi'y\phi'^\dagger. \quad (3.57)$$

This signifies that the y introduced into (3.55) does not change, either seen by the observer of x or by the observer of x' . It is independent on the observer, intrinsic to the wave.

And since ϕ is function of x , the D dilation is also a function of x , and varies from a point to another in space-time: y is not an element of the global space-time, only of the local space-time. So we must see y as the general element of the tangent space-time, at x , to a space-time manifold which depends only on the wave, not on the observer, and that we will name **intrinsic manifold**. On the contrary the dilation depends on the observer, the observer of x sees D , the one of x' sees $D' = R \circ D$.

At each point of the space-time we have then, not only one space-time manifold, but **two space-time manifolds**, and **two different affine connections**: the manifold of the x and x' , for which each relativistic observer is associated to a Lorentzian tangent space-time, and another manifold, this of the y on which we will study farther a few properties. Moreover each tangent space-time is doted of an orthonormal basis allowing to construct a Clifford algebra. The fiber of these manifolds is then a Clifford algebra.²⁴ We shall generalize this result in section 6.

3.4 Wave normalization

The invariance of the Lagrangian under all translations, as with the linear Dirac theory, induces the existence of a conservative impulse-energy tensor, the Tetrode's tensor:

$$T_\nu^\mu = i\frac{\hbar}{2}c(\bar{\psi}\gamma^\mu\partial_\nu\psi - \partial_\nu\bar{\psi}\gamma^\mu\psi) - \delta_\nu^\mu\mathcal{L}. \quad (3.58)$$

Since the wave equation is homogeneous, the Lagrangian is null and we get:

$$T_\nu^\mu = i\frac{\hbar}{2}c(\bar{\psi}\gamma^\mu\partial_\nu\psi - \partial_\nu\bar{\psi}\gamma^\mu\psi). \quad (3.59)$$

For a stationary state with energy E we have:

$$\begin{aligned} \psi &= e^{-i\frac{Et}{\hbar}}\psi(x); \quad \bar{\psi} = e^{i\frac{Et}{\hbar}}\bar{\psi}(x) \\ \partial_0\psi &= -i\frac{E}{\hbar c}\psi; \quad \partial_0\bar{\psi} = i\frac{E}{\hbar c}\bar{\psi}. \end{aligned} \quad (3.60)$$

So we get:

$$T_0^0 = i\frac{\hbar}{2}c(\bar{\psi}\gamma^0(-i\frac{E}{\hbar c})\psi - i\frac{E}{\hbar c}\bar{\psi}\gamma^0\psi) = EJ^0. \quad (3.61)$$

²⁴This was firstly pointed by W. Rodrigues, but with only one manifold. Our work is then completely different.

The condition normalizing the wave functions

$$\iiint J^0 dv = 1 \quad (3.62)$$

is then equivalent, for a bound state, to

$$\iiint T_0^0 dv = E. \quad (3.63)$$

The left term of this equality is the total energy of the wave, whilst the right term is the global energy of the electron. So it is not because we must get a probability density that the wave must be normalized. The wave is physically normalized because the energy has a determined value, not an arbitrary one, and because **the energy of the electron is the energy of its wave**. And it is the same, for the Dirac equation as for the homogeneous nonlinear equation which has the Dirac equation as linear approximation. This normalization applies evidently to solutions in the case of the H atom that we study in appendix C.

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3.5 Charge conjugation

We start now again from the link between the wave of the particle and the wave of the antiparticle, which reads (2.120) in the frame of Dirac matrices and (2.125) in the space algebra. The homogeneous nonlinear equation (3.10) is

$$\nabla \widehat{\phi}_e \sigma_{21} + qA \widehat{\phi}_e + m e^{-i\beta_e} \phi_e = 0. \quad (3.64)$$

We also have

$$\rho_e e^{i\beta_e} = \phi_e \bar{\phi}_e. \quad (3.65)$$

This gives with (2.125) and (2.126)

$$\rho_e e^{i\beta_e} = \phi_e \bar{\phi}_e = \phi_p (-\sigma_1) (\widehat{\phi}_p \sigma_1)^\dagger = -\phi_p \bar{\phi}_p = -\rho_p e^{i\beta_p} \quad (3.66)$$

Therefore (3.64) reads

$$\nabla \widehat{\phi}_p \sigma_1 \sigma_{21} + qA \widehat{\phi}_p \sigma_1 + m (-e^{-i\beta_p}) (-\phi_p \sigma_1) = 0. \quad (3.67)$$

Multiplying by σ_1 on the right this is equivalent to

$$-\nabla \widehat{\phi}_p \sigma_{21} + qA \widehat{\phi}_p + m e^{-i\beta_p} \phi_p = 0. \quad (3.68)$$

Now multiplying by $\bar{\phi}_p$ on the left we get the invariant form of the wave equation for the positron:

$$-\bar{\phi}_p \nabla \widehat{\phi}_p \sigma_{21} + q \bar{\phi}_p A \widehat{\phi}_p + m \rho_p = 0. \quad (3.69)$$

²⁵We must recall that J^0 is not equal to the relativistic invariant ρ . It is the temporal component of a space-time vector. T_0^0 is a component of a non-symmetric tensor, this is totally out of the frame of general relativity, based on the symmetric Ricci's tensor.

This means that only the differential part of the wave equation is changed. Instead of the system (3.21) to (3.28) we get now

$$0 = -w^3 + V^0 + m\rho \quad (3.70)$$

$$0 = -v^2 + V^1 \quad (3.71)$$

$$0 = v^1 + V^2 \quad (3.72)$$

$$0 = -w^0 + V^3 \quad (3.73)$$

$$0 = v^3 \quad (3.74)$$

$$0 = -w^2 \quad (3.75)$$

$$0 = w^1 \quad (3.76)$$

$$0 = v^0 \quad (3.77)$$

So the charge conjugation does not really change the sign of the charge nor the sign of the mass, only the sign of the differential term of the wave equation²⁶. Only v^μ and w^μ change sign. Therefore the Lagrangian density which is the scalar part of the wave equation becomes

$$\begin{aligned} -\mathcal{L} &= -\frac{1}{2}[(\bar{\psi}\gamma^\mu(-i\partial_\mu - qA_\mu)\psi) + (\bar{\psi}\gamma^\mu(-i\partial_\mu - qA_\mu)\psi)^\dagger] + m\rho \\ &= -w^3 + V^0 + m\rho \end{aligned} \quad (3.78)$$

And the normalization of the wave is now equivalent, in the case of a stationary state, to

$$\iiint T_0^0 dv = -E. \quad (3.79)$$

The positive mass-energy of the positron is exactly opposite to the negative energy-coefficient of the stationary state²⁷. The homogeneous nonlinear wave equation solves then the puzzle of the sign of the energy in a much more understandable way than quantum field theory: we have the negative coefficient E necessary to obtain the Fourier transformation, but the true density of energy is T_0^0 which remains positive.

Since (3.69) may be gotten from (3.9) simply by changing x^μ into $-x^\mu$ which is the PT transformation, the CPT theorem of quantum field theory is trivial.

Since the Dirac equation is the linear approximation of the homogeneous nonlinear wave equation, we get the Dirac equation of the positron from the nonlinear equation by changing the mass term. But we must account for the fact that $\beta_p \approx \pi$ and $\rho_p \approx -\Omega_{1p}$. The linear approximation of the homogeneous

²⁶The electric gauge invariance is gotten, in the place of (3.29), as $\phi_p \mapsto \phi'_p = \phi_p e^{ia\sigma_3}$ and $A_\mu \mapsto A'_\mu = A_\mu - \frac{1}{q}(-\partial_\mu a) = A_\mu - \frac{1}{q}\partial_\mu a$. Therefore the positron appears as having a charge opposite to the charge of the electron. In fact it is not q but $\partial_\mu a$ which changes sign.

²⁷The study of plane waves (3.35) in the case of (3.68) gives in the place of (3.37): $\phi_p = -e^{i\beta_p} v \widehat{\phi}_p$, $\widehat{\phi}_p = -e^{-i\beta_p} \widehat{v} \phi_p$, $\phi_p = v \widehat{v} \phi_p$. In the place of (3.42) we get $D_0 = -v\rho_p$ and then $D_0^0 = -\rho_p v^0$. Therefore we get $v^0 = -\sqrt{1 + \widehat{v}^2}$: v^0 and E are then negative.

nonlinear wave equation is then:

$$0 = -\bar{\phi}_p \nabla \hat{\phi}_p \sigma_{21} + q \bar{\phi}_p A \hat{\phi}_p - m \Omega_{1p} \quad (3.80)$$

$$0 = -\nabla \hat{\phi}_p \sigma_{21} + q A \hat{\phi}_p - m \phi_p \quad (3.81)$$

which is (2.108). We have, for the sign of E and T_0^0 , the same results as for the nonlinear equation: E is negative but T_0^0 is positive.

3.6 The Hydrogen atom

Quantum mechanics got quantized energy levels by solving Schrödinger equation in the case of the hydrogen atom, an electron “turning” around a proton. The quantification was a brilliant result, but the other results were not so good. The energy levels were not accurate, and the number of states for a principal quantum number \mathbf{n} was \mathbf{n}^2 when $2\mathbf{n}^2$ states were awaited.

We put the detailed calculation in Appendix C, it is very beautiful but also very difficult. We resume here conclusions. Our study of the solutions for the homogeneous nonlinear equation proves that a family of solutions exists, labeled by the same quantum numbers appearing into the Dirac theory, and that these solutions are very close to the solutions of the linear equation such that the Yvon-Takabayasi angle is everywhere defined and small. Now if ϕ_1 and ϕ_2 are two solutions of this family, $a_1\phi_1$ and $a_2\phi_2$ are also solutions of the nonlinear wave equation, since it is both homogeneous and invariant under the chiral gauge. But the sum $a_1\phi_1 + a_2\phi_2$ has no good luck to be a solution of the nonlinear wave equation, because the determinant giving the Yvon-Takabayasi angle is quadratic. The solutions labeled by the quantum numbers j, κ, λ, n , are plausibly the alone bound states of the hydrogen atom. And this explains why an electron in the H atom is in one of this labeled states, never in a linear combination of these states. This is an experimental fact that only this nonlinear wave equation can explain simply.

The homogeneous nonlinear equation is the only nonlinear wave equation such that quantized energy levels exist with exactly the true energy levels. We encountered another thing that pleads for the homogeneous nonlinear equation. The solutions of the linear equation, for each value of the j, κ, λ and $n > 1$ quantum numbers, depend on two arbitrary complex constants, there are two many solutions. We don't see this when we solve the Dirac equation with the method of operators, because imposing to the wave to be proper value of an ad hoc operator inevitably rules out one constant. With the method separating variables which postulates nothing about the wave, there are too many solutions. But these two complex constants are reduced to only one by (C.114) and (C.118), sufficient conditions allowing to get a Yvon-Takabayasi (C.123) angle everywhere defined and small. We can say we get the true number of solutions by imposing to the solutions to be the linear approximations of the solutions of the homogeneous nonlinear equation.

4 Invariance of electromagnetic laws

The group of Lorentz dilations induced by the Cl_3^* group is also the invariance group of electromagnetic laws. This is established for the electromagnetism of Maxwell-de Broglie, with massive photon, as for the electromagnetism with magnetic monopole.

The laws of Maxwell's electromagnetism are not invariant under the invariance group of mechanics. Putting at the center of his thought the invariance of the light speed, Einstein replaced, for all physics, the invariance group of mechanics by a greater group, containing translations and rotations, but also the Lorentz transformations including space and time. When an invariance group is replaced by another greater group, there are less invariants, for instance the mass and the impulse are no more invariant, only the proper mass remains invariant. And there is a grouping of quantities, for instance the electric field and the magnetic field become parts of a same object, the electromagnetic tensor field. Energy and impulse become parts of a same impulse-energy vector.

The existence of particles with spin 1/2 makes us to know that the group of Lorentz transformations is too small, and we must use another greater group, $SL(2, \mathbb{C})$, itself a subgroup of the group $GL(2, \mathbb{C}) = Cl_3^*$ made of the invertible elements of the space algebra. This induces to think Cl_3^* as the true invariance group, not only of the Dirac equation, but more, of all the electromagnetism, and this is what we will look at now.

4.1 Maxwell-de Broglie electromagnetism

Louis de Broglie worked out a quantum theory for the light [27] [28] where the wave of the photon is built by fusion of two Dirac spinors. The electric field \vec{E} , the magnetic field \vec{H} , the electric potential V , the potential vector \vec{A} follow Maxwell's laws in the void, supplemented by mass terms:

$$\begin{aligned} -\frac{1}{c} \frac{\partial \vec{H}}{\partial t} &= \text{curl} \vec{E} ; \quad \text{div} \vec{H} = 0 ; \quad \vec{H} = \text{curl} \vec{A} \\ \frac{1}{c} \frac{\partial \vec{E}}{\partial t} &= \text{curl} \vec{H} + k_0^2 \vec{A} ; \quad \text{div} \vec{E} = -k_0^2 V ; \quad \vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \text{grad} V \\ \frac{1}{c} \frac{\partial V}{\partial t} + \text{div} \vec{A} &= 0 \end{aligned} \quad (4.1)$$

The $k_0 = \frac{m_0 c}{\hbar}$ term contains the proper mass m_0 of the photon. That term is certainly very small, since there is very few time dispersion for the light emitted since millions of years. But Louis de Broglie answered to those who think the photon mass as exactly null that no physical experiment can prove a quantity to be exactly, with an infinite accuracy, equal to another one. To write these

Maxwell-de Broglie equations into space algebra, we let ²⁸:

$$x^0 = ct ; A^0 = V ; A = A^0 + \vec{A} ; F = \vec{E} + i\vec{H} \quad (4.2)$$

The seven equations (4.1) group together into only two equations:

$$F = \nabla \hat{A} \quad (4.3)$$

$$\widehat{\nabla} F = -k_0^2 \hat{A} \quad (4.4)$$

Because (4.3) reads:

$$\begin{aligned} \vec{E} + i\vec{H} &= (\partial_0 - \vec{\partial})(A^0 - \vec{A}) \\ 0 + \vec{E} + i\vec{H} + 0i &= (\partial_0 A^0 + \vec{\partial} \cdot \vec{A}) + (-\partial_0 \vec{A} - \vec{\partial} A^0) + i\vec{\partial} \times \vec{A} + 0i \end{aligned} \quad (4.5)$$

This equation is equivalent to the system obtained by separating the scalar, vector, pseudo-vector and pseudo-scalar parts:

$$0 = \frac{1}{c} \frac{\partial V}{\partial t} + \text{div} \vec{A} \quad (4.6)$$

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \text{grad} V \quad (4.7)$$

$$\vec{H} = \text{curl} \vec{A} \quad (4.8)$$

As for (4.4), it gives:

$$\begin{aligned} (\partial_0 + \vec{\partial})(\vec{E} + i\vec{H}) &= -k_0^2(A^0 - \vec{A}) \\ \partial_0 \vec{E} + i\partial_0 \vec{H} + \vec{\partial} \cdot \vec{E} + i\vec{\partial} \times \vec{E} + i(\vec{\partial} \cdot \vec{H} + i\vec{\partial} \times \vec{H}) &= -k_0^2(A^0 - \vec{A}) \\ \vec{\partial} \cdot \vec{E} + \partial_0 \vec{E} - \vec{\partial} \times \vec{H} + i(\partial_0 \vec{H} + \vec{\partial} \times \vec{E}) + i\vec{\partial} \cdot \vec{H} &= -k_0^2 A^0 + k_0^2 \vec{A} + i\vec{0} + 0i \end{aligned} \quad (4.9)$$

Separating, as previously, the scalar, vector, pseudo-vector and pseudo-scalar parts, this is equivalent to:

$$\text{div} \vec{E} = -k_0^2 V \quad (4.10)$$

$$\frac{1}{c} \frac{\partial \vec{E}}{\partial t} - \text{curl} \vec{H} = k_0^2 \vec{A} \quad (4.11)$$

$$\frac{1}{c} \frac{\partial \vec{H}}{\partial t} + \text{curl} \vec{E} = \vec{0} \quad (4.12)$$

$$\text{div} \vec{H} = 0 \quad (4.13)$$

These equations reduce to Maxwell equations in the void, plus the Lorentz's gauge condition, if the proper mass of the photon is null. We get then, in the place of (4.4): $\widehat{\nabla} F = 0$.

²⁸To read the electromagnetic field as $\vec{E} + i\vec{H}$ is a very old way. We will remark that the i here is the generator of the chiral gauge, it is not the i of quantum mechanics, generator of the electric gauge.

4.1.1 Invariance under Cl_3^*

With the Maxwell-de Broglie electromagnetism the potential terms V and \vec{A} are not simple tools for calculations, but are parts of physical quantities of the wave of the photon, as much as \vec{E} and \vec{H} . How vary these quantities under a rotation, under a Lorentz dilation with ratio not equal to 1 ?

Since Maxwell's laws of electromagnetism in the void are invariant, not only under the group of Lorentz transformations, but under the conformal group, which contains in addition inversions and dilations, we will suppose that, under a Lorentz dilation R with ratio r , generated by a M matrix satisfying (3.49), the electromagnetic field transforms as²⁹ :

$$F' = MFM^{-1} \quad (4.14)$$

If we write then M as $M = \sqrt{r}e^{i\frac{\theta}{2}}P$, where P is an element of $SL(2, \mathbb{C})$, we have $P^{-1} = \overline{P}$ and we get:

$$F' = \sqrt{r}e^{i\frac{\theta}{2}}PF \frac{1}{\sqrt{r}}e^{-i\frac{\theta}{2}}\overline{P} = PF\overline{P} \quad (4.15)$$

which is the same transformation as if the dilation was induced only by P , that is to say was a Lorentz transformation. So (4.14) is such that the electromagnetic field does depend neither on r , nor on θ : the presence of the Cl_3^* group is as discreet as possible.

The equation (4.4) is form invariant if we have

$$\widehat{\nabla}'F' = -k_0'^2 \widehat{A}'. \quad (4.16)$$

But we have:

$$\nabla = \overline{M}\nabla'\widehat{M}; \quad \widehat{\nabla} = M^\dagger\widehat{\nabla}'M; \quad \widehat{\nabla}' = (M^\dagger)^{-1}\widehat{\nabla}M^{-1} \quad (4.17)$$

We get then:

$$\begin{aligned} -k_0'^2 \widehat{A}' &= \widehat{\nabla}'F' = (M^\dagger)^{-1}\widehat{\nabla}M^{-1}MFM^{-1} \\ &= (M^\dagger)^{-1}\widehat{\nabla}FM^{-1} = (M^\dagger)^{-1}(-k_0^2 \widehat{A})M^{-1} \end{aligned} \quad (4.18)$$

But $k_0 = rk_0'$ since $m = rm'$ is required by the invariance of the homogeneous

²⁹We have not found this relation immediately, and we used in [13] another relation between F and F' , which gave complications to get a full invariance both for electromagnetism and Dirac theory.

nonlinear equation³⁰. We get then:

$$\begin{aligned} -k_0'^2 \widehat{A}' &= (M^\dagger)^{-1} (-r^2 k_0'^2 \widehat{A}) M^{-1} \\ \widehat{A}' &= (M^\dagger)^{-1} r e^{-i\theta} \widehat{A} r e^{i\theta} M^{-1} \\ \widehat{A}' &= (M^\dagger)^{-1} M^\dagger \widehat{M} \widehat{A} \overline{M} M M^{-1} \\ \widehat{A}' &= \widehat{M} \widehat{A} \overline{M} \end{aligned} \quad (4.19)$$

$$A' = M A M^\dagger \quad (4.20)$$

which signifies that, contrarily to qA which transforms as ∇ , A transforms as x , is contravariant. Physically this means that potential terms are linked to and move with sources. How A may be contravariant and qA covariant ? This signifies :

$$qA = \overline{M} q' A' \widehat{M} = q' \overline{M} M A M^\dagger \widehat{M} = q' r e^{i\theta} A r e^{-i\theta} = q' r^2 A \quad (4.21)$$

that is to say³¹:

$$q = q' r^2 \quad (4.22)$$

The electric charge, as the proper mass, is a relativistic invariant. The electric charge, as the mass, is not invariant under the complete group Cl_3^* , and varies when the ratio of the dilation is not equal to 1.

The transformation (4.14), and the contra-variance (4.20) of A which comes from, are compatible with the law (4.3) linking the field to the potentials, because this gives:

$$F' = \nabla' \widehat{A}' \quad (4.23)$$

$$M F M^{-1} = M (\nabla \widehat{A}) M^{-1} = M (\overline{M} \nabla' \widehat{M} \widehat{A}) M^{-1} \quad (4.24)$$

But we have, with (4.20):

$$\widehat{A}' = \widehat{M} \widehat{A} \overline{M} ; \quad \widehat{M} \widehat{A} = \widehat{A}' \overline{M}^{-1} \quad (4.25)$$

and (4.24) gives

$$\begin{aligned} M F M^{-1} &= M \overline{M} \nabla' \widehat{A}' \overline{M}^{-1} M^{-1} \\ &= (M \overline{M}) F' (M \overline{M})^{-1} = \det(M) F' (\det(M))^{-1} = F'. \end{aligned} \quad (4.26)$$

³⁰This is the best indication that the true wave equation for the electron is not the Dirac linear equation, but the homogeneous nonlinear equation. Theory of electromagnetism and wave equation of the electron are Cl_3^* form invariant only with this wave equation.

³¹We get used to go down up indexes and to go up down indexes of tensors. To do that we use the metric, and so we implicitly consider it as invariant. But if the space-time metric is invariant under the Lorentz group, it is not invariant under the greater group of dilations, so we have no more the right to raise or lower indexes of tensors. A covariant vector does not become as a contravariant vector under a dilation. Therefore we are not allowed to treat ∇ , covariant, as x , contravariant and to compute $T(\nabla)$ instead of $T(x)$, a usual way [38] in space-time algebra which we must also avoid.

Dilations are composed of Lorentz rotations and pure homothety with ratio $r > 0$. We know c is invariant under Lorentz rotations. Since a speed is a ratio distance on time, since these two terms are multiplied by the same ratio r of a pure homothety, the ratio distance on time is invariant. So we may suppose that the invariance of the light speed is true not only under the Lorentz rotations, but also under all dilations induced by an element of Cl_3^* . The other essential invariant of the Dirac theory is the fine structure constant α , which is a pure number, and so cannot vary under a dilation, no more than under a Lorentz rotation. But we have:

$$q = \frac{e}{\hbar c} ; qe = \frac{e^2}{\hbar c} = \alpha = q'e' ; qe = q'r^2e = q'e'. \quad (4.27)$$

We get then

$$e' = r^2e. \quad (4.28)$$

We have now:

$$\alpha = \frac{e^2}{\hbar c} = \frac{e'^2}{\hbar'c} = \frac{r^4e^2}{\hbar'c} ; e^2\hbar'c = \hbar cr^4e^2 \quad (4.29)$$

which gives:³²

$$\hbar' = r^4\hbar. \quad (4.30)$$

We have finally:

$$\frac{m_0c}{\hbar} = m = rm' = r \frac{m'_0c}{\hbar'} = r \frac{m'_0c}{r^4\hbar} = \frac{m'_0c}{r^3\hbar} \quad (4.31)$$

which gives:

$$m'_0 = r^3m_0. \quad (4.32)$$

Then a proper mass does not vary as an electric charge under a Lorentz dilation. An electric charge varies as a surface, a proper mass varies as a volume. There is a geometrical difference between a mass and a charge, which explains very well the difficulties to unify electromagnetism and gravitation.

Since a speed is multiplied by r^0 , an acceleration is multiplied by r^{-1} and a force is multiplied by r^2 . This is coherent with the Lorentz force since the electromagnetic field is multiplied by r^0 and the charge by r^2 .

4.2 Electromagnetism with monopoles

When Maxwell wrote his laws for magnetism, he supposed that magnetic fields come from magnetic charges, we name them now magnetic monopoles. Later this was forgotten, because during decades nobody was able to prove the existence of such monopoles. Finally teachers have presented the laws to their students as if magnetic monopoles could not exist. Nevertheless the laws of electromagnetism can easily be completed, if magnetic monopoles exist. On top

³²We must then see \hbar as a variable term under a dilation with ratio $r \neq 1$. To let $\hbar = 1$ is then a very bad habit that we must get rid of as soon as possible.

of the electric charge density ρ_e and the current density \vec{j} , a density of magnetic charge ρ_m and a density of magnetic current \vec{k} exist. On top of the electric potential V and of the potential vector \vec{A} , a magnetic potential W and a magnetic potential vector \vec{B} exist. The laws of electromagnetism with monopoles read:

$$\begin{aligned}\vec{E} &= -\text{grad}V - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} + \text{curl} \vec{B} ; \quad \vec{H} = \text{curl} \vec{A} + \text{grad}W + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \\ 0 &= \partial_\mu A^\mu = \frac{1}{c} \frac{\partial V}{\partial t} + \text{div} \vec{A} ; \quad 0 = \partial_\mu B^\mu = \frac{1}{c} \frac{\partial W}{\partial t} + \text{div} \vec{B} \\ \text{curl} \vec{H} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} &= \frac{4\pi}{c} \vec{j} ; \quad \text{div} \vec{E} = 4\pi \rho_e \\ \text{curl} \vec{E} + \frac{1}{c} \frac{\partial \vec{H}}{\partial t} &= \frac{4\pi}{c} \vec{k} ; \quad \text{div} \vec{H} = -4\pi \rho_m.\end{aligned}\tag{4.33}$$

We can see, the calculation is identical to that made to establish (4.3) and (4.4), that these equations are equivalent to:

$$F = \nabla(\widehat{A + iB})\tag{4.34}$$

$$\widehat{\nabla} F = \frac{4\pi}{c} (\widehat{j + ik})\tag{4.35}$$

where we have let:

$$B = W + \vec{B} ; \quad k = \rho_m + \vec{k}.\tag{4.36}$$

So it is very simple to go from the electromagnetism without monopole to the electromagnetism with monopoles: it is enough to add to the space-time vector made of the electric potential and the potential vector, a pseudo-vector, made of the magnetic potential and the magnetic potential vector, and to add to the space-time vector made of the density of charge and density of current, a space-time pseudo-vector made of the density of magnetic charge and the density of magnetic current. The laws are exactly the same, and we cannot see why such potentials and current should be prohibited.

The form invariance of the law (4.34) under the Cl_3^* group has evidently the same consequence for the two potentials, so B must be, as A , a contravariant vector:

$$B' = MBM^\dagger.\tag{4.37}$$

To look at what is implied by (4.35), we remark that we have:

$$\widehat{\nabla} = M^\dagger \widehat{\nabla}' M ; \quad F' M = M F\tag{4.38}$$

so we have:

$$\begin{aligned}\frac{4\pi}{c} (\widehat{j + ik}) &= \widehat{\nabla} F = M^\dagger \widehat{\nabla}' M F = M^\dagger \widehat{\nabla}' F' M \\ \frac{4\pi}{c} (\widehat{j + ik}) &= M^\dagger \frac{4\pi}{c} (\widehat{j' + ik'}) M \\ j + ik &= \overline{M} (j' + ik') \widehat{M}\end{aligned}\tag{4.39}$$

$$j = \overline{M} j' \widehat{M} ; \quad k = \overline{M} k' \widehat{M}\tag{4.40}$$

which signifies that the j and k vectors are covariant, transform as ∇ . And this is consistent with the electrostatic, because a charge density is the quotient of a charge e on a volume dv , and because we have, under a dilation with ratio r :

$$\rho_e = \frac{e}{dv} ; \quad \rho'_e = \frac{e'}{dv'} = \frac{r^2 e}{r^3 dv} = \frac{\rho_e}{r} ; \quad \rho_e = r \rho'_e. \quad (4.41)$$

We may say consequently that the choice made for the transformation of the electromagnetic field under a dilation, even if it gives surprising results, with the variation of the charge, the proper mass and the Planck term, is consistent with the elementary laws of electricity and magnetism.

4.3 Back to space-time

Until now we mainly used the space algebra, because the relativistic invariance of the Dirac theory leads inevitably to this algebra and because the even sub-algebra of the space-time algebra is isomorphic to the space algebra. Nevertheless the natural mathematical frame of any relativistic theory is the space-time algebra. It is underlying in sections 2, 3 and 4. Several reasons lead back to space-time algebra:

- 1 - The variance in r^4 of the action is the variance of a volume of space-time.
- 2 - The two manifolds relied by the Dirac wave, with (3.55), are space-time manifolds, not space manifolds.
- 3 - We shall see farther that electro-weak interactions need the space-time algebra.

4.3.1 From Cl_3 to $Cl_{1,3}$

To go from the space-time algebra to the space algebra, all you need is to use only even terms.³³ To go from the space algebra to the space-time algebra the most easy way is to use the matrix representation (1.75). The wave, noted ϕ in space algebra, is noted Ψ in space-time algebra. We have gotten in (2.76) $\Psi = \begin{pmatrix} \phi & 0 \\ 0 & \widehat{\phi} \end{pmatrix}$. N of (1.80) is similarly an even element of the space-time algebra and also the electromagnetic field that we note \mathbf{F} :

$$\begin{aligned} \mathbf{F} &= \begin{pmatrix} F & 0 \\ 0 & \widehat{F} \end{pmatrix} = \begin{pmatrix} \vec{E} + i\vec{H} & 0 \\ 0 & \vec{E} - i\vec{H} \end{pmatrix} \\ &= \begin{pmatrix} \vec{E} & 0 \\ 0 & \vec{E} \end{pmatrix} + \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} \begin{pmatrix} \vec{H} & 0 \\ 0 & \vec{H} \end{pmatrix} = \mathbf{E} + \gamma_{0123}\mathbf{H} \end{aligned} \quad (4.42)$$

³³In a Clifford algebra on a n -dimensional linear space, the linear space of even elements and the linear space of odd elements are linear spaces with dimension 2^{n-1} . Since the product of two odd elements is even the linear space of odd elements is not a sub-algebra. Since the product of two even elements is even, the linear space of even elements is a sub-algebra. In the case of $Cl_{1,3}$ the even sub-algebra is isomorphic to Cl_3 .

We can shorten the notations with:

$$\mathbf{i} = \gamma_{0123} ; \mathbf{F} = \mathbf{E} + \mathbf{iH} \quad (4.43)$$

Odd elements of the space-time algebra are the product by γ_0 of an even element, then read

$$\begin{pmatrix} P & 0 \\ 0 & \widehat{P} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & P \\ \widehat{P} & 0 \end{pmatrix} \quad (4.44)$$

We have also used, in (1.76) and (2.76):

$$\boldsymbol{\partial} = \gamma^\mu \partial_\mu = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix} ; \mathbf{A} = \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix}$$

Similarly the magnetic potential reads:

$$\mathbf{B} = \begin{pmatrix} 0 & B \\ \widehat{B} & 0 \end{pmatrix} \quad (4.45)$$

The reverse of an even element is:

$$\widetilde{N} = \begin{pmatrix} \overline{M} & 0 \\ 0 & M^\dagger \end{pmatrix} ; \widetilde{\Psi} = \begin{pmatrix} \overline{\phi} & 0 \\ 0 & \phi^\dagger \end{pmatrix} \quad (4.46)$$

The reverse of an odd element is:

$$\widetilde{\mathbf{B}} = \begin{pmatrix} 0 & B^\dagger \\ \overline{B} & 0 \end{pmatrix} \quad (4.47)$$

With tensorial densities without derivative of 2.2.2 we have:

$$\begin{aligned} \Psi \widetilde{\Psi} &= \begin{pmatrix} \phi & 0 \\ 0 & \widehat{\phi} \end{pmatrix} \begin{pmatrix} \overline{\phi} & 0 \\ 0 & \phi^\dagger \end{pmatrix} = \begin{pmatrix} \phi \overline{\phi} & 0 \\ 0 & \widehat{\phi} \phi^\dagger \end{pmatrix} = \begin{pmatrix} \rho e^{i\beta} & 0 \\ 0 & \rho e^{-i\beta} \end{pmatrix} \\ &= \begin{pmatrix} \Omega_1 + i\Omega_2 & 0 \\ 0 & \Omega_1 - i\Omega_2 \end{pmatrix} = \Omega_1 + \mathbf{i}\Omega_2 = \rho e^{i\beta} \end{aligned} \quad (4.48)$$

$$\begin{aligned} \mathbf{D}_\mu &= \begin{pmatrix} 0 & D_\mu \\ \widehat{D}_\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & \phi \sigma_\mu \phi^\dagger \\ \widehat{\phi \sigma_\mu \phi^\dagger} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \phi & 0 \\ 0 & \widehat{\phi} \end{pmatrix} \begin{pmatrix} 0 & \sigma_\mu \\ \widehat{\sigma}_\mu & 0 \end{pmatrix} \begin{pmatrix} \overline{\phi} & 0 \\ 0 & \phi^\dagger \end{pmatrix} = \Psi \gamma_\mu \widetilde{\Psi} \end{aligned} \quad (4.49)$$

$$\mathbf{S}_k = \begin{pmatrix} \phi \sigma_k \overline{\phi} & 0 \\ 0 & -\widehat{\phi} \sigma_k \phi^\dagger \end{pmatrix} = \Psi \gamma_{k0} \widetilde{\Psi} \quad (4.50)$$

We must also notice

$$\widetilde{\Psi} \Psi = \Psi \widetilde{\Psi} = \Omega_1 + \mathbf{i}\Omega_2 \quad (4.51)$$

We saw in (2.79) how Hestenes reads the Dirac equation. Since we have:

$$\Psi \gamma_0 e^{\beta \mathbf{i}} = \begin{pmatrix} 0 & e^{-\beta i} \phi \\ e^{\beta i} \widehat{\phi} & 0 \end{pmatrix} \quad (4.52)$$

the homogeneous nonlinear wave equation reads in space-time algebra:

$$\partial\Psi\gamma_{21} = m\Psi\gamma_0e^{\beta i} + q\mathbf{A}\Psi \quad (4.53)$$

For a space-time dilation, with:

$$\mathbf{x} = \begin{pmatrix} 0 & x \\ \hat{x} & 0 \end{pmatrix} ; \quad \mathbf{x}' = \begin{pmatrix} 0 & x' \\ \hat{x}' & 0 \end{pmatrix} \quad (4.54)$$

and with (1.80) and (2.45), we get equalities

$$\mathbf{x}' = N\mathbf{x}\tilde{N} ; \quad \Psi' = N\Psi. \quad (4.55)$$

4.3.2 Electromagnetism

Laws of the electromagnetism of Maxwell-de Broglie become:

$$\mathbf{F} = \partial\mathbf{A} \quad (4.56)$$

$$\partial\mathbf{F} = -k_0^2\mathbf{A} \quad (4.57)$$

because

$$\partial\mathbf{A} = \begin{pmatrix} 0 & \nabla \\ \hat{\nabla} & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ \hat{A} & 0 \end{pmatrix} = \begin{pmatrix} \nabla\hat{A} & 0 \\ 0 & \hat{\nabla}A \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & \hat{F} \end{pmatrix} = \mathbf{F} \quad (4.58)$$

$$\partial\mathbf{F} = \begin{pmatrix} 0 & \nabla \\ \hat{\nabla} & 0 \end{pmatrix} \begin{pmatrix} F & 0 \\ 0 & \hat{F} \end{pmatrix} = \begin{pmatrix} 0 & \nabla\hat{F} \\ \hat{\nabla}F & 0 \end{pmatrix} = -k_0^2\mathbf{A} \quad (4.59)$$

Laws of electromagnetism with magnetic monopoles become:

$$\mathbf{F} = \partial(\mathbf{A} + i\mathbf{B}) \quad (4.60)$$

$$\partial\mathbf{F} = \frac{4\pi}{c}(\mathbf{j} + i\mathbf{k}) \quad (4.61)$$

Since space-time algebra was well known and gave simpler results, what was the purpose of developing calculations in space algebra ? We may think that space-time algebra is too simple, there is too much symmetry between space and time. We can use services of space-time algebra as long as it is not necessary to distinguish space and time, as long as there are none zero space-time length. But it is also necessary to never forget that time is not space. Time flows only from past to future while we can go away and go back in space. Under dilations generated by elements of the Cl_3^* group the orientation of the time and the orientation of the space cannot change. There is no physical way to change the time orientation, there is no physical way to change the space orientation. P and T transformations of quantum fields are purely theoretical.

4.4 Four photons

The beginning of quantum physics was the invention by Einstein in 1905 of a theory of the light with quanta of impulse-energy. After the Newton's corpuscular theory, the Huyghens' undulatory theory was imposed by Fresnel with his transversal waves. This undulatory theory allowed a synthesis including electromagnetism and optics. The next page of this story was the discovery of the wave associated to any particle move by Louis de Broglie. When he had the Dirac equation, he returned to the initial problem of the wave of a corpuscular photon. A photon with a proper mass $m_0 < 10^{-52}$ kg gives, for all observable radiations, a non-observable dispersion of the light. The corpuscular nature of the light explains the Compton diffusion and it is compatible with the absorption and emission of the light by electrons of atoms. It allows to understand the pressure of radiation and to calculate completely all kinds of Doppler effects. In the same time the light has also the undulatory aspects of the Fresnel's waves and we know since Einstein that density of photons and intensity of the electromagnetic wave are proportional. Louis de Broglie firstly tried to associate a Dirac wave to the photon, but it was impossible to associate an electromagnetic wave. From this first attempt he understood that the electromagnetic field of the photon must be associated to the change of state of the electron interacting with the photon. And the only processes of interaction between photons and matter are absorption and photoelectric effect.

For his construction of the wave of a photon Louis de Broglie started from two Dirac spinors, one of a particle and one of an anti-particle, able to annihilate, giving then all impulse-energy to the exterior. He established also that electromagnetic quantities must be linear combinations of the wave components. In the frame of the initial formalism used by de Broglie his two spinors read

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} ; \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}. \quad (4.62)$$

They are solutions of the Dirac wave equation for a particle without charge, like a neutrino

$$\partial_0 \psi = (\alpha_1 \partial_1 + \alpha_2 \partial_2 + \alpha_3 \partial_3 + i \frac{m}{2} \alpha_4) \psi \quad (4.63)$$

and of the wave equation for its antiparticle, similar to an antineutrino

$$\partial_0 \varphi = (\alpha_1 \partial_1 - \alpha_2 \partial_2 + \alpha_3 \partial_3 - i \frac{m}{2} \alpha_4) \varphi \quad (4.64)$$

where

$$x^0 = ct ; \quad \partial_\mu = \frac{\partial}{\partial x^\mu} ; \quad m = \frac{m_0 c}{\hbar} \quad (4.65)$$

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}. \quad (4.66)$$

It is well known that these matrix relations are not enough to define uniquely α_μ . We can choose different sets of α_μ matrices. We choose here a set working with the Weyl spinors and the relativistic invariance:

$$\alpha_j = \begin{pmatrix} -\sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}, \quad j = 1, 2, 3; \quad \alpha_4 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.67)$$

where σ_j are the Pauli matrices and we let

$$\begin{aligned} \xi &= \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}; \quad \eta = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \\ \zeta^* &= \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \zeta_1^* \\ \zeta_2^* \end{pmatrix}; \quad \lambda^* = \begin{pmatrix} \varphi_3 \\ \varphi_4 \end{pmatrix} = \begin{pmatrix} \lambda_1^* \\ \lambda_2^* \end{pmatrix} \end{aligned} \quad (4.68)$$

where a^* is the complex conjugate of a . With

$$\begin{aligned} \vec{\partial} &= \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3 \\ \vec{\partial}^* &= \sigma_1 \partial_1 - \sigma_2 \partial_2 + \sigma_3 \partial_3 \end{aligned} \quad (4.69)$$

the wave equation (4.63) is equivalent to the system

$$(\partial_0 + \vec{\partial})\xi + i\frac{m}{2}\eta = 0 \quad (4.70)$$

$$(\partial_0 - \vec{\partial})\eta + i\frac{m}{2}\xi = 0. \quad (4.71)$$

ξ and η are the Weyl spinors of the wave ψ and the wave equation of the anti-particle (4.64) is equivalent to the system

$$\begin{aligned} (\partial_0 + \vec{\partial}^*)\zeta^* - i\frac{m}{2}\lambda^* &= 0 \\ (\partial_0 - \vec{\partial}^*)\lambda^* - i\frac{m}{2}\zeta^* &= 0. \end{aligned} \quad (4.72)$$

By complex conjugation we get

$$(\partial_0 + \vec{\partial})\zeta + i\frac{m}{2}\lambda = 0 \quad (4.73)$$

$$(\partial_0 - \vec{\partial})\lambda + i\frac{m}{2}\zeta = 0. \quad (4.74)$$

This system is identical to (4.70)-(4.71) if we replace ζ by ξ and λ by η . We let

$$\phi_1 = \sqrt{2} \begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix}; \quad \phi_2 = \sqrt{2} \begin{pmatrix} \zeta_1 & -\lambda_2^* \\ \zeta_2 & \lambda_1^* \end{pmatrix} \quad (4.75)$$

which have their value in the Pauli algebra. Comparing the system (4.70)-(4.71) to the system (2.31)-(2.32) we see from (2.42) that this system is equivalent to the equation

$$\nabla \hat{\phi}_1 + \frac{m}{2} \phi_1 \sigma_{12} = 0. \quad (4.76)$$

Similarly the system (4.73)-(4.74) is equivalent to

$$\nabla \widehat{\phi}_2 + \frac{m}{2} \phi_2 \sigma_{12} = 0. \quad (4.77)$$

The two spinors follow the same wave equation. This is coherent with the Dirac theory where the charge conjugation changes the sign of the charge and do not change the sign of the mass.

De Broglie had no theory for the wave of a relativistic system of particles nor for the interaction between its two spinors. So he simply supposed that his two half-photons ψ and φ are linked, have the same energy and the same impulse [27]. They satisfy

$$\varphi_k \partial_\mu \psi_i = (\partial_\mu \varphi_k) \psi_i = \frac{1}{2} \partial_\mu (\varphi_k \psi_i), \quad k, j = 1, 2, 3, 4; \quad \mu = 0, 1, 2, 3. \quad (4.78)$$

This is equivalent, with (4.62) and (4.68), to

$$\begin{aligned} \xi_k (\partial_\mu \zeta_i^*) &= (\partial_\mu \xi_k) \zeta_i^* = \frac{1}{2} \partial_\mu (\xi_k \zeta_i^*) \\ \xi_k (\partial_\mu \lambda_i^*) &= (\partial_\mu \xi_k) \lambda_i^* = \frac{1}{2} \partial_\mu (\xi_k \lambda_i^*) \\ \eta_k (\partial_\mu \zeta_i^*) &= (\partial_\mu \eta_k) \zeta_i^* = \frac{1}{2} \partial_\mu (\eta_k \zeta_i^*) \\ \eta_k (\partial_\mu \lambda_i^*) &= (\partial_\mu \eta_k) \lambda_i^* = \frac{1}{2} \partial_\mu (\eta_k \lambda_i^*). \end{aligned} \quad (4.79)$$

Wave equations (4.76) and (4.77) are form invariant under the Lorentz dilation D defined by (1.42) and satisfy

$$\phi'_1 = M \phi_1; \quad \phi'_2 = M \phi_2 \quad (4.80)$$

4.4.1 The electromagnetism of the photon

We start here from the fact seen in (4.20) that the electromagnetic potential A is a contravariant space-time vector, that is a vector transforming as x :

$$A' = M A M^\dagger. \quad (4.81)$$

We know in addition that the Pauli's principle rules that products must be antisymmetric. We also know that the σ_3 term is privileged with the Dirac equation³⁴. We then must consider a space-time vector A and an electromagnetic field F_e so defined :

$$A = \phi_1 i \sigma_3 \phi_2^\dagger - \phi_2 i \sigma_3 \phi_1^\dagger \quad (4.82)$$

$$F_e = \nabla \widehat{A} \quad (4.83)$$

³⁴We shall develop in section 5.

The variance of A and the variance of the electromagnetic field F_e under Cl_3^* are expected variances because

$$\begin{aligned}
A' &= \phi'_1 i\sigma_3 \phi'_2{}^\dagger - \phi'_2 i\sigma_3 \phi'_1{}^\dagger \\
&= (M\phi_1) i\sigma_3 (M\phi_2)^\dagger - (M\phi_2) i\sigma_3 (M\phi_1)^\dagger \\
&= M(\phi_1 i\sigma_3 \phi_2^\dagger - \phi_2 i\sigma_3 \phi_1^\dagger) M^\dagger = MAM^\dagger
\end{aligned} \tag{4.84}$$

$$\begin{aligned}
F_e &= \nabla \widehat{A} = \overline{M} \nabla' \widehat{M} \widehat{A} = \overline{M} \nabla' \widehat{M} \widehat{A} M^\dagger (\widehat{M}^\dagger)^{-1} \\
&= \overline{M} (\nabla' \widehat{A}') \overline{M}^{-1} = M^{-1} M \overline{M} F'_e \overline{M}^{-1} M^{-1} M \\
&= M^{-1} \det(M) F'_e \det(M^{-1}) M = M^{-1} F'_e M \\
F'_e &= M F_e M^{-1}.
\end{aligned} \tag{4.85}$$

A is actually a space-time vector because

$$\begin{aligned}
A^\dagger &= (\phi_1 i\sigma_3 \phi_2^\dagger - \phi_2 i\sigma_3 \phi_1^\dagger)^\dagger \\
&= \phi_2 (-i\sigma_3) \phi_1^\dagger - \phi_1 (-i\sigma_3) \phi_2^\dagger = A.
\end{aligned} \tag{4.86}$$

The calculation of A with (4.75) and the Pauli matrices (1.19) gives

$$\widehat{A} = 2i \begin{pmatrix} \eta_1 \lambda_1^* - \xi_2^* \zeta_2 - \lambda_1 \eta_1^* + \zeta_2^* \xi_2 & \eta_1 \lambda_2^* + \xi_2^* \zeta_1 - \lambda_1 \eta_2^* - \zeta_2^* \xi_1 \\ \eta_2 \lambda_1^* + \xi_1^* \zeta_2 - \lambda_2 \eta_1^* - \zeta_1^* \xi_2 & \eta_2 \lambda_2^* - \xi_1^* \zeta_1 - \lambda_2 \eta_2^* + \zeta_1^* \xi_1 \end{pmatrix}. \tag{4.87}$$

We then remark that each product is one of the products in (4.79) and this gives

$$\begin{aligned}
\partial_\mu \widehat{A} &= \partial_\mu (\widehat{\phi}_1 i\sigma_3 \overline{\phi}_2 - \widehat{\phi}_2 i\sigma_3 \overline{\phi}_1) = 2(\partial_\mu \widehat{\phi}_1) i\sigma_3 \overline{\phi}_2 - 2(\partial_\mu \widehat{\phi}_2) i\sigma_3 \overline{\phi}_1 \\
\nabla \widehat{A} &= 2[(\nabla \widehat{\phi}_1) i\sigma_3 \overline{\phi}_2 - (\nabla \widehat{\phi}_2) i\sigma_3 \overline{\phi}_1].
\end{aligned} \tag{4.88}$$

The Dirac equations (4.76) and (4.77) give then

$$\begin{aligned}
F_e &= m\phi_1 (-i\sigma_3) i\sigma_3 \overline{\phi}_2 - m\phi_2 (-i\sigma_3) i\sigma_3 \overline{\phi}_1 \\
&= m(\phi_1 \overline{\phi}_2 - \phi_2 \overline{\phi}_1).
\end{aligned} \tag{4.89}$$

Any element in the Cl_3 algebra as F_e is a sum

$$F_e = s + \vec{E} + i\vec{H} + ip \tag{4.90}$$

where s is a scalar, \vec{E} is a vector, $i\vec{H}$ is a pseudo-vector and ip is a pseudo-scalar. But we get

$$\begin{aligned}
\overline{F}_e &= s - \vec{E} - i\vec{H} + ip = \overline{m(\phi_1 \overline{\phi}_2 - \phi_2 \overline{\phi}_1)} = m(\phi_2 \overline{\phi}_1 - \phi_1 \overline{\phi}_2) \\
&= -m(\phi_1 \overline{\phi}_2 - \phi_2 \overline{\phi}_1) = -F_e = -s - \vec{E} - i\vec{H} - ip.
\end{aligned} \tag{4.91}$$

F_e is therefore a pure bivector³⁵:

$$s = 0 ; \quad p = 0 ; \quad F_e = \vec{E} + i\vec{H}. \quad (4.92)$$

This agrees with all we know about electromagnetism and optics. Now (4.83) reads

$$\vec{E} + i\vec{H} = (\partial_0 - \vec{\partial})(A^0 - \vec{A}) = \partial_0 A^0 - \vec{\partial} A^0 - \partial_0 \vec{A} + \vec{\partial} \vec{A} \quad (4.93)$$

and since

$$\vec{\partial} \vec{A} = \vec{\partial} \cdot \vec{A} + i\vec{\partial} \times \vec{A} ; \quad \partial_0 A^0 + \vec{\partial} \cdot \vec{A} = \partial_\mu A^\mu \quad (4.94)$$

(4.93) is equivalent to the system

$$0 = \partial_\mu A^\mu \quad (4.95)$$

$$\vec{E} = -\vec{\partial} A^0 - \partial_0 \vec{A} \quad (4.96)$$

$$\vec{H} = \vec{\partial} \times \vec{A}. \quad (4.97)$$

(4.96) and (4.97) are the well known relations (4.7) and (4.8) between electric and magnetic fields and the potential terms. (4.95) is the relation (4.6) known as the Lorentz gauge which is in the frame of the theory of light a necessary condition. With (4.89) we get

$$\widehat{\nabla} F_e = m \widehat{\nabla} (\phi_1 \bar{\phi}_2 - \phi_2 \bar{\phi}_1). \quad (4.98)$$

A detailed calculation of the matrices shows as in (4.87) only products present in (4.79) and this gives, similarly to (4.88)

$$\widehat{\nabla} F_e = 2m [(\widehat{\nabla} \phi_1) \bar{\phi}_2 - (\widehat{\nabla} \phi_2) \bar{\phi}_1] \quad (4.99)$$

And we get with wave equations (4.76) and (4.77)

$$\widehat{\nabla} \phi_1 = \frac{m}{2} \widehat{\phi}_1 \sigma_{21} ; \quad \widehat{\nabla} \phi_2 = \frac{m}{2} \widehat{\phi}_2 \sigma_{21} \quad (4.100)$$

$$\begin{aligned} \square \widehat{A} &= \widehat{\nabla} \nabla \widehat{A} = \widehat{\nabla} F_e = m^2 [\widehat{\phi}_1 (-i\sigma_3) \bar{\phi}_2 - \widehat{\phi}_2 (-i\sigma_3) \bar{\phi}_1] \\ \widehat{\nabla} F_e &= -m^2 \widehat{A} \end{aligned} \quad (4.101)$$

This is the expected law (4.4) and gives (4.10) to (4.12). So we get the seven laws of the electromagnetism of Maxwell in the void, completed by the terms found by Louis de Broglie containing the very small proper mass $m_0 = \frac{m\hbar}{c}$ of the photon. The seven laws are exactly the same, but the quantities are here only real or with real components, F_e is therefore exactly the electromagnetic field of the classical electromagnetism and optics. The definition (4.82)-(4.83)

³⁵We have previously supposed that the electromagnetic field F is a pure bivector, without scalar or pseudo-scalar part, for instance in (4.2). It is necessary to get the Maxwell laws without supplementary non-physical terms. Here we have nothing to suppose, the pure bivector nature of the electromagnetic field is a consequence of the antisymmetric building from two spinors and of Dirac wave equations.

allows to get a theory of a massive photon with a wave which includes real components of an electromagnetic space-time potential vector A , contravariant, and an electromagnetic bivector field F_e . This is an improvement in the theory of light, coming from the use of Cl_3 allowing an antisymmetric building, instead of Dirac matrices. Moreover the potential term is directly linked to the two spinors as much as the field bivector. It is an important difference with classical electromagnetism where potential terms are often considered as non-physical. This difference comes with quantum physics, potential terms are the electromagnetic terms present in the Dirac or Schrödinger wave equations.

The differential laws (4.83) and (4.101) are form invariant under the dilations defined by (1.42). This invariance under Cl_3^* induces that they are invariant under the restricted Lorentz group. But this larger group induces constraints which restrict the possibilities of building from two spinors.

4.4.2 Three other photons of Lochak

Following the example of (4.82) seven other space-time vectors should be possible on the model $\phi_1 X \phi_2^\dagger - \phi_2 X \phi_1^\dagger$ since the Cl_3 algebra is 8-dimensional. Only three of these seven choices: $X = -\sigma_3$, $X = i$, $X = 1$ are compatible with (4.59)³⁶ and we have established [16] that this gives the three other photons of G. Lochak [44][46]. Firstly if $X = -\sigma_3$

$$iB = \phi_1 \widehat{\sigma}_3 \phi_2^\dagger - \phi_2 \widehat{\sigma}_3 \phi_1^\dagger ; F_m = \nabla \widehat{iB} \quad (4.102)$$

gives the magnetic photon. As with the electric photon each quantity is real or with real components. It is possible to consider a total field $F = F_e + F_m$ satisfying

$$F = \nabla(\widehat{A + iB}) \quad (4.103)$$

$$\widehat{\nabla} F = -m^2(\widehat{A + iB}) \quad (4.104)$$

which are laws of the electromagnetism with electric charges and magnetic monopoles and densities of electric current j and magnetic current k satisfying

$$j = -\frac{c}{4\pi} m^2 A ; k = -\frac{c}{4\pi} m^2 B \quad (4.105)$$

very small since m_0 is very small. Even if A and B are contra-variant vectors, the variance of m allows j and k to be covariant vectors under Cl_3^* , varying as ∇ , not as x .

$$A_{(i)} = \phi_1 i \phi_2^\dagger - \phi_2 i \phi_1^\dagger ; s = \nabla \widehat{A}_{(i)} \quad (4.106)$$

defines an invariant scalar field s while

$$iB_{(1)} = \phi_1 \phi_2^\dagger - \phi_2 \phi_1^\dagger ; ip = \nabla \widehat{iB}_{(1)} \quad (4.107)$$

³⁶This comes from the non-commutative product in Cl_3 . Since σ_{12} is present in the Dirac equation, only terms commuting with σ_{12} work here.

defines an invariant pseudo-scalar field ip . We can put together cases $X = i$ and $X = 1$. We let

$$P = A_{(i)} + iB_{(1)} ; F_0 = \nabla \widehat{P} = s + ip \quad (4.108)$$

and we get

$$\widehat{\nabla} F_0 = -m^2 \widehat{P} \quad (4.109)$$

So it is possible to get in the frame of Cl_3 all four photons of the theory of de Broglie enlarged by Lochak and the whole thing is form invariant under Cl_3^* . There are differences in comparison with the construction based on the Dirac matrices : Physical quantities are real or have real components and they are obtained by antisymmetric products of spinors. This is very easy to get with the internal multiplication of the Cl_3 algebra and was very difficult to make from the complex uni-column matrices. These two differences are advantageous because vectors and tensors of the classical electromagnetism and optics have only real components. And de Broglie had understood very early that antisymmetric products are enough to get the Bose-Einstein statistics for bosons made of an even number of fermions. The scalar field of G. Lochak and the pseudo-scalar field for which de Broglie was cautious are perhaps to bring together with the scalar Higgs boson that physicists think they have identified today. Since s and p fields are obtained here independently from the field of the electric photon and the magnetic photon, their mass is not necessarily very small and may be huge. Curiously it was the first idea of de Broglie about the non-Maxwellian part of his theory. Were the Higgs bosons seen as soon as 1934 ?

4.5 Uniqueness of the electromagnetic field

The Dirac equation contains a privileged σ_3 which can be generalized as σ_j , $j = 1, 2, 3$. We generalize then (3.62)-(3.63) if we let

$$A_{(j)} = \phi_1 i\sigma_j \phi_2^\dagger - \phi_2 i\sigma_j \phi_1^\dagger \quad (4.110)$$

$$F_e = \nabla \widehat{A}_{(j)} \quad (4.111)$$

with the ϕ_1 and ϕ_2 waves following

$$\nabla \widehat{\phi}_1 = \frac{m}{2} \phi_1 (-i\sigma_j) \quad (4.112)$$

$$\nabla \widehat{\phi}_2 = \frac{m}{2} \phi_2 (-i\sigma_j) \quad (4.113)$$

We can also start from fields and get from them potentials. The electromagnetic field is then defined by (3.69)

$$F_e = m(\phi_1 \bar{\phi}_2 - \phi_2 \bar{\phi}_1) \quad (4.114)$$

Potentials terms are linked to this field by Dirac equations (3.92) and (3.93), they satisfy

$$F_e = \nabla \widehat{A}_{(j)} ; \widehat{\nabla} F_e = -m^2 \widehat{A}_{(j)} \quad (4.115)$$

It is interesting to note that F_e is independent of the index j , then the electromagnetic field is unique, a fact that will be very useful in section 5.

4.6 Remarks

The quantum wave of a photon is actually an electromagnetic wave, with field and potential term with real components.

All objects were defined from antisymmetric products of spinors. They can disappear as soon as the two spinors are equal. They can appear as soon as the two spinors are not equal.

The main difference in comparison with classical electromagnetism is about the potentials. They are not convenient vectors coming from adequate calculation. They are essential quantities. Fields are computed from them. It is a reinforcement of the position of the potentials in quantum theory. We must recall that the wave equations of quantum mechanics contain the potentials. Fields are second.

The result of the conditions (4.78) of Louis de Broglie is a linearization of the derivation of products which gives linear equations for the bosons built from the fermions. This is how the linear operator ∇ acts both in the Dirac equation and in the Maxwell equations. This linearization gives the Maxwell laws.

The form invariance of physical laws under Cl_3^* is general and rules the building of bosons from fermions. The gauge symmetries are partial, because the proper mass of the photon limits this symmetry.

5 Consequences

We study a first consequence of the two space-time manifolds and of the dilations between these two manifolds: the non isotropy of the intrinsic manifold. We link this with the existence of three kinds of leptons. We present new possibilities for the wave of systems of identical particles. We study a wave equation without possibility of Lagrangian mechanism.

The wave of the electron induces, in each space-time point, a geometric transformation from the tangent space-time to an intrinsic manifold linked to the wave, onto the usual space-time of the restricted relativity. The intrinsic space-time, contrarily to the usual space-time, is not isotropic, and we study now this anisotropy.

5.1 Anisotropy

The fact that there exists, in the Dirac theory, a privileged direction was remarked by par Louis de Broglie as soon as his first work on the Dirac equation [26] p.138 ³⁷: *"Les fonctions ψ_i solutions de ces équations sont donc intimement liées au choix des axes comme dans la théorie de Pauli; elles doivent servir à calculer des probabilités pour lesquelles l'axe des z joue un rôle particulier"*.

³⁷Translation: "The ψ_i functions solutions of these equations are then completely linked to the choice of axis as into the Pauli theory ; They must serve to calculate probabilities for which the z axis plays a particular role".

The solution to this difficulty is that with a rotation it is always possible to bring the z axis onto any direction of the space.

The solution uses then a conveniently chosen element of Cl_3^* , which generates a spatial rotation and rotates the third axis onto the chosen direction. There is always two solutions, and then the final space-time, the relative space-time, is isotropic, has no privileged direction. But the initial space-time, the intrinsic space-time, on the contrary, remains perfectly non-isotropic: before as after the rotation, it is always σ_3 which is privileged. We have remarked previously that with the Lorentz rotations of the complex formalism the γ_μ matrices are invariant. They are identical before or after the rotation. Whatever formalism is used it is always the third component of the spin that is measured and the square of the spin vector, never the first or the second component of the spin. The reason is evident if we regard the wave equation or the Lagrangian in the Clifford algebra of space. This third direction is present into the wave equation and into the Lagrangian which both contain a $i\sigma_3$.

Now, and this is the first concrete consequence of calculations with the Pauli algebra, it is perfectly possible to write two other Lagrangian densities, two other wave equations similar to the Dirac equation:

$$\nabla\widehat{\phi} + qA\widehat{\phi}\sigma_{23} + me^{-i\beta}\phi\sigma_{23} = 0 \quad (5.1)$$

$$\nabla\widehat{\phi} + qA\widehat{\phi}\sigma_{31} + me^{-i\beta}\phi\sigma_{31} = 0 \quad (5.2)$$

The invariant wave equations obtained by multiplying on the left by $\bar{\phi}$ are

$$\bar{\phi}(\nabla\widehat{\phi})\sigma_{32} + \bar{\phi}qA\widehat{\phi} + m\rho = 0 \quad (5.3)$$

$$\bar{\phi}(\nabla\widehat{\phi})\sigma_{13} + \bar{\phi}qA\widehat{\phi} + m\rho = 0 \quad (5.4)$$

With the wave equation (5.1)(5.3), it is the first axis which is privileged. The conservative space-time vectors are D_0 and D_1 . To solve the wave equation (5.1) for the hydrogen atom, we shall take again the method of separation of variables of Appendix C, making a circular permutation p on indexes 1, 2, 3 of matrices σ : $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 1$, and on indexes of formula (C.1). Since it is the only thing that changes, results will be similar.

With the wave equation (5.2), it is the second axis which is privileged. The conservative space-time vectors are D_0 and D_2 . To solve the wave equation (5.2) for the hydrogen atom, we shall take again the method of separation of variables of Appendix C, making a circular permutation $p^2 = p^{-1}$ on indexes 1, 2, 3 of matrices σ , and on indexes of formula (C.1). Since it is the only thing that changes, results will be similar.

In all what we know today about experimental physics there is something very similar. Beside electrons exist also muons and tauons. Three kinds of objects are similar and nevertheless different. Muons are known since more than 70 years, and until now without a simple explanation saying why they exist, nor what distinguishes them from electrons. We shall associate here to each category, that is said to each of the three generations, one of the three wave equations (3.9), (5.3) and (5.4). The similarity between the wave equations

allows to explain why electrons, muons and tauons have the same properties, behave in the same way into an electromagnetic field, and have the same energy levels in a Coulombian potential. In fact, to see a difference between these three equations, it is necessary to go past the wave equation of a single particle and to enter the question of a system made of different kinds of particles.³⁸

We cannot be objected that the third direction **or** the first direction may be put after a rotation in any direction: a rotation cannot turn both the third direction **and** the first direction into a given direction. So in this direction it is impossible to measure both the spin of an electron following (3.9) **and** the spin of a muon following (5.3).

In addition we know that a muon, even if it is a particle with spin $\frac{1}{2}$ as the electron, cannot spontaneously disintegrate into an alone electron. Its disintegration gives an electron plus a muonic neutrino and an electronic antineutrino. This may be understood in the following way: The wave of the muonic neutrino, as the wave of the muon, has the property to have a measurable spin in the first direction and takes away the muon's spin. The spin of the electron which is measurable in the third direction is brought by the antineutrino with a spin opposed to the spin of the electron.

Just before we supposed arbitrarily that the electron follows (3.9) and that the muon follows (5.3). One or the other could evidently follow (5.4), nothing allows us to choose. On the other hand, the choice made by the Nature of one or another equation justifies the fact that the physical space is oriented: Consider in the intrinsic space three space vectors having respectively the third direction and the wavelength of the electron, the first direction and the wavelength of the muon, the second direction and the wavelength of a tauon. Those three vectors form a basis of the intrinsic space. If we exchange now the second and the third vectors, we get another basis, with another orientation.

These three equations (3.9), (5.3) and 5.4) are equivalent only if the mass terms are equal into the different equations. But the experiment shows that these masses are completely different from a generation to another. This difference, of unknown origin, contributes to differentiate the three generations of leptons.

5.1.1 Torsion

We have calculated the affine connection of the intrinsic manifold [16]. In the case of plane waves studied in sections 2 and 3 only two terms are not zero and give a torsion. This terms of torsion are linked to the proper mass of the particle.

³⁸We know for instance that a muon within the electronic cloud of an atom, does not respect the Pauli principle of exclusion. This is rather easy to justify if that exclusion principle is linked to the spin of the different particles, because the spin of an electron following the equation (3.10) is always measured in the third direction and cannot be added or subtracted to the spin of a muon following the wave equation (5.1) which is always measured in the first direction.

5.2 Systems of electrons

The non-relativistic theory for a particle system gives to a system of two particles without spin a wave $\psi = \psi_1\psi_2$ which is the product of the two waves of each particle, when it is possible to neglect the interaction between these particles. We cannot transpose $\psi_1\psi_2$ into $\phi_1\phi_2$ which should transform into $M\phi_1M\phi_2$ under the dilation R defined in (1.42), because M does not commute with ϕ . Another product is suggested by (4.14) because if $\phi_{12} = \phi_1\phi_2^{-1}$ we get

$$\phi'_{12} = \phi'_1\phi'^{-1}_2 = M\phi_1\phi_2^{-1}M^{-1} = M\phi_{12}M^{-1}. \quad (5.5)$$

And ϕ_{12} transforms under a dilation as the electromagnetic field. But the factor $e^{-i\frac{E}{\hbar}t}$ of the non-relativistic quantum mechanics becomes in the case of the electron $e^{-\frac{E}{\hbar c}x^0\sigma_{12}}$ with the Cl_3 algebra, and with $\phi_1\phi_2^{-1}$ the energies are not added but subtracted. To get the addition of energies we can consider terms as $\phi_1\sigma_1\phi_2^{-1}$ or $\phi_1\sigma_2\phi_2^{-1}$ because σ_1 and σ_2 anti-commute with σ_{12} and

$$\sigma_1 e^{-\frac{E}{\hbar c}x^0\sigma_{12}} = e^{\frac{E}{\hbar c}x^0\sigma_{12}}\sigma_1. \quad (5.6)$$

Since we have

$$\sigma_2 = \sigma_1\sigma_{12} = \sigma_1 e^{\frac{\pi}{2}\sigma_{12}} \quad (5.7)$$

σ_1 and σ_2 differ only by a constant gauge factor and we can choose σ_1 . Since we know that two electrons are identical we can consider only terms such as $\phi_1\sigma_1\phi_2^{-1} \pm \phi_2\sigma_1\phi_1^{-1}$. The Pauli principle invites us to consider for the wave of a system of two electrons

$$\phi_{12} = \phi_1\sigma_1\phi_2^{-1} - \phi_2\sigma_1\phi_1^{-1} \quad (5.8)$$

which is antisymmetric:

$$\phi_{21} = -\phi_{12} \quad (5.9)$$

and transforms under a dilation R of dilator M as F [15]

$$\phi'_{12} = M\phi_{12}M^{-1}. \quad (5.10)$$

For a system of three electrons whose respective waves are ϕ_1, ϕ_2, ϕ_3 we consider

$$\phi_{123} = \phi_{12}\phi_3 + \phi_{23}\phi_1 + \phi_{31}\phi_2 \quad (5.11)$$

which satisfies

$$\phi_{123} = \phi_{231} = \phi_{312} = -\phi_{132} = -\phi_{321} = -\phi_{123}, \quad (5.12)$$

$$\phi'_{123} = M\phi_{123}. \quad (5.13)$$

The Pauli principle is satisfied and ϕ_{123} transforms as a unique electronic wave³⁹. Then for four electrons we consider

$$\phi_{1234} = \phi_{12}\phi_{34} + \phi_{23}\phi_{14} + \phi_{31}\phi_{24} + \phi_{34}\phi_{12} + \phi_{14}\phi_{23} + \phi_{31}\phi_{24} \quad (5.14)$$

³⁹If a similar construction is possible for quarks, this could explain why a proton or a neutron containing three quarks is seen also as a unique spinor, transforming under a Lorentz rotation as the wave of a unique electron.

which is antisymmetric, and transforms also as the electromagnetic field

$$\phi'_{1234} = M\phi_{1234}M^{-1}. \quad (5.15)$$

We can easily generalize to n electrons. We get $n + 1$ wave equations, one for each electronic wave

$$\nabla\widehat{\phi}_k + qA_k\widehat{\phi}_k\sigma_{12} + me^{-i\beta_k}\phi_k\sigma_{12} = 0 \quad (5.16)$$

where A_k is the sum of the exterior potential A and the potential created by the $n - 1$ other electrons, β_k is the Yvon-Takabayasi angle of the k th electron. And the wave of the system is antisymmetric. The wave equation of this wave is determined by the n wave equations of each particle. If n is even $\phi_{12\dots n}$ transforms under a dilation as the electromagnetic field F . The wave of an even system appears as a boson wave. Even systems compose greater systems symmetrically as in (5.14). This is the source of the Bose-Einstein statistics. If n is odd $\phi_{12\dots n}$ transforms under a dilation as a spinor ϕ . The wave of an odd system of electrons transforms under a dilation as the wave of a unique electron.

The wave of a system propagates, as the waves of each electron, in the usual space-time. It is not necessary to use configuration spaces. Difficulties coming from the difference between a unique time and several spaces disappear. Space is, as the time, unique in this model. The wave of a system is not very different from the waves of its individual parts, they continue to exist and to propagate.

This model can also explain why a muon in an electronic cloud does not follow the Pauli's principle of exclusion: with a wave equation (5.1) for instance the phase contains not a σ_{12} factor, but instead a σ_{23} factor, and the muon cannot add its impulse-energy and so cannot enter the process of construction of a wave of system described here.

5.3 Equation without Lagrangian formalism

We have seen in 2.4 that the Lagrangian density of the Dirac wave is exactly the scalar part of the invariant wave equation. The Lagrangian formalism is a consequence, not the cause of the Dirac equation. Therefore, if we modify the wave equation without changing its scalar part, we shall get a wave equation which cannot result from a Lagrangian mechanism, since the scalar part gives the Dirac equation without change [20].

Such a wave may be non-physical, since the Lagrangian formalism works very well for each physical situation. We shall nevertheless study a case, as failures and accidents are scrutinized in the industry or as counter-examples are used in mathematics. We consider the invariant wave equation

$$\overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\overline{\phi}\phi(1 + \epsilon\sigma_3) = 0. \quad (5.17)$$

where ϵ is a very small real constant. Only the mass term is changed from the invariant equation (2.81) equivalent to the Dirac equation. Computation of first

terms is unchanged, the mass term is

$$\begin{aligned} m\bar{\phi}\phi(1 + \epsilon\sigma_3) &= m(\Omega_1 + i\Omega_2)(1 + \epsilon\sigma_3) \\ &= m\Omega_1 + m\epsilon\Omega_1\sigma_3 + m\epsilon\Omega_2i\sigma_3 + im\Omega_2. \end{aligned} \quad (5.18)$$

and the system (2.118) to (2.125) becomes

$$0 = w^3 + V^0 + m\Omega_1 \quad (5.19)$$

$$0 = v^2 + V^1 \quad (5.20)$$

$$0 = -v^1 + V^2 \quad (5.21)$$

$$0 = w^0 + V^3 + m\epsilon\Omega_1 \quad (5.22)$$

$$0 = -v^3 + m\Omega_2 \quad (5.23)$$

$$0 = w^2 \quad (5.24)$$

$$0 = -w^1 \quad (5.25)$$

$$0 = -v^0 + m\epsilon\Omega_2 \quad (5.26)$$

This last equation signifies that the current of probability is no more conservative, so this wave equation is certainly unusual. Now it is easy to escape the problem of the conservation of probabilities: we start from the homogeneous non-linear equation (3.9) and we add the same mass term

$$\bar{\phi}(\nabla\hat{\phi})\sigma_{21} + \bar{\phi}qA\hat{\phi} + m\rho(1 + \epsilon\sigma_3) = 0. \quad (5.27)$$

The system (3.21) to (3.28) becomes

$$0 = w^3 + V^0 + m\rho \quad (5.28)$$

$$0 = v^2 + V^1 \quad (5.29)$$

$$0 = -v^1 + V^2 \quad (5.30)$$

$$0 = w^0 + V^3 + m\epsilon\rho \quad (5.31)$$

$$0 = -v^3 \quad (5.32)$$

$$0 = w^2 \quad (5.33)$$

$$0 = -w^1 \quad (5.34)$$

$$0 = -v^0 \quad (5.35)$$

And as previously we have two conservative currents, $J = D_0$ and $K = D_3$. It is easy to see that (5.27) is invariant under Cl_3^* , there are two gauge invariances (see 3.1). The angular momentum operators of the Dirac theory are still available, but there is no Hamiltonian to commute with them. This wave equation cannot come from a Lagrangian density since such a density should modify (5.28), which gives (3.9), not (5.27).

5.3.1 Plane waves

We consider a plane wave with a phase φ satisfying (3.35) with the vector v defined in (3.35). Without exterior electromagnetic field we get in the place of (3.39)

$$-mv\hat{\phi} + me^{-i\beta}\phi(1 + \epsilon\sigma_3) = 0. \quad (5.36)$$

This gives

$$\phi(1 + \epsilon\sigma_3) = e^{i\beta}v\hat{\phi} \quad (5.37)$$

Conjugating, we then get

$$\hat{\phi}(1 - \epsilon\sigma_3) = e^{-i\beta}\hat{v}\phi \quad (5.38)$$

Together we have

$$\begin{aligned} \phi(1 + \epsilon\sigma_3)(1 - \epsilon\sigma_3) &= e^{i\beta}v\hat{\phi}(1 - \epsilon\sigma_3) \\ \phi(1 - \epsilon^2) &= e^{i\beta}ve^{-i\beta}\hat{v}\phi \\ (1 - \epsilon^2)\phi &= v\hat{v}\phi \end{aligned} \quad (5.39)$$

and we get

$$v \cdot v = v\hat{v} = 1 - \epsilon^2 \quad (5.40)$$

$$\|v\| = \sqrt{1 - \epsilon^2} \quad (5.41)$$

We let then

$$c' = c\sqrt{1 - \epsilon^2} ; \quad v = v'\sqrt{1 - \epsilon^2}. \quad (5.42)$$

And we get

$$\|v'\| = 1. \quad (5.43)$$

First consequence: c' , not c is the velocity limit of this unusual quantum object. The present study has no known physical application, but this wave equation indicates that the limit speed c is not so general [20] than we thought. ⁴⁰

6 Electro-weak and strong interactions

We firstly use the Clifford algebra of space-time to study electro-weak interactions. We begin with weak interactions of the electron with its neutrino and their charge conjugate waves. Next we study invariances of these interactions. We extend the gauge group to the quark sector, using $Cl_{5,1}$. We present in this frame the $SU(3)$ group of chromodynamics. We study the geometric transformation generated by the complete wave in the frame of the 6-dimensional space-time. We get two remarkable identities which make the wave often invertible. We get wave equations with mass term, that are form invariant and gauge invariant.

⁴⁰More, if ϵ tends to 1 the limit speed tends to 0 and may be very small.

6.1 The Weinberg-Salam model for the electron

An extension of the Dirac equation up to electro-weak interactions [54] was tried by D. Hestenes [33] and by R. Boudet [3] [4] in the frame of the Clifford algebra $Cl_{1,3}$ of the space-time. We used in [18] another start which implies the use of the greater frame $Cl_{2,3}$. A greater frame was necessary because we wanted to use no supplementary condition. Now, the study that we shall make in this section necessitates to use the condition (2.120) or (2.125) which is used in the standard model to link the wave of the antiparticle to the wave of the particle. Therefore the mathematical frame remains the space-time algebra which has 16 dimensions, enough to accommodate 8 real parameters of the wave of the electron and 8 parameters of its neutrino.⁴¹ We saw in 3.5 that the condition (2.125) is compatible with the nonlinear equation and that it solves the puzzle of negative energies.

We begin with the electron case and we follow [31]. We change nothing to the Dirac wave of the electron, noted as ψ_e in the Dirac formalism and as ϕ_e with space algebra. We use the same notations as previously for Weyl spinors. The electron wave is noted as ψ_e , the wave of the electronic neutrino as ψ_n , the wave of the positron as ψ_p and the wave of the electronic anti-neutrino as ψ_a . As previously right spinors are ξ Weyl spinors and left ones are η spinors.

$$\psi_e = \begin{pmatrix} \xi_e \\ \eta_e \end{pmatrix} ; \psi_n = \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} ; \psi_p = \begin{pmatrix} \xi_p \\ \eta_p \end{pmatrix} ; \psi_a = \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix}. \quad (6.1)$$

We have

$$\phi_e = \sqrt{2} \begin{pmatrix} \xi_e & -i\sigma_2\eta_e^* \end{pmatrix} ; \widehat{\phi}_e = \sqrt{2} \begin{pmatrix} \eta_e & -i\sigma_2\xi_e^* \end{pmatrix} \quad (6.2)$$

$$\phi_n = \sqrt{2} \begin{pmatrix} \xi_n & -i\sigma_2\eta_n^* \end{pmatrix} ; \widehat{\phi}_n = \sqrt{2} \begin{pmatrix} \eta_n & -i\sigma_2\xi_n^* \end{pmatrix} \quad (6.3)$$

$$\widehat{\phi}_p = \widehat{\phi}_e\sigma_1 ; \widehat{\phi}_a = \widehat{\phi}_n\sigma_1 \quad (6.4)$$

which gives

$$\widehat{\phi}_p = \sqrt{2} \begin{pmatrix} \eta_p & -i\sigma_2\xi_p^* \end{pmatrix} ; \phi_p = \sqrt{2} \begin{pmatrix} \xi_p & -i\sigma_2\eta_p^* \end{pmatrix} \quad (6.5)$$

$$\widehat{\phi}_a = \sqrt{2} \begin{pmatrix} \eta_a & -i\sigma_2\xi_a^* \end{pmatrix} ; \phi_a = \sqrt{2} \begin{pmatrix} \xi_a & -i\sigma_2\eta_a^* \end{pmatrix} \quad (6.6)$$

$$\begin{aligned} \xi_{1p} &= \eta_{2e}^*, & \xi_{2p} &= -\eta_{1e}^*; & \eta_{1p} &= -\xi_{2e}^*; & \eta_{2p} &= \xi_{1e}^* \\ \xi_{1a} &= \eta_{2n}^*, & \xi_{2a} &= -\eta_{1n}^*; & \eta_{1a} &= -\xi_{2n}^*; & \eta_{2a} &= \xi_{1n}^*. \end{aligned} \quad (6.7)$$

We used in [18] a wave Ψ function of the space-time with value into the $Cl_{2,3} = M_4(\mathbb{C})$ algebra. We disposed waves of particle on the above line and waves of antiparticle on the second line to get correct transformations of left and right waves under Lorentz dilations, as we will see in 6.5. We used a σ_1 factor which was a necessary factor exchanging ξ and η terms. This allows to get a wave for these four particles of the electronic sector⁴² and with the link (2.125) between

⁴¹We shall see in 6.6 that only four of them are non zero.

⁴²We could exchange the places of ϕ_e and ϕ_n . With (6.8) the wave of the electron has value in the even sub-algebra and the neutrino has value in the odd part of the algebra. The other choice is possible if we adapt the definition of projectors in (6.12) to (6.16)

the wave of the particle and the wave of the antiparticle we have

$$\Psi = \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_a \sigma_1 & \widehat{\phi}_p \sigma_1 \end{pmatrix} = \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_e \end{pmatrix} \quad (6.8)$$

Now with (6.4) and (6.8) the wave is a function of space-time with value in the Clifford algebra of space-time. The Weinberg-Salam model uses ξ_e, η_e, η_n and supposes $\xi_n = 0$. This hypothesis will be used further in 6.6. To separate ξ_e, η_e and η_n the Weinberg-Salam model uses projectors $\frac{1}{2}(1 \pm \gamma_5)$, which read with our choice (1.75) of Dirac matrices:

$$\frac{1}{2}(1 - \gamma_5)\psi = \psi_L = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ \eta \end{pmatrix} \quad (6.9)$$

$$\frac{1}{2}(1 + \gamma_5)\psi = \psi_R = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix}. \quad (6.10)$$

Then for particles left waves are η waves and right waves are ξ waves. This is Cl_3^* invariant, consequently relativistic invariant, since under a Lorentz dilation D defined by $D : x \mapsto x' = MxM^\dagger$ we have (1.30) : $\xi' = M\xi, \eta' = \widehat{M}\eta$. The γ_5 matrix is not included in the space-time algebra⁴³, but this is not a problem here, because the projectors separating ξ and η are in space algebra $\frac{1}{2}(1 \pm \sigma_3)$:

$$\begin{aligned} \phi_R &= \sqrt{2} \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} = \phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \phi \frac{1}{2}(1 + \sigma_3) \\ \phi_L &= \sqrt{2} \begin{pmatrix} 0 & -i\sigma_2 \eta^* \\ 0 & 1 \end{pmatrix} = \phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \phi \frac{1}{2}(1 - \sigma_3) \\ \widehat{\phi}_L &= \sqrt{2} \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix} = \widehat{\phi} \frac{1}{2}(1 + \sigma_3); \quad \widehat{\phi}_R = \widehat{\phi} \frac{1}{2}(1 - \sigma_3). \end{aligned} \quad (6.11)$$

We define now two projectors P_\pm and four operators P_0, P_1, P_2, P_3 acting in the space-time algebra as follows

$$P_\pm(\Psi) = \frac{1}{2}(\Psi \pm \mathbf{i}\Psi\gamma_{21}); \quad \mathbf{i} = \gamma_{0123} \quad (6.12)$$

$$P_0(\Psi) = \Psi\gamma_{21} + \frac{1}{2}\Psi\mathbf{i} + \frac{1}{2}\mathbf{i}\Psi\gamma_{30} = \Psi\gamma_{21} + P_-(\Psi)\mathbf{i} \quad (6.13)$$

$$P_1(\Psi) = \frac{1}{2}(\mathbf{i}\Psi\gamma_0 + \Psi\gamma_{012}) = P_+(\Psi)\gamma_3\mathbf{i} \quad (6.14)$$

$$P_2(\Psi) = \frac{1}{2}(\Psi\gamma_3 - \mathbf{i}\Psi\gamma_{123}) = P_+(\Psi)\gamma_3 \quad (6.15)$$

$$P_3(\Psi) = \frac{1}{2}(-\Psi\mathbf{i} + \mathbf{i}\Psi\gamma_{30}) = P_+(\Psi)(-\mathbf{i}). \quad (6.16)$$

⁴³This was wrongly considered as a reason to forbid the use of space-time algebra.

Noting $P_\mu P_\nu(\Psi) = P_\mu[P_\nu(\Psi)]$ they satisfy

$$\begin{aligned}
P_1 P_2 &= P_3 = -P_2 P_1 \\
P_2 P_3 &= P_1 = -P_3 P_2 \\
P_3 P_1 &= P_2 = -P_1 P_3 \\
P_1^2 &= P_2^2 = P_3^2 = -P_+ \\
P_0 P_j &= P_j P_0 = -\mathbf{i} P_j, \quad j = 1, 2, 3.
\end{aligned} \tag{6.17}$$

The Weinberg-Salam model replaces partial derivatives ∂_μ by covariant derivatives

$$D_\mu = \partial_\mu - ig_1 \frac{Y}{2} B_\mu - ig_2 T_j W_\mu^j \tag{6.18}$$

with $T_j = \frac{\tau_j}{2}$ for a doublet of left-handed particles and $T_j = 0$ for a singlet of right-handed particle. Y is the weak hypercharge, $Y_L = -1$, $Y_R = -2$ for the electron. To transpose into space-time algebra, we let

$$D = \sigma^\mu D_\mu; \quad \mathbf{D} = \gamma^\mu D_\mu = \begin{pmatrix} 0 & D \\ \widehat{D} & 0 \end{pmatrix} \tag{6.19}$$

$$B = \sigma^\mu B_\mu; \quad \mathbf{B} = \gamma^\mu B_\mu = \begin{pmatrix} 0 & B \\ \widehat{B} & 0 \end{pmatrix} \tag{6.20}$$

$$W^j = \sigma^\mu W_\mu^j; \quad \mathbf{W}^j = \gamma^\mu W_\mu^j = \begin{pmatrix} 0 & W^j \\ \widehat{W}^j & 0 \end{pmatrix} \tag{6.21}$$

We will prove now that (6.18) comes from

$$\mathbf{D} = \boldsymbol{\partial} + \frac{g_1}{2} \mathbf{B} P_0 + \frac{g_2}{2} (\mathbf{W}^1 P_1 + \mathbf{W}^2 P_2 + \mathbf{W}^3 P_3). \tag{6.22}$$

Firstly we have in space-time algebra (see 1.4.1)

$$\boldsymbol{\partial} \Psi = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix} \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_a \sigma_1 & \widehat{\phi}_p \sigma_1 \end{pmatrix} = \begin{pmatrix} \nabla \widehat{\phi}_a \sigma_1 & \nabla \widehat{\phi}_p \sigma_1 \\ \widehat{\nabla} \phi_e & \widehat{\nabla} \phi_n \end{pmatrix} \tag{6.23}$$

while we get with (6.19)

$$\mathbf{D} \Psi = \begin{pmatrix} 0 & D \\ \widehat{D} & 0 \end{pmatrix} \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_a \sigma_1 & \widehat{\phi}_p \sigma_1 \end{pmatrix} = \begin{pmatrix} D \widehat{\phi}_a \sigma_1 & D \widehat{\phi}_p \sigma_1 \\ \widehat{D} \phi_e & \widehat{D} \phi_n \end{pmatrix}. \tag{6.24}$$

To compute $P_0(\Psi)$ we use

$$P_0(\Psi) = \begin{pmatrix} p_0(\phi_e) & p_0(\phi_n) \\ p_0(\widehat{\phi}_a) \sigma_1 & p_0(\widehat{\phi}_p) \sigma_1 \end{pmatrix}. \tag{6.25}$$

And we get

$$\Psi\gamma_{21} = i \begin{pmatrix} \phi_e\sigma_3 & \phi_n\sigma_3 \\ -\widehat{\phi}_a\sigma_3\sigma_1 & -\widehat{\phi}_p\sigma_3\sigma_1 \end{pmatrix} \quad (6.26)$$

$$\frac{1}{2}\Psi\mathbf{i} = \frac{i}{2} \begin{pmatrix} \phi_e & -\phi_n \\ \widehat{\phi}_a\sigma_1 & -\widehat{\phi}_p\sigma_1 \end{pmatrix} \quad (6.27)$$

$$\frac{1}{2}\mathbf{i}\Psi\gamma_{30} = \frac{i}{2} \begin{pmatrix} \phi_e\sigma_3 & -\phi_n\sigma_3 \\ \widehat{\phi}_a\sigma_3\sigma_1 & -\widehat{\phi}_p\sigma_3\sigma_1 \end{pmatrix}. \quad (6.28)$$

Then we get

$$P_0(\Psi) = i \begin{pmatrix} \phi_e \frac{1+3\sigma_3}{2} & \phi_n \frac{-1+\sigma_3}{2} \\ \widehat{\phi}_a \frac{1-\sigma_3}{2}\sigma_1 & \widehat{\phi}_p \frac{-1-3\sigma_3}{2}\sigma_1 \end{pmatrix} \quad (6.29)$$

$$p_0(\phi_e) = i\phi_e \frac{1+3\sigma_3}{2} = i(2\phi_{eR} - \phi_{eL}) \quad (6.30)$$

$$p_0(\phi_n) = i\phi_n \frac{-1+\sigma_3}{2} = -i\phi_{nL} \quad (6.31)$$

$$p_0(\widehat{\phi}_p) = i\widehat{\phi}_p \frac{-1-3\sigma_3}{2} = -i(2\widehat{\phi}_{pL} - \widehat{\phi}_{pR}) \quad (6.32)$$

$$p_0(\widehat{\phi}_a) = i\widehat{\phi}_a \frac{1-\sigma_3}{2} = i\widehat{\phi}_{aR} \quad (6.33)$$

with

$$\phi_{eL} = \phi_e \frac{1-\sigma_3}{2}; \quad \phi_{eR} = \phi_e \frac{1+\sigma_3}{2} \quad (6.34)$$

$$\phi_{nL} = \phi_n \frac{1-\sigma_3}{2}; \quad \phi_{nR} = \phi_n \frac{1+\sigma_3}{2} \quad (6.35)$$

$$\widehat{\phi}_{pL} = \widehat{\phi}_p \frac{1+\sigma_3}{2}; \quad \widehat{\phi}_{pR} = \widehat{\phi}_p \frac{1-\sigma_3}{2} \quad (6.36)$$

$$\widehat{\phi}_{aL} = \widehat{\phi}_a \frac{1+\sigma_3}{2}; \quad \widehat{\phi}_{aR} = \widehat{\phi}_a \frac{1-\sigma_3}{2} \quad (6.37)$$

which gives

$$\begin{aligned} \mathbf{B}P_0(\Psi) &= \begin{pmatrix} 0 & B \\ \widehat{B} & 0 \end{pmatrix} \begin{pmatrix} p_0(\phi_e) & p_0(\phi_n) \\ p_0(\widehat{\phi}_a)\sigma_1 & p_0(\widehat{\phi}_p)\sigma_1 \end{pmatrix} \\ &= i \begin{pmatrix} B\widehat{\phi}_{aR}\sigma_1 & -B(2\widehat{\phi}_{pL} - \widehat{\phi}_{pR})\sigma_1 \\ \widehat{B}(2\phi_{eR} - \phi_{eL}) & -\widehat{B}\phi_{nL} \end{pmatrix}. \end{aligned} \quad (6.38)$$

Next we let

$$P_j(\Psi) = \begin{pmatrix} p_j(\phi_e) & p_j(\phi_n) \\ p_j(\widehat{\phi}_a)\sigma_1 & p_j(\widehat{\phi}_p)\sigma_1 \end{pmatrix}, \quad j = 1, 2, 3. \quad (6.39)$$

We get for $j = 1$

$$\begin{aligned} \mathbf{i}\Psi\gamma_0 &= i \begin{pmatrix} \phi_n & \phi_e \\ -\widehat{\phi}_p\sigma_1 & -\widehat{\phi}_a\sigma_1 \end{pmatrix}; \quad \Psi\gamma_{012} = i \begin{pmatrix} -\phi_n\sigma_3 & -\phi_e\sigma_3 \\ \widehat{\phi}_p\sigma_3\sigma_1 & \widehat{\phi}_a\sigma_3\sigma_1 \end{pmatrix} \\ P_1(\Psi) &= i \begin{pmatrix} \phi_n \frac{1-\sigma_3}{2} & \phi_e \frac{1-\sigma_3}{2} \\ \widehat{\phi}_p \frac{-1+\sigma_3}{2}\sigma_1 & \widehat{\phi}_a \frac{-1+\sigma_3}{2}\sigma_1 \end{pmatrix} = i \begin{pmatrix} \phi_{nL} & \phi_{eL} \\ -\widehat{\phi}_{pR}\sigma_1 & -\widehat{\phi}_{aR}\sigma_1 \end{pmatrix} \end{aligned} \quad (6.40)$$

$$\begin{aligned} p_1(\phi_e) &= i\phi_{nL}; \quad p_1(\phi_n) = i\phi_{eL} \\ p_1(\widehat{\phi}_a) &= -i\widehat{\phi}_{pR}; \quad p_1(\widehat{\phi}_p) = -i\widehat{\phi}_{aR}. \end{aligned} \quad (6.41)$$

We get for $j = 2$

$$\begin{aligned} \Psi\gamma_3 &= \begin{pmatrix} -\phi_n\sigma_3 & \phi_e\sigma_3 \\ \widehat{\phi}_p\sigma_3\sigma_1 & -\widehat{\phi}_a\sigma_3\sigma_1 \end{pmatrix}; \quad -\mathbf{i}\Psi\gamma_{123} = \begin{pmatrix} \phi_n & -\phi_e \\ -\widehat{\phi}_p\sigma_1 & \widehat{\phi}_a\sigma_1 \end{pmatrix} \\ P_2(\Psi) &= \begin{pmatrix} \phi_n \frac{1-\sigma_3}{2} & \phi_e \frac{-1+\sigma_3}{2} \\ \widehat{\phi}_p \frac{-1+\sigma_3}{2}\sigma_1 & \widehat{\phi}_a \frac{1-\sigma_3}{2}\sigma_1 \end{pmatrix} = \begin{pmatrix} \phi_{nL} & -\phi_{eL} \\ -\widehat{\phi}_{pR}\sigma_1 & \widehat{\phi}_{aR}\sigma_1 \end{pmatrix} \end{aligned} \quad (6.42)$$

$$\begin{aligned} p_2(\phi_e) &= \phi_{nL}; \quad p_2(\phi_n) = -\phi_{eL} \\ p_2(\widehat{\phi}_a) &= -\widehat{\phi}_{pR}; \quad p_2(\widehat{\phi}_p) = \widehat{\phi}_{aR}. \end{aligned} \quad (6.43)$$

We get for $j = 3$

$$\begin{aligned} -\mathbf{i}\Psi &= i \begin{pmatrix} -\phi_e & \phi_n \\ -\widehat{\phi}_a\sigma_1 & \widehat{\phi}_p\sigma_1 \end{pmatrix}; \quad \mathbf{i}\Psi\gamma_{30} = i \begin{pmatrix} \phi_e\sigma_3 & -\phi_n\sigma_3 \\ \widehat{\phi}_a\sigma_3\sigma_1 & -\widehat{\phi}_p\sigma_3\sigma_1 \end{pmatrix} \\ P_3(\Psi) &= i \begin{pmatrix} \phi_e \frac{-1+\sigma_3}{2} & \phi_n \frac{1-\sigma_3}{2} \\ \widehat{\phi}_a \frac{-1+\sigma_3}{2}\sigma_1 & \widehat{\phi}_p \frac{1-\sigma_3}{2}\sigma_1 \end{pmatrix} = i \begin{pmatrix} -\phi_{eL} & \phi_{nL} \\ -\widehat{\phi}_{aR}\sigma_1 & \widehat{\phi}_{pR}\sigma_1 \end{pmatrix} \end{aligned} \quad (6.44)$$

$$\begin{aligned} p_3(\phi_e) &= -i\phi_{eL}; \quad p_3(\phi_n) = i\phi_{nL} \\ p_3(\widehat{\phi}_a) &= -i\widehat{\phi}_{aR}; \quad p_3(\widehat{\phi}_p) = i\widehat{\phi}_{pR}. \end{aligned} \quad (6.45)$$

We also have

$$\begin{aligned} \mathbf{W}^j P_j(\Psi) &= \begin{pmatrix} 0 & W^j \\ \widehat{W}^j & 0 \end{pmatrix} \begin{pmatrix} p_j(\phi_e) & p_j(\phi_n) \\ p_j(\widehat{\phi}_a)\sigma_1 & p_j(\widehat{\phi}_p)\sigma_1 \end{pmatrix} \\ &= \begin{pmatrix} W^j p_j(\widehat{\phi}_a)\sigma_1 & W^j p_j(\widehat{\phi}_p)\sigma_1 \\ \widehat{W}^j p_j(\phi_e) & \widehat{W}^j p_j(\phi_n) \end{pmatrix}. \end{aligned} \quad (6.46)$$

Therefore (6.22) gives the system

$$D\widehat{\phi}_a = \nabla\widehat{\phi}_a + \frac{g_1}{2} B p_0(\widehat{\phi}_a) + \frac{g_2}{2} W^j p_j(\widehat{\phi}_a) \quad (6.47)$$

$$D\widehat{\phi}_p = \nabla\widehat{\phi}_p + \frac{g_1}{2} B p_0(\widehat{\phi}_p) + \frac{g_2}{2} W^j p_j(\widehat{\phi}_p) \quad (6.48)$$

$$\widehat{D}\phi_e = \widehat{\nabla}\phi_e + \frac{g_1}{2} \widehat{B} p_0(\phi_e) + \frac{g_2}{2} \widehat{W}^j p_j(\phi_e) \quad (6.49)$$

$$\widehat{D}\phi_n = \widehat{\nabla}\phi_n + \frac{g_1}{2} \widehat{B} p_0(\phi_n) + \frac{g_2}{2} \widehat{W}^j p_j(\phi_n). \quad (6.50)$$

With (6.30) to (6.33), (6.41), (6.43) and (6.45) this gives

$$D\widehat{\phi}_a = \nabla\widehat{\phi}_a + i\frac{g_1}{2}B\widehat{\phi}_{aR} + \frac{g_2}{2}[(-iW^1 - W^2)\widehat{\phi}_{pR} - iW^3\widehat{\phi}_{aR}] \quad (6.51)$$

$$D\widehat{\phi}_p = \nabla\widehat{\phi}_p + i\frac{g_1}{2}B(-2\widehat{\phi}_{pL} + \widehat{\phi}_{pR}) + \frac{g_2}{2}[(-iW^1 + W^2)\widehat{\phi}_{aR} + iW^3\widehat{\phi}_{pR}] \quad (6.52)$$

$$\widehat{D}\widehat{\phi}_e = \widehat{\nabla}\widehat{\phi}_e + i\frac{g_1}{2}\widehat{B}(2\widehat{\phi}_{eR} - \widehat{\phi}_{eL}) + \frac{g_2}{2}[(i\widehat{W}^1 + \widehat{W}^2)\widehat{\phi}_{nL} - i\widehat{W}^3\widehat{\phi}_{eL}] \quad (6.53)$$

$$\widehat{D}\widehat{\phi}_n = \widehat{\nabla}\widehat{\phi}_n - i\frac{g_1}{2}\widehat{B}\widehat{\phi}_{nL} + \frac{g_2}{2}[(i\widehat{W}^1 - \widehat{W}^2)\widehat{\phi}_{eL} + i\widehat{W}^3\widehat{\phi}_{nL}]. \quad (6.54)$$

Using the conjugation $M \mapsto \widehat{M}$ in (6.53) and (6.54) this gives

$$D\widehat{\phi}_a = \nabla\widehat{\phi}_a + i\frac{g_1}{2}B\widehat{\phi}_{aR} + i\frac{g_2}{2}[(-W^1 + iW^2)\widehat{\phi}_{pR} - W^3\widehat{\phi}_{aR}] \quad (6.55)$$

$$D\widehat{\phi}_p = \nabla\widehat{\phi}_p + i\frac{g_1}{2}B(-2\widehat{\phi}_{pL} + \widehat{\phi}_{pR}) + i\frac{g_2}{2}[-(W^1 + iW^2)\widehat{\phi}_{aR} + W^3\widehat{\phi}_{pR}] \quad (6.56)$$

$$D\widehat{\phi}_e = \nabla\widehat{\phi}_e + i\frac{g_1}{2}B(-2\widehat{\phi}_{eR} + \widehat{\phi}_{eL}) + i\frac{g_2}{2}[-(W^1 + iW^2)\widehat{\phi}_{nL} + W^3\widehat{\phi}_{eL}] \quad (6.57)$$

$$D\widehat{\phi}_n = \nabla\widehat{\phi}_n + i\frac{g_1}{2}B\widehat{\phi}_{nL} + i\frac{g_2}{2}[(-W^1 + iW^2)\widehat{\phi}_{eL} - W^3\widehat{\phi}_{nL}]. \quad (6.58)$$

We study firstly the case of the electron and its neutrino. We have with (6.34)

$$\widehat{\phi}_{eL} = \widehat{\phi}_e \frac{1 + \sigma_3}{2} ; \widehat{\phi}_{eL}\sigma_3 = \widehat{\phi}_{eL} \quad (6.59)$$

$$\widehat{\phi}_{eR} = \widehat{\phi}_e \frac{1 - \sigma_3}{2} ; \widehat{\phi}_{eR}\sigma_3 = -\widehat{\phi}_{eR} \quad (6.60)$$

$$-2\widehat{\phi}_{eR} + 2\widehat{\phi}_{eL} = 2(\widehat{\phi}_{eR} + \widehat{\phi}_{eL})\sigma_3 = 2\widehat{\phi}_e\sigma_3 \quad (6.61)$$

and we get for (6.57) and (6.58)

$$D\widehat{\phi}_e = \nabla\widehat{\phi}_e + g_1B\widehat{\phi}_e i\sigma_3 + \frac{i}{2}(-g_1B + g_2W^3)\widehat{\phi}_{eL} - i\frac{g_2}{2}(W^1 + iW^2)\widehat{\phi}_{nL} \quad (6.62)$$

$$D\widehat{\phi}_n = \nabla\widehat{\phi}_n - \frac{i}{2}(-g_1B + g_2W^3)\widehat{\phi}_{nL} + i\frac{g_2}{2}(-W^1 + iW^2)\widehat{\phi}_{eL}. \quad (6.63)$$

We separate left and right parts of the wave:

$$D\widehat{\phi}_{nR} = \nabla\widehat{\phi}_{nR} ; \widehat{D}\widehat{\phi}_{nR} = \widehat{\nabla}\widehat{\phi}_{nR} \quad (6.64)$$

$$D\widehat{\phi}_{nL} = \nabla\widehat{\phi}_{nL} + \frac{i}{2}(g_1B - g_2W^3)\widehat{\phi}_{nL} + i\frac{g_2}{2}(-W^1 + iW^2)\widehat{\phi}_{eL} \quad (6.65)$$

$$D\widehat{\phi}_{eR} = \nabla\widehat{\phi}_{eR} - ig_1B\widehat{\phi}_{eR} ; \widehat{D}\widehat{\phi}_{eR} = \widehat{\nabla}\widehat{\phi}_{eR} + ig_1\widehat{B}\widehat{\phi}_{eR} \quad (6.66)$$

$$D\widehat{\phi}_{eL} = \nabla\widehat{\phi}_{eL} + \frac{i}{2}(g_1B + g_2W^3)\widehat{\phi}_{eL} - i\frac{g_2}{2}(W^1 + iW^2)\widehat{\phi}_{nL}. \quad (6.67)$$

which is equivalent to⁴⁴

$$D_\mu \xi_n = \partial_\mu \xi_n \quad (6.68)$$

$$D_\mu \eta_n = \partial_\mu \eta_n + i \frac{g_1}{2} B_\mu \eta_n - i \frac{g_2}{2} [(W_\mu^1 - iW_\mu^2) \eta_e + W_\mu^3 \eta_n] \quad (6.69)$$

$$D_\mu \xi_e = \partial_\mu \xi_e + i g_1 B_\mu \xi_e \quad (6.70)$$

$$D_\mu \eta_e = \partial_\mu \eta_e + i \frac{g_1}{2} B_\mu \eta_e - i \frac{g_2}{2} [(W_\mu^1 + iW_\mu^2) \eta_n - W_\mu^3 \eta_e]. \quad (6.71)$$

(6.69) and (6.71) give for the “lepton doublet” $\psi_L = \begin{pmatrix} \eta_n \\ \eta_e \end{pmatrix}$ with weak isospin $Y = -1$:

$$D_\mu \psi_L = \partial_\mu \psi_L - i g_1 \frac{Y}{2} B_\mu \psi_L - i \frac{g_2}{2} W_\mu^j \tau_j \psi_L$$

$$\tau_1 = \gamma_0 ; \tau_2 = \gamma_{123} ; \tau_3 = \gamma_5 \quad (6.72)$$

With (6.68) we see that the right part of the wave of the neutrino does not interact. (6.70) is interpreted as a SU(2) singlet $\psi_R = \xi$ with weak isospin $Y = -2$:

$$D_\mu \psi_R = \partial_\mu \psi_R - i g_1 \frac{Y}{2} B_\mu \psi_R \quad (6.73)$$

Finally we see here that all features of weak interactions, with a doublet of left waves, a singlet of right wave, a non-interacting right neutrino, a charge conjugation exchanging right and left waves are obtained here from simple hypothesis :

1 - The wave of all components of the lepton sector, electron, positron, electronic neutrino and anti-neutrino, is the function (6.8) of space-time with value into the Clifford algebra of the space-time.

2 - Four operators P_0, P_1, P_2, P_3 are defined by (6.13) to (6.16).

3 - A covariant derivative is defined by (6.22).

It is now easy to use the system (6.55) to (6.58) to get all other features of the Weinberg-Salam model. It considers the “charged currents” W^+ and W^- defined by

$$W_\mu^+ = W_\mu^1 + iW_\mu^2 ; \quad W_\mu^- = -W_\mu^1 + iW_\mu^2$$

$$W^+ = W^1 + iW^2 ; \quad W^- = -W^1 + iW^2 \quad (6.74)$$

where $i = \sigma_{123}$ is the generator of the chiral gauge⁴⁵, not the i_3 of the electric gauge. We will use (6.61) and similarly

$$\widehat{\phi}_{pR} = \widehat{\phi}_p \frac{1 - \sigma_3}{2} ; \quad \widehat{\phi}_{pR} \sigma_3 = -\widehat{\phi}_{pR} \quad (6.75)$$

$$\widehat{\phi}_{pL} = \widehat{\phi}_p \frac{1 + \sigma_3}{2} ; \quad \widehat{\phi}_{pL} \sigma_3 = \widehat{\phi}_{pL} \quad (6.76)$$

$$\widehat{\phi}_{pL} - \widehat{\phi}_{pR} = \widehat{\phi}_{pL} \sigma_3 + \widehat{\phi}_{pR} \sigma_3 = (\widehat{\phi}_{pL} + \widehat{\phi}_{pR}) \sigma_3 = \widehat{\phi}_p \sigma_3. \quad (6.77)$$

⁴⁴Since $\phi_{eR} = \sqrt{2}(\xi_e \cdot 0)$ we must use the second equality (6.66) to get (6.70).

⁴⁵This is another sufficient reason to abandon the formalism of Dirac matrices, which uses a unique i . It is therefore unable to discriminate the different gauges at work.

Then (6.55) to (6.58) reads

$$D\hat{\phi}_a = \nabla\hat{\phi}_a + \frac{i}{2}(g_1B - g_2W^3)\hat{\phi}_{aR} + i\frac{g_2}{2}W^-\hat{\phi}_{pR} \quad (6.78)$$

$$D\hat{\phi}_p = \nabla\hat{\phi}_p + ig_1B(\hat{\phi}_{pR} - \hat{\phi}_{pL}) + \frac{i}{2}(-g_1B + g_2W^3)\hat{\phi}_{pR} - i\frac{g_2}{2}W^+\hat{\phi}_{aR} \quad (6.79)$$

$$D\hat{\phi}_n = \nabla\hat{\phi}_n + \frac{i}{2}(g_1B - g_2W^3)\hat{\phi}_{nL} + \frac{i}{2}g_2W^-\hat{\phi}_{eL} \quad (6.80)$$

$$D\hat{\phi}_e = \nabla\hat{\phi}_e + ig_1B(\hat{\phi}_{eL} - \hat{\phi}_{eR}) + \frac{i}{2}(-g_1B + g_2W^3)\hat{\phi}_{eL} - \frac{i}{2}g_2W^+\hat{\phi}_{nL}. \quad (6.81)$$

The Weinberg-Salam model uses the electromagnetic potential A , a θ_W angle and a Z^0 term satisfying⁴⁶

$$g_1 = \frac{q}{\cos(\theta_W)} ; \quad g_2 = \frac{q}{\sin(\theta_W)} ; \quad q = \frac{e}{\hbar c} \quad (6.82)$$

$$-g_1B + g_2W^3 = \sqrt{g_1^2 + g_2^2}Z^0 = \frac{2q}{\sin(2\theta_W)}Z^0 \quad (6.83)$$

$$B = \cos(\theta_W)A - \sin(\theta_W)Z^0 ; \quad W^3 = \sin(\theta_W)A + \cos(\theta_W)Z^0 \quad (6.84)$$

$$B + iW^3 = e^{i\theta_W}(A + iZ^0) ; \quad A + iZ^0 = e^{-i\theta_W}(B + iW^3). \quad (6.85)$$

Using (6.77) this gives for the system (6.78) to (6.81)

$$D\hat{\phi}_a = \nabla\hat{\phi}_a - \frac{iq}{\sin(2\theta_W)}Z^0\hat{\phi}_{aR} + i\frac{g_2}{2}W^-\hat{\phi}_{pR} \quad (6.86)$$

$$D\hat{\phi}_p = \nabla\hat{\phi}_p - qA\hat{\phi}_p\sigma_{12} + q\tan(\theta_W)Z^0\hat{\phi}_p\sigma_{12} + i\frac{q}{\sin(2\theta_W)}Z^0\hat{\phi}_{pR} - i\frac{g_2}{2}W^+\hat{\phi}_{aR} \quad (6.87)$$

$$D\hat{\phi}_e = \nabla\hat{\phi}_e + qA\hat{\phi}_e\sigma_{12} - q\tan(\theta_W)Z^0\hat{\phi}_e\sigma_{12} + i\frac{q}{\sin(2\theta_W)}Z^0\hat{\phi}_{eL} - i\frac{g_2}{2}W^+\hat{\phi}_{nL} \quad (6.88)$$

$$D\hat{\phi}_n = \nabla\hat{\phi}_n - \frac{iq}{\sin(2\theta_W)}Z^0\hat{\phi}_{nL} + i\frac{g_2}{2}W^-\hat{\phi}_{eL}. \quad (6.89)$$

Equation (6.88) contains first and second terms $\nabla\hat{\phi} + qA\hat{\phi}\sigma_{12}$ of the Dirac equation, giving the electromagnetic interaction of the electron. Equation (6.87) contains first and second terms $-\nabla\hat{\phi} + qA\hat{\phi}\sigma_{12}$ of the Dirac equation for a positron. There is no potential A term in (6.86) nor (6.89), since anti-neutrino and neutrino have no electromagnetic interaction. As we have

$$\hat{\phi}_e\sigma_{12} = i(-\hat{\phi}_{eR} + \hat{\phi}_{eL}) \quad (6.90)$$

⁴⁶(6.85) indicates that Z^0 is similar to Cabibbo-Ferrari's B of (4.36).

we can read (6.89) and (6.88) as

$$D\widehat{\phi}_{nR} = \nabla\widehat{\phi}_{nR} \quad (6.91)$$

$$D\widehat{\phi}_{nL} = \nabla\widehat{\phi}_{nL} - i\frac{q}{\sin(2\theta_W)}Z^0\widehat{\phi}_{nL} + i\frac{q}{2\sin(\theta_W)}W^-\widehat{\phi}_{eL} \quad (6.92)$$

$$D\widehat{\phi}_{eR} = \nabla\widehat{\phi}_{eR} + qA\widehat{\phi}_{eR}\sigma_{12} + iq\tan(\theta_W)Z^0\widehat{\phi}_{eR} \quad (6.93)$$

$$D\widehat{\phi}_{eL} = \nabla\widehat{\phi}_{eL} + qA\widehat{\phi}_{eL}\sigma_{12} + iq[-\tan(\theta_W) + \frac{1}{\sin(2\theta_W)}]Z^0\widehat{\phi}_{eL} - i\frac{q}{2\sin(\theta_W)}W^+\widehat{\phi}_{nL} \quad (6.94)$$

Terms containing W^+ and W^- which couple left electron to left neutrino generate “charged currents”, terms containing Z^0 generate “neutral currents”. The Z^0 boson is linked to ϕ_L , ϕ_{nL} and ϕ_R , not to ϕ_{nR} . Similarly we can read (6.86) and (6.87) as

$$D\widehat{\phi}_{aL} = \nabla\widehat{\phi}_{aL} \quad (6.95)$$

$$D\widehat{\phi}_{aR} = \nabla\widehat{\phi}_{aR} - i\frac{q}{\sin(2\theta_W)}Z^0\widehat{\phi}_{aR} + i\frac{q}{2\sin(\theta_W)}W^-\widehat{\phi}_{pR} \quad (6.96)$$

$$D\widehat{\phi}_{pL} = \nabla\widehat{\phi}_{pL} - qA\widehat{\phi}_{pL}\sigma_{12} + iq\tan(\theta_W)Z^0\widehat{\phi}_{pL} \quad (6.97)$$

$$D\widehat{\phi}_{pR} = \nabla\widehat{\phi}_{pR} - qA\widehat{\phi}_{pR}\sigma_{12} + iq[-\tan(\theta_W) + \frac{1}{\sin(2\theta_W)}]Z^0\widehat{\phi}_{pR} - i\frac{q}{2\sin(\theta_W)}W^+\widehat{\phi}_{aR}. \quad (6.98)$$

(6.95) signifies that the left anti-neutrino does not interact by electro-weak forces. The electric charge of the positron is opposite to the charge of the electron. But the comparison with the same relation for the electron shows that, contrarily to what is said about charge conjugation, thought as changing the sign of any quantum number, only the exchange between left and right waves, plus the multiplication on the right by σ_3 give a change of sign. Other coefficients are conserved when passing from electron to positron or from neutrino to anti-neutrino. Charge conjugation must be seen as a pure quantum transformation acting only on the wave, as described in 3.5.⁴⁷ A similar result was obtained by G. Lochak [42] for the magnetic monopole: charge conjugation does not change the sign of magnetic charges, and there is no polarization of the void resulting from spontaneous creation of pairs. It is the same for neutrinos, there is no creation of pairs of neutrino-anti-neutrino similar to the creation of pairs particle-antiparticle with opposite electric charges.⁴⁸

6.2 Invariances

As with the electromagnetism, we can enlarge the relativistic invariance to the greater group Cl_3^* . With the Lorentz dilation R defined by a M element in Cl_3^*

⁴⁷More, if we try to build a charge conjugation changing other signs, we get instead of (6.17) relations which do not give a $U(1) \times SU(2)$ gauge invariance.

⁴⁸This is also consistent with (4.76)-(4.77) where charge conjugation in the neutrino case gives the same wave equation.

satisfying $x \mapsto x' = MxM^\dagger$ we have

$$\begin{aligned} \phi'_e &= M\phi_e ; \quad \phi'_n = M\phi_n ; \quad \widehat{\phi}'_p = \widehat{M}\widehat{\phi}_p ; \quad \widehat{\phi}'_a = \widehat{M}\widehat{\phi}_a ; \quad \Psi' = N\Psi \\ N &= \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} ; \quad \widetilde{N} = \begin{pmatrix} \overline{M} & 0 \\ 0 & M^\dagger \end{pmatrix} \end{aligned} \quad (6.99)$$

We may consider g_1B and g_2W^j , linked to qA , as covariants vectors:

$$\begin{aligned} g_1B &= \overline{M}g'_1B'\widehat{M} ; \quad g_2W^j = \overline{M}g'_2W^{j'}\widehat{M} \\ g_1\mathbf{B} &= \widetilde{N}g'_1\mathbf{B}'N ; \quad g_2\mathbf{W}^j = \widetilde{N}g'_2\mathbf{W}^{j'}N. \end{aligned} \quad (6.100)$$

This allows D to be a covariant vector, varying as ∇ ;

$$\begin{aligned} D &= \overline{M}D'\widehat{M} ; \quad \nabla = \overline{M}\nabla'\widehat{M} \\ \mathbf{D} &= \widetilde{N}\mathbf{D}'N. \end{aligned} \quad (6.101)$$

That also gives for the Weinberg-Salam angle:

$$B' + iW'^3 = e^{i\theta_W}(A' + iZ'^0) \quad (6.102)$$

which means that the θ_W angle is Cl_3^* invariant, then is a relativistic invariant. We get

$$D\widehat{\phi}'_e = \overline{M}D'\widehat{\phi}'_e ; \quad D\widehat{\phi}'_n = \overline{M}D'\widehat{\phi}'_n \quad (6.103)$$

$$D\widehat{\phi}'_p = \overline{M}D'\widehat{\phi}'_p ; \quad D\widehat{\phi}'_a = \overline{M}D'\widehat{\phi}'_a \quad (6.104)$$

$$\mathbf{D}\Psi = \widetilde{N}\mathbf{D}'\Psi' \quad (6.105)$$

and the Cl_3^* invariance of electro-weak interactions is completely similar to the invariance of the electromagnetism.

Operators P_0, P_1, P_2 and P_3 are built from projectors and have no inverse. They are not directly elements of a gauge group. Nevertheless we can build a Yang-Mills gauge group by using the exponential function. With four real numbers a^0, a^1, a^2, a^3 , we define

$$\exp(a^0P_0) = \sum_{n=0}^{\infty} \frac{(a^0P_0)^n}{n!} \quad (6.106)$$

$$\exp(a^jP_j) = \sum_{n=0}^{\infty} \frac{(a^1P_1 + a^2P_2 + a^3P_3)^n}{n!} \quad (6.107)$$

We get with (6.25), (6.30) to (6.33)

$$\exp(a^0 P_0)(\Psi) = \begin{pmatrix} \exp(a^0 p_0)(\phi_e) & \exp(a^0 p_0)(\phi_n) \\ \exp(a^0 p_0)(\hat{\phi}_a)\sigma_1 & \exp(a^0 p_0)(\hat{\phi}_p)\sigma_1 \end{pmatrix} \quad (6.108)$$

$$\exp(a^0 p_0)(\phi_{nL}) = e^{-ia^0} \phi_{nL} ; \quad \exp(a^0 p_0)(\phi_{nR}) = \phi_{nR} \quad (6.109)$$

$$\exp(a^0 p_0)(\phi_{eL}) = e^{-ia^0} \phi_{eL} ; \quad \exp(a^0 p_0)(\phi_{eR}) = e^{2ia^0} \phi_{eR} \quad (6.110)$$

$$\exp(a^0 p_0)(\phi_{aR}) = e^{ia^0} \phi_{aR} ; \quad \exp(a^0 p_0)(\phi_{aL}) = \phi_{aL} \quad (6.111)$$

$$\exp(a^0 p_0)(\phi_{pR}) = e^{ia^0} \phi_{pR} ; \quad \exp(a_0 p_0)(\phi_{pL}) = e^{-2ia^0} \phi_{pL} \quad (6.112)$$

$$\exp(-a^0 P_0) = [\exp(a^0 P_0)]^{-1} \quad (6.113)$$

Next we let

$$a = \sqrt{(a^1)^2 + (a^2)^2 + (a^3)^2} ; \quad S = a^j P_j \quad (6.114)$$

and we get

$$[\exp(S)](\Psi) = \Psi + [-1 + \cos(a)]P_+(\Psi) + \frac{\sin(a)}{a}S(\Psi) \quad (6.115)$$

$$[\exp(-S)](\Psi) = \Psi + [-1 + \cos(a)]P_+(\Psi) - \frac{\sin(a)}{a}S(\Psi) \quad (6.116)$$

which gives

$$\exp(-S) = [\exp(S)]^{-1}. \quad (6.117)$$

Since P_0 commutes with S (see (6.16)) we get

$$\exp(a^0 P_0 + S) = \exp(a^0 P_0) \exp(S) = \exp(S) \exp(a^0 P_0) \quad (6.118)$$

The set of the operators $\exp(a^0 P_0 + S)$ is a $U(1) \times SU(2)$ Lie group. The local gauge invariance under this group comes from the derivation of products. If we use

$$\Psi' = [\exp(a^0 P_0 + S)](\Psi) ; \quad \mathbf{D} = \gamma^\mu D_\mu \quad (6.119)$$

then $D_\mu \Psi$ is replaced by $D'_\mu \Psi'$ where

$$D'_\mu \Psi' = \exp(a^0 P_0 + S) D_\mu \Psi \quad (6.120)$$

$$B'_\mu = B_\mu - \frac{2}{g_1} \partial_\mu a^0 \quad (6.121)$$

$$W'^j{}_\mu P_j = \left[\exp(S) W^j{}_\mu P_j - \frac{2}{g_2} \partial_\mu [\exp(S)] \right] \exp(-S). \quad (6.122)$$

6.3 The quark sector

For the first generation of fundamental fermions the standard model includes 16 fermions, 8 particles and their antiparticles. We studied previously the case of the electron, its neutrino, its antiparticle the positron and its anti-neutrino. We put these waves into a unique wave (6.8) that will be named now as Ψ_l .

Each generation includes also two quarks with three states, so we get six waves similar to ϕ_e or ϕ_n . Quarks of the first generation are named u and d and the couple d-u is similar to n-e for electro-weak interactions but with differences since the electric charge of u is $\frac{2}{3}|e|$, the charge of d is $-\frac{1}{3}|e|$. Similarly to the lepton sector, electric charges of antiparticles are opposite to charges of particles. Three states of “color” are named r, g, b (red, green, blue). So we build a wave with all fermions of the first generation as

$$\Psi = \begin{pmatrix} \Psi_l & \Psi_r \\ \Psi_g & \Psi_b \end{pmatrix} \quad (6.123)$$

where Ψ_l is defined by (6.8) and Ψ_r, Ψ_g, Ψ_b are defined on the same model:

$$\Psi_r = \begin{pmatrix} \phi_{dr} & \phi_{ur} \\ \widehat{\phi}_{\bar{u}r}\sigma_1 & \widehat{\phi}_{\bar{d}r}\sigma_1 \end{pmatrix} = \begin{pmatrix} \phi_{dr} & \phi_{ur} \\ \widehat{\phi}_{ur} & \widehat{\phi}_{dr} \end{pmatrix} \quad (6.124)$$

$$\Psi_g = \begin{pmatrix} \phi_{dg} & \phi_{ug} \\ \widehat{\phi}_{\bar{u}g}\sigma_1 & \widehat{\phi}_{\bar{d}g}\sigma_1 \end{pmatrix} = \begin{pmatrix} \phi_{dg} & \phi_{ug} \\ \widehat{\phi}_{ug} & \widehat{\phi}_{dg} \end{pmatrix} \quad (6.125)$$

$$\Psi_b = \begin{pmatrix} \phi_{db} & \phi_{ub} \\ \widehat{\phi}_{\bar{u}b}\sigma_1 & \widehat{\phi}_{\bar{d}b}\sigma_1 \end{pmatrix} = \begin{pmatrix} \phi_{db} & \phi_{ub} \\ \widehat{\phi}_{ub} & \widehat{\phi}_{db} \end{pmatrix}. \quad (6.126)$$

The wave is a function of space-time with value into $Cl_{5,1}$ which is a sub-algebra of $Cl_{5,2} = M_8(\mathbb{C})$ (see 1.5). As previously, electro-weak interactions are obtained by replacing partial derivatives by covariant derivatives. Now we use notations of 1.5 and we let

$$\underline{W}^j = \Lambda^\mu W_\mu^j, \quad j = 1, 2, 3; \quad \underline{D} = \Lambda^\mu D_\mu; \quad \Lambda^0 = -\Lambda_0; \quad \Lambda^j = \Lambda_j \quad (6.127)$$

for $j = 1, 2, 3$. The covariant derivative reads now

$$\underline{D}(\Psi) = \underline{\partial}(\Psi) + \frac{g_1}{2} \underline{B} \underline{P}_0(\Psi) + \frac{g_2}{2} \underline{W}^j \underline{P}_j(\Psi). \quad (6.128)$$

We use two projectors \underline{P}_\pm satisfying

$$\underline{P}_\pm(\Psi) = \frac{1}{2}(\Psi \mp \mathbf{i}\Psi\Lambda_{21}); \quad \mathbf{i} = \Lambda_{0123} \quad (6.129)$$

Three operators act on the quark sector as on the lepton sector :

$$\underline{P}_1 = \underline{P}_+(\Psi)\Lambda_{53} \quad (6.130)$$

$$\underline{P}_2 = \underline{P}_+(\Psi)\Lambda_{0125} \quad (6.131)$$

$$\underline{P}_3 = \underline{P}_+(\Psi)\Lambda_{0132}. \quad (6.132)$$

The fourth operator acts differently on the lepton wave and on the quark sector⁴⁹ :

$$\underline{P}_0 = \begin{pmatrix} P_0(\Psi_l) & P'_0(\Psi_r) \\ P'_0(\Psi_g) & P'_0(\Psi_b) \end{pmatrix} \quad (6.133)$$

$$P_0(\Psi_l) = \Psi_l \gamma_{21} + P_-(\Psi_l) \mathbf{i} = \Psi_l \gamma_{21} + \frac{1}{2}(\Psi_l \mathbf{i} + \mathbf{i} \Psi_l \gamma_{30}) \quad (6.134)$$

$$P'_0(\Psi_r) = -\frac{1}{3} \Psi_r \gamma_{21} + P_-(\Psi_r) \mathbf{i} = -\frac{1}{3} \Psi_r \gamma_{21} + \frac{1}{2}(\Psi_r \mathbf{i} + \mathbf{i} \Psi_r \gamma_{30}) \quad (6.135)$$

And we get two identical formulas by replacing r index by g and b. Now we can abbreviate and we remove indexes r, g, b to study the electro-weak covariant derivative. We let

$$P'_0(\Psi) = \begin{pmatrix} p'_0(\phi_d) & p'_0(\phi_u) \\ p'_0(\widehat{\phi}_{\bar{u}}) \sigma_1 & p'_0(\widehat{\phi}_{\bar{d}}) \sigma_1 \end{pmatrix} \quad (6.136)$$

which gives with (6.135):

$$\begin{aligned} P'_0(\Psi) = & -\frac{i}{3} \begin{pmatrix} \phi_d \sigma_3 & \phi_u \sigma_3 \\ -\widehat{\phi}_{\bar{u}} \sigma_3 \sigma_1 & -\widehat{\phi}_{\bar{d}} \sigma_3 \sigma_1 \end{pmatrix} \\ & + \frac{i}{2} \begin{pmatrix} \phi_d & -\phi_u \\ \widehat{\phi}_{\bar{u}} \sigma_1 & -\widehat{\phi}_{\bar{d}} \sigma_1 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} \phi_d \sigma_3 & -\phi_u \sigma_3 \\ \widehat{\phi}_{\bar{u}} \sigma_3 \sigma_1 & -\widehat{\phi}_{\bar{d}} \sigma_3 \sigma_1 \end{pmatrix} \end{aligned} \quad (6.137)$$

We then get the system :

$$\begin{aligned} p'_0(\phi_d) &= -\frac{i}{3} \phi_d \sigma_3 + \frac{i}{2} \phi_d + \frac{i}{2} \phi_d \sigma_3 = \frac{i}{3} (2\phi_{dR} + \phi_{dL}) \\ p'_0(\phi_u) &= -\frac{i}{3} \phi_u \sigma_3 - \frac{i}{2} \phi_u - \frac{i}{2} \phi_u \sigma_3 = \frac{i}{3} (-4\phi_{uR} + \phi_{uL}) \\ p'_0(\widehat{\phi}_{\bar{u}}) &= \frac{i}{3} \widehat{\phi}_{\bar{u}} \sigma_3 + \frac{i}{2} \widehat{\phi}_{\bar{u}} + \frac{i}{2} \widehat{\phi}_{\bar{u}} \sigma_3 = \frac{i}{3} (4\widehat{\phi}_{\bar{u}L} - \widehat{\phi}_{\bar{u}R}) \\ p'_0(\widehat{\phi}_{\bar{d}}) &= \frac{i}{3} \widehat{\phi}_{\bar{d}} \sigma_3 - \frac{i}{2} \widehat{\phi}_{\bar{d}} - \frac{i}{2} \widehat{\phi}_{\bar{d}} \sigma_3 = \frac{i}{3} (-2\widehat{\phi}_{\bar{d}L} - \widehat{\phi}_{\bar{d}R}). \end{aligned} \quad (6.138)$$

Since P_1 , P_2 and P_3 are unchanged in the quark sector, we get from (6.41), (6.43) and (6.45)

$$p_1(\phi_d) = i\phi_{uL} ; p_1(\phi_u) = i\phi_{dL} ; p_1(\widehat{\phi}_{\bar{u}}) = -i\widehat{\phi}_{\bar{d}R} ; p_1(\widehat{\phi}_{\bar{d}}) = -i\widehat{\phi}_{\bar{u}R} \quad (6.139)$$

$$p_2(\phi_d) = \phi_{uL} ; p_2(\phi_u) = -\phi_{dL} ; p_2(\widehat{\phi}_{\bar{u}}) = -\widehat{\phi}_{\bar{d}R} ; p_2(\widehat{\phi}_{\bar{d}}) = \widehat{\phi}_{\bar{u}R} \quad (6.140)$$

$$p_3(\phi_d) = -i\phi_{dL} ; p_3(\phi_u) = i\phi_{uL} ; p_3(\widehat{\phi}_{\bar{u}}) = -i\widehat{\phi}_{\bar{u}R} ; p_3(\widehat{\phi}_{\bar{d}}) = i\widehat{\phi}_{\bar{d}R}. \quad (6.141)$$

Now (6.129) gives

$$\mathbf{D}\Psi_r = \boldsymbol{\partial}\Psi_r + \frac{g_1}{2} \mathbf{B}P'_0(\Psi_r) + \frac{g_2}{2} \mathbf{W}^j P_j(\Psi_r) \quad (6.142)$$

⁴⁹This is very important, since it is the reason explaining how a lepton is not a quark. If all four operators were identical, we should get four states and a SU(4) group for chromodynamics and the electron should be sensitive to strong interactions. Since only three parts of the wave are similar, we will get in 6.4 a SU(3) group for chromodynamics.

and we get, similarly to (6.47) to (6.50)

$$D\widehat{\phi}_{\bar{u}} = \nabla\widehat{\phi}_{\bar{u}} + \frac{g_1}{2}Bp'_0(\widehat{\phi}_{\bar{u}}) + \frac{g_2}{2}W^j p_j(\widehat{\phi}_{\bar{u}}) \quad (6.143)$$

$$D\widehat{\phi}_{\bar{d}} = \nabla\widehat{\phi}_{\bar{d}} + \frac{g_1}{2}Bp'_0(\widehat{\phi}_{\bar{d}}) + \frac{g_2}{2}W^j p_j(\widehat{\phi}_{\bar{d}}) \quad (6.144)$$

$$\widehat{D}\phi_d = \widehat{\nabla}\phi_d + \frac{g_1}{2}\widehat{B}p'_0(\phi_d) + \frac{g_2}{2}\widehat{W}^j p_j(\phi_d) \quad (6.145)$$

$$\widehat{D}\phi_u = \widehat{\nabla}\phi_u + \frac{g_1}{2}\widehat{B}p'_0(\phi_u) + \frac{g_2}{2}\widehat{W}^j p_j(\phi_u). \quad (6.146)$$

With (6.138) to (6.141) this gives

$$\begin{aligned} D\widehat{\phi}_{\bar{u}} &= \nabla\widehat{\phi}_{\bar{u}} + \frac{g_1}{2}B\frac{i}{3}(4\widehat{\phi}_{\bar{u}L} - \widehat{\phi}_{\bar{u}R}) \\ &\quad + \frac{g_2}{2}[W^1(-i\widehat{\phi}_{\bar{d}R}) + W^2(-\widehat{\phi}_{\bar{d}R}) + W^3(-i\widehat{\phi}_{\bar{u}R})] \end{aligned} \quad (6.147)$$

$$\begin{aligned} D\widehat{\phi}_{\bar{d}} &= \nabla\widehat{\phi}_{\bar{d}} + \frac{g_1}{2}B\frac{i}{3}(-2\widehat{\phi}_{\bar{d}L} - \widehat{\phi}_{\bar{d}R}) \\ &\quad + \frac{g_2}{2}[W^1(-i\widehat{\phi}_{\bar{u}R}) + W^2\widehat{\phi}_{\bar{u}R}) + W^3i\widehat{\phi}_{\bar{d}R}] \end{aligned} \quad (6.148)$$

$$\begin{aligned} \widehat{D}\phi_d &= \widehat{\nabla}\phi_d + \frac{g_1}{2}\widehat{B}\frac{i}{3}(2\phi_{dR} + \phi_{dL}) \\ &\quad + \frac{g_2}{2}[\widehat{W}^1(i\phi_{uL}) + \widehat{W}^2(\phi_{uL}) + \widehat{W}^3(-i\phi_{dL})] \end{aligned} \quad (6.149)$$

$$\begin{aligned} \widehat{D}\phi_u &= \widehat{\nabla}\phi_u + \frac{g_1}{2}\widehat{B}\frac{i}{3}(-4\phi_{uR} + \phi_{uL}) \\ &\quad + \frac{g_2}{2}[\widehat{W}^1(i\phi_{dL}) + \widehat{W}^2(-\phi_{dL}) + \widehat{W}^3(i\phi_{uL})] \end{aligned} \quad (6.150)$$

We separate right and left waves, this gives

$$D\widehat{\phi}_{\bar{u}L} = \nabla\widehat{\phi}_{\bar{u}L} - i(-\frac{2}{3})g_1B\widehat{\phi}_{\bar{u}L} ; D_\mu\eta_{\bar{u}} = \partial_\mu\eta_{\bar{u}} - i(-\frac{2}{3})g_1B_\mu\eta_{\bar{u}} \quad (6.151)$$

$$D\widehat{\phi}_{\bar{d}L} = \nabla\widehat{\phi}_{\bar{d}L} - i(+\frac{1}{3})g_1B\widehat{\phi}_{\bar{d}L} ; D_\mu\eta_{\bar{d}} = \partial_\mu\eta_{\bar{d}} - i(+\frac{1}{3})g_1B_\mu\eta_{\bar{d}} \quad (6.152)$$

$$\widehat{D}\phi_{dR} = \widehat{\nabla}\phi_{dR} - i(-\frac{1}{3})g_1\widehat{B}\phi_{dR} ; D_\mu\xi_d = \partial_\mu\xi_d - i(-\frac{1}{3})g_1B_\mu\xi_d \quad (6.153)$$

$$\widehat{D}\phi_{uR} = \widehat{\nabla}\phi_{uR} - i(+\frac{2}{3})g_1\widehat{B}\phi_{uR} ; D_\mu\xi_u = \partial_\mu\xi_u - i(+\frac{2}{3})g_1B_\mu\xi_u. \quad (6.154)$$

Comparison with 6.1 shows that quarks and anti-quarks have awaited electric charges : $-\frac{2}{3}|e|$ for anti-quark \bar{u} , $+\frac{1}{3}|e|$ for anti-quark \bar{d} , $+\frac{2}{3}|e|$ for the u quark and $-\frac{1}{3}|e|$ for the d quark.⁵⁰ Separation of right and left waves from (6.147) to

⁵⁰ Another mechanism giving the $\pm\frac{e}{3}$ and $\pm\frac{2e}{3}$ charges of quarks was proposed in 5.3 of [16].

(6.150) gives also

$$D\widehat{\phi}_{\bar{u}R} = \nabla\widehat{\phi}_{\bar{u}R} - i\frac{g_1}{6}B\widehat{\phi}_{\bar{u}R} + \frac{g_2}{2}[-iW^1\widehat{\phi}_{\bar{d}R}) - W^2\widehat{\phi}_{\bar{d}R} - iW^3\widehat{\phi}_{\bar{u}R}] \quad (6.155)$$

$$D\widehat{\phi}_{\bar{d}R} = \nabla\widehat{\phi}_{\bar{d}R} - i\frac{g_1}{6}B\widehat{\phi}_{\bar{d}R} + \frac{g_2}{2}[-iW^1\widehat{\phi}_{\bar{u}R}) + W^2\widehat{\phi}_{\bar{u}R}) + iW^3\widehat{\phi}_{\bar{d}R}] \quad (6.156)$$

$$\widehat{D}\phi_{dL} = \widehat{\nabla}\phi_{dL} + i\frac{g_1}{6}\widehat{B}\phi_{dL} + \frac{g_2}{2}[i\widehat{W}^1\phi_{uL}) + \widehat{W}^2\phi_{uL} - i\widehat{W}^3\phi_{dL}] \quad (6.157)$$

$$\widehat{D}\phi_{uL} = \widehat{\nabla}\phi_{uL} + i\frac{g_1}{6}\widehat{B}\phi_{uL} + \frac{g_2}{2}[i\widehat{W}^1\phi_{dL}) - \widehat{W}^2\phi_{dL} + i\widehat{W}^3\phi_{uL}]. \quad (6.158)$$

Using the conjugation $\phi \mapsto \widehat{\phi}$ we get

$$\widehat{D}\phi_{\bar{u}R} = \widehat{\nabla}\phi_{\bar{u}R} + i\frac{g_1}{6}\widehat{B}\phi_{\bar{u}R} + \frac{g_2}{2}[+i\widehat{W}^1\phi_{\bar{d}R}) - \widehat{W}^2\phi_{\bar{d}R} + i\widehat{W}^3\phi_{\bar{u}R}] \quad (6.159)$$

$$\widehat{D}\phi_{\bar{d}R} = \widehat{\nabla}\phi_{\bar{d}R} + i\frac{g_1}{6}\widehat{B}\phi_{\bar{d}R} + \frac{g_2}{2}[+i\widehat{W}^1\phi_{\bar{u}R}) + \widehat{W}^2\phi_{\bar{u}R}) - i\widehat{W}^3\phi_{\bar{d}R}] \quad (6.160)$$

$$D\widehat{\phi}_{dL} = \nabla\widehat{\phi}_{dL} - i\frac{g_1}{6}B\widehat{\phi}_{dL} + \frac{g_2}{2}[-iW^1\widehat{\phi}_{uL}) + W^2\widehat{\phi}_{uL} + iW^3\widehat{\phi}_{dL}] \quad (6.161)$$

$$D\widehat{\phi}_{uL} = \nabla\widehat{\phi}_{uL} - i\frac{g_1}{6}B\widehat{\phi}_{uL} + \frac{g_2}{2}[-iW^1\widehat{\phi}_{dL}) - W^2\widehat{\phi}_{dL} - iW^3\widehat{\phi}_{uL}]. \quad (6.162)$$

This gives a left doublet of particles and a right doublet of antiparticle. With (6.72) and

$$\psi_L = \begin{pmatrix} \eta_u \\ \eta_d \end{pmatrix} ; \quad \psi_R = \begin{pmatrix} \xi_{\bar{u}} \\ \xi_{\bar{d}} \end{pmatrix} \quad (6.163)$$

we get

$$D_\mu\psi_L = \partial_\mu\psi_L - i\frac{g_1}{6}B_\mu\psi_L - i\frac{g_2}{2}(W_\mu^1\tau_1 + W_\mu^2\tau_2 + W_\mu^3\tau_3)\psi_L \quad (6.164)$$

$$D_\mu\psi_R = \partial_\mu\psi_R + i\frac{g_1}{6}B_\mu\psi_R - i\frac{g_2}{2}(W_\mu^1\tau_1 - W_\mu^2\tau_2 + W_\mu^3\tau_3)\psi_R. \quad (6.165)$$

We can then say that charge conjugation is not only a changing of signs of electric charges, but it exchanges the right and the left waves. It also changes the orientation of the space of the τ_j , where a direct basis (τ_1, τ_2, τ_3) , is replaced by an inverse basis $(\tau_1, -\tau_2, \tau_3)$. We encounter this basis both here and in the wave of antiparticle (3.44) used by de Broglie.

6.4 Chromodynamics

We start from generators λ_k of the $SU(3)$ gauge group of chromodynamics

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{6.166}$$

To simplify notations we use now l, r, g, b instead $\Psi_l, \Psi_r, \Psi_g, \Psi_b$. So we have $\Psi = \begin{pmatrix} l & r \\ g & b \end{pmatrix}$. Then (6.166) gives

$$\begin{aligned}
\lambda_1 \begin{pmatrix} r \\ g \\ b \end{pmatrix} &= \begin{pmatrix} g \\ r \\ 0 \end{pmatrix}, \lambda_2 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} -ig \\ ir \\ 0 \end{pmatrix}, \lambda_3 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} r \\ -g \\ 0 \end{pmatrix} \\
\lambda_4 \begin{pmatrix} r \\ g \\ b \end{pmatrix} &= \begin{pmatrix} b \\ 0 \\ r \end{pmatrix}, \lambda_5 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} -ib \\ 0 \\ ir \end{pmatrix}, \lambda_6 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ g \end{pmatrix} \\
\lambda_7 \begin{pmatrix} r \\ g \\ b \end{pmatrix} &= \begin{pmatrix} 0 \\ -ib \\ ig \end{pmatrix}, \lambda_8 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} r \\ g \\ -2b \end{pmatrix}.
\end{aligned} \tag{6.167}$$

We name Γ_k operators corresponding to λ_k acting on Ψ . We get with (1.93)

$$\Gamma_1(\Psi) = \frac{1}{2}(\Lambda_4\Psi\Lambda_4 + \Lambda_{01235}\Psi\Lambda_{01235}) = \begin{pmatrix} 0 & g \\ r & 0 \end{pmatrix} \tag{6.168}$$

$$\Gamma_2(\Psi) = \frac{1}{2}(\Lambda_5\Psi\Lambda_4 - \Lambda_{01234}\Psi\Lambda_{01235}) = \begin{pmatrix} 0 & -ig \\ ir & 0 \end{pmatrix} \tag{6.169}$$

$$\Gamma_3(\Psi) = P^+\Psi P^- - P^-\Psi P^+ = \begin{pmatrix} 0 & r \\ -g & 0 \end{pmatrix} \tag{6.170}$$

$$\Gamma_4(\Psi) = \Lambda_4\Psi P^- = \begin{pmatrix} 0 & b \\ 0 & r \end{pmatrix}; \Gamma_5(\Psi) = \Lambda_5\Psi P^- = \begin{pmatrix} 0 & -ib \\ 0 & ir \end{pmatrix} \tag{6.171}$$

$$\Gamma_6(\Psi) = P^-\Psi\Lambda_4 = \begin{pmatrix} 0 & 0 \\ b & g \end{pmatrix}; \Gamma_7(\Psi) = \mathbf{i}P^-\Psi\Lambda_{01235} = \begin{pmatrix} 0 & 0 \\ -ib & ig \end{pmatrix} \tag{6.172}$$

$$\Gamma_8(\Psi) = -\frac{1}{\sqrt{3}}(P^-\Psi\Lambda_{012345} + \Lambda_{012345}\Psi P^-) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & r \\ g & -2b \end{pmatrix}. \tag{6.173}$$

Everywhere the left up term is 0, so all Γ_k project the wave Ψ on its quark sector.

We can extend the covariant derivative of electro-weak interactions (6.127):

$$\underline{D}(\Psi) = \underline{\partial}(\Psi) + \frac{g_1}{2} \underline{B} \underline{P}_0(\Psi) + \frac{g_2}{2} \underline{W}^j \underline{P}_j(\Psi) + \frac{g_3}{2} \underline{G}^k \underline{\mathbf{i}}\Gamma_k(\Psi). \quad (6.174)$$

where g_3 is another constant and \underline{G}^k are eight terms called ‘‘gluons’’. Since I_4 commute with any element of $Cl_{1,3}$ and since $P_j(\mathbf{i}\Psi_{ind}) = \mathbf{i}P_j(\Psi_{ind})$ for $j = 0, 1, 2, 3$ and $ind = l, r, g, b$ each operator $\underline{\mathbf{i}}\Gamma_k$ commutes with all operators \underline{P}_j .

Now we use 12 real numbers $a^0, a^j, j = 1, 2, 3, b^k, k = 1, 2, \dots, 8$, we let

$$S_1 = \sum_{j=1}^{j=3} a^j \underline{P}_j; \quad S_2 = \sum_{k=1}^{k=8} b^k \underline{\mathbf{i}}\Gamma_k \quad (6.175)$$

and we get, using exponentiation (see 6.2)

$$\exp(a^0 \underline{P}_0 + S_1 + S_2) = \exp(a^0 \underline{P}_0) \exp(S_1) \exp(S_2) \quad (6.176)$$

The set of these operators is a $U(1) \times SU(2) \times SU(3)$ Lie group. Only difference with the standard model the structure of this group is not postulated but calculated. The invariance under Cl_3^* (and particularly the relativistic invariance) of this covariant derivative is similar to (6.105) with underlined terms. The gauge invariance reads with

$$\Psi' = [\exp(a^0 \underline{P}_0 + S_1 + S_2)](\Psi); \quad \underline{\mathbf{D}} = \Lambda^\mu \underline{D}_\mu; \quad \underline{\mathbf{D}}' = \Lambda^\mu \underline{D}'_\mu \quad (6.177)$$

$$\underline{D}'_\mu \Psi' = \exp(a^0 \underline{P}_0 + S_1 + S_2) \underline{D}_\mu \Psi \quad (6.178)$$

$$B'_\mu = B_\mu - \frac{2}{g_1} \partial_\mu a^0 \quad (6.179)$$

$$W'^j_\mu \underline{P}_j = \left[\exp(S_1) W^j_\mu \underline{P}_j - \frac{2}{g_2} \partial_\mu [\exp(S_1)] \right] \exp(-S_1) \quad (6.180)$$

$$\underline{G}'^k_\mu \underline{\mathbf{i}}\Gamma_k = \left[\exp(S_2) \underline{G}^k_\mu \underline{\mathbf{i}}\Gamma_k - \frac{2}{g_3} \partial_\mu [\exp(S_2)] \right] \exp(-S_2). \quad (6.181)$$

The $SU(3)$ group generated by projectors on the quark sector acts only on this sector of the wave :

$$P^+ [\exp(b^k \underline{\mathbf{i}}\Gamma_k)](\Psi) P^+ = P^+ \Psi P^+ = \begin{pmatrix} \Psi_l & 0 \\ 0 & 0 \end{pmatrix} \quad (6.182)$$

We get then a $U(1) \times SU(2) \times SU(3)$ gauge group for a wave including all fermions of the first generation. This group acts on the lepton sector only by its $U(1) \times SU(2)$ part. The physical translation is: leptons do not strongly interact, they have only electromagnetic and weak interactions. This is fully satisfied in experiments. The novelty here is that this comes from the structure itself of the quantum wave. Since it is independent on the energy scale, we understand why great unified theories do not work.

6.4.1 Three generations, four neutrinos

The aim of theoretical physics is to understand experimental facts. Today we have to understand both why we get only three kinds of leptons and quarks and a fourth neutrino, without electro-weak interactions. Actual experiments show both the limitation to three kinds of light leptons from the study of the Z^0 and the possible existence of a fourth neutrino without electro-weak interactions. We explained the existence of three kinds of leptons in section 5. This is easily generalized to the three generations of the standard model. Two other generations are gotten by replacing the privileged third direction σ_3 by σ_1 or σ_2 , everywhere this direction is used. The passage from one to another generation must be seen as a circular permutation of indexes $1 \mapsto 2 \mapsto 3 \mapsto 1$ or $1 \mapsto 3 \mapsto 2 \mapsto 1$ for the other. For instance the σ_3 in (6.11) which defines left and right projectors must be replaced by σ_1 or σ_2 . The σ_1 in (6.8) which links the wave of the particle to the wave of the antiparticle must be replaced by σ_2 or σ_3 . These changes imply to treat separately each generation, and it is the reason of this separate treatment in the standard model. Now for a fourth generation we have no other similar possibility since the Cl_3 algebra is based on the 3-dimensional physical space. We cannot get a fourth set of operators similar to the P_μ .

But the existence of a fourth neutrino [20] is possible because Cl_3 has four generators with square -1 . The wave equation of the electron includes one of these four generators, $i\sigma_3 = \sigma_{12}$. Now $i\sigma_1 = \sigma_{23}$ and $i\sigma_2 = \sigma_{31}$ explain why two other kinds of leptons exist. We can also build an invariant wave equation with the fourth generator, $i = \sigma_{123}$:

$$\bar{\phi}(\nabla\hat{\phi})\sigma_{123} + m\rho = 0. \quad (6.183)$$

Multiplying by the left by $\bar{\phi}^{-1}$ we get with $\rho = e^{-i\beta}\bar{\phi}\phi$ the equivalent equation

$$\nabla\hat{\phi}i + me^{-i\beta}\phi = 0; \quad \nabla\hat{\phi} = ime^{-i\beta}\phi. \quad (6.184)$$

Contrarily to our homogeneous non-linear wave equation (3.9) which has the Dirac equation as linear approximation, this wave equation cannot come from the linear quantum theory, it has no linear approximation because the β angle is not small, it is now the angle of the phase of the wave⁵¹. We can nevertheless get plane waves. We search now solutions satisfying

$$\phi = e^{-i\varphi}\phi_0; \quad \varphi = mv_\mu x^\mu; \quad v = \sigma^\mu v_\mu. \quad (6.185)$$

where v is a fixed reduced speed and ϕ_0 is also a fixed term, we get:

$$\nabla\hat{\phi} = \sigma^\mu \partial_\mu (e^{i\varphi}\hat{\phi}_0) = imve^{i\varphi}\hat{\phi}_0. \quad (6.186)$$

And we have

$$\phi\bar{\phi} = e^{-i\varphi}\phi_0 e^{-i\varphi}\bar{\phi}_0 = e^{-2i\varphi}\phi_0\bar{\phi}_0. \quad (6.187)$$

⁵¹This is another reason to think that the homogeneous non-linear equation is better than its linear approximation.

Then if we let

$$\phi_0 \bar{\phi}_0 = \rho_0 e^{i\beta_0} \quad (6.188)$$

we get

$$\beta = \beta_0 - 2\varphi ; e^{-i\beta} \phi = e^{-i(\beta_0-2\varphi)} e^{-i\varphi} \phi_0 = e^{-i(\beta_0-\varphi)} \phi_0 \quad (6.189)$$

Then (6.184) is equivalent to

$$imve^{i\varphi} \hat{\phi}_0 = ime^{-i(\beta_0-\varphi)} \phi_0 \quad (6.190)$$

$$v \hat{\phi}_0 = e^{-i\beta_0} \phi_0$$

$$e^{i\beta_0} v \hat{\phi}_0 = \phi_0. \quad (6.191)$$

Conjugating we get

$$e^{-i\beta_0} \hat{v} \phi_0 = \hat{\phi}_0. \quad (6.192)$$

So we get

$$\phi_0 = e^{i\beta_0} v \hat{\phi}_0 = e^{i\beta_0} v [e^{-i\beta_0} \hat{v} \phi_0] = v \hat{v} \phi_0. \quad (6.193)$$

Then if $\phi_0 \neq 0$ we get

$$1 = v \hat{v} \quad (6.194)$$

which gives (2.73) or (3.40) and since (6.191) implies (3.43) we get the same results as with our non-linear wave equation: existence of plane waves with only positive energy. Developing (6.183) we get a system of eight equations similar to the system (2.92) to (2.99) and four of these equations are the conservation of the D_μ currents ($\partial_\nu D_\mu^\nu = 0$) [20]. Then the density of probability is conservative and there is no possible disintegration of such a particle. Without a set of operators P_μ there are no electro-weak forces. Therefore only gravitational interactions remain possible. Such an object could be necessary part of the black matter, since it is unable to emit photons.

6.5 Geometric transformation linked to the wave

We saw in 3.3 that the wave of the electron defines in each point of space-time a geometric transformation (3.55) from the tangent space-time of an intrinsic manifold into the tangent space-time to our space-time manifold. What becomes this transformation when we consider the wave Ψ_l of a couple electron-neutrino, or the complete wave Ψ of the first generation?

6.5.1 In space-time algebra

Any element M in Cl_3 is sum of a scalar s , a vector \vec{v} , a bivector $i\vec{w}$ and a pseudo-scalar ip . We have

$$\begin{aligned} M &= s + \vec{v} + i\vec{w} + ip; \quad \widehat{M} = s - \vec{v} + i\vec{w} - ip \\ M^\dagger &= s + \vec{v} - i\vec{w} - ip; \quad \overline{M} = s - \vec{v} - i\vec{w} + ip. \end{aligned} \quad (6.195)$$

With the matrix representation of the space-time algebra studied in 1.4.1 and the N in (1.68) we associate to $x = x^\mu \sigma_\mu$ in Cl_3 the space-time vector

$$\mathbf{x} = x^\mu \gamma_\mu = \begin{pmatrix} 0 & x \\ \widehat{x} & 0 \end{pmatrix} \quad (6.196)$$

Then the dilation R defined by (1.42) associates to the space-time vector \mathbf{x} the space-time vector \mathbf{x}' satisfying

$$\mathbf{x}' = N\mathbf{x}\widetilde{N} \quad (6.197)$$

while the differential operator $\boldsymbol{\partial} = \gamma^\mu \partial_\mu$ satisfies

$$\boldsymbol{\partial} = \widetilde{N}\boldsymbol{\partial}'N. \quad (6.198)$$

And the dilation D defined by (3.55) associates to the space-time vector \mathbf{y} , element of the tangent space-time to the intrinsic manifold linked to the wave, a space-time vector \mathbf{x} in the usual space-time, satisfying

$$\mathbf{x} = \Psi\mathbf{y}\widetilde{\Psi}; \quad \Psi = \begin{pmatrix} \phi & 0 \\ 0 & \widehat{\phi} \end{pmatrix}; \quad \widetilde{\Psi} = \begin{pmatrix} \overline{\phi} & 0 \\ 0 & \phi^\dagger \end{pmatrix}; \quad \mathbf{y} = y^\mu \gamma_\mu = \begin{pmatrix} 0 & y \\ \widehat{y} & 0 \end{pmatrix}. \quad (6.199)$$

Now we consider the wave of the lepton case Ψ_l which reads

$$\Psi_l = \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_e \end{pmatrix}; \quad \widetilde{\Psi}_l = \begin{pmatrix} \overline{\phi}_e & \phi_n^\dagger \\ \overline{\phi}_n & \phi_e^\dagger \end{pmatrix}. \quad (6.200)$$

The generalization of (6.199) is

$$\mathbf{x} = \Psi_l\mathbf{y}\widetilde{\Psi}_l. \quad (6.201)$$

But, since

$$\widetilde{\mathbf{x}} = \Psi_l\mathbf{y}\widetilde{\Psi}_l = \mathbf{x} \quad (6.202)$$

then \mathbf{x} is the sum of a scalar, a vector and a pseudo-scalar.⁵² To get only a vector, we must separate the vector part. Noting $\langle M \rangle_1$ the vector part of the multivector M , we then let instead of (6.201)

$$\mathbf{x} = \langle \Psi_l\mathbf{y}\widetilde{\Psi}_l \rangle_1 \quad (6.203)$$

We have

$$\begin{aligned} \Psi_l\mathbf{y}\widetilde{\Psi}_l &= \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_e \end{pmatrix} \begin{pmatrix} 0 & y \\ \widehat{y} & 0 \end{pmatrix} \begin{pmatrix} \overline{\phi}_e & \phi_n^\dagger \\ \overline{\phi}_n & \phi_e^\dagger \end{pmatrix} \\ &= \begin{pmatrix} \phi_n\widehat{y}\overline{\phi}_e + \phi_e y\overline{\phi}_n & \phi_e y\phi_e^\dagger + \phi_n\widehat{y}\phi_n^\dagger \\ \widehat{\phi}_e\widehat{y}\overline{\phi}_e + \widehat{\phi}_n y\overline{\phi}_n & \widehat{\phi}_n y\phi_e^\dagger + \widehat{\phi}_e\widehat{y}\phi_n^\dagger \end{pmatrix} \end{aligned} \quad (6.204)$$

⁵²The same property in Cl_3 proves that x is the sum of a scalar and a vector and this is exact for a space-time vector.

which gives

$$\mathbf{x} = \langle \Psi_l \mathbf{y} \tilde{\Psi}_l \rangle_1 = \begin{pmatrix} 0 & x \\ \hat{x} & 0 \end{pmatrix}; \quad x = \phi_e y \phi_e^\dagger + \phi_n \hat{y} \phi_n^\dagger. \quad (6.205)$$

We let

$$D = D_e + D_n; \quad D_e(y) = \phi_e y \phi_e^\dagger; \quad D_n(y) = \phi_n \hat{y} \phi_n^\dagger \quad (6.206)$$

D_e is a direct dilation, conserving the orientation of the time and the space. D_n is an inverse dilation, conserving the orientation of the time and changing the orientation of the space. The geometric transformation $D : y \mapsto x$ is the sum of these two dilations.

The element \mathbf{y} is independent on the relative observer: if M is any element of Cl_3^* and N is given by (1.68), the dilation R defined in (1.42) satisfies

$$x' = M x M^\dagger; \quad \phi'_e = M \phi_e; \quad \phi'_n = M \phi_n \quad (6.207)$$

$$\Psi'_l = \begin{pmatrix} \phi'_e & \phi'_n \\ \hat{\phi}'_n & \hat{\phi}'_e \end{pmatrix} = \begin{pmatrix} M \phi_e & M \phi_n \\ \widehat{M \phi_n} & \widehat{M \phi_e} \end{pmatrix} = N \Psi_l \quad (6.208)$$

$$\begin{aligned} \mathbf{x}' &= \begin{pmatrix} 0 & x' \\ \hat{x}' & 0 \end{pmatrix} = N \mathbf{x} \tilde{N} = N \langle \Psi_l \mathbf{y} \tilde{\Psi}_l \rangle_1 \tilde{N} = \langle N \Psi_l \mathbf{y} \tilde{\Psi}_l \tilde{N} \rangle_1 \\ &= \langle \Psi'_l \mathbf{y} \tilde{\Psi}'_l \rangle_1. \end{aligned} \quad (6.209)$$

We then have

$$\mathbf{x} = \langle \Psi_l \mathbf{y} \tilde{\Psi}_l \rangle_1; \quad \mathbf{x}' = \langle \Psi'_l \mathbf{y} \tilde{\Psi}'_l \rangle_1 \quad (6.210)$$

with the same \mathbf{y} for the observer of \mathbf{x} and for the observer of \mathbf{x}' .

6.5.2 Extension to the complete wave

The complete wave Ψ containing the wave of leptons and quarks of the first generation defined in (6.123) satisfies (the proof is in Appendix A)

$$\tilde{\Psi} = \begin{pmatrix} \tilde{\Psi}_b & \tilde{\Psi}_r \\ \tilde{\Psi}_g & \tilde{\Psi}_l \end{pmatrix} \quad (6.211)$$

The wave has value in the Clifford algebra $Cl_{5,1}$. Each element reads

$$\Psi = \sum_{n=0}^{n=6} \langle \Psi \rangle_n \quad (6.212)$$

where $\langle \Psi \rangle_n$ is named a n-vector. The reverse satisfies

$$\tilde{\Psi} = \langle \Psi \rangle_0 + \langle \Psi \rangle_1 - \langle \Psi \rangle_2 - \langle \Psi \rangle_3 + \langle \Psi \rangle_4 + \langle \Psi \rangle_5 - \langle \Psi \rangle_6. \quad (6.213)$$

We define the v-part A_v of any multivector A as the sum of the vectorial part and of the pseudo-vectorial part in the complete space-time:

$$A_v = \langle A \rangle_v = \langle A \rangle_1 + \langle A \rangle_5 \quad (6.214)$$

because it is this vectorial part which replaces in $Cl_{5,1}$ the vectorial part $\langle M \rangle_1$ of the space-time algebra. It is linked to the fact that vectors of $Cl_{1,5}$ are 5-vectors of $Cl_{5,1}$ and vice-versa. We should then get the same definition of the vectorial part by using $Cl_{1,5}$. We use

$$\mathbf{y} = \sum_{\mu=0}^{\mu=3} y^\mu \gamma_\mu ; \quad \mathbf{y}_5 = \sum_{\mu=0}^{\mu=3} y_5^\mu \gamma_\mu \quad (6.215)$$

$$\langle y \rangle_1 = \sum_{a=0}^{a=5} y^a \Lambda_a = \sum_{\mu=0}^{\mu=3} y^\mu \Lambda_\mu + y^4 \Lambda_4 + y^5 \Lambda_5 \quad (6.216)$$

$$\langle y \rangle_5 = \sum_{a=0}^{a=5} y_5^a \Lambda_a \Lambda_{0123} = \left(\sum_{\mu=0}^{\mu=3} y_5^\mu \Lambda_\mu + y_5^4 \Lambda_4 + y_5^5 \Lambda_5 \right) \Lambda_{0123}. \quad (6.217)$$

This gives with (1.72) and (1.75)

$$\langle y \rangle_1 = \begin{pmatrix} 0 & -\mathbf{y} + y^4 - y^5 \mathbf{i} \\ \mathbf{y} + y^4 + y^5 \mathbf{i} & 0 \end{pmatrix} \quad (6.218)$$

$$\langle y \rangle_5 = \begin{pmatrix} 0 & -\mathbf{y}_5 + y_5^4 - y_5^5 \mathbf{i} \\ \mathbf{y}_5 + y_5^4 + y_5^5 \mathbf{i} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} \quad (6.219)$$

$$y_v = \langle y \rangle_v = \begin{pmatrix} 0 & -\mathbf{y}_v + y_v^4 - y_v^5 \\ \mathbf{y}_v + y_v^4 + y_v^5 & 0 \end{pmatrix} \quad (6.220)$$

with

$$\mathbf{y}_v = \mathbf{y} + \mathbf{y}_5 \mathbf{i} ; \quad y_v^4 = y^4 + y_5^4 \mathbf{i} ; \quad y_v^5 = -y_5^5 + y^5 \mathbf{i} \quad (6.221)$$

The generalization of (6.203) is the transformation:

$$f : y_v \mapsto x_v = \langle \Psi y_v \tilde{\Psi} \rangle_v \quad (6.222)$$

We use for x similar notations as in (6.215), (6.216), (6.217), (6.221), with x in the place of y and we get

$$\Psi y_v \tilde{\Psi} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} ; \quad x_v = \langle \Psi y_v \tilde{\Psi} \rangle_v = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

$$\mathbf{x}_v + x_v^4 + x_v^5 = C = \Psi_b (\mathbf{y}_v + y_v^4 + y_v^5) \tilde{\Psi}_b + \Psi_g (-\mathbf{y}_v + y_v^4 - y_v^5) \tilde{\Psi}_g \quad (6.223)$$

$$-\mathbf{x}_v + x_v^4 - x_v^5 = B = \Psi_r (\mathbf{y}_v + y_v^4 + y_v^5) \tilde{\Psi}_r + \Psi_l (-\mathbf{y}_v + y_v^4 - y_v^5) \tilde{\Psi}_l. \quad (6.224)$$

This equation contains the term $\Psi_l \mathbf{y}_v \tilde{\Psi}_l$ and similar terms coming from the wave of quarks. We remark that the supplementary dimensions are mixed with the ordinary ones. We remark also that x_v or y_v are now sum of a vector and a 5-vector, which is in $Cl_{5,1}$ a pseudo-vector. Components of these sums may be expressed as

$$x_v^a = x^a + x_5^a \mathbf{i} \quad (6.225)$$

The complete space-time has then a natural structure of complex linear space with dimension 6, with the \mathbf{i} of the chiral gauge.

6.5.3 Invariance

The dilation R induced by any element M of Cl_3 satisfying (1.42) reads in space-time algebra, with the N of (1.68):

$$\mathbf{x}' = N\mathbf{x}\tilde{N} ; \mathbf{x} = \sum_{\mu=0}^{\mu=3} x^\mu \gamma_\mu ; \mathbf{x}' = \sum_{\mu=0}^{\mu=3} x'^\mu \gamma_\mu ; \Psi'_l = N\Psi_l \quad (6.226)$$

and we await the same transformation for the waves of quarks:

$$\Psi'_r = N\Psi_r ; \Psi'_g = N\Psi_g ; \Psi'_b = N\Psi_b \quad (6.227)$$

We let

$$\mathbf{N} = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix} = \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & \widehat{M} & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & \widehat{M} \end{pmatrix}. \quad (6.228)$$

We then get

$$\tilde{\mathbf{N}} = \begin{pmatrix} \tilde{N} & 0 \\ 0 & \tilde{N} \end{pmatrix} = \begin{pmatrix} \overline{M} & 0 & 0 & 0 \\ 0 & M^\dagger & 0 & 0 \\ 0 & 0 & \overline{M} & 0 \\ 0 & 0 & 0 & M^\dagger \end{pmatrix} \quad (6.229)$$

With

$$\mathbf{x}_5 = \sum_{\mu=0}^{\mu=3} x_5^\mu \gamma_\mu ; \mathbf{x}'_5 = \sum_{\mu=0}^{\mu=3} x'^\mu_5 \gamma_\mu \quad (6.230)$$

$$\mathbf{x}_v = \mathbf{x} + \mathbf{x}_5 \mathbf{i} ; \mathbf{x}'_v = \mathbf{x}' + \mathbf{x}'_5 \mathbf{i} \quad (6.231)$$

$$x_v^4 = x^4 + x_5^4 \mathbf{i} ; x_v^5 = -x_5^5 + x^5 \mathbf{i} ; x'^4_v = x'^4 + x_5^4 \mathbf{i} ; x'^5_v = -x_5^5 + x'^5 \mathbf{i} \quad (6.232)$$

$$x_v = \begin{pmatrix} 0 & -\mathbf{x}_v + x_v^4 - x_v^5 \\ \mathbf{x}_v + x_v^4 + x_v^5 & 0 \end{pmatrix} \quad (6.233)$$

$$x'_v = \begin{pmatrix} 0 & -\mathbf{x}'_v + x'^4_v - x'^5_v \\ \mathbf{x}'_v + x'^4_v + x'^5_v & 0 \end{pmatrix}$$

the generalization of (6.209) in $Cl_{5,1}$ reads

$$x'_v = \mathbf{N}x_v\tilde{\mathbf{N}} ; \Psi' = \mathbf{N}\Psi \quad (6.234)$$

which gives

$$\tilde{\Psi}' = \widetilde{\mathbf{N}\Psi} = \tilde{\Psi}\tilde{\mathbf{N}}. \quad (6.235)$$

Then the first equality (6.234) is equivalent to the system:

$$\mathbf{x}'_v + x'^5_v = N(\mathbf{x}_v + x_v^5)\tilde{N} \quad (6.236)$$

$$x'^4_v = x_v^4 N\tilde{N}. \quad (6.237)$$

And since we can separate, in (6.236), the different multivector parts, it is equivalent to the system:

$$\mathbf{x}' = N\mathbf{x}\tilde{N} \quad (6.238)$$

$$\mathbf{x}'_5 = N\mathbf{x}_5\tilde{N} \quad (6.239)$$

$$\begin{aligned} x'_v{}^4 &= x_v{}^4 N\tilde{N} \\ x'_v{}^5 &= x_v{}^5 N\tilde{N}. \end{aligned} \quad (6.240)$$

With (1.43) (6.237) and (6.240) read

$$x'^4 + ix'^4_5 = re^{i\theta}(x^4 + ix^4_5) \quad (6.241)$$

$$-x'^5_5 + ix'^5 = re^{i\theta}(-x^5_5 + ix^5) \quad (6.242)$$

This separation between the different components of the global space-time explains why we usually see only the real components of the 4-dimensional space-time vector \mathbf{x} . Only the usual space-time has real components. (6.241) and (6.242) indicates both that these two supplementary dimensions are complex dimensions and that they separate completely the usual space-time in the global space-time. A space-time with one or two supplementary conditions has been used early [52]. The problem was always to explain why classical physics do not see these supplementary dimensions. Here this problem is automatically solved by the difference coming from the invariance group of physical laws.

The form invariance of the geometric transformation f results from (6.231) and (6.232) which give

$$f : y_v \mapsto x_v = \langle \Psi y_v \tilde{\Psi} \rangle_v \quad (6.243)$$

$$\begin{aligned} f : y_v \mapsto x'_v &= \langle \Psi' y_v \tilde{\Psi}' \rangle_v = \langle \mathbf{N}\Psi y_v \tilde{\Psi}\tilde{\mathbf{N}} \rangle_v \\ &= \mathbf{N} \langle \Psi y_v \tilde{\Psi} \rangle_v \tilde{\mathbf{N}} = \mathbf{N}x_v \tilde{\mathbf{N}}. \end{aligned} \quad (6.244)$$

Similarly to what we said in 3.3, y_v is independent on the observer, intrinsic to the wave.

6.6 Existence of the inverse

Our study in 5.2 of systems of electrons has introduced the inverse ϕ^{-1} which is defined only where $\det \phi \neq 0$. We can see that this condition is satisfied everywhere for each bound state of the H atom (see Appendix C). We saw previously that the wave of the electron is a part of the wave Ψ_l with value in $Cl_{1,3}$ which must be also invertible. We must then get $\det(\Psi_l) \neq 0$.

We have not yet used one of the features of the standard model, because it was not useful until now: the right part of the neutrino wave does not interact, and the standard model can do anything without ξ_n . We can then suppose

$$\xi_{1n} = \xi_{2n} = 0 \quad (6.245)$$

We then have with (6.2), (6.3) and (6.8):

$$\Psi_l = \sqrt{2} \begin{pmatrix} \xi_{1e} & -\eta_{2e}^* & 0 & -\eta_{2n}^* \\ \xi_{2e} & \eta_{1e}^* & 0 & \eta_{1n}^* \\ \eta_{1n} & 0 & \eta_{1e} & -\xi_{2e}^* \\ \eta_{2n} & 0 & \eta_{2e} & \xi_{1e}^* \end{pmatrix} \quad (6.246)$$

We let

$$\rho_e = |\det(\phi_e)| = 2|\xi_{1e}\eta_{1e}^* + \xi_{2e}\eta_{2e}^*| \quad (6.247)$$

$$\rho_L = |\det(\phi_L)| ; \phi_L = \sqrt{2} \begin{pmatrix} \eta_{1n} & \eta_{1e} \\ \eta_{2n} & \eta_{2e} \end{pmatrix} \quad (6.248)$$

$$\rho_l = [\det(\Psi_l)]^{1/2} \quad (6.249)$$

The calculation of the determinant of the matrix (6.246) gives the remarkable result:

$$\rho_l = \sqrt{\rho_e^2 + \rho_L^2}. \quad (6.250)$$

It is then very easy to get an invertible Ψ_l , it happens as soon as ϕ_e is invertible (for instance for each bound state of the H atom), or as soon as η_e and η_n are linearly independent. This is a very interesting use of the condition $\xi_n = 0$, and means that all features of the standard model are important. It also means that the true mathematical frame is the Clifford algebras and that the existence of an inverse wave in each point is physically useful. Moreover we can extend this to the complete wave of all fermions of the first generation. The standard model uses only left waves for the quarks, we get then for the color r :

$$\Psi_r = \sqrt{2} \begin{pmatrix} 0 & -\eta_{2dr}^* & 0 & -\eta_{2ur}^* \\ 0 & \eta_{1dr}^* & 0 & \eta_{1ur}^* \\ \eta_{1ur} & 0 & \eta_{1dr} & 0 \\ \eta_{2ur} & 0 & \eta_{2dr} & 0 \end{pmatrix} \quad (6.251)$$

and two similar equalities for colors g and b . Now we consider two matrices:

$$L = \sqrt{2} \begin{pmatrix} \eta_{1e} & \eta_{1n} & \eta_{1dr} & \eta_{1ur} \\ \eta_{2e} & \eta_{2n} & \eta_{2dr} & \eta_{2ur} \\ \eta_{1dg} & \eta_{1ug} & \eta_{1db} & \eta_{1ub} \\ \eta_{2dg} & \eta_{2ug} & \eta_{2db} & \eta_{2ub} \end{pmatrix} ; M = \sqrt{2} \begin{pmatrix} -\xi_{2e}^* & \eta_{1e} & \eta_{1dr} & \eta_{1ur} \\ \xi_{1e}^* & \eta_{2e} & \eta_{2dr} & \eta_{2ur} \\ 0 & \eta_{1dg} & \eta_{1db} & \eta_{1ub} \\ 0 & \eta_{2dg} & \eta_{2db} & \eta_{2ub} \end{pmatrix} \quad (6.252)$$

And we get with (6.123) the identity

$$\det(\Psi) = |\det(L)|^2 + |\det(M)|^2 \quad (6.253)$$

We can then see the waves of the standard model as having the maximum number of degrees of freedom compatible with the existence of an inverse wave Ψ^{-1} .

In the M matrix of (6.252) the green color is less present than red and blue colors, which seems a priori abnormal. Technically the reason is simple: since

the only right term, ξ , is on the same column as the ug wave, when we suppress all terms of a column and of a line in the calculation of a determinant, the ug term necessarily disappears.

More important, the fundamental mathematical tool is Clifford algebra, not complex matrix algebra. This comes from the fact that space-time algebra is not identical to the algebra of 4×4 complex matrices, or the $Cl_{5,1}$ Clifford algebra of the 6-dimensional space-time is not identical to the algebra of 8×8 complex matrices. These Clifford algebras are only sub-algebras of the complex matrix algebras. Moreover they are sub-algebras only as real algebras. The use of complex linear spaces is then a kind of accident, a fortuitous coincidence: the possible identification between the Clifford algebra of the 3-dimensional space and the algebra of 2×2 complex matrices, as algebras on the real field.

Consequently, for instance, Ψ matrices are not at all symmetric. Null terms are on same columns, not in lines. Lines 1, 2 and 5, 6 of Ψ are multiplied by a M factor while lines 3, 4 and 7, 8 are multiplied by a \widehat{M} factor of the form invariance group. We may also remark that all terms of L and M matrices in (6.252) are left terms, multiplied by a \widehat{M} factor in the form invariance group. When we look at operators of the electro-weak gauge group we can also see that they act on columns of matrices, not on lines.

Other consequence, the change of a matrix into its adjoint matrix is not the most important transformation. This should be the case if the theory of hermitian spaces and unitary matrices was fundamental. The most important transformation is the reversion, (A becoming \widetilde{A}), as we can see throughout this text. This reversion makes sense in any Clifford algebra, and it is this alone reversion that appears in calculations. It happens that the reverse is identical to the adjoint matrix when we identify space algebra and Pauli algebra. But this happens only in the Clifford algebra of the physical space. With the space-time algebra, or with the algebra of the 6-dimensional space-time the reverse is not the adjoint matrix. Reversion exchanges the left-up matrix-bloc and the right-down matrix-bloc, while the right-up and the left-down matrix-blocs stay in the same place, being exchanged two times. If the $SU(3)$ group exchanging r, g, b states was fundamental, it should be effectively necessary that the green color should play the same role as red and blue colors.

Since this is not true, we see that the $U(1) \times SU(2) \times SU(3)$ structure is widely "accidental" on the mathematical point of view. This means that the necessity of unitary gauge groups is not justified. The important structure is in fact the $Cl_{5,1}$ algebra and its left and right multiplicative automorphisms.

6.7 Wave equation

The mass term of the Dirac equation links the right wave to the left wave, we can read this in (2.31) and (2.32). Terms containing W^1 and W^2 in the electro-weak theory link left waves of the electron and of its neutrino. Terms containing B and W^3 act differently on left and right waves. The Weinberg-Salam model took advantage of the very small mass of the electron to neglect and cancel this

mass term⁵³. The proper mass is consequently missing in 6.1 to 6.6.

When we consider the three spinors, a right one and two left ones, necessary to get the electro-weak gauge group, we have much more tensorial densities than with the two spinors of the Dirac theory: we change the 36 tensorial densities of 2.2.2 into $78 = 12 \times 13/2$ tensorial densities. We get three complex densities \mathbf{a} , \mathbf{b} , \mathbf{c} (see B.1) in the place of the alone density $\mathbf{a} = \Omega_1 + i\Omega_2$. The remarkable identity (6.250) uses two of the three terms that replace, in the case of the electron-neutrino pair, the unique ρ density in (2.16):

$$\mathbf{a} = \det(\phi_e) ; \quad \mathbf{b} = \det(\phi_L). \quad (6.254)$$

The third term \mathbf{c} allows to construct the density

$$\boldsymbol{\rho} = \sqrt{\mathbf{a}\mathbf{a}^* + \mathbf{b}\mathbf{b}^* + \mathbf{c}\mathbf{c}^*}. \quad (6.255)$$

This allows a mass term for a wave equation which is both form invariant and gauge invariant in the case of the Ψ_l wave of the electron and its neutrino [21]:

$$\tilde{\Psi}_l(\mathbf{D}\Psi_l)\gamma_{012} + m\boldsymbol{\rho}\tilde{\Psi}_l\chi = 0 \quad (6.256)$$

where χ is a term depending on Ψ_l , defined by (B.89). To get the form invariance (consequently to get the relativistic invariance) we establish in (B.41) the invariance of $m\boldsymbol{\rho}$ and in (B.106) the invariance of the mass term. The wave equation (6.256) is then form invariant under the transformation R defined by M in (1.42) and N in (1.80), the equation becomes [21]:

$$\tilde{\Psi}'_l(\mathbf{D}'\Psi'_l)\gamma_{012} + m'\boldsymbol{\rho}'\tilde{\Psi}'_l\chi' = 0 ; \quad m\boldsymbol{\rho} = m'\boldsymbol{\rho}' ; \quad \tilde{\Psi}'_l\chi' = \tilde{\Psi}_l\chi. \quad (6.257)$$

This wave equation could not come from the linear Dirac equation because relation (B.41) that links χ to Ψ_l makes the wave equation non linear. The homogeneous non linear wave equation studied in chapter 3 acts as go-between: the wave equation (3.9) is what remains in (6.256) when we cancel the wave of the neutrino and the Dirac equation is the linear approximation of (3.9).

In any domain of space-time where the wave of the electron is null or negligible we get $\boldsymbol{\rho} = 0$ therefore the wave equation of the neutrino is reduced to $\nabla\eta_n = 0$ that is the usual wave equation of the neutrino traveling at the velocity of light. This wave is then without interaction, since the neutrino does not interact with the electron and its ϕ_e wave.

Under the gauge transformation defined by (6.119) to (6.122) we get (the detailed calculation is in B.3) :

$$\tilde{\Psi}'_l(\mathbf{D}'\Psi'_l)\gamma_{012} + m\boldsymbol{\rho}\tilde{\Psi}'_l\chi' = 0. \quad (6.258)$$

Therefore it is not necessary to use a complicated mechanism of spontaneously broken symmetry to reconcile the electro-weak gauge invariance and the mass term of the electron.

⁵³This approximation is a posteriori satisfied by the huge mass of the Z^0 which is 180,000 times the mass of the electron. This approximation was inevitable because the mass term of the linear Dirac equation cannot be compatible with the electro-weak gauge.

The generalization of (6.256) to the complete wave including the quarks of the first generation will need to account for the $36 = 9 \times 8/2$ densities generalizing the three **a**, **b**, **c** densities in (6.255).

7 Magnetic monopoles

We present here the recent experimental works on magnetic monopoles. Next we apply to the magnetic monopole our study of electro-weak interactions.

7.1 Russian experimental works

Recent experimental works about magnetic monopoles began with V.F. Mikhailov [53]. He was taking up works made fifty years ago by F. Ehrenhaft. An electric arc produces ferromagnetic dusts that are conducted by a Ar gas into a chamber where a laser lights them up. Into the chamber the ferromagnetic particles are moved by a magnetic field and an electric field orthogonal to the magnetic field. The direction of the fields may be reversed. Movements are observed, under the light of the laser, with an optic microscope.

The measure of the magnetic charge of these particles took advantage of the fact that some of them have also an electric charge and the move of an electric charge into an electric field is well known. Mikhailov observed an elementary magnetic charge $g = n\alpha \frac{e}{6}$. The fine structure constant α is small ($\alpha \approx \frac{1}{137}$).

But the expected value is completely different [31]. A calculation made by Dirac, obtained again in a very smart way by G. Lochak from his theory of the monopole [42] gives, for the elementary magnetic charge

$$\frac{eg}{\hbar c} = \frac{n}{2} \tag{7.1}$$

where n is an integer. The elementary magnetic charge observed by Mikhailov was much smaller than the theoretical charge. We may ask if there is a reason to refute the theoretical calculation, or if there exists an experimental reason to this divergence. The two things are possible: each process allowing to get (7.1) includes a calculation of the potentials created by charges, and we can doubt its validity. Magnetic charges observed by Mikhailov were visible only during the illumination by the strong light of a laser, and may have been second order effects coming from this illumination. Mikhailov realized also an experiment where the ferromagnetic particles were included into water droplets, with spherical symmetry. Then he measured magnetic charges compatible with the elementary magnetic charge calculated by Dirac. The value of such a charge is then a question that must be solved experimentally.

The experimental work of L. Urutskoev had in common with Mikhailov's work only the use of an electric arc. To shatter concrete, little holes were made and filled with water, an electric wire was put in each hole and an electric condenser was discharged into the wires. The discharge produced an explosion

and this explosion shattered the concrete. The first astonishing fact was the great speed of the pieces of concrete smashed by the explosion, this induced a need to better study what was going there.

The continuation of experiments was to shoot into pure water, without concrete. An intense glowing was found to appear above the device. The duration of this phenomenon, about 5ms, was much greater than the duration of the discharge, 0.15ms. A spectral analysis of the emitted light was performed. Spectral lines of nitrogen or oxygen were very weak, while the glowing was emitted into the air. and the strongest spectral lines showed the presence of Ti, Fe, Cu, Zn, Cr, Ni, Ca, Na. The presence of Cu and Zn could come from the electric wires, the presence of Ti signified that the Ti foils used in discharges spread above the device, in spite of the cover. The presence of the other elements was enigmatic. This induced to analyze more finely the metal powder resulting from the explosion of the Ti foil in water. Observations made were still stranger. While the foil was made of 99.7% Ti the ratio of Ti in powder could go down to 92%. The amount of disappeared titanium corresponded to the amount of new elements appearing, Fe, Si, Al, Ca, Na, Cu, Zn, principally. In addition, an isotopic analysis showed that the isotopic composition of Ti was changed, with a significant decline of the ratio of ^{48}Ti . Experiments were repeated many times, with all necessary precautions. Other metals were used, in particular zirconium. Ratios of different outside elements changed on composition of the exploded foil. For instance there was much more Cr with the zirconium than with titanium, and much less Si and Al.

Since the transformation from an element to another is usually associated to radioactivity, an intensive search of radioactive emission was made. There was no X- γ rays detected, in spite of 10^{19} - 10^{20} transformed atoms at each shot. Detection of neutrons was also performed. Scintillator detectors indicated a pulse that allowed to estimate the speed of the radiation to 20 - 40 m/s. Such a low speed could not match a neutron flux, because neutrons should be ultra cold. A detection of the radiation with photo emulsions was applied, for lack of better means. We will come back farther on what is seen with these films. Urutskoev saw next that the presence of a strong magnetic field changed the aspect of these traces, and he deduced that the radiation going out his shots had magnetic properties. He led then experiments to trap the radiation with strong magnets and he used the Moessbauer effect to prove the reality of these captures.

Urutskoev noted also that the transformations come principally from even-even kernels, that is to say from kernels with an even number of protons and an even number of neutrons. He noticed that the mean binding energy of produced kernels was very few different from the mean binding energy of initial kernels: there was no nuclear energy emitted or absorbed in significant amount. And all the produced kernels were in the ground state, there was no radioactivity.

Experiments made by N. G. Ivoilov [36] indicated that it was possible to get similar traces on photographic films with much less energy: he used an electric arc into water, with a current exceeding not 40 A with a 80 V tension. He got traces complying to properties of magnetic monopoles predicted by the G.

Lochak's theory.

7.2 Works at E.C.N.

Works made at the Ecole Centrale de Nantes, in the laboratory of Guillaume Racineux by Didier Priem and Claude Daviau [48] with the help of Henri Lehn and of the Fondation Louis de Broglie, had the aim to satisfy and to continue the Urutskoev's work. This seemed necessary in view of the extraordinary nature of yet obtained results.

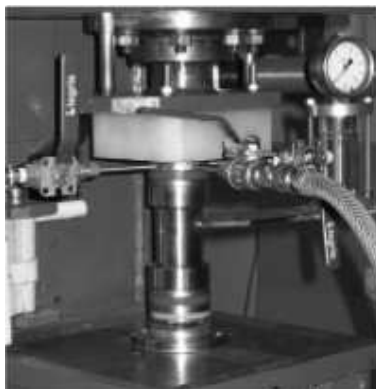


Figure 1: Vessel

The experimental device is dependent on the available equipment at the E.C.N. and then is different, even if it is as few as possible, from this used by Urutskoev. The generator is an American one, Maxwell type, maximum power 12 kJ at 8.4 kV, capacity 360 μ F and a vessel (Figure 1). The first containment vessel was made of aluminum, it was replaced by a second vessel to allow to collect the gas produced during a shot. Experiments made by Urutskoev allow him to see that the gas is almost totally hydrogen. This second vessel was made of stainless steel, it contained a tank with an internal diameter of 20 mm covered by polyurethane. The internal diameter was then reduced to 16 mm which has improved the yield. A third vessel was made when the second was tired. The current coming from the generator is distributed into two electrodes, one up and one down. They are linked by a fuse made of Ti40.

After a shot, the gas is collected, its volume is measured. Powders are collected with the liquid which contains them, and are placed during 24 hours under a photographic plate exposed to the radiation coming out of powders. This photographic film is then developed and examined with an optic microscope. Powders are dessicated and examined with the electronic microscope of the E.C.N.. This allows us to get three kinds of results, about powders, gas and traces on the photographic films.

7.2.1 Results about powder and gas

Our observations confirm very widely the results obtained by L. Urutskoev, even if our ratios of production are lower than those he got. The energy of the discharge being lower than this of Urutskoev, and the discharge being shorter, this is not astonishing. But outside this, strange elements whose we get spectrograms into the electronic microscope have a composition very near that obtained by Urutskoev. In the same time our observations make the things still stranger: when we notice the presence of one per cent of iron into our powders, this iron is not dispersed a little everywhere. On the contrary what we notice is: one per cent of the particles are made of so much iron than titanium is quasi missing. It is often iron which is dominating but there are shots where we find more copper than iron. The particles made of copper have any scale, some are numerous and have length of about one micrometer, others much rarer are greater and even visible to the naked eye. Those particles contain very few titanium. The composition of the exotic particles may be more complicated: we observe particles of iron-chromium, of copper-zinc. Iron is rarely alone, it is most of time with chromium, a little nickel, sometimes 1% manganese, and with carbon and oxygen. The composition of particles is often not homogeneous, a particle may have not transformed titanium at places and titanium may have been nearly all replaced at another place.

Figure 2 shows a particle with an evident continuity, which has dark places and one light place, in addition to many holes. On the left and above, titanium remains intact. At the center, the spectral analysis indicates the following mass composition : Fe 69,8%, Ti 10,81%, Ni 7.28%, Cr 4,33%, O 3,98%, C 3,8%. Holes are also significant, because they indicate a gas production just before the solidification caused by the intense cooling in water.

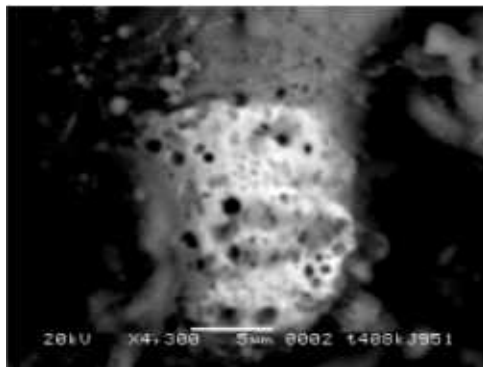


Figure 2: Particle with an iron place

The fact that iron is rarely alone, and that it is associated to chromium and nickel has much complicated our work, because the stainless steel of our tank is made of those three metals, and we could be objected that stainless steel of our tank contaminated powders. Stainless steel was therefore removed from the inside of the tank, it contains now only titanium and polyurethane.

The suppression of the stainless steel has changed nothing actually, there is also iron into powders when the alone metal inside the tank is titanium. This was predictable since the composition indicated above is not this of the stainless steel of our vessel. We can also easily satisfy that the Ti40 used to make our fuse does not contain the ratio of iron, copper and other materials found into powders.

Extraordinary results obtained by Urutskoev are therefore well real. The one who should say them impossible only has to reproduce the experiment. If he is honest, he will be obliged to see that something really happens.

But nothing should happen: conditions of the experiment move an energy measured in kJ, this furnish only a hundred of eV at uppermost for each concerned atom of titanium. This is ridiculously small in comparison with nuclear binding energies. In addition interactions known until then work in a completely different way. For instance weak interactions allow to transform one proton of a kernel into a neutron, or vice versa, and that is submitted to general laws of quantum mechanics, where random plays a obligatory and permanent role. If the kernel of a titanium atom was transformed by weak interaction, it could give a kernel of scandium or vanadium. Neither of those metals was seen until then. We saw vanadium rays not only once, and vanadium is an obligatory way if you want to go, with weak interactions, from titanium to iron or copper. And if weak interactions were acting, transformed kernels should arrive at random, in time and in space, not into macroscopic bundles.

We must not forget that our experiment is an explosion and an explosion is not precisely the best way to assemble into a packet some dispersed atoms. It is on the contrary a very good way to disperse a concentrated matter. Since we see particles made of iron, or of copper, or of nickel, or of iron-chromium, with very few titanium, these elements were produced together. We do not understand how it is possible, but that changes nothing to the reality of the phenomenon.

In addition there are energy constraints. The mass of the elements found into our powders and which should not be there is 10^{10} times greater than the mass of the energy brought by the electric discharge. With an excellent precision we can then say that the total energy of the produced atoms is equal to the total energy of the destroyed atoms. This conservation of the total energy restricts considerably the possibilities of reaction. We cannot get for instance vanadium. The isotopes of vanadium are heavier than these of titanium, which allows to ^{48}V to be β^+ radioactive and to disintegrate into ^{48}Ti . And as we have no radioactivity linked to these transformations, it is necessary that the total number of electrons, of protons and of neutrons are also conserved. So strange as it may be, all these conditions of conservation do not forbid the observed transformations. As Urutskoev said, all is just like if for instance 100 kernels of ^{48}Ti go together for some reason to form a big "kernel", then reallocate their nucleons to form in the same time lighter and heavier kernels. Doing so they also respect the conservation laws of energy, electric charge, baryonic charge, leptonic charge... And in addition this magical transformation is accompanied by no significant radioactivity!

Now we do not know enough to have an idea about mechanisms of trans-

formation. We do not know why after one shot we get many iron particles while after the following shot we will get instead copper particles. To begin to understand what happens will necessitate probably a very fine analysis of the obtained particles and of the physical conditions into our vessel. In view of the brevity of the discharge ($72\mu\text{s}$) and the intense pressures linked to the shock wave, it will be not easy to learn more.

Some gas was always produced during the metallurgical works made at the Ecole Centrale. The presence of this gas was besides considered as a nuisance, limiting the repetitiveness of the shots. The device that we use allows to measure easily the quantity of produced gas. This gas is quasi totally made of hydrogen. As titanium heated to a very high temperature is a reducing agent, this is not surprising. It is also difficult to estimate the quantity of oxygen going into the powders as oxide or dioxide of titanium, or dissolving into water. We have estimations indicating that a part of the hydrogen does not come from the dissociation of water. To satisfy this a shot into heavy water has been done with success by L. Urutskoev. He got not only D₂, but also HD and H₂. And this hydrogen cannot come from the water. Transformations of titanium can leave isolated protons and electrons which form hydrogen atoms. This hydrogen, either from chemical origin or not, is formed inside particles, which are often so much spongy that they float on the water in which we collect the powders.

7.2.2 Stains

After each trial, the titanium powders from the fusible are collected along with the water contained in the trial chamber and are placed under a photographic plate. The traces are produced, not immediately in the electric arc, but by what is in the water and powders, and leaving several hours after the electric arc. Sometimes, something go out of the water, it is not only the things that make the traces, but also a part of the powders on the surface of the water. They emerge from the water, despite the gravity and the surface tension of water, and are glued on the wrapping paper of the photographic plate:

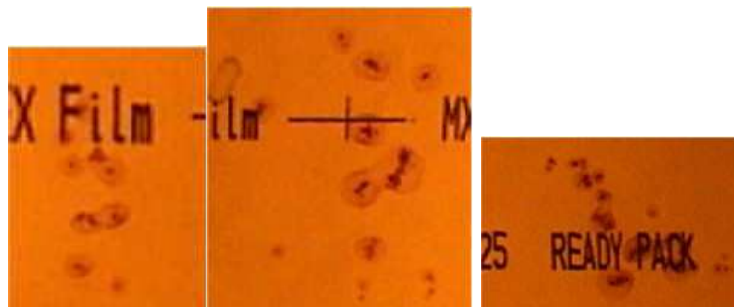


Figure 1: Stained paper, experiments 103, 62, 79.

7.2.3 Traces

Powders collected after a shot and the photographic plate, following both indications of Urutskoev and Lochak, are placed between two metallic plates forming a plane condenser, under a low 10V tension. The move of a magnetic monopole in a fixed uniform electric field is analogous to the move of an electric charge in a fixed uniform magnetic field. The Laplace force is

$$\vec{F} = g(\vec{H} - \frac{\vec{v}}{c} \times \vec{E}) \quad (7.2)$$

where g is the charge of the magnetic monopole. In a constant electric field orthogonal to the plane of the plate, a monopole must have a circular move. We expect rotations into the plane of the photographic plate, and it is what happens rather often, as figures 3 to 7 show.

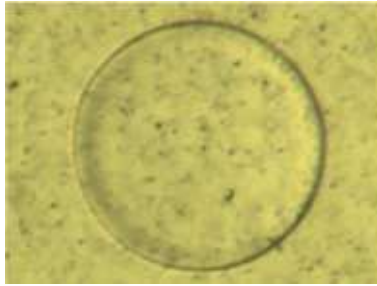


Figure 3: Circle, diameter: 0,2 mm



Figure 4: Circle, length of the picture: 2,6 mm



Figure 5: circle, length of the picture: 2,47 mm



Figure 6: Circle, length of the picture: 0,95 mm



Figure 7: Circle, length of the picture: 1,45 mm

We must not expect all traces to be circular, because the presence of glass dishes between plates induces a certainly non-uniform electric field. We must also notice we do not know a priori what we seek, we see probably only a little

part of traces, in the absence of knowing completely the dynamics of magnetic monopoles. We also do not know how monopoles interact with the photographic plate. It is easier to see the very long and stark traces, more difficult to see the short and weak traces. Circles are not the only curved traces, we obtain also horseshoes:



Figure 8: Horseshoe, length of the picture: 0,19 mm

We must expect not perfect circles, notably because the loss of energy gives a smaller radius. This is visible on the following pictures



Figure 9: Braking, length of the picture: 1,78 mm



Figure 10: Braking, length of the picture: 1,9 mm



Figure 11: Braking, length of the picture: 0,57 mm



Figure 12: Braking, length of the picture: 0,2 mm

Large traces, as in figures 8 or 12, are actually double traces. This doubling of traces is more visible when the two traces are well separated:



Figure 13: Double trace, length of the picture: 2,67 mm



Figure 14: Double trace, length of the picture: 0,58 mm

The magnetic monopole of G. Lochak is a chiral object, built from an angle which is pseudo-scalar. The simpler object of our usual world explaining what is chirality is a screw. There are left screws and right screws. This property is verified for several observed traces⁵⁴. We can see spirals, often with difficulty. Sometimes the spiral is very visible, as on this trace and its enlargements :

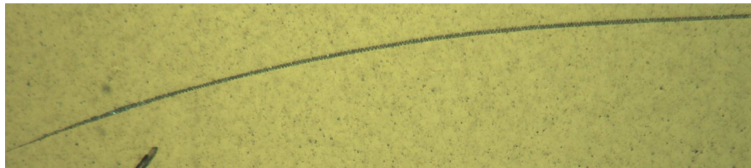


Figure 15: Spiral trace (length 2mm)



⁵⁴This is at the moment the best proof of the predictive power of the Lochak's idea of a Dirac wave for the magnetic monopole.

Figures 16 and 17: Enlargements of figure 15

Undulations are often seen on enlargements of our pictures:



Figure 18: Wave, length of the picture: 2,67 mm

A wavelength is directly measurable on this picture, where we count 30 wavelengths, this gives a $89\mu\text{m}$ wavelength. A wavelength is also directly measurable on the following picture:



Figure 19: Wave, length of the picture: 1,54 mm

Considering the four undulations in the middle we can estimate the wavelength: $130\mu\text{m}$. Moreover a second thing is visible on this picture, a double pattern with alternatively rising and descending traces.

The Lochak's theory of the magnetic monopole can account for this double pattern: the wave is a Dirac spinor made of two Weyl spinors, a right one and a left one. If the proper mass of the monopole is null these two Weyl spinors are independent and may move one without the other. If the proper mass is not null the two Weyl spinors are coupled by the mass term. Perhaps what we see on figure 19 is exactly that, a left wave and a right wave, whose we see only pieces. They are superposed at ends and successively seen in the middle. A double pattern is rather common, we can see this in the following figures:

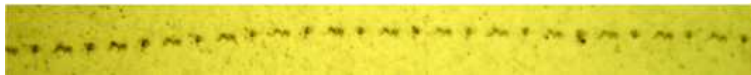


Figure 20: Double pattern, length of the picture: 2 mm

The wavelength is estimated to $143\mu\text{m}$.



Figure 21: Double pattern, length of the picture: 1,64 mm

Here the wavelength is estimated to $65\mu\text{m}$.

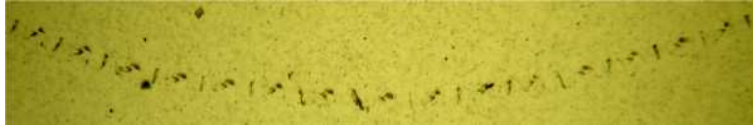


Figure 22: Double pattern, length of the picture: 2,65 mm

The wavelength is estimated to $177\mu\text{m}$. If the wavelength is the de Broglie's wavelength, not an artifact⁵⁵, it is possible to calculate the impulse:

$$p = mv = \frac{h}{\lambda} \quad (7.3)$$

For the wave of figure 21 where the wavelength is the shortest the impulse is about 10^{-29}kgm/s . The big question is then the velocity of the magnetic monopole. If it is the light speed the energy is very small. Can a wave with only $0,02\text{eV}c^{-2}$ make the visible effects on figure 21 ? This is dubious. The only experimental velocity was given by Urutskoev and it is very low: 20-40 m/s. A velocity of 20 m/s gives then a mass: 5.10^{-31}kg , similar to the proper mass of the electron. A velocity still lower is possible since it is perhaps at the end of the braking that we saw this trace. Another theoretical possibility is given by (5.42) where the limit speed has a null limit when ϵ is near 1.

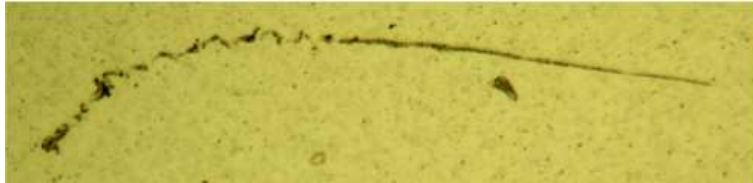


Figure 23: Continuous-broken trace, length of the picture: 1,47 mm.

Continuity of many traces is only an appearance coming from a blurred picture. We can see this on the next picture, where a numeric enlargement allows to count grains and to estimate the distance between two grains: $8\mu\text{m}$.



Figure 24: Enlarged trace, length of the picture: 0,38 mm.

Another frequent aspect of our traces is the quasi-parallelism of very long traces, as on the following figure:

⁵⁵G. Lochak thinks that what we see is not the de Broglie's wavelength, but only a scale corresponding to the response of the plate to the move of the wave. But then why two patterns ?



Figure 25: Multiple traces, length of the picture: 1,97 mm.

Figure 25 shows only a part of each trace which extends on the two sides of the picture. We see five traces nearly parallel and we guess two other ones. We can suppose the double character of these traces is linked to the double character of the wave, with a left and a right part. Following this hypothesis we can think single traces due to superimposed left and right parts. The parallelism of some traces can come from a weak separation of divergent traces, as on the following figure:

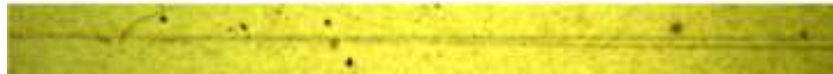


Figure 26: Divergent traces, length of the picture: 2,67 mm.

These traces present obviously a granular structure, with distances between grains very similar: this pleads strongly for the hypothesis of a unique wave, left and right. The wavelength is estimated to $19,6\mu\text{m}$. A few branchings between traces may be seen:



Figure 27: Branching, length of the picture: 1,87 mm.



Figure 28: Branching, length of the picture: 0,77 mm.

One trace favoring best the hypothesis of the left and right spinors is the following, with an enlargement of the upper trace and another of the down trace:



Figure 29: Double spiral, length of the picture: 1,94 mm.

The two enlargements are similar to two screws turning inversely.

All these traces show stark differences from physics of particles with an electric charge. To see the left or right nature of a trace will necessitate a three-dimensional observation of these traces. Such observations show that monopoles make depressions on the surface of the plate [22].

7.3 Electrons and monopoles

The invariant wave equation (3.9) of the electron was obtained from the wave equation of the Lochak's magnetic monopole (3.11) in the particular case (3.12) where the wave equation is homogeneous. To do this we replaced the local chiral gauge by the local electric gauge. We shall then get the invariant wave equation of the magnetic monopole by using the inverse transformation, replacing the electric gauge by the chiral gauge. We read this gauge in space algebra as :

$$\phi' = e^{ia} \phi ; QB' = QB - \nabla a ; Q = \frac{g}{\hbar c} \quad (7.4)$$

where a is a real number and where g is the charge of the magnetic monopole. iB is the pseudo-vector of space-time magnetic potential, which is also the Cabibbo-Ferrari's potential of the theory of the monopole and which is also the potential term that is multiplied by the projector P_0 in (6.13). The invariant wave equation of the magnetic monopole reads then :

$$\bar{\phi}(\nabla\hat{\phi})\sigma_{21} + \bar{\phi}QiB\hat{\phi}\sigma_{21} + m\rho = 0. \quad (7.5)$$

First difference with the case of the electron : this wave equation has none linear approximation. It is not allowed to add a $e^{-i\beta}$ term into the mass term because β is not chiral gauge invariant.

To get the 8 numeric equations of this invariant wave equation we use a space-time vector U satisfying

$$\bar{\phi}QB\hat{\phi} = U^\mu \sigma_\mu \quad (7.6)$$

and we get in the place of (3.21) to (3.28) the system

$$0 = w^3 - U^3 + m\rho \quad (7.7)$$

$$0 = v^2 \quad (7.8)$$

$$0 = -v^1 \quad (7.9)$$

$$0 = w^0 - U^0 \quad (7.10)$$

$$0 = -v^3 \quad (7.11)$$

$$0 = w^2 + U^1 \quad (7.12)$$

$$0 = -w^1 - U^2 \quad (7.13)$$

$$0 = -v^0 \quad (7.14)$$

As with the electron the scalar part of the invariant wave (7.7) is the Lagrangian density. Lochak immediately remarked that in this Lagrangian density the current $J = D_0$ is replaced by $K = D_3$. From there comes in 7.7 the 3 index instead of a 0 index. We can say that the invariant wave equation is somewhere simpler than the invariant wave of the electron : all four v^μ terms are zero. This means that the four D_μ vectors are conservative. We recall that the density D_0^0 gives in the case of the electron that quantum theory see, from the Schrödinger equation, as a probability density. Lochak has proved that $K = D_3$ is the conservative current linked to the invariance of the Lagrangian density (7.7) under the chiral gauge (7.4). Vectors D_1 and D_2 , equally conservative, are unknown of the formalism of Dirac matrices. We have seen in (2.103)-(2.104) that the electric gauge gives a rotation in the (D_1, D_2) plane. With the chiral gauge all four D_μ are invariant. They are with (2.56) the elements of an orthogonal basis, and their components are the elements of the matrix of the dilatation D in (3.55).

7.3.1 Charge conjugation

We use again the link between the wave of the particle and the wave of the antiparticle. We note the wave of the antimonopole ϕ_a :

$$\widehat{\phi} = \widehat{\phi}_a \sigma_1 ; \phi_a = -\phi \sigma_1 ; \bar{\phi} = \sigma_1 \bar{\phi}_a \quad (7.15)$$

The invariant wave equation is then read as

$$\sigma_1 \bar{\phi}_a (\nabla \widehat{\phi}_a) \sigma_1 \sigma_{21} + \sigma_1 \bar{\phi} Q i B \widehat{\phi}_a \sigma_1 \sigma_{21} + m\rho = 0. \quad (7.16)$$

Multiplying on the right and on the left by σ_1 we get

$$-\bar{\phi}_a (\nabla \widehat{\phi}_a) \sigma_{21} - \bar{\phi}_a Q i B \widehat{\phi}_a \sigma_{21} + m\rho = 0. \quad (7.17)$$

This is usually simplified into

$$\bar{\phi}_a (\nabla \widehat{\phi}_a) \sigma_{21} + \bar{\phi}_a Q i B \widehat{\phi}_a \sigma_{21} - m\rho = 0. \quad (7.18)$$

Therefore Lochak remarked immediately that the charge conjugation does not change the sign of the magnetic charge, contrarily to the case of the electric charge. Then there is no polarization of the void from magnetic charges.[40] [41] [42]. But the form invariance of the wave equation indicates that the true wave equation is (7.17), not (7.18). It should then be more correct to say that, contrarily to the case of the electron, the charge conjugation changes here not only the differential term, but also the charge, then it does not change the gauge nor the sign of the mass-energy.

7.3.2 The interaction electron-monopole

The space-time vector B is, like the vector electromagnetic potential A , a contravariant vector, this is correct because O. Costa de Beauregard explained [24] why potential terms are moving with sources that are electric and magnetic charges. The QB vector, similar to the qA vector, is a covariant vector (see section 4). This allows the interaction by gauge invariance. We have seen in section 6 that the Weinberg-Salam θ_W angle is invariant under the group of dilations. An electric charge creating a A potential creates then also, with (6.85), a potential :

$$B = \cos(\theta_W)A \quad (7.19)$$

Since this B potential is present in the wave equation of the magnetic monopole, it is able to interact with the electric charge. This interaction was detailed by Lochak. The basis of his calculation is the continuity of the wave function under the group of rotations. The continuity of the wave being comforted by the continuity of the potential, it is not necessary to review the calculation and we can use [40] [42]. The B potential used there was questionable because it is not continuous in each point of the z axis. It is why the result, even if the physical reasoning was perfect, is a little too short. In the case of a potential created by an electric charge we have

$$A^0 = -\frac{e}{r} ; B^0 = \cos(\theta_W)A^0 = \cos(\theta_W)\left(-\frac{e}{r}\right) = -\frac{e \cos(\theta_W)}{r} = -\frac{e'}{r} \quad (7.20)$$

where $e' = e \cos(\theta_W)$. The Dirac formula giving the magnetic charge that Lochak obtained by the only condition of continuity of the wave under the group of rotations becomes then

$$\frac{e'g}{\hbar c} = \frac{n}{2} \quad (7.21)$$

where n is an integer, this gives a magnetic charge which is a multiple of :

$$g = \frac{\hbar c}{2e \cos(\theta_W)}. \quad (7.22)$$

We get then a lightly greater charge, 1.134 times the charge calculated by the Dirac formula. This charge has been gotten by numerous ways, for instance from the angular momentum of the electromagnetic field, or from the movement of

an electric charge in the field of a magnetic monopole. In this case it is a iB potential that is created by the magnetic charge. The electron sees the A potential that comes from (6.85) and we get also

$$A = \cos(\theta_W)B \quad (7.23)$$

Then all is as if the charge of the magnetic monopole should be $g' = g \cos(\theta_W)$. In the place of (7.21) we get

$$\frac{eg'}{\hbar c} = \frac{n}{2} \quad (7.24)$$

which gives again the modified Dirac formula (7.22). This small change to the value of the magnetic charge is the only change. The Poincaré's equation giving the trajectory of an electron upon a magnetic monopole [35] is unchanged, as the cone that he introduced. Lochak proved that this cone is the Poincaré cone of a quantum top [43].

The presence of a σ_{21} term in the invariant wave equation implies similarly to the electron case, the existence of two other wave equations obtained by a circular permutation of indexes 1, 2, 3 in Pauli matrices (see section 5). A fourth kind of magnetic monopole comes from the wave equation of a fourth neutrino (6.183) by adding a gauge term. We can then think that four kinds of magnetic monopoles may exist, three of them similarly to the fact that there are electrons but also muons and tauons. These three generations must be treated separately in the electro-weak interactions that we look at now

7.3.3 Electro-weak interactions with monopoles

We want to get an identity similar to (6.250) allowing to Ψ^{-1} to exist everywhere, we suppose then that the wave of the monopole interacting is

$$\Psi = \begin{pmatrix} \phi_L & \phi_n \\ \hat{\phi}_n & \hat{\phi}_L \end{pmatrix}; \quad \phi_n = \phi_{nL} + \phi_{nR} \quad (7.25)$$

where ϕ_n is the wave of the magnetic monopole. We use here the idea of Lochak of the monopole as an excited state of the neutrino, and we place the wave of the monopole where was the place of the neutrino. The supplementary left spinor ϕ_L may be seen as a part of an electric wave. We conserve the form (6.22) of the covariant derivative. Since only P_0 was changed when we went from the lepton case to the quark case, we shall use the same projectors P_{\pm} of (6.12) and we use again projectors P_j in (6.14) to (6.16). In the place of (6.13) we let

$$P_0(\Psi) = a\Psi\gamma_{21} + bP_-(\Psi)\mathbf{i} \quad (7.26)$$

where a and b are real numbers. We get the same commutation relations as in (6.17), except the last equality which must be replaced by :

$$P_0P_j = P_jP_0 = -a\mathbf{i}P_j. \quad (7.27)$$

Therefore the gauge group has the same structure $U(1) \times SU(2)$. We get with (7.25)

$$P_+(\Psi) = \begin{pmatrix} \phi_L & \phi_{nL} \\ \widehat{\phi}_{nL} & \widehat{\phi}_L \end{pmatrix}; \quad P_-(\Psi) = \begin{pmatrix} 0 & \phi_{nR} \\ \widehat{\phi}_{nR} & 0 \end{pmatrix}. \quad (7.28)$$

We recall that

$$\gamma_{21} = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}; \quad \phi_R\sigma_3 = \phi_R; \quad \phi_L\sigma_3 = -\phi_L. \quad (7.29)$$

We then have

$$\begin{aligned} \Psi\gamma_{21} &= i \begin{pmatrix} -\phi_L & \phi_{nR} - \phi_{nL} \\ -\widehat{\phi}_{nR} + \widehat{\phi}_{nL} & \widehat{\phi}_L \end{pmatrix}; \quad P_-(\Psi) = i \begin{pmatrix} 0 & -\phi_{nR} \\ \widehat{\phi}_{nR} & 0 \end{pmatrix} \\ P_0(\Psi) &= i \begin{pmatrix} -a\phi_L & (a-b)\phi_{nR} - a\phi_{nL} \\ (-a+b)\widehat{\phi}_{nR} + a\widehat{\phi}_{nL} & a\widehat{\phi}_L \end{pmatrix} \end{aligned} \quad (7.30)$$

$$\mathbf{B}P_0(\Psi) = i \begin{pmatrix} (-a+b)B\widehat{\phi}_{nR} + aB\widehat{\phi}_{nL} & aB\widehat{\phi}_L \\ -a\widehat{B}\phi_L & (a-b)\widehat{B}\phi_{nR} - a\widehat{B}\phi_{nL} \end{pmatrix} \quad (7.31)$$

We use (7.29) and (6.15), then we get

$$P_2(\Psi) = \begin{pmatrix} \phi_{nL} & -\phi_L \\ -\widehat{\phi}_L & \widehat{\phi}_{nL} \end{pmatrix} \quad (7.32)$$

$$P_1(\Psi) = P_2(\Psi)\mathbf{i} = i \begin{pmatrix} \phi_{nL} & \phi_L \\ -\widehat{\phi}_L & -\widehat{\phi}_{nL} \end{pmatrix} \quad (7.33)$$

$$(\mathbf{W}^1P_1 + \mathbf{W}^2P_2)(\Psi) = i \begin{pmatrix} (-W^1 + iW^2)\widehat{\phi}_L & (-W^1 - iW^2)\widehat{\phi}_{nL} \\ (\widehat{W}^1 - i\widehat{W}^2)\phi_{nL} & (\widehat{W}^1 + i\widehat{W}^2)\phi_L \end{pmatrix} \quad (7.34)$$

Using W^+ and W^- defined in (6.74) we get

$$(\mathbf{W}^1P_1 + \mathbf{W}^2P_2)(\Psi) = i \begin{pmatrix} W^-\widehat{\phi}_L & -W^+\widehat{\phi}_{nL} \\ \widehat{W}^+\phi_{nL} & -\widehat{W}^-\phi_L \end{pmatrix} \quad (7.35)$$

We have also :

$$P_3(\Psi) = P_+(\Psi)(-\mathbf{i}) = i \begin{pmatrix} -\phi_L & \phi_{nL} \\ -\widehat{\phi}_{nL} & \widehat{\phi}_L \end{pmatrix} \quad (7.36)$$

$$\mathbf{W}^3P_3(\Psi) = i \begin{pmatrix} -W^3\widehat{\phi}_{nL} & W^3\widehat{\phi}_L \\ -\widehat{W}^3\phi_L & \widehat{W}^3\phi_{nL} \end{pmatrix} \quad (7.37)$$

The gauge derivative (6.22) is then equivalent to the system

$$D\widehat{\phi}_n = \nabla\widehat{\phi}_n + i\frac{g_1}{2}[aB\widehat{\phi}_{nL} + (-a+b)B\widehat{\phi}_{nR}] + i\frac{g_2}{2}(W^-\widehat{\phi}_L - W^3\widehat{\phi}_{nL}) \quad (7.38)$$

$$D\widehat{\phi}_L = \nabla\widehat{\phi}_L + i\frac{g_1}{2}aB\widehat{\phi}_L + i\frac{g_2}{2}(-W^+\widehat{\phi}_{nL} + W^3\widehat{\phi}_L) \quad (7.39)$$

With(6.83) we get

$$g_2 W^3 = \sqrt{g_1^2 + g_2^2} Z^0 + g_1 B \quad (7.40)$$

Then (7.38) reads

$$\begin{aligned} D\hat{\phi}_n &= \nabla\hat{\phi}_n + i\frac{g_1}{2}[(a-1)B\hat{\phi}_{nL} + (-a+b)B\hat{\phi}_{nR}] \\ &\quad + i\frac{g_2}{2}W^-\hat{\phi}_L - \frac{i}{2}\sqrt{g_1^2 + g_2^2}Z^0\hat{\phi}_{nL} \end{aligned} \quad (7.41)$$

We want to get $\nabla\hat{\phi}_n + iQB\hat{\phi}_n$ then we must have

$$g_1(a-1) = g_1(b-a) = 2Q \quad (7.42)$$

The first equality gives $b = 2a - 1$ and if this condition is satisfied we get

$$\begin{aligned} g_1 &= \frac{q}{\cos(\theta_W)} = \frac{e}{\hbar c \cos(\theta_W)} \\ \frac{e(a-1)}{\hbar c \cos(\theta_W)} &= 2Q = \frac{2g}{\hbar c} \\ \frac{e(a-1)}{\cos(\theta_W)} &= 2g = \frac{\hbar c}{e \cos(\theta_W)} \\ a-1 &= \frac{\hbar c}{e^2} = \frac{1}{\alpha} \end{aligned} \quad (7.43)$$

where α is the fine structure constant and we must take

$$a = 1 + \frac{1}{\alpha}; \quad b = 1 + \frac{2}{\alpha} \quad (7.44)$$

This gives in (7.39)

$$D\hat{\phi}_L = \nabla\hat{\phi}_L + i\frac{g_1}{2}\left(1 + \frac{1}{\alpha}\right)B\hat{\phi}_L + i\frac{g_2}{2}(-W^+\hat{\phi}_{nL} + W^3\hat{\phi}_L) \quad (7.45)$$

which is not the derivative term of an electron and remains to interpret. Since terms containing a are much bigger than other terms the magnetic charge term seems dominant in (7.41) and (7.45).

This third spinor is not only a theoretical invention: it is perfectly visible on figure 29. We can enlarge this image and then see two interlaced spirals in the strong trace and an alone spiral in the weak trace.

7.3.4 Form and gauge invariant wave equation

The couple wave of the monopole - ϕ_L wave is similar to the couple electron-neutrino, we can see this by letting:

$$\Psi_m = \Psi_{\gamma_0} = \begin{pmatrix} \phi_n & \phi_L \\ \hat{\phi}_L & \hat{\phi}_n \end{pmatrix}. \quad (7.46)$$

The wave equation for the Ψ in (7.25) is then similar to (6.256):

$$\tilde{\Psi}(\mathbf{D}\Psi)\gamma_{012} + m\rho\tilde{\Psi}\chi_m = 0 ; \quad \chi_m = \chi\gamma_0. \quad (7.47)$$

The ρ term and χ are obtained by the replacement of Ψ_l by Ψ_m in formulas of B. The wave equation is form invariant under the transformation R in (1.42) induced by M because we have (6.105) and:

$$\Psi' = N\Psi ; \quad N = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} \quad (7.48)$$

$$\tilde{\Psi}' = \tilde{\Psi}\tilde{N} \quad (7.49)$$

$$\rho' = r\rho \quad (7.50)$$

$$m'\rho' = m'r\rho = m\rho. \quad (7.51)$$

The existence of one mass term, then of one impulse-energy vector, implies the same wavelength for each of the three spinors, this may be seen on figure 29.

The wave equation (7.46) is also gauge invariant under the gauge transformation defined by (6.119) to (6.122), because P_0 has the general form studied in B and we then get:

$$\tilde{\Psi}'(\mathbf{D}'\Psi')\gamma_{012} + m\rho\tilde{\Psi}'\chi'_m = 0. \quad (7.52)$$

The mechanism of the spontaneously broken gauge symmetry is not necessary, neither for the electron nor for the magnetic monopole since the wave equations are simply gauge invariant.

8 Conclusion

Starting from old flaws of the relativistic quantum mechanics, we resume the new insights of the standard model that are allowed by our new way with Clifford algebras. Physics using a principle of minimum is only a part of undulatory physics. Beyond the confrontation between theory and experiment, beyond future applications, the standard model appears both comforted and essential. Only novelties are the leptonic magnetic monopoles.

8.1 Old flaws

The discovery of the spin of the electron goes back to 1926 and was not predicted by the physical theory. Physicists have very naturally begun to get round the novelty by trying to reduce spinorial waves to tensors that were better known. The study was difficult, the field was cleared by the students of Louis de Broglie, mainly O. Costa de Beauregard [23] and T. Takabayasi [51] who was able to give a set of tensorial equations equivalent to the Dirac equation. These tensorial equations however act on quantities which are quadratic on the wave. But when

we add the waves these tensors do not add. Therefore the spinorial wave itself is essential on the physical point of view, propagating and interfering. Only the solutions of the spinorial wave explain quanta, the true quantum numbers, the true number of bound states and the true energy levels. Let us go to the end of the Takabayasi's attempt, let us replace completely spinors by a set of tensors and let us solve completely the tensorial equations in the case of the hydrogen atom. Should we get the true results, the true number of bound states, the true quantum numbers and the true energy levels? The answer is: no, because true representations of the rotation group $SO(3)$ use only integer numbers, not the half-integer numbers which are necessary to get the true results. These true results are obtained only by taking the representations of $SL(2, \mathbb{C})$, but then we are in Cl_3 .

The second reason why scientists did not understand the novelty of the spinorial wave was the difficulty of the mathematical tools. Two different groups may be similar in the vicinity of their neutral element. $SL(2, \mathbb{C})$ and \mathcal{L}_+^\uparrow or their subgroups $SU(2)$ and $SO(3)$ are globally different but locally identical. The present study does not use infinitesimal operators, then it is able to see the difference between a Lie group and its Lie algebra.

Physical waves imply the use of trigonometric functions, then imply the complex exponential function that simplifies calculations. Going into a very unusual axiomatization, the quantum theory has been locked on the only use of complex numbers. This is equivalent to work only with plane geometry, with a unique i with square -1 that is the generator of all rotations of the Euclidean plane. It is somewhere a "2D software". The basic tool of the present study is a "3D software", the Clifford algebra of the 3-dimensional physical space. Next the building of Clifford algebras by recursion on the dimension allows to use this basic tool in the algebra of space-time as in the algebra of the 6-dimensional space-time which is necessary for physics of the standard model. These algebras present all abilities of the linear spaces built on the complex field, because they are also linear spaces. But they also allow to use products. The exponential function is then everywhere defined and allows to study a large variety of undulatory phenomenons. These algebras also allow to use the inverse, when it exists. Indisputable mathematical rules replace then the not well defined tensorial products of hermitian spaces and the operators operating on undefined linear spaces.

8.2 Our work

Two kinds of particles, fermions and bosons, are used by the standard model. Each kind of fermion is a quantum object with a wave following the Dirac equation. This is the starting point of our work. Following the initial de Broglie's idea of a physical wave linked to the move of any particle, we have introduced a change in the wave equation which concerns only the mass term. This wave equation is nonlinear, homogeneous and has the Dirac equation as linear approximation.

First interesting result, the true sign of the mass-energy comes directly from

the wave equation, and from the charge conjugation, which changes the sign of the derivative terms of the wave equation. This form of the charge conjugation was firstly gotten by the standard model itself, which uses, for the Dirac equation, the old frame of Dirac matrices.

The second result was very difficult to get, because the resolution of the Dirac equation is very accurate in the case of bound states of the H atom, and any change in terms of this wave equation could imply disaster. But for each bound state a solution of the linear equation exists such that the Yvon-Takabayasi β angle is everywhere defined and small. This result is very accurate and surprising. It means that physical bound states are the rare solutions of the homogeneous nonlinear equation (see Appendix C). We can therefore understand why there are privileged bound states and why an electron in a H atom is always in one of these states, never in a linear combination of states.

A second frame for the Dirac wave was introduced by D. Hestenes, the Clifford algebra of space-time, which is the second starting point of our work. A comparison between old and new frame is easy if we use the Dirac matrices as a matrix representation of the space-time vectors. We have reviewed in section 1 and 2 how the relativistic invariance is gotten for fermion waves. These waves appear very different, they are not vectors or tensors of the space-time, but a different kind of object, spinors.

The spinorial form of the fermion wave is included in the standard model, it is one of its main features. What we have done here is only to fully account for consequences of this fact. The form invariance of the Dirac equation necessitates the use of the $SL(2, \mathbb{C})$ group that is a sub-set of the Clifford algebra of the physical space, Cl_3 . We have learned to read all the Dirac theory in this frame. This algebra is isomorphic to the matrix algebra $M_2(\mathbb{C})$ of 2×2 complex matrices. This algebra allows to see its multiplicative group $GL(2, \mathbb{C}) = Cl_3^*$ as the true group of form invariance of the Dirac theory.

Then we have explained in a simple way how this form invariance, that is a reinforcement of relativistic invariance, rules not only the Dirac wave equation, but all the electromagnetism (section 4). This is well hidden in the case of the electromagnetic field itself because only the $SL(2, \mathbb{C})$ part of the Cl_3^* group acts upon this boson field (and this is true of any other boson field). The electromagnetic field has properties resulting from its antisymmetric building from a pair of spinors. It is a pure bivector, sum $\vec{E} + i\vec{H}$ of a vector \vec{E} and a pseudo-vector $i\vec{H}$, without scalar nor pseudo-scalar term. It rotates under a Lorentz rotation but it is insensitive to the ratio r of a dilation nor to the chiral angle. This behaviour is imposed by the form of its invariance under Cl_3^* .

More generally boson fields may be gotten by antisymmetric product of an even number of fermions. Their wave is then a physical wave, a function of space and time with value in the even sub-algebra of the Clifford algebra of the usual 4-dimensional space-time, or of the complete 6-dimensional space-time. Such a construction was impossible in the old formalism of Dirac matrices, because the wave had values into a linear space which is not an algebra. The frame of Clifford algebras allows to use the internal multiplication and the inverse to build waves of systems of particles. This has interesting physical consequences:

all waves are true functions of the space-time into well defined sets. These sets of functions are the Hilbert spaces whose existence is supposed in the standard model. Clifford algebras have matrix representations, and their elements can always be considered as operators. The use of creation and annihilation operators is a consequence of the fact that products of an even number of elements belong to the same linear space.

The form invariance of the electromagnetism uses a group which fully accounts for two main aspects of the modern physical results: the conservation of the orientation of the time, the conservation of the orientation of the space. These two orientations are not a consequence of the invariance group. The form invariance is only compatible with such conserved orientations, which are a true experimental discovery of the second part of the twentieth century. The oriented space is fully compatible with a gauge group which acts differently on left and right waves. The conservation of an oriented time is compatible with laws of thermodynamics.

The extended form invariance allows a better understanding of old questions as: why there is a Planck constant? What is a charge, or a mass, what is the difference between a charge and a mass? Charges appear in terms necessary to link a contravariant vector as A to a covariant vector as qA or to link a $\nabla\hat{A}_j$ term to a $q(A_k\hat{A}_l - A_l\hat{A}_k)$ term of a Yang-Mills gauge group. The invariance under Cl_3^* necessitates $q = r^2q'$. Mass appears in a term necessary to link a differential term to a constant term in the invariant wave equation. The invariance under Cl_3^* necessitates $m = rm'$. Therefore a charge is not a mass, a mass is not a charge, they have a different behaviour under Cl_3^* . These new aspects of old concepts comes from the supplementary strains added by a greater invariance group. They are fully compatible with classical and relativistic mechanics and electromagnetism.

Anything in the previous review is compatible with the standard model, particularly with the CPT theorem which is now trivially satisfied. Nevertheless several habits must be abandoned. For instance the Planck factor linking proper mass to frequency is variable if we consider the full invariance group. This is completely hidden if the Planck factor is changed into a constant number. Tensors constructed from the Dirac waves with Dirac matrices do not have a behaviour allowing the full invariance group. This implies to use Clifford algebras, only frames where the full invariance group acts. Another bad habit to abandon is the habit to go up or down an index of tensor, because covariant and contravariant vectors vary differently under the full invariance group.

If you have paid the preceding prices to account for the full invariance group you are able to get many awards. The first one is the possibility to read the electro-weak theory in a much simpler way, with a wave which is a function of space-time into a Clifford algebra: firstly of the usual space-time if you account only for the electron-neutrino case and secondly of a 6-dimensional space-time to account for all fermions of one generation. In this second case the gauge group is exactly the $U(1) \times SU(2) \times SU(3)$ group of the standard model, the lepton part of the complete wave sees only the $U(1) \times SU(2)$ part of the gauge group. Then

electron and neutrino are automatically unable to see strong interactions and the right wave of the neutrino does not interact at all. A greater gauge group is not available, this account for the fact that no way exists to transform a quark into a lepton. Then this justifies the empiric construction, in the standard model, of conservative quantum numbers as the baryonic number. Another award is the comprehension of both the existence of exactly three generations of fermions, completely similar and having nevertheless a separate behaviour in the gauge invariance, and of four kinds of neutrinos (see 6.4.1)

The generalization to the complete wave of the geometric dilation linked to the electron wave is possible if only two additional dimensions are added to the three usual dimensions of the physical space⁵⁶. The reward to this new strain on the theory is that this construction includes all parts of the standard model in a closed frame. Moreover the usual space-time is a well bounded part of the complete space-time, bounded by the fact that usual space-time is real in a complete space-time with a natural complex structure (see 6.5). A second award is a unique dimension for the time in the complete space-time, this time is then our usual oriented time, oriented from past to future. Another award is a much more general geometric transformation between the intrinsic space-time and the usual space-time, as soon as neutrino and quarks are considered. The study of the wave of all fermions of the first generation introduces, in each point of the space-time, two Clifford algebras and a geometric transformation from the intrinsic manifold into the relative manifold. Then the corresponding mathematical tool is two bundles whose fiber is $Cl_{5,1}$.

8.3 Principle of minimum

Modern physics always uses a principle of minimum. This was first seen by Fermat. He understood that light goes in such a way that the duration of the travel between two points is minimal. This was secondly seen in Hamiltonian mechanics, where the move of any object is made in a way such that a quantity called action is minimal. These two principles of minimum were united by de Broglie and his discovery of the wave linked to the move of any material particle. So the Dirac wave equation of an electron or of other fermions may also be gotten from such a principle of minimum. But why?

Clifford algebra and the form invariance group give a strange answer. The true wave equation of a unique fermion is the invariant form of the wave equation. And the scalar part of this wave equation is exactly the Lagrangian equation. This Lagrangian density is not truly minimal. In fact, it is exactly zero, because the wave equation is homogeneous. The second part of the answer comes from the fact that you can get the seven other equations by using the calculus of variations. This is probably not very correctly done, because an assumption is made that the infinitesimal variation of the wave is null on a

⁵⁶We used in [18] two greater Clifford algebras, $Cl_{2,3}$ and $Cl_{3,4}$ which cannot allow the relation (6.211) between the reverse in space-time algebra and the reverse in the complete algebra. Relation (6.211) implies to use only $Cl_{5,1}$, or $Cl_{1,5}$ but it happens to be the same algebra.

boundary of the integration volume. It is easy to get this assumption in the case of a bound state, but nobody proved that it is always possible for a propagating wave. Since the true link between the Lagrangian density and the wave equation is not what we thought, an error there is not very important. The most important consequence is that the wave is more general than the principle of minimum: it is easy to study wave equations which cannot be gotten from a Lagrangian density (see 5.3).

Another part of this strange answer is the non equality between the light speed as limit speed of any Dirac wave and the limit speed of other waves. We have gotten a limit speed different from the light speed in the frame of a wave equation coming not from a Lagrangian density (see 5.3.1).

8.4 Theory versus experiment

The world of the particles with magnetic charges is different and we are, in the knowledge of this new world, probably no more aware than Columbus after his first journey. We know it exists and this is already something.

The two parts of this article, the theoretical part formed by the six first sections and three appendices, the experimental part in one modest section, seem in the same time disconnected and disproportionate. We can expect that more experiments will bring many new properties of the magnetic world.

These two parts are however doubly linked. Firstly probabilities, essential in quantum theory, seem absent here. In our powders, random seems not to play a very big role. When titanium is transformed at some place, all that can be transformed is changed. Random may be the reason why results are variable, but to get copper instead of iron, chromium instead of manganese can also come from differences in temperature, pressure, duration of the discharge and so on. We know nothing until now about that.

Second link between our two parts, space-time is different on what we thought it was. In the theoretical part we explained how a second space-time manifold appears, anisotropic, and how the spinor wave makes a bridge between these two manifolds. This bridge can be extended to the complete space-time which allows to include electro-weak and strong interactions. The usual space-time is not the fundamental entity, it is only a part, well defined, of a complete space-time where complex numbers appear naturally. The geometric transformation linked to the wave contains a sum of direct and of inverse dilations. It is then very different from simple Lorentz rotations. Geometric transformations are only induced by elements of a group coming from the space algebra. They appear secondary objects, the fundamental one being the wave.

In the experimental part also space seems very different on what we thought, since kernels of atoms that we think separated by huge distances in comparison with their own size, seem able to put together their nucleons and to reallocate them as if the distance between them was cancelled.

8.5 Future applications

What will be the applications of magnetic monopoles is also a premature question and this we are able to imagine today will probably have few to share with the very practical applications which will go out laboratories in the future.

Urutskoev who is a nuclear physicist thinks to new nuclear reactors, intrinsically safe, driven by very intense magnetic fields. Another exciting possibility is that magnetism may be linked to gravitation. There are some theoretical indexes there, because G. Lochak [45] has explained that it is a magnetic photon which is linked to the graviton in the fusion theory of Louis de Broglie.

A third kind of possible practical consequence is about geology. The magnetic monopole of G. Lochak is a kind of excited neutrino. The Sun can produce magnetic monopoles which will arrive on our Earth, mainly at the magnetic poles. This was experimentally satisfied [1]. These monopoles are likely to induce the same transformations we see at the laboratory, notably to produce hydrogen. This may change the process of fossilization and the creation of deposits. The future of the dynamo creating the Earth's magnetic field may also be involved if the life-time of monopoles is sufficient.

Evidently if magnetic monopoles are produced in the heart of the Sun, they are able to produce effects on magnetic fields. The study of the magnetic monopoles is perhaps the best way to progress in the comprehension of our Sun.

It is easier, it needs less imagination, to prospect theoretical applications. When Lorentz studied the electron particle he used a classical model of extended electron, with a mass-energy resulting from the energy of the electric field created by the electron's charge. If the electron is a point this energy is infinite, which is unfortunate. If the electron is extended, forces coming from the distribution of charges are repulsive, then must be offset by forces of attraction. These forces may only be weak forces, since the electron knows only the electro-weak forces.

We can also have a good idea of what will not allow the double space-time manifolds. The existence of a second space-time manifold changes nothing to properties of the first manifold, this into which we move and observe our universe. We must then not to dream to things we know forbidden by physical laws, as to overtake the limit speed. Accelerate until this limit speed remains impossible because the mass goes to infinite when the speed approaches the limit speed, even if this limit speed is not necessarily equal to the light speed in the void. Another restriction which has no chance to change is the one linked to the time arrow. Any journey back into past will stay forbidden since the invariance group conserves the orientation of our time.

8.6 Improved standard model

The different parts of the work that we present here are strongly interacting and reinforcing one another. The Cl_3^* group is easier to see from the invariant form (3.10) of the wave equation. The behaviour of the mass term used in section

4 is a sufficient reason to prefer the nonlinear homogeneous wave equation to its linear approximation, the Dirac equation. We got also the electro-weak gauge group in a much simpler way. It was easy to extend this model from the lepton sector to the quark sector. We can explain not only why there are three completely similar generations (and four neutrinos⁵⁷), but also why the different generations must be separately treated in the gauge theory. We can explain why the complete gauge group is the $U(1) \times SU(2) \times SU(3)$ gauge group found from experiments and why the $SU(3)$ gauge group does not act upon the lepton part of the complete wave, which was before postulated. The link between the wave and the space-time geometry is reinforced by the fact that this link survives to the extension of the wave to all fermions of a generation, which necessitates the use of two supplementary dimensions of space.

A greater invariance group implies strong new strains. These news strains imply a better understanding of old concepts and induce the only way to go further. For instance the relation $\phi' = M\phi$ is not new, it was used under equivalent form since the Pauli equation in the twenty's. But it indicates to us that the relation between the ϕ wave and the Weyl's spinors ξ and η is invariant, that the right and left parts of the wave are invariant, that M and ϕ are similar. When you know that, it is evident that the wave of a pair electron-

neutrino must read $\Psi_l = \begin{pmatrix} \hat{\phi}_e & \hat{\phi}_n \\ \hat{\phi}_n & \hat{\phi}_e \end{pmatrix}$, and this equality gives then the form

of the projectors P_μ . Next these projectors have naturally the $U(1) \times SU(2)$ structure of the electroweak gauge group. It is also easy to get the charges of quarks u and d simply by changing P_0 to P'_0 , changing only one coefficient from 1 into $-1/3$. Strains coming from the invariance group imply also that you have only one simple way to get a wave with all fermions of one generation, which

is $\Psi = \begin{pmatrix} \Psi_l & \Psi_r \\ \Psi_g & \Psi_b \end{pmatrix}$. But after that if you want again to have the same link

as before between the wave and the geometry of space-time, it is necessary to dispose of (6.211). This in fact requires to use the link (2.125), well known in the standard model, existing between the wave of the particle and the wave of the antiparticle. This link restrict the value of the wave from $Cl_{2,3} = M_4(\mathbb{C})$ to its sub-algebra $Cl_{1,3}$ and from $Cl_{2,5} = M_8(\mathbb{C})$ to its sub-algebra $Cl_{1,5}$. It happens that this Clifford algebra is isomorphic to $Cl_{5,1}$ and this isomorphism is both the reason why the non-isomorphic sub-algebras $Cl_{1,3}$ and $Cl_{3,1}$ are equally used, and a reason to be more confident on the standard model and its precepts issued from a long building out of numerous experiments. Another reason to be confident both in the standard model and in the use of Clifford algebra is the link 6.6 between the cancellation of right waves, except the electron wave, and the existence of a mathematical inverse, used to build waves of systems. This should allow to build the wave of a proton or a neutron from their internal quarks.

Questions of the initial quantum theory are today nearly forgotten. Why

⁵⁷The fourth neutrino is not able to interact by electro-weak or strong forces, it is then a part of the **black matter**.

there is a Planck constant and why there are complex numbers were two of them. Since nobody had a clear and simple answer, these questions were put “under the table”. The existence of the Planck factor, that links proper mass to frequency of the wave, is directly linked to supplementary strains of the invariance group. Complex numbers are also simply explained, first by the isomorphism between Cl_3 and the algebra generated by Pauli matrices, next by the matrix representation of $Cl_{1,3}$, finally by the matrix representation of $Cl_{5,1}$. This algebra is isomorphic to a sub-algebra of the algebra $M_8(\mathbb{C})$ of 8×8 matrices on the complex field. The complete wave is then a function of the space-time with value into $Cl_{5,1}$, this justifies most of the mathematical apparatus of the standard model. On the physical point of view, this allows to build the boson fields by antisymmetric products of fermions in even number.

New strains used here also explain why old attempts were not successful. We think to the numerous attempts made in the thirty’s to unify mechanics and electromagnetism. Such an unification is limited by the fact that a charge is not a mass when you use the full invariance group. Another attempt, made to unify the different parts of the $U(1) \times SU(2) \times SU(3)$ gauge group as subgroups of $SU(5)$ or $SO(10)$ had no more success, predicting a possible disintegrating proton which was not experimentally found. The structure of the gauge group comes from the structure of the complete wave and does not change when you increase the energy. The structure of the wave is then fully compatible with protons without disintegration, and more generally with all known aspects of modern physics.

The standard model shall remain so more essential because the wave equations with a mass term are compatible both with the form invariance and with the gauge invariance. We have then no new particles to await, with the only exception of the leptonic magnetic monopoles that we have yet observed.

A Calculations in Clifford algebras

A.1 Invariant equation and Lagrangian

Let M be an invertible matrix, element of Cl_3^* , with determinant $re^{i\theta}$. Let R and \bar{R} be Lorentz dilations such as :

$$R : x \mapsto x' = R(x) = MxM^\dagger ; \quad \bar{R} : x \mapsto x' = \bar{R}(x) = \bar{M}x\widehat{M}. \quad (\text{A.1})$$

Let P be the matrix such as:

$$M = \sqrt{r}e^{i\frac{\theta}{2}}P \quad (\text{A.2})$$

and let L and \bar{L} be dilations such as :

$$L : x \mapsto x' = L(x) = PxP^\dagger ; \quad \bar{L} : x \mapsto x' = \bar{L}(x) = \bar{P}x\widehat{P}. \quad (\text{A.3})$$

We have:

$$re^{i\theta} = \det(M) = M\bar{M} = \sqrt{r}e^{i\frac{\theta}{2}}P\sqrt{r}e^{i\frac{\theta}{2}}\bar{P} = re^{i\theta}P\bar{P} \quad (\text{A.4})$$

we get then

$$P\bar{P} = 1 ; \quad \bar{P} = P^{-1} ; \quad \bar{L} = L^{-1} \quad (\text{A.5})$$

P is then an element of $SL(2, \mathbb{C})$ and L is a Lorentz rotation. We know, for such a rotation, noting (L) the matrix of L in an orthonormal basis and g the signature-matrix :

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (\text{A.6})$$

that we have, M^t being the transposed matrix⁵⁸ of M :

$$(L)^{-1} = g(L)^t g ; \quad (\bar{L})g = g(L)^t. \quad (\text{A.7})$$

But we have also :

$$R(x) = MxM^\dagger = \sqrt{r}e^{i\frac{\theta}{2}}Px\sqrt{r}e^{-i\frac{\theta}{2}}P^\dagger = rPxP^\dagger = rL(x) \quad (\text{A.8})$$

therefore

$$R = rL ; \quad (R) = r(L). \quad (\text{A.9})$$

We have also:

$$\bar{R}(x) = \bar{M}x\widehat{M} = \sqrt{r}e^{i\frac{\theta}{2}}\bar{P}x\sqrt{r}e^{-i\frac{\theta}{2}}\widehat{P} = r\bar{P}x\widehat{P} = r\bar{L}(x) \quad (\text{A.10})$$

$$\bar{R} = r\bar{L} ; \quad (\bar{R}) = r(\bar{L}). \quad (\text{A.11})$$

Multiplying (A.7) by r we get :

$$(\bar{R})g = g(R)^t ; \quad (\bar{R}) = g(R)^t g. \quad (\text{A.12})$$

which gives for $j = 1, 2, 3$ and $k = 1, 2, 3$:

$$\bar{R}_0^0 = R_0^0 ; \quad \bar{R}_0^j = -R_0^j ; \quad \bar{R}_j^0 = -R_0^j ; \quad \bar{R}_j^k = R_k^j \quad (\text{A.13})$$

Consequently lines as columns of the matrix R_μ^ν are orthogonal, because we have, for R and \bar{R} , with:

$$R_\mu = M\sigma_\mu M^\dagger = R_\mu^\nu \sigma_\nu ; \quad \bar{R}_\mu = \bar{M}\sigma_\mu \widehat{M} = \bar{R}_\mu^\nu \sigma_\nu \quad (\text{A.14})$$

$$R_\mu \cdot R_\nu = \bar{R}_\mu \cdot \bar{R}_\nu = \delta_{\mu\nu} \rho^2 \quad (\text{A.15})$$

where $\delta_{00} = 1, \delta_{11} = \delta_{22} = \delta_{33} = -1, \delta_{\mu\nu} = 0$ if $\mu \neq \nu$. We have

$$\begin{aligned} \bar{\phi}A\widehat{\phi} &= A^\mu \bar{\phi} \sigma_\mu \widehat{\phi} = A_0 \bar{D}_0 - \sum_{j=1}^{j=3} A_j \bar{D}_j \\ &= A_0 (\bar{D}_0^\mu \sigma_\mu) - \sum_{j=1}^{j=3} A_j (\bar{D}_j^\mu \sigma_\mu) \end{aligned} \quad (\text{A.16})$$

⁵⁸The transposition exchanges lines and columns of matrices : if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $M^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. We have, for any matrices A and B , $(AB)^t = B^t A^t$ and $\det(A^t) = \det(A)$

But the link between the D_μ and the \bar{D}_μ is the same as between the R_μ and the \bar{R}_μ and we get with (A.13) for $j = 1, 2, 3$ and $k = 1, 2, 3$:

$$\bar{D}_0^0 = D_0^0 ; \quad \bar{D}_0^j = -D_j^0 ; \quad \bar{D}_j^0 = -D_0^j ; \quad \bar{D}_j^k = D_k^j \quad (\text{A.17})$$

which gives

$$\begin{aligned} \bar{\phi}A\hat{\phi} &= A_0(\bar{D}_0^0 + \sum_{j=1}^{j=3} \bar{D}_0^j \sigma_j) - \sum_{j=1}^{j=3} A_j(\bar{D}_j^0 + \sum_{k=1}^{k=3} \bar{D}_j^k \sigma_k) \\ &= A_0(D_0^0 - \sum_{j=1}^{j=3} D_j^0 \sigma_j) - \sum_{j=1}^{j=3} A_j(-D_0^j + \sum_{k=1}^{k=3} D_k^j \sigma_k) \\ &= A_\nu D_\mu^\nu \sigma^\mu \end{aligned} \quad (\text{A.18})$$

The scalar part is then

$$\langle \bar{\phi}A\hat{\phi} \rangle = D_0^\nu A_\nu = A_\mu J^\mu \quad (\text{A.19})$$

The corresponding term with the Dirac matrices is

$$\begin{aligned} &\frac{1}{2}[(\bar{\psi}\gamma^\mu q A_\mu \psi) + (\bar{\psi}\gamma^\mu q A_\mu \psi)^\dagger] \\ &= \frac{q}{2} A_\mu [\bar{\psi}\gamma^\mu \psi + (\bar{\psi}\gamma^\mu \psi)] = q A_\mu \bar{\psi}\gamma^\mu \psi \\ &= q A_\mu J^\mu. \end{aligned} \quad (\text{A.20})$$

We get next

$$\begin{aligned} &\frac{1}{2}[(\bar{\psi}\gamma^\mu (-i)\partial_\mu \psi) + (\bar{\psi}\gamma^\mu (-i)\partial_\mu \psi)^\dagger] \\ &= \frac{i}{2}(-\bar{\psi}\gamma^\mu \partial_\mu \psi + \partial_\mu \bar{\psi}\gamma^\mu \psi) \\ &= \frac{i}{2}[-\xi^\dagger \partial_0 \xi - \eta^\dagger \partial_0 \eta + (\partial_0 \xi^\dagger)\xi + (\partial_0 \eta^\dagger)\eta] \\ &+ \frac{i}{2} \sum_{j=1}^{j=3} [-\xi^\dagger \sigma_j \partial_j \xi + \eta^\dagger \sigma_j \partial_j \eta + (\partial_j \xi^\dagger)\sigma_j \xi - (\partial_j \eta^\dagger)\sigma_j \eta] \end{aligned} \quad (\text{A.21})$$

which gives

$$\begin{aligned} &\frac{1}{2}[(\bar{\psi}\gamma^\mu (-i)\partial_\mu \psi) + (\bar{\psi}\gamma^\mu (-i)\partial_\mu \psi)^\dagger] \\ &= \frac{i}{2}(\xi_1 \partial_0 \xi_1^* + \xi_2 \partial_0 \xi_2^* + \eta_1 \partial_0 \eta_1^* + \eta_2 \partial_0 \eta_2^* - \xi_1^* \partial_0 \xi_1 - \xi_2^* \partial_0 \xi_2 - \eta_1^* \partial_0 \eta_1 - \eta_2^* \partial_0 \eta_2) \\ &+ \frac{i}{2}(\xi_1 \partial_1 \xi_2^* + \xi_2 \partial_1 \xi_1^* - \eta_1 \partial_1 \eta_2^* - \eta_2 \partial_1 \eta_1^* - \xi_1^* \partial_1 \xi_2 - \xi_2^* \partial_1 \xi_1 + \eta_1^* \partial_1 \eta_2 + \eta_2^* \partial_1 \eta_1) \\ &+ \frac{1}{2}(-\xi_1 \partial_2 \xi_2^* + \xi_2 \partial_2 \xi_1^* + \eta_1 \partial_2 \eta_2^* - \eta_2 \partial_2 \eta_1^* - \xi_1^* \partial_2 \xi_2 + \xi_2^* \partial_2 \xi_1 + \eta_1^* \partial_2 \eta_2 - \eta_2^* \partial_2 \eta_1) \\ &+ \frac{i}{2}(\xi_1 \partial_3 \xi_1^* - \xi_2 \partial_3 \xi_2^* - \eta_1 \partial_3 \eta_1^* + \eta_2 \partial_3 \eta_2^* - \xi_1^* \partial_3 \xi_1 + \xi_2^* \partial_3 \xi_2 + \eta_1^* \partial_3 \eta_1 - \eta_2^* \partial_3 \eta_2). \end{aligned} \quad (\text{A.22})$$

In the Pauli algebra we have

$$\begin{aligned} & \bar{\phi}(\nabla\widehat{\phi})\sigma_{21} \\ &= 2i \begin{pmatrix} \eta_1^* & \eta_2^* \\ -\xi_2 & \xi_1 \end{pmatrix} \begin{pmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{pmatrix} \begin{pmatrix} -\eta_1 & -\xi_2^* \\ -\eta_2 & \xi_1^* \end{pmatrix} \end{aligned} \quad (\text{A.23})$$

and with (2.91) we get

$$\begin{aligned} \bar{\phi}(\nabla\widehat{\phi})\sigma_{21} &= \begin{pmatrix} w^3 + w^0 - iv^3 - iv^0 & v^2 + iv^1 + iw^2 - w^1 \\ v^2 - iv^1 + iw^2 + w^1 & w^3 - w^0 - iv^3 + iv^0 \end{pmatrix} = \\ & 2i \begin{pmatrix} \eta_1^*(-\partial_0\eta_1 + \partial_1\eta_2 - i\partial_2\eta_2 + \partial_3\eta_1) & \eta_1^*(-\partial_0\xi_2^* - \partial_1\xi_1^* + i\partial_2\xi_1^* + \partial_3\xi_2^*) \\ +\eta_2^*(-\partial_0\eta_2 + \partial_1\eta_1 + i\partial_2\eta_1 - \partial_3\eta_2) & +\eta_2^*(\partial_0\xi_1^* + \partial_1\xi_2^* + i\partial_2\xi_2^* + \partial_3\xi_1^*) \\ -\xi_2(-\partial_0\eta_1 + \partial_1\eta_2 - i\partial_2\eta_2 + \partial_3\eta_1) & -\xi_2(-\partial_0\xi_2^* - \partial_1\xi_1^* + i\partial_2\xi_1^* + \partial_3\xi_2^*) \\ +\xi_1(-\partial_0\eta_2 + \partial_1\eta_1 + i\partial_2\eta_1 - \partial_3\eta_2) & +\xi_1(\partial_0\xi_1^* + \partial_1\xi_2^* + i\partial_2\xi_2^* + \partial_3\xi_1^*) \end{pmatrix}. \end{aligned} \quad (\text{A.24})$$

This gives

$$w^3 + w^0 - iv^3 - iv^0 = \quad (\text{A.25})$$

$$2i(-\eta_1^*\partial_0\eta_1 - \eta_2^*\partial_0\eta_2 + \eta_1^*\partial_1\eta_2 + \eta_2^*\partial_1\eta_1 - i\eta_1^*\partial_2\eta_2 + i\eta_2^*\partial_2\eta_1 + \eta_1^*\partial_3\eta_1 - \eta_2^*\partial_3\eta_2)$$

$$w^3 - w^0 - iv^3 + iv^0 = \quad (\text{A.26})$$

$$2i(\xi_2\partial_0\xi_2^* + \xi_1\partial_0\xi_1^* + \xi_2\partial_1\xi_1^* + \xi_1\partial_1\xi_2^* - i\xi_2\partial_2\xi_1^* + i\xi_1\partial_2\xi_2^* - \xi_2\partial_3\xi_2^* + \xi_1\partial_3\xi_1^*)$$

$$v^2 - iv^1 + iw^2 + w^1 = \quad (\text{A.27})$$

$$2i(\xi_2\partial_0\eta_1 - \xi_1\partial_0\eta_2 - \xi_2\partial_1\eta_2 + \xi_1\partial_1\eta_1 + i\xi_2\partial_2\eta_2 + i\xi_1\partial_2\eta_1 - \xi_2\partial_3\eta_1 - \xi_1^*\partial_3\eta_2)$$

$$v^2 + iv^1 + iw^2 - w^1 = \quad (\text{A.28})$$

$$2i(-\eta_1^*\partial_0\xi_2^* - \eta_2^*\partial_0\xi_1^* - \eta_1^*\partial_1\xi_1^* + \eta_2^*\partial_1\xi_2^* + i\eta_1^*\partial_2\xi_1^* + i\eta_2^*\partial_2\xi_2^* + \eta_1^*\partial_3\xi_2^* + \eta_2^*\partial_3\xi_1^*).$$

Adding and subtracting (A.25) and (A.26) we get

$$\begin{aligned} w^3 - iv^3 &= -i\eta_1^*\partial_0\eta_1 - i\eta_2^*\partial_0\eta_2 + i\xi_2\partial_0\xi_2^* + i\xi_1\partial_0\xi_1^* \\ &+ i\eta_1^*\partial_1\eta_2 + i\eta_2^*\partial_1\eta_1 + i\xi_2\partial_1\xi_1^* + i\xi_1\partial_1\xi_2^* \\ &+ \eta_1^*\partial_2\eta_2 - \eta_2^*\partial_2\eta_1 + \xi_2\partial_2\xi_1^* - \xi_1\partial_2\xi_2^* \\ &+ i\eta_1^*\partial_3\eta_1 - i\eta_2^*\partial_3\eta_2 - i\xi_2\partial_3\xi_2^* + i\xi_1\partial_3\xi_1^* \end{aligned} \quad (\text{A.29})$$

$$\begin{aligned} w^0 - iv^0 &= -i\eta_1^*\partial_0\eta_1 - i\eta_2^*\partial_0\eta_2 - i\xi_2\partial_0\xi_2^* - i\xi_1\partial_0\xi_1^* \\ &+ i\eta_1^*\partial_1\eta_2 + i\eta_2^*\partial_1\eta_1 - i\xi_2\partial_1\xi_1^* - i\xi_1\partial_1\xi_2^* \\ &+ \eta_1^*\partial_2\eta_2 - \eta_2^*\partial_2\eta_1 - \xi_2\partial_2\xi_1^* + \xi_1\partial_2\xi_2^* \\ &+ i\eta_1^*\partial_3\eta_1 - i\eta_2^*\partial_3\eta_2 + i\xi_2\partial_3\xi_2^* - i\xi_1\partial_3\xi_1^*. \end{aligned} \quad (\text{A.30})$$

Separating the real and the imaginary part of (A.29) we get

$$\begin{aligned} \frac{2}{i}w^3 = & \quad (A.31) \\ & \xi_1\partial_0\xi_1^* + \xi_2\partial_0\xi_2^* + \eta_1\partial_0\eta_1^* + \eta_2\partial_0\eta_2^* - \xi_1^*\partial_0\xi_1 - \xi_2^*\partial_0\xi_2 - \eta_1^*\partial_0\eta_1 - \eta_2^*\partial_0\eta_2 \\ & + \xi_1\partial_1\xi_2^* + \xi_2\partial_1\xi_1^* - \eta_1\partial_1\eta_2^* - \eta_2\partial_1\eta_1^* - \xi_1^*\partial_1\xi_2 - \xi_2^*\partial_1\xi_1 + \eta_1^*\partial_1\eta_2 + \eta_2^*\partial_1\eta_1 \\ & - i(-\xi_1\partial_2\xi_2^* + \xi_2\partial_2\xi_1^* + \eta_1\partial_2\eta_2^* - \eta_2\partial_2\eta_1^* - \xi_1^*\partial_2\xi_2 + \xi_2^*\partial_2\xi_1 + \eta_1^*\partial_2\eta_2 - \eta_2^*\partial_2\eta_1) \\ & + \xi_1\partial_3\xi_1^* - \xi_2\partial_3\xi_2^* - \eta_1\partial_3\eta_1^* + \eta_2\partial_3\eta_2^* - \xi_1^*\partial_3\xi_1 + \xi_2^*\partial_3\xi_2 + \eta_1^*\partial_3\eta_1 - \eta_2^*\partial_3\eta_2 \end{aligned}$$

This gives with (A.22)

$$\frac{1}{2}[(\bar{\psi}\gamma^\mu(-i)\partial_\mu\psi) + (\bar{\psi}\gamma^\mu(-i)\partial_\mu\psi)^\dagger] = w^3 \quad (A.32)$$

and with (A.20) we get (2.100). The Tetrad's impulse-energy tensor coming from the invariance of the Lagrangian density under translations satisfies

$$-T_\lambda^\mu = \frac{1}{2}[(\bar{\psi}\gamma^\mu(-i)\partial_\lambda + qA_\lambda)\psi + ((\bar{\psi}\gamma^\mu(-i)\partial_\lambda + qA_\lambda)\psi)^\dagger] \quad (A.33)$$

We get then from (A.32)

$$\begin{aligned} w^3 &= -T_\mu^\mu - q\bar{\psi}\gamma^\mu A_\mu\psi \\ w^3 &= -T_\mu^\mu - V^0 \\ w^3 + V^0 &= -\text{tr}(T) \end{aligned} \quad (A.34)$$

Now the imaginary part of (A.29) gives

$$-2v^3 = \partial_0(\xi_1\xi_1^* + \xi_2\xi_2^* - \eta_1\eta_1^* - \eta_2\eta_2^*) \quad (A.35)$$

$$\begin{aligned} & + \partial_1(\xi_1\xi_2^* + \xi_2\xi_1^* + \eta_1\eta_2^* + \eta_2\eta_1^*) \\ & + \partial_2i(\xi_1\xi_2^* - \xi_2\xi_1^* + \eta_1\eta_2^* - \eta_2\eta_1^*) \\ & + \partial_3(\xi_1\xi_1^* - \xi_2\xi_2^* + \eta_1\eta_1^* - \eta_2\eta_2^*) \\ & = \partial_\mu D_3^\mu = \nabla \cdot D_3 \end{aligned} \quad (A.36)$$

Now the imaginary part of (A.30) gives

$$2v^0 = \partial_0(\xi_1\xi_1^* + \xi_2\xi_2^* + \eta_1\eta_1^* + \eta_2\eta_2^*) \quad (A.37)$$

$$\begin{aligned} & + \partial_1(\xi_1\xi_2^* + \xi_2\xi_1^* - \eta_1\eta_2^* - \eta_2\eta_1^*) \\ & + \partial_2i(\xi_1\xi_2^* - \xi_2\xi_1^* - \eta_1\eta_2^* + \eta_2\eta_1^*) \\ & + \partial_3(\xi_1\xi_1^* - \xi_2\xi_2^* - \eta_1\eta_1^* + \eta_2\eta_2^*) \\ & = \partial_\mu D_0^\mu = \nabla \cdot D_0 \end{aligned} \quad (A.38)$$

and we get the conservation of the current of probability. From (A.18) we get

$$\begin{aligned} qA_\nu D_\mu^\nu \sigma^\mu &= \bar{\phi}qA\hat{\phi} = V = V^\mu \sigma_\mu \\ &= V^0 - V^1\sigma^1 - V^2\sigma^2 - V^3\sigma^3 \end{aligned} \quad (A.39)$$

$$V^j = -qA_\nu D_j^\nu = -qA \cdot D_j ; j = 1, 2, 3. \quad (A.40)$$

The real part of (A.30) gives with (2.95)

$$\begin{aligned} \frac{2}{i}w^0 = 2iV^3 = \frac{2}{i}qA \cdot D_3 = & \quad (A.41) \\ & -\xi_1\partial_0\xi_1^* - \xi_2\partial_0\xi_2^* + \eta_1\partial_0\eta_1^* + \eta_2\partial_0\eta_2^* + \xi_1^*\partial_0\xi_1 + \xi_2^*\partial_0\xi_2 - \eta_1^*\partial_0\eta_1 - \eta_2^*\partial_0\eta_2 \\ & -\xi_1\partial_1\xi_2^* - \xi_2\partial_1\xi_1^* - \eta_1\partial_1\eta_2^* - \eta_2\partial_1\eta_1^* + \xi_1^*\partial_1\xi_2 + \xi_2^*\partial_1\xi_1 + \eta_1^*\partial_1\eta_2 + \eta_2^*\partial_1\eta_1 \\ & -i(\xi_1\partial_2\xi_2^* - \xi_2\partial_2\xi_1^* + \eta_1\partial_2\eta_2^* - \eta_2\partial_2\eta_1^* + \xi_1^*\partial_2\xi_2 - \xi_2^*\partial_2\xi_1 + \eta_1^*\partial_2\eta_2 - \eta_2^*\partial_2\eta_1) \\ & -\xi_1\partial_3\xi_1^* + \xi_2\partial_3\xi_2^* - \eta_1\partial_3\eta_1^* + \eta_2\partial_3\eta_2^* + \xi_1^*\partial_3\xi_1 - \xi_2^*\partial_3\xi_2 + \eta_1^*\partial_3\eta_1 - \eta_2^*\partial_3\eta_2 \end{aligned}$$

Now adding and subtracting (A.27) and (A.28) we get

$$\begin{aligned} v^2 + iw^2 = & i\xi_2\partial_0\eta_1 - i\xi_1\partial_0\eta_2 - i\eta_1^*\partial_0\xi_2^* + i\eta_2^*\partial_0\xi_1^* \\ & -i\xi_2\partial_1\eta_2 + i\xi_1\partial_1\eta_1 - i\eta_1^*\partial_1\xi_1^* + i\eta_2^*\partial_1\xi_2^* \quad (A.42) \\ & -\xi_2\partial_2\eta_2 - \xi_1\partial_2\eta_1 - \eta_1^*\partial_2\xi_1^* - \eta_2^*\partial_2\xi_2^* \\ & -i\xi_2\partial_3\eta_1 - i\xi_1\partial_3\eta_2 + i\eta_1^*\partial_3\xi_2^* + i\eta_2^*\partial_3\xi_1^* \end{aligned}$$

$$\begin{aligned} w^1 - iv^1 = & i\xi_2\partial_0\eta_1 - i\xi_1\partial_0\eta_2 + i\eta_1^*\partial_0\xi_2^* - i\eta_2^*\partial_0\xi_1^* \\ & -i\xi_2\partial_1\eta_2 + i\xi_1\partial_1\eta_1 + i\eta_1^*\partial_1\xi_1^* - i\eta_2^*\partial_1\xi_2^* \quad (A.43) \\ & -\xi_2\partial_2\eta_2 - \xi_1\partial_2\eta_1 + \eta_1^*\partial_2\xi_1^* + \eta_2^*\partial_2\xi_2^* \\ & -i\xi_2\partial_3\eta_1 - i\xi_1\partial_3\eta_2 - i\eta_1^*\partial_3\xi_2^* - i\eta_2^*\partial_3\xi_1^*. \end{aligned}$$

The real part of (A.42) gives

$$\begin{aligned} 2v^2 = & \partial_0i(-\xi_1\eta_2 + \xi_2\eta_1 + \xi_1^*\eta_2^* - \xi_2^*\eta_1^*) \quad (A.44) \\ & + \partial_1i(\xi_1\eta_1 - \xi_2\eta_2 - \xi_1^*\eta_1^* + \xi_2^*\eta_2^*) \\ & + \partial_2(-\xi_1\eta_1 - \xi_2\eta_2 - \xi_1^*\eta_1^* - \xi_2^*\eta_2^*) \\ & + \partial_3i(-\xi_1\eta_2 - \xi_2\eta_1 + \xi_1^*\eta_2^* + \xi_2^*\eta_1^*) \\ = & -\partial_\mu D_2^\mu = -\nabla \cdot D_2 \quad (A.45) \end{aligned}$$

which gives, with (2.93)

$$\nabla \cdot D_2 = -2v^2 = 2V^1 \quad (A.46)$$

and we get with (A.40)

$$\nabla \cdot D_2 + 2qA \cdot D_1 = 0 \quad (A.47)$$

which is (2.104). The imaginary part of (A.42) gives with (2.97)

$$\begin{aligned} 2w^2 = 0 = & \quad (A.48) \\ & -\xi_1\partial_0\eta_2 + \xi_2\partial_0\eta_1 - \eta_1\partial_0\xi_2 - \eta_2\partial_0\xi_1 - \xi_1^*\partial_0\eta_2^* + \xi_2^*\partial_0\eta_1^* - \eta_1^*\partial_0\xi_2^* + \eta_2^*\partial_0\xi_1^* \\ & + \xi_1\partial_1\eta_1 - \xi_2\partial_1\eta_2 - \eta_1\partial_1\xi_1 + \eta_2\partial_1\xi_2 + \xi_1^*\partial_1\eta_1^* - \xi_2^*\partial_1\eta_2^* - \eta_1^*\partial_1\xi_1^* + \eta_2^*\partial_1\xi_2^* \\ & + i(\xi_1\partial_2\eta_1 + \xi_2\partial_2\eta_2 - \eta_1\partial_2\xi_1 - \eta_2\partial_2\xi_2 - \xi_1^*\partial_2\eta_1^* - \xi_2^*\partial_2\eta_2^* + \eta_1^*\partial_2\xi_1^* + \eta_2^*\partial_2\xi_2^*) \\ & -\xi_1\partial_3\eta_2 - \xi_2\partial_3\eta_1 + \eta_1\partial_3\xi_2 - \eta_2\partial_3\xi_1 - \xi_1^*\partial_3\eta_2^* - \xi_2^*\partial_3\eta_1^* + \eta_1^*\partial_3\xi_2^* + \eta_2^*\partial_3\xi_1^*. \end{aligned}$$

The real part of (A.43) gives with (2.98)

$$2w^1 = 0 = \tag{A.49}$$

$$\begin{aligned} & i(-\xi_1 \partial_0 \eta_2 + \xi_2 \partial_0 \eta_1 - \eta_1 \partial_0 \xi_2 + \eta_2 \partial_0 \xi_1 + \xi_1^* \partial_0 \eta_2^* - \xi_2^* \partial_0 \eta_1^* + \eta_1^* \partial_0 \xi_2^* - \eta_2^* \partial_0 \xi_1^*) \\ & + i(\xi_1 \partial_1 \eta_1 - \xi_2 \partial_1 \eta_2 - \eta_1 \partial_1 \xi_1 + \eta_2 \partial_1 \xi_2 - \xi_1^* \partial_1 \eta_1^* + \xi_2^* \partial_1 \eta_2^* + \eta_1^* \partial_1 \xi_1^* - \eta_2^* \partial_1 \xi_2^*) \\ & - \xi_1 \partial_2 \eta_1 - \xi_2 \partial_2 \eta_2 + \eta_1 \partial_2 \xi_1 + \eta_2 \partial_2 \xi_2 - \xi_1^* \partial_2 \eta_1^* - \xi_2^* \partial_2 \eta_2^* + \eta_1^* \partial_2 \xi_1^* + \eta_2^* \partial_2 \xi_2^* \\ & + i(-\xi_1 \partial_3 \eta_2 - \xi_2 \partial_3 \eta_1 + \eta_1 \partial_3 \xi_2 + \eta_2 \partial_3 \xi_1 + \xi_1^* \partial_3 \eta_2^* + \xi_2^* \partial_3 \eta_1^* - \eta_1^* \partial_3 \xi_2^* - \eta_2^* \partial_3 \xi_1^*) \end{aligned}$$

The imaginary part of (A.43) gives

$$-2v^1 = \partial_0(-\xi_1 \eta_2 + \xi_2 \eta_1 - \xi_1^* \eta_2^* + \xi_2^* \eta_1^*) \tag{A.50}$$

$$\begin{aligned} & + \partial_1(\xi_1 \eta_1 - \xi_2 \eta_2 + \xi_1^* \eta_1^* - \xi_2^* \eta_2^*) \\ & + \partial_2 i(\xi_1 \eta_1 + \xi_2 \eta_2 - \xi_1^* \eta_1^* - \xi_2^* \eta_2^*) \\ & + \partial_3(-\xi_1 \eta_2 - \xi_2 \eta_1 - \xi_1^* \eta_2^* - \xi_2^* \eta_1^*) \\ & = \partial_\mu D_1^\mu = \nabla \cdot D_1 \end{aligned} \tag{A.51}$$

which gives, with (2.94)

$$\nabla \cdot D_1 = -2v^1 = -2V^2 \tag{A.52}$$

and we get with (A.40)

$$\nabla \cdot D_1 - 2qA \cdot D_2 = 0 \tag{A.53}$$

which is (2.103).

A.2 Calculation of the reverse in $Cl_{5,1}$

Here, as in 1.5, indexes $\mu, \nu, \rho \dots$ have value 0, 1, 2, 3 and indexes a, b, c, d, e have value 0, 1, 2, 3, 4, 5. We get⁵⁹

$$\Lambda_{\mu\nu} = \Lambda_\mu \Lambda_\nu = \begin{pmatrix} 0 & -\gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & -\gamma_\nu \\ \gamma_\nu & 0 \end{pmatrix} = \begin{pmatrix} -\gamma_{\mu\nu} & 0 \\ 0 & -\gamma_{\mu\nu} \end{pmatrix} \tag{A.54}$$

$$\Lambda_{\mu\nu\rho} = \Lambda_{\mu\nu} \Lambda_\rho = \begin{pmatrix} -\gamma_{\mu\nu} & 0 \\ 0 & -\gamma_{\mu\nu} \end{pmatrix} \begin{pmatrix} 0 & -\gamma_\rho \\ \gamma_\rho & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma_{\mu\nu\rho} \\ -\gamma_{\mu\nu\rho} & 0 \end{pmatrix} \tag{A.55}$$

$$\Lambda_{0123} = \Lambda_{01} \Lambda_{23} = \begin{pmatrix} \gamma_{0123} & 0 \\ 0 & \gamma_{0123} \end{pmatrix} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix} \tag{A.56}$$

We get also

$$\Lambda_{45} = \Lambda_4 \Lambda_5 = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} = -\Lambda_{54} \tag{A.57}$$

$$\Lambda_{012345} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} = \begin{pmatrix} -I_4 & 0 \\ 0 & I_4 \end{pmatrix} \tag{A.58}$$

$$\Lambda_{01235} = \Lambda_{0123} \Lambda_5 = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}. \tag{A.59}$$

⁵⁹ I_2, I_4, I_8 are unit matrices. The identification process allowing to include \mathbb{R} in each real Clifford algebra allows to read a instead of aI_n for any complex number a .

Similarly we get⁶⁰

$$\Lambda_{\mu 4} = \begin{pmatrix} -\gamma_\mu & 0 \\ 0 & \gamma_\mu \end{pmatrix}; \quad \Lambda_{\mu 5} = \begin{pmatrix} \mathbf{i}\gamma_\mu & 0 \\ 0 & \mathbf{i}\gamma_\mu \end{pmatrix} \quad (\text{A.60})$$

$$\Lambda_{\mu\nu 4} = \begin{pmatrix} 0 & -\gamma_{\mu\nu} \\ -\gamma_{\mu\nu} & 0 \end{pmatrix}; \quad \Lambda_{\mu\nu 5} = \begin{pmatrix} 0 & \gamma_{\mu\nu}\mathbf{i} \\ -\gamma_{\mu\nu}\mathbf{i} & 0 \end{pmatrix} \quad (\text{A.61})$$

$$\Lambda_{\mu\nu\rho 4} = \begin{pmatrix} \gamma_{\mu\nu\rho} & 0 \\ 0 & -\gamma_{\mu\nu\rho} \end{pmatrix}; \quad \Lambda_{\mu\nu\rho 5} = \begin{pmatrix} \gamma_{\mu\nu\rho}\mathbf{i} & 0 \\ 0 & \gamma_{\mu\nu\rho}\mathbf{i} \end{pmatrix} \quad (\text{A.62})$$

$$\Lambda_{\mu 45} = \begin{pmatrix} 0 & \gamma_\mu\mathbf{i} \\ \gamma_\mu\mathbf{i} & 0 \end{pmatrix}; \quad \Lambda_{\mu\nu 45} = \begin{pmatrix} -\gamma_{\mu\nu}\mathbf{i} & 0 \\ 0 & \gamma_{\mu\nu}\mathbf{i} \end{pmatrix} \quad (\text{A.63})$$

$$\Lambda_{\mu\nu\rho 45} = \begin{pmatrix} 0 & -\gamma_{\mu\nu\rho}\mathbf{i} \\ -\gamma_{\mu\nu\rho}\mathbf{i} & 0 \end{pmatrix}; \quad \Lambda_{01234} = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} \quad (\text{A.64})$$

Scalar and pseudo-scalar terms read

$$\alpha I_8 + \omega \Lambda_{012345} = \begin{pmatrix} (\alpha - \omega)I_4 & 0 \\ 0 & (\alpha + \omega)I_4 \end{pmatrix} \quad (\text{A.65})$$

$$\alpha I_8 - \omega \Lambda_{012345} = \begin{pmatrix} (\alpha + \omega)I_4 & 0 \\ 0 & (\alpha - \omega)I_4 \end{pmatrix} \quad (\text{A.66})$$

For the calculation of the 1-vector term

$$N^a \Lambda_a = N^4 \Lambda_4 + N^5 \Lambda_5 + N^\mu \Lambda_\mu$$

we let

$$\beta = N^4; \quad \delta = N^5; \quad \mathbf{a} = N^\mu \gamma_\mu. \quad (\text{A.67})$$

This gives

$$N^a \Lambda_a = \begin{pmatrix} 0 & \beta I_4 - \delta \mathbf{i} - \mathbf{a} \\ \beta I_4 + \delta \mathbf{i} + \mathbf{a} & 0 \end{pmatrix}. \quad (\text{A.68})$$

For the calculation of the 2-vector term

$$N^{ab} \Lambda_{ab} = N^{45} \Lambda_{45} + N^{\mu 4} \Lambda_{\mu 4} + N^{\mu 5} \Lambda_{\mu 5} + N^{\mu\nu} \Lambda_{\mu\nu}$$

we let

$$\epsilon = N^{45}; \quad \mathbf{b} = N^{\mu 4} \gamma_\mu; \quad \mathbf{c} = N^{\mu 5} \gamma_\mu; \quad \mathbf{A} = N^{\mu\nu} \gamma_{\mu\nu} \quad (\text{A.69})$$

This gives with (A.54) and (A.60)

$$N^{ab} \Lambda_{ab} = \begin{pmatrix} \epsilon \mathbf{i} - \mathbf{b} + \mathbf{ic} - \mathbf{A} & 0 \\ 0 & -\epsilon \mathbf{i} + \mathbf{b} + \mathbf{ic} - \mathbf{A} \end{pmatrix}. \quad (\text{A.70})$$

For the calculation of the 3-vector term

$$N^{abc} \Lambda_{abc} = N^{\mu 45} \Lambda_{\mu 45} + N^{\mu\nu 4} \Lambda_{\mu\nu 4} + N^{\mu\nu 5} \Lambda_{\mu\nu 5} + N^{\mu\nu\rho} \Lambda_{\mu\nu\rho}$$

⁶⁰ \mathbf{i} anti-commutes with any odd element in space-time algebra and commutes with any even element.

we let

$$\mathbf{d} = N^{\mu 45} \gamma_{\mu} ; \quad \mathbf{B} = N^{\mu \nu 4} \gamma_{\mu \nu} ; \quad \mathbf{C} = N^{\mu \nu 5} \gamma_{\mu \nu} ; \quad \mathbf{ie} = N^{\mu \nu \rho} \gamma_{\mu \nu \rho} \quad (\text{A.71})$$

This gives with (A.55) and (A.61)

$$N^{abc} \Lambda_{abc} = \begin{pmatrix} 0 & \mathbf{di} - \mathbf{B} + \mathbf{iC} + \mathbf{ie} \\ -\mathbf{id} - \mathbf{B} - \mathbf{iC} - \mathbf{ie} & 0 \end{pmatrix}. \quad (\text{A.72})$$

For the calculation of the 4-vector term

$$N^{abcd} \Lambda_{abcd} = N^{\mu \nu 45} \Lambda_{\mu \nu 45} + N^{\mu \nu \rho 4} \Lambda_{\mu \nu \rho 4} + N^{\mu \nu \rho 5} \Lambda_{\mu \nu \rho 5} + N^{0123} \Lambda_{0123}$$

we let

$$\mathbf{D} = N^{\mu \nu 45} \gamma_{\mu \nu} ; \quad \mathbf{if} = N^{\mu \nu \rho 4} \gamma_{\mu \nu \rho} ; \quad \mathbf{ig} = N^{\mu \nu \rho 5} \gamma_{\mu \nu \rho} ; \quad \zeta = N^{0123} \quad (\text{A.73})$$

This gives with (A.56) and (A.62)

$$N^{abcd} \Lambda_{abcd} = \begin{pmatrix} -\mathbf{iD} + \mathbf{if} + \mathbf{g} + \zeta \mathbf{i} & 0 \\ 0 & \mathbf{iD} - \mathbf{if} + \mathbf{g} + \zeta \mathbf{i} \end{pmatrix}. \quad (\text{A.74})$$

For the calculation of the pseudo-vector term

$$N^{abcde} \Lambda_{abcde} = N^{\mu \nu \rho 45} \Lambda_{\mu \nu \rho 45} + N^{01234} \Lambda_{01234} + N^{01235} \Lambda_{01235}$$

we let

$$\mathbf{ih} = N^{\mu \nu \rho 45} \gamma_{\mu \nu \rho} ; \quad \eta = N^{01234} ; \quad \theta = N^{01235} \quad (\text{A.75})$$

This gives with (A.59) and (A.64)

$$N^{abcde} \Lambda_{abcde} = \begin{pmatrix} 0 & -\mathbf{h} + \eta \mathbf{i} + \theta I_4 \\ -\mathbf{h} + \eta \mathbf{i} - \theta I_4 & 0 \end{pmatrix}. \quad (\text{A.76})$$

We then get

$$\begin{aligned} \Psi &= \begin{pmatrix} \Psi_l & \Psi_r \\ \Psi_g & \Psi_b \end{pmatrix} \quad (\text{A.77}) \\ &= \begin{pmatrix} (\alpha - \omega)I_4 + (-\mathbf{b} + \mathbf{g}) - (\mathbf{A} + \mathbf{iD}) & (\beta + \theta)I_4 - (\mathbf{a} + \mathbf{h}) + (-\mathbf{B} + \mathbf{iC}) \\ \mathbf{i(c + f)} + (\zeta + \epsilon)\mathbf{i} & \mathbf{i(-d + e)} + (-\delta + \eta)\mathbf{i} \\ (\beta - \theta)I_4 + (\mathbf{a} - \mathbf{h}) - (\mathbf{B} + \mathbf{iC}) & (\alpha + \omega)I_4 + (\mathbf{b} + \mathbf{g}) + (-\mathbf{A} + \mathbf{iD}) \\ -\mathbf{i(d + e)} + (\delta + \eta)\mathbf{i} & \mathbf{i(c - f)} + (\zeta - \epsilon)\mathbf{i} \end{pmatrix} \end{aligned}$$

This implies

$$\Psi_l = (\alpha - \omega)I_4 + (-\mathbf{b} + \mathbf{g}) - (\mathbf{A} + \mathbf{iD}) + \mathbf{i(c + f)} + (\zeta + \epsilon)\mathbf{i} \quad (\text{A.78})$$

$$\Psi_r = (\beta + \theta)I_4 - (\mathbf{a} + \mathbf{h}) + (-\mathbf{B} + \mathbf{iC}) + \mathbf{i(-d + e)} + (-\delta + \eta)\mathbf{i} \quad (\text{A.79})$$

$$\Psi_g = (\beta - \theta)I_4 + (\mathbf{a} - \mathbf{h}) - (\mathbf{B} + \mathbf{iC}) - \mathbf{i(d + e)} + (\delta + \eta)\mathbf{i} \quad (\text{A.80})$$

$$\Psi_b = (\alpha + \omega)I_4 + (\mathbf{b} + \mathbf{g}) + (-\mathbf{A} + \mathbf{iD}) + \mathbf{i(c - f)} + (\zeta - \epsilon)\mathbf{i} \quad (\text{A.81})$$

In $Cl_{1,3}$ the reverse of

$$A = \langle A \rangle_0 + \langle A \rangle_1 + \langle A \rangle_2 + \langle A \rangle_3 + \langle A \rangle_4$$

is

$$\tilde{A} = \langle A \rangle_0 + \langle A \rangle_1 - \langle A \rangle_2 - \langle A \rangle_3 + \langle A \rangle_4$$

we must change the sign of bivectors \mathbf{A} , \mathbf{B} , \mathbf{iC} , \mathbf{iD} , and trivectors \mathbf{ic} , \mathbf{id} , \mathbf{ie} , \mathbf{if} and we then get

$$\tilde{\Psi}_l = (\alpha - \omega)I_4 + (-\mathbf{b} + \mathbf{g}) + (\mathbf{A} + \mathbf{iD}) - \mathbf{i}(\mathbf{c} + \mathbf{f}) + (\zeta + \epsilon)\mathbf{i} \quad (\text{A.82})$$

$$\tilde{\Psi}_r = (\beta + \theta)I_4 - (\mathbf{a} + \mathbf{h}) + (\mathbf{B} - \mathbf{iC}) + \mathbf{i}(\mathbf{d} - \mathbf{e}) + (-\delta + \eta)\mathbf{i} \quad (\text{A.83})$$

$$\tilde{\Psi}_g = (\beta - \theta)I_4 + (\mathbf{a} - \mathbf{h}) + (\mathbf{B} + \mathbf{iC}) + \mathbf{i}(\mathbf{d} + \mathbf{e}) + (\delta + \eta)\mathbf{i} \quad (\text{A.84})$$

$$\tilde{\Psi}_b = (\alpha + \omega)I_4 + (\mathbf{b} + \mathbf{g}) + (\mathbf{A} - \mathbf{iD}) + \mathbf{i}(-\mathbf{c} + \mathbf{f}) + (\zeta - \epsilon)\mathbf{i} \quad (\text{A.85})$$

The reverse, in $Cl_{5,1}$ now, of

$$A = \langle A \rangle_0 + \langle A \rangle_1 + \langle A \rangle_2 + \langle A \rangle_3 + \langle A \rangle_4 + \langle A \rangle_5 + \langle A \rangle_6$$

is

$$\tilde{A} = \langle A \rangle_0 + \langle A \rangle_1 - \langle A \rangle_2 - \langle A \rangle_3 + \langle A \rangle_4 + \langle A \rangle_5 - \langle A \rangle_6$$

Only terms which change sign, with (A.65), (A.70) and (A.72), are scalars ϵ and ω , vectors \mathbf{b} , \mathbf{c} , \mathbf{d} , \mathbf{e} and bivectors \mathbf{A} , \mathbf{B} , \mathbf{C} ⁶¹. We then get from (A.77)

$$\begin{aligned} \tilde{\Psi} &= \begin{pmatrix} (\alpha + \omega)I_4 + (\mathbf{b} + \mathbf{g}) + (\mathbf{A} - \mathbf{iD}) & (\beta + \theta)I_4 - (\mathbf{a} + \mathbf{h}) + (\mathbf{B} - \mathbf{iC}) \\ \mathbf{i}(-\mathbf{c} + \mathbf{f}) + (\zeta - \epsilon)\mathbf{i} & \mathbf{i}(\mathbf{d} - \mathbf{e}) + (-\delta + \eta)\mathbf{i} \\ (\beta - \theta)I_4 + (\mathbf{a} - \mathbf{h}) + (\mathbf{B} + \mathbf{iC}) & (\alpha - \omega)I_4 + (-\mathbf{b} + \mathbf{g}) + (\mathbf{A} + \mathbf{iD}) \\ \mathbf{i}(\mathbf{d} + \mathbf{e}) + (\delta + \eta)\mathbf{i} & -\mathbf{i}(\mathbf{c} + \mathbf{f}) + (\zeta + \epsilon)\mathbf{i} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{\Psi}_b & \tilde{\Psi}_r \\ \tilde{\Psi}_g & \tilde{\Psi}_l \end{pmatrix}. \end{aligned} \quad (\text{A.86})$$

And we have proved (6.211).

B Calculations for the gauge invariance

The operators P_0 in (6.13) and (7.26) and P'_0 in (6.135) have the form (7.26). They satisfy

$$P_0(\Psi) = a\Psi\gamma_{21} + bP_-(\Psi)\mathbf{i}.$$

⁶¹These changes of sign are not the same in $Cl_{5,1}$ as in $Cl_{1,3}$. Differences are corrected by the fact that the reversion in $Cl_{5,1}$ also exchanges the place of Ψ_l and Ψ_b terms.

Applied to

$$\Psi = \begin{pmatrix} \phi_e & \phi_n \\ \hat{\phi}_n & \hat{\phi}_e \end{pmatrix} = \sqrt{2} \begin{pmatrix} \xi_{1e} & -\eta_{2e}^* & 0 & -\eta_{2n}^* \\ \xi_{2e} & \eta_{1e}^* & 0 & \eta_{1n}^* \\ \eta_{1n} & 0 & \eta_{1e} & -\xi_{2e}^* \\ \eta_{2n} & 0 & \eta_{2e} & \xi_{1e}^* \end{pmatrix} \quad (\text{B.1})$$

this gives

$$P_0(\Psi) = ia \begin{pmatrix} \phi_e \sigma_3 & \phi_n \sigma_3 \\ \hat{\phi}_n \sigma_3 & \hat{\phi}_e \sigma_3 \end{pmatrix} + ib \begin{pmatrix} \phi_{eR} & 0 \\ 0 & -\hat{\phi}_{eR} \end{pmatrix} \quad (\text{B.2})$$

$$P_0(\Psi) = i\sqrt{2} \begin{pmatrix} (a+b)\xi_{1e} & (-a)(-\eta_{2e}^*) & 0 & (-a)(-\eta_{2n}^*) \\ (a+b)\xi_{2e} & (-a)\eta_{1e}^* & 0 & (-a)\eta_{1n}^* \\ a\eta_{1n} & 0 & a\eta_{1e} & -(a+b)(-\xi_{2e}^*) \\ a\eta_{2n} & 0 & a\eta_{2e} & -(a+b)\xi_{1e}^* \end{pmatrix} \quad (\text{B.3})$$

$$P_0(\Psi) = \frac{b\mathbf{i}}{2}\Psi + \Psi(a + \frac{b}{2})\gamma_{21} \quad (\text{B.4})$$

Since by $\exp(a^0 P_0)$

$$\xi_e \mapsto e^{ia^0(a+b)}\xi_e; \quad \eta_e \mapsto e^{ia^0 a}\eta_e; \quad \eta_n \mapsto e^{ia^0 a}\eta_n \quad (\text{B.5})$$

we get

$$[\exp(a^0 P_0)](\Psi) = e^{a^0 \frac{b}{2}\mathbf{i}}\Psi e^{a^0(a+\frac{b}{2})\gamma_{21}} \quad (\text{B.6})$$

We then get

$$\begin{aligned} & \partial_\mu \left[[\exp(a^0 P_0)](\Psi) \right] \\ &= \partial_\mu a^0 \frac{b}{2} \mathbf{i} e^{a^0 \frac{b}{2} \mathbf{i}} \Psi e^{a^0(a+\frac{b}{2})\gamma_{21}} + e^{a^0 \frac{b}{2} \mathbf{i}} \partial_\mu \Psi e^{a^0(a+\frac{b}{2})\gamma_{21}} \\ & \quad + e^{a^0 \frac{b}{2} \mathbf{i}} \Psi \partial_\mu a^0 (a + \frac{b}{2}) \gamma_{21} e^{a^0(a+\frac{b}{2})\gamma_{21}} \\ &= \partial_\mu a^0 e^{a^0 \frac{b}{2} \mathbf{i}} \left[\frac{b\mathbf{i}}{2} \Psi + \Psi(a + \frac{b}{2}) \gamma_{21} \right] e^{a^0(a+\frac{b}{2})\gamma_{21}} + e^{a^0 \frac{b}{2} \mathbf{i}} \partial_\mu \Psi e^{a^0(a+\frac{b}{2})\gamma_{21}} \\ &= \partial_\mu a^0 e^{a^0 \frac{b}{2} \mathbf{i}} P_0(\Psi) e^{a^0(a+\frac{b}{2})\gamma_{21}} + e^{a^0 \frac{b}{2} \mathbf{i}} \partial_\mu \Psi e^{a^0(a+\frac{b}{2})\gamma_{21}} \\ &= e^{a^0 \frac{b}{2} \mathbf{i}} [\partial_\mu a^0 P_0(\Psi) + \partial_\mu \Psi] e^{a^0(a+\frac{b}{2})\gamma_{21}}. \end{aligned} \quad (\text{B.7})$$

The gauge transformation defined such as

$$B'_\mu = B_\mu - \frac{2}{g_1} \partial_\mu a^0 \quad (\text{B.8})$$

$$\Psi' = [\exp(a^0 P_0)](\Psi) = e^{a^0 \frac{b}{2} \mathbf{i}} \Psi e^{a^0(a+\frac{b}{2})\gamma_{21}} \quad (\text{B.9})$$

gives:

$$D_\mu \Psi = \partial_\mu \Psi + \frac{g_1}{2} B_\mu P_0(\Psi) \quad (\text{B.10})$$

$$D'_\mu \Psi' = \partial_\mu \Psi' + \left(\frac{g_1}{2} B_\mu - \partial_\mu a^0\right) P_0(\Psi') \quad (\text{B.11})$$

$$= \partial_\mu \left[\exp(a^0 P_0) (\Psi) \right] + \left(\frac{g_1}{2} B_\mu - \partial_\mu a^0\right) \left[\frac{b}{2} \mathbf{i} \Psi' + \Psi' \left(a + \frac{b}{2}\right) \gamma_{21} \right] \quad (\text{B.12})$$

$$= e^{a^0 \frac{b}{2} \mathbf{i}} \left[\partial_\mu a^0 P_0(\Psi) + \partial_\mu \Psi + \left(\frac{g_1}{2} B_\mu - \partial_\mu a^0\right) P_0(\Psi) \right] e^{a^0 \left(a + \frac{b}{2}\right) \gamma_{21}} \\ = e^{a^0 \frac{b}{2} \mathbf{i}} (D_\mu \Psi) e^{a^0 \left(a + \frac{b}{2}\right) \gamma_{21}}. \quad (\text{B.13})$$

We deduce:

$$\mathbf{D}' \Psi' = \gamma^\mu D'_\mu \Psi' = \gamma^\mu e^{a^0 \frac{b}{2} \mathbf{i}} (D_\mu \Psi) e^{a^0 \left(a + \frac{b}{2}\right) \gamma_{21}} \\ = e^{-a^0 \frac{b}{2} \mathbf{i}} (\mathbf{D} \Psi) e^{a^0 \left(a + \frac{b}{2}\right) \gamma_{21}} \quad (\text{B.14})$$

because \mathbf{i} anti-commutes with each γ^μ . Next we have

$$\tilde{\Psi}' = e^{a^0 \left(a + \frac{b}{2}\right) \gamma_{21}} \tilde{\Psi} e^{a^0 \frac{b}{2} \mathbf{i}} \\ = e^{-a^0 \left(a + \frac{b}{2}\right) \gamma_{21}} \tilde{\Psi} e^{a^0 \frac{b}{2} \mathbf{i}} \quad (\text{B.15})$$

$$\tilde{\Psi}' \mathbf{D}' \Psi' = e^{-a^0 \left(a + \frac{b}{2}\right) \gamma_{21}} \tilde{\Psi} e^{a^0 \frac{b}{2} \mathbf{i}} e^{-a^0 \frac{b}{2} \mathbf{i}} (\mathbf{D} \Psi) e^{a^0 \left(a + \frac{b}{2}\right) \gamma_{21}} \\ = e^{-a^0 \left(a + \frac{b}{2}\right) \gamma_{21}} \tilde{\Psi} (\mathbf{D} \Psi) e^{a^0 \left(a + \frac{b}{2}\right) \gamma_{21}}. \quad (\text{B.16})$$

In the case of the magnetic monopole the leptonic wave is:

$$\Psi = \begin{pmatrix} \hat{\phi}_L & \hat{\phi}_n \\ \hat{\phi}_n & \hat{\phi}_L \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 & -\eta_{2L}^* & \xi_{1n} & -\eta_{2n}^* \\ 0 & \eta_{1L}^* & \xi_{2n} & \eta_{1n}^* \\ \eta_{1n} & -\xi_{2n}^* & \eta_{1L} & 0 \\ \eta_{2n} & \xi_{1n}^* & \eta_{2L} & 0 \end{pmatrix} \quad (\text{B.17})$$

this gives

$$P_0(\Psi) = a \Psi \gamma_{21} + b P_-(\Psi) \mathbf{i} \\ = i\sqrt{2} \begin{pmatrix} 0 & -a(-\eta_{2L}^*) & (a-b)\xi_{1n} & -a(-\eta_{2n}^*) \\ 0 & -a\eta_{1L}^* & (a-b)\xi_{2n} & -a\eta_{1n}^* \\ a\eta_{1n} & (-a+b)(-\xi_{2n}^*) & a\eta_{1L} & 0 \\ a\eta_{2n} & (-a+b)\xi_{1n}^* & a\eta_{2L} & 0 \end{pmatrix} \quad (\text{B.18})$$

$$= -\frac{b}{2} \mathbf{i} \Psi + \Psi \left(a - \frac{b}{2}\right) \gamma_{21}. \quad (\text{B.19})$$

The rest of the calculation is the same, with only the change of b into $-b$.

B.1 Tensors

In the case of the pair electron-neutrino, as in the case of the magnetic monopole, each one among the three spinors has 4 real parameters, each one gives then

$4 \times 5/2 = 10$ components of tensors: a space-time vector (4 components) and a space-time bivector (6 components). With the alone right spinor ϕ_R of the electron (or of the magnetic monopole) ⁶²

$$\phi_R = \sqrt{2} \begin{pmatrix} \xi_1 & 0 \\ \xi_2 & 0 \end{pmatrix} \quad (\text{B.20})$$

we get the space-time vector D_R and the bivector S_R ⁶³ satisfying

$$D_R = \phi_R \phi_R^\dagger ; \quad S_R = \phi_R \sigma_1 \bar{\phi}_R. \quad (\text{B.21})$$

D_R is a space-time vector, because it satisfies $D_R^\dagger = D_R$. Similarly with the left spinor ϕ_L

$$\phi_L = \sqrt{2} \begin{pmatrix} 0 & -\eta_2^* \\ 0 & \eta_1^* \end{pmatrix} ; \quad \phi_e = \phi_R + \phi_L \quad (\text{B.22})$$

we get the vector D_L and the bivector S_L satisfying

$$D_L = \phi_L \phi_L^\dagger ; \quad S_L = \phi_L \sigma_1 \bar{\phi}_L. \quad (\text{B.23})$$

It is well-known [40] that the currents D_R, D_L are fundamental in the Dirac theory and that the usual currents $J = D_0$ and $K = D_3$ are simply the sum and the difference of these chiral currents:

$$D_0 = D_R + D_L ; \quad D_3 = D_R - D_L. \quad (\text{B.24})$$

With the spinor $\phi_n = \phi_{nL}$ of the electronic neutrino, that we write here

$$\phi_n = \sqrt{2} \begin{pmatrix} 0 & -\zeta_2^* \\ 0 & \zeta_1^* \end{pmatrix} \quad (\text{B.25})$$

we get the space-time vector D_n and the bivector S_n such as

$$D_n = \phi_n \phi_n^\dagger ; \quad S_n = \phi_n \sigma_1 \bar{\phi}_n. \quad (\text{B.26})$$

Next with two of the three spinors we get 16 densities. We begin with ϕ_R and ϕ_L . We let

$$\begin{aligned} P &= 2\phi_R \bar{\phi}_L = \mathbf{a} + S_{RL} \\ \bar{P} &= 2\phi_L \bar{\phi}_R = \mathbf{a} - S_{RL} \end{aligned} \quad (\text{B.27})$$

$$\begin{aligned} I &= D_{RL} + id_{RL} = 2\phi_R \sigma_1 \phi_L^\dagger \\ I^\dagger &= D_{RL} - id_{RL} = 2\phi_L \sigma_1 \phi_R^\dagger \end{aligned} \quad (\text{B.28})$$

⁶²Since we want to study both the case of the electron and the case of the magnetic monopole, we shall note without e index the components of the electron wave. We shall note ζ_j what we have previously noted η_{jn} in the case of the left wave of the electron and η_{jL} in the case of the supplementary left wave of the magnetic monopole.

⁶³A detailed calculation of components is in B of [21].

\mathbf{a} and S_{RL} are well-known in the Dirac theory:

$$\mathbf{a} = \det(\phi_e) = \Omega_1 + i\Omega_2 = 2(\xi_1\eta_1^* + \xi_2\eta_2^*) \quad (\text{B.29})$$

where Ω_1, Ω_2 are the relativistic invariants of (2.12) and (2.15). The bivector $S_3 = S_{RL}$ is the bivector defined in (2.59) which with Ω_1, Ω_2, D_0 and D_3 , gives the 16 densities which were the only densities known by the complex formalism. They are the invariant ones under the electric gauge. Vectors $D_{RL} = D_1$ and $d_{RL} = D_2$ are the space-time vectors defined in (2.54). Under the dilation R defined in (1.42) \mathbf{a} is changed into \mathbf{a}' such as

$$\mathbf{a}' = M\phi_e\bar{\phi}_e\bar{M} = M\mathbf{a}\bar{M} = \mathbf{a}M\bar{M} = re^{i\theta}\mathbf{a}. \quad (\text{B.30})$$

Therefore

$$\mathbf{a}'\mathbf{a}'^* = re^{i\theta}\mathbf{a}re^{-i\theta}\mathbf{a}^* = r^2\mathbf{a}\mathbf{a}^*. \quad (\text{B.31})$$

Next with ϕ_L and ϕ_n we let

$$\begin{aligned} P &= 2\hat{\phi}_n\sigma_1\phi_L^\dagger = \mathbf{b} + S_{Ln} \\ \bar{P} &= 2\hat{\phi}_L\bar{\sigma}_1\phi_n^\dagger = \mathbf{b} - S_{Ln} \\ I &= D_{Ln} + id_{Ln} = 2\phi_n\phi_L^\dagger \\ I^\dagger &= D_{Ln} - id_{Ln} = 2\phi_L\phi_n^\dagger. \end{aligned} \quad (\text{B.32})$$

Vectors D_{Ln} and d_{Ln} are contravariant space-time vectors, S_{Ln} is a bivector. We shall need:

$$\mathbf{b} = \hat{\phi}_n\sigma_1\phi_L^\dagger + \hat{\phi}_L\bar{\sigma}_1\phi_n^\dagger = 2(\eta_1\zeta_2 - \eta_2\zeta_1) \quad (\text{B.34})$$

$$\begin{aligned} \mathbf{b}' &= \widehat{M}\mathbf{b}M^\dagger = \widehat{M}M^\dagger\mathbf{b} = re^{-i\theta}\mathbf{b} \\ \mathbf{b}'\mathbf{b}'^* &= r^2\mathbf{b}\mathbf{b}^*. \end{aligned} \quad (\text{B.35})$$

Finally with ϕ_R and ϕ_n we let

$$\begin{aligned} P &= 2\phi_R\bar{\phi}_n = \mathbf{c} + S_{Rn} \\ \bar{P} &= 2\phi_n\bar{\phi}_R = \mathbf{c} - S_{Rn} \end{aligned} \quad (\text{B.36})$$

$$\begin{aligned} I &= D_{Rn} + id_{Rn} = 2\phi_R\sigma_1\phi_n^\dagger \\ I^\dagger &= D_{Rn} - id_{Rn} = 2\phi_n\sigma_1\phi_R^\dagger. \end{aligned} \quad (\text{B.37})$$

Vectors D_{Rn} and d_{Rn} are also contravariant, S_{Rn} is a bivector. We shall use

$$\mathbf{c} = \phi_R\bar{\phi}_n + \phi_n\bar{\phi}_R = 2(\xi_1\zeta_1^* + \xi_2\zeta_2^*) \quad (\text{B.38})$$

$$\mathbf{c}' = M\mathbf{c}\bar{M} = M\bar{M}\mathbf{c} = re^{i\theta}\mathbf{c}; \quad \mathbf{c}'\mathbf{c}'^* = r^2\mathbf{c}\mathbf{c}^*. \quad (\text{B.39})$$

We have established with (3.53) that the main invariant of the wave of the electron is $m\rho$. Since we have not only one but three similar terms, the natural generalization, which is necessary to get the gauge invariance, uses:

$$\boldsymbol{\rho} = \sqrt{\mathbf{a}\mathbf{a}^* + \mathbf{b}\mathbf{b}^* + \mathbf{c}\mathbf{c}^*} \quad (\text{B.40})$$

$$m\rho = m'r\rho = m'\boldsymbol{\rho}'. \quad (\text{B.41})$$

B.2 Getting the wave equation

Since this term is the generalization of the invariant term of the Dirac wave, that is also the mass term of the Lagrangian density, this density in the case of the pair electron-neutrino, is the scalar real part $\mathcal{L} = \langle L \rangle$ of

$$L = \tilde{\Psi} \mathbf{D} \Psi \gamma_{012} + m \rho \quad (\text{B.42})$$

where \mathbf{D} is the covariant (6.22) derivative. We shall use for the electron-neutrino wave:

$$\Psi_L = P_+(\Psi) = \begin{pmatrix} \phi_L & \phi_n \\ \hat{\phi}_n & \hat{\phi}_L \end{pmatrix} \quad (\text{B.43})$$

$$\Psi_R = P_-(\Psi) = \begin{pmatrix} \phi_R & 0 \\ 0 & \hat{\phi}_R \end{pmatrix}$$

$$\Psi = \Psi_R + \Psi_L \quad (\text{B.44})$$

$$P_0(\Psi) = (\Psi_L + 2\Psi_R)\gamma_{21} \quad (\text{B.45})$$

and we have

$$P_0(\Psi) = (\Psi_R + \Psi_L)\gamma_{21} + \Psi_R\gamma_{21} = (\Psi_L + 2\Psi_R)\gamma_{21} \quad (\text{B.46})$$

$$\Psi_L + 2\Psi_R = \begin{pmatrix} \phi_L + 2\phi_R & \phi_n \\ \hat{\phi}_n & \hat{\phi}_L + 2\hat{\phi}_R \end{pmatrix}. \quad (\text{B.47})$$

We also have

$$\begin{aligned} \mathbf{D}\Psi &= \partial\Psi + \frac{g_1}{2} \mathbf{B}P_0(\Psi) + \frac{g_2}{2} \mathbf{W}^j P_j(\Psi) \\ &= \partial\Psi + \frac{g_1}{2} \mathbf{B}(\Psi_L + 2\Psi_R)\gamma_{21} + \frac{g_2}{2} [\mathbf{W}^1 \Psi_L \gamma_3 \mathbf{i} + \mathbf{W}^2 \Psi_L \gamma_3 + \mathbf{W}^3 \Psi_L (-\mathbf{i})] \end{aligned} \quad (\text{B.48})$$

which gives

$$\begin{aligned} \mathbf{D}\Psi \gamma_{012} &= \partial\Psi \gamma_{012} + \frac{g_1}{2} \mathbf{B}(\Psi_L + 2\Psi_R)\gamma_0 \\ &\quad - \frac{g_2}{2} [\mathbf{W}^1 \Psi_L + \mathbf{W}^2 \Psi_L \mathbf{i} + \mathbf{W}^3 \Psi_L \gamma_3]. \end{aligned} \quad (\text{B.49})$$

Next we get

$$\partial\Psi \gamma_{012} = \begin{pmatrix} 0 & \nabla \\ \hat{\nabla} & 0 \end{pmatrix} \begin{pmatrix} \phi_e & \phi_n \\ \hat{\phi}_n & \hat{\phi}_e \end{pmatrix} \begin{pmatrix} 0 & -i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix} \quad (\text{B.50})$$

$$= \begin{pmatrix} -i\nabla(\hat{\phi}_R + \hat{\phi}_L)\sigma_3 & -i\nabla\hat{\phi}_n\sigma_3 \\ -i\hat{\nabla}\phi_n\sigma_3 & -i\hat{\nabla}(\phi_R + \phi_L)\sigma_3 \end{pmatrix}$$

$$\partial\Psi \gamma_{012} = -i \begin{pmatrix} \nabla(-\hat{\phi}_R + \hat{\phi}_L) & \nabla\hat{\phi}_n \\ -\hat{\nabla}\phi_n & \hat{\nabla}(\phi_R - \phi_L) \end{pmatrix}. \quad (\text{B.51})$$

And we get for B

$$\begin{aligned}\mathbf{B}(\Psi_L + 2\Psi_R)\gamma_0 &= \begin{pmatrix} 0 & B \\ \widehat{B} & 0 \end{pmatrix} \begin{pmatrix} \phi_L + 2\phi_R & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_L + 2\widehat{\phi}_R \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & B \\ \widehat{B} & 0 \end{pmatrix} \begin{pmatrix} \phi_n & \phi_e + \phi_R \\ \widehat{\phi}_e + \widehat{\phi}_R & \widehat{\phi}_n \end{pmatrix} \end{aligned} \quad (\text{B.52})$$

$$\mathbf{B}P_0(\Psi)\gamma_{012} = \begin{pmatrix} B(\widehat{\phi}_e + \widehat{\phi}_R) & B\widehat{\phi}_n \\ \widehat{B}\phi_n & \widehat{B}(\phi_e + \phi_R) \end{pmatrix}. \quad (\text{B.53})$$

Similarly we have

$$\mathbf{W}^1\Psi_L = \begin{pmatrix} 0 & W^1 \\ \widehat{W}^1 & 0 \end{pmatrix} \begin{pmatrix} \phi_L & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_L \end{pmatrix} = \begin{pmatrix} W^1\widehat{\phi}_n & W^1\widehat{\phi}_L \\ \widehat{W}^1\phi_L & \widehat{W}^1\phi_n \end{pmatrix} \quad (\text{B.54})$$

$$\mathbf{W}^2\Psi_L\mathbf{i} = \begin{pmatrix} 0 & W^2 \\ \widehat{W}^2 & 0 \end{pmatrix} \begin{pmatrix} \phi_L & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_L \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} iW^2\widehat{\phi}_n & -iW^2\widehat{\phi}_L \\ i\widehat{W}^2\phi_L & -i\widehat{W}^2\phi_n \end{pmatrix} \quad (\text{B.55})$$

$$\begin{aligned}\mathbf{W}^3\Psi_L\gamma_3 &= \begin{pmatrix} 0 & W^3 \\ \widehat{W}^3 & 0 \end{pmatrix} \begin{pmatrix} \phi_L & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_L \end{pmatrix} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -W^3\widehat{\phi}_L\sigma_3 & W^3\widehat{\phi}_n\sigma_3 \\ -\widehat{W}^3\phi_n\sigma_3 & \widehat{W}^3\phi_L\sigma_3 \end{pmatrix} = \begin{pmatrix} -W^3\widehat{\phi}_L & W^3\widehat{\phi}_n \\ \widehat{W}^3\phi_n & -\widehat{W}^3\phi_L \end{pmatrix}. \end{aligned} \quad (\text{B.56})$$

We then get

$$\begin{aligned}\mathbf{W}^1\Psi_L + \mathbf{W}^2\Psi_L\mathbf{i} + \mathbf{W}^3\Psi_L\gamma_3 \\ = \begin{pmatrix} (W^1 + iW^2)\widehat{\phi}_n - W^3\widehat{\phi}_L & (W^1 - iW^2)\widehat{\phi}_L + W^3\widehat{\phi}_n \\ (\widehat{W}^1 + i\widehat{W}^2)\phi_L + \widehat{W}^3\phi_n & (\widehat{W}^1 - i\widehat{W}^2)\phi_n - \widehat{W}^3\phi_L \end{pmatrix}. \end{aligned} \quad (\text{B.57})$$

Next we get

$$\begin{aligned}\widetilde{\Psi}\mathbf{D}\Psi\gamma_{012} &= \widetilde{\Psi}\boldsymbol{\partial}\Psi\gamma_{012} + \frac{g_1}{2}\widetilde{\Psi}\mathbf{B}(\Psi_L + 2\Psi_R)\gamma_0 \\ &\quad - \frac{g_2}{2}\widetilde{\Psi}[\mathbf{W}^1\Psi_L + \mathbf{W}^2\Psi_L\mathbf{i} + \mathbf{W}^3\Psi_L\gamma_3] \end{aligned} \quad (\text{B.58})$$

$$\begin{aligned}\widetilde{\Psi}\boldsymbol{\partial}\Psi\gamma_{012} &= -i \begin{pmatrix} \bar{\phi}_e & \phi_n^\dagger \\ \bar{\phi}_n & \phi_e^\dagger \end{pmatrix} \begin{pmatrix} \nabla(-\widehat{\phi}_R + \widehat{\phi}_L) & \nabla\widehat{\phi}_n \\ -\widehat{\nabla}\phi_n & \widehat{\nabla}(\phi_R - \phi_L) \end{pmatrix} \\ &= -i \begin{pmatrix} \bar{\phi}_e\nabla(-\widehat{\phi}_R + \widehat{\phi}_L) - \phi_n^\dagger\widehat{\nabla}\phi_n & \bar{\phi}_e\nabla\widehat{\phi}_n + \phi_n^\dagger\widehat{\nabla}(\phi_R - \phi_L) \\ \bar{\phi}_n\nabla(-\widehat{\phi}_R + \widehat{\phi}_L) - \phi_e^\dagger\widehat{\nabla}\phi_n & \bar{\phi}_n\nabla\widehat{\phi}_n + \phi_e^\dagger\widehat{\nabla}(\phi_R - \phi_L) \end{pmatrix} \end{aligned} \quad (\text{B.59})$$

With the matrix representation (1.75) the real part of a multivector in space-time algebra is the real part of the scalar part of the matrix. We then get

$$\Re(\tilde{\Psi}\boldsymbol{\theta}\Psi\gamma_{012}) = \Re[i\bar{\phi}_e\nabla(\hat{\phi}_R - \hat{\phi}_L) + i\phi_n^\dagger\widehat{\nabla}\phi_n]. \quad (\text{B.60})$$

Next we get

$$\tilde{\Psi}\mathbf{B}P_0(\Psi)\gamma_{012} = \begin{pmatrix} \bar{\phi}_e & \phi_n^\dagger \\ \bar{\phi}_n & \phi_e^\dagger \end{pmatrix} \begin{pmatrix} B(\hat{\phi}_R + \hat{\phi}_e) & B\hat{\phi}_n \\ \widehat{B}\phi_n & \widehat{B}(\phi_R + \phi_e) \end{pmatrix} \quad (\text{B.61})$$

$$= \begin{pmatrix} \bar{\phi}_e B(\hat{\phi}_R + \hat{\phi}_e) + \phi_n^\dagger \widehat{B}\phi_n & \bar{\phi}_e B\hat{\phi}_n + \phi_n^\dagger \widehat{B}(\phi_R + \phi_e) \\ \bar{\phi}_n B(\hat{\phi}_R + \hat{\phi}_e) + \phi_e^\dagger \widehat{B}\phi_n & \bar{\phi}_n B\hat{\phi}_n + \phi_e^\dagger \widehat{B}(\phi_R + \phi_e) \end{pmatrix}$$

$$\Re[\tilde{\Psi}\mathbf{B}P_0(\Psi)\gamma_{012}] = \Re[\bar{\phi}_e B(\hat{\phi}_R + \hat{\phi}_e) + \phi_n^\dagger \widehat{B}\phi_n]. \quad (\text{B.62})$$

We also get

$$\tilde{\Psi}[\mathbf{W}^1\Psi_L + \mathbf{W}^2\Psi_L\mathbf{i} + \mathbf{W}^3\Psi_L\gamma_3] \quad (\text{B.63})$$

$$= \begin{pmatrix} \bar{\phi}_e & \phi_n^\dagger \\ \bar{\phi}_n & \phi_e^\dagger \end{pmatrix} \begin{pmatrix} (W^1 + iW^2)\hat{\phi}_n - W^3\hat{\phi}_L & (W^1 - iW^2)\hat{\phi}_L + W^3\hat{\phi}_n \\ (\widehat{W}^1 + i\widehat{W}^2)\phi_L + \widehat{W}^3\phi_n & (\widehat{W}^1 - i\widehat{W}^2)\phi_n - \widehat{W}^3\phi_L \end{pmatrix} = \begin{pmatrix} U & V \\ \widehat{V} & \widehat{U} \end{pmatrix}$$

$$U = \bar{\phi}_e[(W^1 + iW^2)\hat{\phi}_n - W^3\hat{\phi}_L] + \phi_n^\dagger[(\widehat{W}^1 + i\widehat{W}^2)\phi_L + \widehat{W}^3\phi_n] \quad (\text{B.64})$$

which gives

$$\begin{aligned} \Re[\tilde{\Psi}[\mathbf{W}^1\Psi_L + \mathbf{W}^2\Psi_L\mathbf{i} + \mathbf{W}^3\Psi_L\gamma_3]] &= \Re[\bar{\phi}_e W^1\hat{\phi}_n + \phi_n^\dagger \widehat{W}^1\phi_L] \\ &+ \Re[i\bar{\phi}_e W^2\hat{\phi}_n + i\phi_n^\dagger \widehat{W}^2\phi_L] \\ &+ \Re[-\bar{\phi}_e W^3\hat{\phi}_L + \phi_n^\dagger \widehat{W}^3\phi_n]. \end{aligned} \quad (\text{B.65})$$

We get now

$$\begin{aligned} &2\Re[i\bar{\phi}_e\nabla(\hat{\phi}_R - \hat{\phi}_L) + i\phi_n^\dagger\widehat{\nabla}\phi_n] \\ &= -i\eta_1^*\partial_0\eta_1 + i\eta_1\partial_0\eta_1^* - i\eta_2^*\partial_0\eta_2 + i\eta_2\partial_0\eta_2^* + i\xi_2\partial_0\xi_2^* - i\xi_2^*\partial_0\xi_2 \\ &+ i\xi_1\partial_0\xi_1^* - i\xi_1^*\partial_0\xi_1 + i\xi_2\partial_0\xi_2^* - i\xi_2^*\partial_0\xi_2 + i\xi_1\partial_0\xi_1^* - i\xi_1^*\partial_0\xi_1 \\ &+ i\eta_1^*\partial_1\eta_2 - i\eta_1\partial_1\eta_2^* + i\eta_2^*\partial_1\eta_1 - i\eta_2\partial_1\eta_1^* + i\xi_2\partial_1\xi_1^* - i\xi_2^*\partial_1\xi_1 \\ &+ i\xi_1\partial_1\xi_2^* - i\xi_1^*\partial_1\xi_2 - i\xi_2\partial_1\xi_1^* + i\xi_2^*\partial_1\xi_1 - i\xi_1\partial_1\xi_2^* + i\xi_1^*\partial_1\xi_2 \\ &+ \eta_1^*\partial_2\eta_2 + \eta_1\partial_2\eta_2^* - \eta_2^*\partial_2\eta_1 - \eta_2\partial_2\eta_1^* + \xi_2\partial_2\xi_1^* + \xi_2^*\partial_2\xi_1 \\ &- \xi_1\partial_2\xi_2^* - \xi_1^*\partial_2\xi_2 - \xi_2\partial_2\xi_1^* - \xi_2^*\partial_2\xi_1 + \xi_1\partial_2\xi_2^* + \xi_1^*\partial_2\xi_2 \\ &+ i\eta_1^*\partial_3\eta_1 - i\eta_1\partial_3\eta_1^* - i\eta_2^*\partial_3\eta_2 + i\eta_2\partial_3\eta_2^* - i\xi_2\partial_3\xi_2^* + i\xi_2^*\partial_3\xi_2 \\ &+ i\xi_1\partial_3\xi_1^* - i\xi_1^*\partial_3\xi_1 + i\xi_2\partial_3\xi_2^* - i\xi_2^*\partial_3\xi_2 - i\xi_1\partial_3\xi_1^* + i\xi_1^*\partial_3\xi_1. \end{aligned} \quad (\text{B.66})$$

For (B.65) we have

$$\begin{aligned} &\Re[\bar{\phi}_e B(\hat{\phi}_R + \hat{\phi}_e) + \phi_n^\dagger \widehat{B}\phi_n] \\ &= B_0(\eta_1\eta_1^* + \eta_2\eta_2^* + 2\xi_1\xi_1^* + 2\xi_2\xi_2^* + \zeta_1\zeta_1^* + \zeta_2\zeta_2^*) \\ &+ B_1(-\eta_1\eta_2^* - \eta_2\eta_1^* + 2\xi_1\xi_2^* + 2\xi_2\xi_1^* - \zeta_1\zeta_2^* - \zeta_2\zeta_1^*) \\ &+ B_2i(-\eta_1\eta_2^* + \eta_2\eta_1^* + 2\xi_1\xi_2^* - 2\xi_2\xi_1^* - \zeta_1\zeta_2^* + \zeta_2\zeta_1^*) \\ &+ B_3(-\eta_1\eta_1^* + \eta_2\eta_2^* + 2\xi_1\xi_1^* - 2\xi_2\xi_2^* - \zeta_1\zeta_1^* + \zeta_2\zeta_2^*) \end{aligned} \quad (\text{B.67})$$

and for (B.68) we have

$$\begin{aligned}
\Re(\bar{\phi}_e W^1 \widehat{\phi}_n + \phi_n^\dagger \widehat{W}^1 \phi_L) &= W_0^1(\eta_1 \zeta_1^* + \eta_2 \zeta_2^* + \zeta_1 \eta_1^* + \zeta_2 \eta_2^*) \\
&\quad + W_1^1(-\eta_1 \zeta_2^* - \eta_2 \zeta_1^* - \zeta_1 \eta_2^* - \zeta_2 \eta_1^*) \\
&\quad + W_2^1 i(-\eta_1 \zeta_2^* + \eta_2 \zeta_1^* - \zeta_1 \eta_2^* + \zeta_2 \eta_1^*) \\
&\quad + W_3^1(-\eta_1 \zeta_1^* + \eta_2 \zeta_2^* - \zeta_1 \eta_1^* + \zeta_2 \eta_2^*) \tag{B.68}
\end{aligned}$$

$$\begin{aligned}
\Re(i\bar{\phi}_e W^2 \widehat{\phi}_n + i\phi_n^\dagger \widehat{W}^2 \phi_L) &= W_0^2 i(-\eta_1 \zeta_1^* - \eta_2 \zeta_2^* + \zeta_1 \eta_1^* + \zeta_2 \eta_2^*) \\
&\quad + W_1^2 i(\eta_1 \zeta_2^* + \eta_2 \zeta_1^* - \zeta_1 \eta_2^* - \zeta_2 \eta_1^*) \\
&\quad + W_2^2(-\eta_1 \zeta_2^* + \eta_2 \zeta_1^* + \zeta_1 \eta_2^* - \zeta_2 \eta_1^*) \\
&\quad + W_3^2 i(-\eta_1 \zeta_1^* + \eta_2 \zeta_2^* - \zeta_1 \eta_1^* + \zeta_2 \eta_2^*) \tag{B.69}
\end{aligned}$$

$$\Re(-\bar{\phi}_e W^3 \widehat{\phi}_L + \phi_n^\dagger \widehat{W}^3 \phi_n) \tag{B.70}$$

$$\begin{aligned}
&= W_0^3(-\eta_1 \eta_1^* - \eta_2 \eta_2^* + \zeta_1 \zeta_1^* + \zeta_2 \zeta_2^*) \\
&\quad + W_1^3(+\eta_1 \eta_2^* + \eta_2 \eta_1^* - \zeta_1 \zeta_2^* - \zeta_2 \zeta_1^*) \\
&\quad + W_2^3 i(+\eta_1 \eta_2^* - \eta_2 \eta_1^* - \zeta_1 \zeta_2^* + \zeta_2 \zeta_1^*) \\
&\quad + W_3^3(\eta_1 \eta_1^* - \eta_2 \eta_2^* - \zeta_1 \zeta_1^* + \zeta_2 \zeta_2^*). \tag{B.71}
\end{aligned}$$

Therefore the Lagrangian density is

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2} \left(\begin{aligned}
&-i\eta_1^* \partial_0 \eta_1 + i\eta_1 \partial_0 \eta_1^* - i\eta_2^* \partial_0 \eta_2 + i\eta_2 \partial_0 \eta_2^* + i\xi_2 \partial_0 \xi_2^* - i\xi_2^* \partial_0 \xi_2 \\
&+ i\xi_1 \partial_0 \xi_1^* - i\xi_1^* \partial_0 \xi_1 + i\zeta_2 \partial_0 \zeta_2^* - i\zeta_2^* \partial_0 \zeta_2 + i\zeta_1 \partial_0 \zeta_1^* - i\zeta_1^* \partial_0 \zeta_1 \\
&+ i\eta_1^* \partial_1 \eta_2 - i\eta_1 \partial_1 \eta_2^* + i\eta_2^* \partial_1 \eta_1 - i\eta_2 \partial_1 \eta_1^* + i\xi_2 \partial_1 \xi_1^* - i\xi_2^* \partial_1 \xi_1 \\
&+ i\xi_1 \partial_1 \xi_2^* - i\xi_1^* \partial_1 \xi_2 - i\zeta_2 \partial_1 \zeta_1^* + i\zeta_2^* \partial_1 \zeta_1 - i\zeta_1 \partial_1 \zeta_2^* + i\zeta_1^* \partial_1 \zeta_2 \\
&\quad + \eta_1^* \partial_2 \eta_2 + \eta_1 \partial_2 \eta_2^* - \eta_2^* \partial_2 \eta_1 - \eta_2 \partial_2 \eta_1^* + \xi_2 \partial_2 \xi_1^* + \xi_2^* \partial_2 \xi_1 \\
&\quad - \xi_1 \partial_2 \xi_2^* - \xi_1^* \partial_2 \xi_2 - \zeta_2 \partial_2 \zeta_1^* - \zeta_2^* \partial_2 \zeta_1 + \zeta_1 \partial_2 \zeta_2^* + \zeta_1^* \partial_2 \zeta_2 \\
&+ i\eta_1^* \partial_3 \eta_1 - i\eta_1 \partial_3 \eta_1^* - i\eta_2^* \partial_3 \eta_2 + i\eta_2 \partial_3 \eta_2^* - i\xi_2 \partial_3 \xi_2^* + i\xi_2^* \partial_3 \xi_2 \\
&+ i\xi_1 \partial_3 \xi_1^* - i\xi_1^* \partial_3 \xi_1 + i\zeta_2 \partial_3 \zeta_2^* - i\zeta_2^* \partial_3 \zeta_2 - i\zeta_1 \partial_3 \zeta_1^* + i\zeta_1^* \partial_3 \zeta_1
\end{aligned} \right) \\
&\quad + \frac{g_1}{2} \left(\begin{aligned}
&B_0(\eta_1 \eta_1^* + \eta_2 \eta_2^* + 2\xi_1 \xi_1^* + 2\xi_2 \xi_2^* + \zeta_1 \zeta_1^* + \zeta_2 \zeta_2^*) \\
&+ B_1(-\eta_1 \eta_2^* - \eta_2 \eta_1^* + 2\xi_1 \xi_2^* + 2\xi_2 \xi_1^* - \zeta_1 \zeta_2^* - \zeta_2 \zeta_1^*) \\
&+ B_2 i(-\eta_1 \eta_2^* + \eta_2 \eta_1^* + 2\xi_1 \xi_2^* - 2\xi_2 \xi_1^* - \zeta_1 \zeta_2^* + \zeta_2 \zeta_1^*) \\
&+ B_3(-\eta_1 \eta_1^* + \eta_2 \eta_2^* + 2\xi_1 \xi_1^* - 2\xi_2 \xi_2^* - \zeta_1 \zeta_1^* + \zeta_2 \zeta_2^*)
\end{aligned} \right) \tag{B.72} \\
&\quad - \frac{g_2}{2} \left(\begin{aligned}
&W_0^1(\eta_1 \zeta_1^* + \eta_2 \zeta_2^* + \zeta_1 \eta_1^* + \zeta_2 \eta_2^*) + W_0^2 i(-\eta_1 \zeta_1^* - \eta_2 \zeta_2^* + \zeta_1 \eta_1^* + \zeta_2 \eta_2^*) \\
&\quad + W_0^3(-\eta_1 \eta_1^* - \eta_2 \eta_2^* + \zeta_1 \zeta_1^* + \zeta_2 \zeta_2^*) \\
&+ W_1^1(-\eta_1 \zeta_2^* - \eta_2 \zeta_1^* - \zeta_1 \eta_2^* - \zeta_2 \eta_1^*) + W_1^2 i(\eta_1 \zeta_2^* + \eta_2 \zeta_1^* - \zeta_1 \eta_2^* - \zeta_2 \eta_1^*) \\
&\quad + W_1^3(\eta_1 \eta_2^* + \eta_2 \eta_1^* - \zeta_1 \zeta_2^* - \zeta_2 \zeta_1^*) \\
&+ W_2^2 i(-\eta_1 \zeta_2^* + \eta_2 \zeta_1^* - \zeta_1 \eta_2^* + \zeta_2 \eta_1^*) + W_2^2(-\eta_1 \zeta_2^* + \eta_2 \zeta_1^* + \zeta_1 \eta_2^* - \zeta_2 \eta_1^*) \\
&\quad + W_2^3 i(\eta_1 \eta_2^* - \eta_2 \eta_1^* - \zeta_1 \zeta_2^* + \zeta_2 \zeta_1^*) \\
&+ W_3^1(-\eta_1 \zeta_1^* + \eta_2 \zeta_2^* - \zeta_1 \eta_1^* + \zeta_2 \eta_2^*) + W_3^2 i(-\eta_1 \zeta_1^* + \eta_2 \zeta_2^* - \zeta_1 \eta_1^* + \zeta_2 \eta_2^*) \\
&\quad + W_3^3(\eta_1 \eta_1^* - \eta_2 \eta_2^* - \zeta_1 \zeta_1^* + \zeta_2 \zeta_2^*)
\end{aligned} \right) \\
&\quad + m\sqrt{\mathbf{a}\mathbf{a}^* + \mathbf{b}\mathbf{b}^* + \mathbf{c}\mathbf{c}^*}.
\end{aligned}$$

The Lagrange equation $\frac{\partial \mathcal{L}}{\partial \xi_1^*} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \xi_1^*)} \right)$ gives

$$0 = -i[(\partial_0 + \partial_3)\xi_1 + (\partial_1 - i\partial_2)\xi_2] + g_1[(B_0 + B_3)\xi_1 + (B_1 - iB_2)\xi_2] + \frac{m}{\boldsymbol{\rho}}(\mathbf{a}\eta_1 + \mathbf{c}\zeta_1). \quad (\text{B.73})$$

The Lagrange equation $\frac{\partial \mathcal{L}}{\partial \xi_2^*} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \xi_2^*)} \right)$ gives

$$0 = -i[(\partial_1 + i\partial_2)\xi_1 + (\partial_0 - \partial_3)\xi_2] + g_1[(B_1 + iB_2)\xi_1 + (B_0 - B_3)\xi_2] + \frac{m}{\boldsymbol{\rho}}(\mathbf{a}\eta_2 + \mathbf{c}\zeta_2). \quad (\text{B.74})$$

Together these equations read

$$0 = -i \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix} \begin{pmatrix} \xi_1 & 0 \\ \xi_2 & 0 \end{pmatrix} + g_1 \begin{pmatrix} B_0 + B_3 & B_1 - iB_2 \\ B_1 + iB_2 & B_0 - B_3 \end{pmatrix} \begin{pmatrix} \xi_1 & 0 \\ \xi_2 & 0 \end{pmatrix} + \frac{m}{\boldsymbol{\rho}} \left[\mathbf{a} \begin{pmatrix} \eta_1 & 0 \\ \eta_2 & 0 \end{pmatrix} + \mathbf{c} \begin{pmatrix} \zeta_1 & 0 \\ \zeta_2 & 0 \end{pmatrix} \right]. \quad (\text{B.75})$$

Multiplying by $\sqrt{2}$ we get

$$-i\widehat{\nabla}\phi_R + g_1\widehat{B}\phi_R + \frac{m}{\boldsymbol{\rho}}(\mathbf{a}\widehat{\phi}_L + \mathbf{c}\widehat{\phi}_n) = 0. \quad (\text{B.76})$$

Since $\phi_R\sigma_3 = \phi_R$ and $\widehat{\phi}_L\sigma_3 = \widehat{\phi}_L$, this reads

$$\widehat{\nabla}\phi_R\sigma_{21} + g_1\widehat{B}\phi_R + \frac{m}{\boldsymbol{\rho}}(\mathbf{a}\widehat{\phi}_L + \mathbf{c}\widehat{\phi}_n) = 0. \quad (\text{B.77})$$

Then by using the conjugation $M \mapsto \widehat{M}$ we get

$$\widehat{\nabla}\widehat{\phi}_R\sigma_{21} + g_1\widehat{B}\widehat{\phi}_R + \frac{m}{\boldsymbol{\rho}}(\mathbf{a}^*\phi_L + \mathbf{c}^*\phi_n) = 0. \quad (\text{B.78})$$

The Lagrange equation $\frac{\partial \mathcal{L}}{\partial \eta_1^*} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \eta_1^*)} \right)$ gives

$$0 = -i[(\partial_0 - \partial_3)\eta_1 + (-\partial_1 + i\partial_2)\eta_2] + \frac{g_1}{2}[(B_0 - B_3)\eta_1 + (-B_1 + iB_2)\eta_2] - \frac{g_2}{2} \begin{pmatrix} (W_0^1 - W_3^1)\zeta_1 + (-W_1^1 + iW_2^1)\zeta_2 \\ +i[(W_0^2 - W_3^2)\zeta_1 + (-W_1^2 + iW_2^2)\zeta_2] \\ -(W_0^3 - W_3^3)\eta_1 - (-W_1^3 + iW_2^3)\eta_2 \end{pmatrix} + \frac{m}{\boldsymbol{\rho}}(\mathbf{a}^*\xi_1 + \mathbf{b}\zeta_2^*). \quad (\text{B.79})$$

The Lagrange equation $\frac{\partial \mathcal{L}}{\partial \eta_2^*} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \eta_2^*)} \right)$ gives

$$0 = -i[(-\partial_1 - i\partial_2)\eta_1 + (\partial_0 + \partial_3)\eta_2] + \frac{g_1}{2}[(-B_1 - iB_2)\eta_1 + (B_0 + B_3)\eta_2] - \frac{g_2}{2} \begin{pmatrix} (-W_1^1 - iW_2^1)\zeta_1 + (W_0^1 + W_3^1)\zeta_2 \\ +i[(-W_1^2 - iW_2^2)\zeta_1 + (W_0^2 + W_3^2)\zeta_2] \\ -(-W_1^3 - iW_2^3)\eta_1 - (W_0^3 + W_3^3)\eta_2 \end{pmatrix} + \frac{m}{\boldsymbol{\rho}}(\mathbf{a}^*\xi_2 - \mathbf{b}\zeta_1^*). \quad (\text{B.80})$$

Together these equations give

$$\begin{aligned}
0 = & -i \begin{pmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{pmatrix} \begin{pmatrix} \eta_1 & 0 \\ \eta_2 & 0 \end{pmatrix} \\
& + \frac{g_1}{2} \begin{pmatrix} B_0 - B_3 & -B_1 + iB_2 \\ -B_1 - iB_2 & B_0 + B_3 \end{pmatrix} \begin{pmatrix} \eta_1 & 0 \\ \eta_2 & 0 \end{pmatrix} \\
& - \frac{g_2}{2} \left[\begin{pmatrix} W_0^1 - W_3^1 & -W_1^1 + iW_2^1 \\ -W_1^1 - iW_2^1 & W_0^1 + W_3^1 \end{pmatrix} \begin{pmatrix} \zeta_1 & 0 \\ \zeta_2 & 0 \end{pmatrix} \right. \\
& + i \begin{pmatrix} W_0^2 - W_3^2 & -W_1^2 + iW_2^2 \\ -W_1^2 - iW_2^2 & W_0^2 + W_3^2 \end{pmatrix} \begin{pmatrix} \zeta_1 & 0 \\ \zeta_2 & 0 \end{pmatrix} \\
& \left. - \begin{pmatrix} W_0^3 - W_3^3 & -W_1^3 + iW_2^3 \\ -W_1^3 - iW_2^3 & W_0^3 + W_3^3 \end{pmatrix} \begin{pmatrix} \eta_1 & 0 \\ \eta_2 & 0 \end{pmatrix} \right] \\
& + \frac{m}{\boldsymbol{\rho}} \left[\mathbf{a}^* \begin{pmatrix} \xi_1 & 0 \\ \xi_2 & 0 \end{pmatrix} + \mathbf{b} \begin{pmatrix} \zeta_2^* & 0 \\ -\zeta_1^* & 0 \end{pmatrix} \right].
\end{aligned} \tag{B.81}$$

Multiplying by $\sqrt{2}$ this reads

$$\begin{aligned}
0 = & \nabla \widehat{\phi}_L \sigma_{21} + \frac{g_1}{2} B \widehat{\phi}_L + \frac{g_2}{2} [-(W^1 + iW^2) \widehat{\phi}_n + W^3 \widehat{\phi}_L] \\
& + \frac{m}{\boldsymbol{\rho}} (\mathbf{a}^* \phi_R - \mathbf{b} \phi_n \sigma_1).
\end{aligned} \tag{B.82}$$

Adding (B.78) and (B.82) we get the wave equation

$$\begin{aligned}
0 = & \nabla \widehat{\phi}_e \sigma_{21} + \frac{g_1}{2} B (\widehat{\phi}_L + 2\widehat{\phi}_R) + \frac{g_2}{2} [-(W^1 + iW^2) \widehat{\phi}_n + W^3 \widehat{\phi}_L] \\
& + \frac{m}{\boldsymbol{\rho}} (\mathbf{a}^* \phi_e - \mathbf{b} \phi_n \sigma_1 + \mathbf{c}^* \phi_n)
\end{aligned} \tag{B.83}$$

This equation contains the covariant derivative (6.57) of the electron.

The Lagrange equation $\frac{\partial \mathcal{L}}{\partial \zeta_1^*} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \zeta_1^*)} \right)$ gives

$$\begin{aligned}
0 = & -i[(\partial_0 - \partial_3)\zeta_1 + (-\partial_1 + i\partial_2)\zeta_2] + \frac{g_1}{2} [(B_0 - B_3)\zeta_1 + (-B_1 + iB_2)\zeta_2] \\
& + \frac{g_2}{2} \begin{pmatrix} -[(W_0^1 - W_3^1)\eta_1 + (-W_1^1 + iW_2^1)\eta_2] \\ +i[(W_0^2 - W_3^2)\eta_1 + (-W_1^2 + iW_2^2)\eta_2] \\ -(W_0^3 - W_3^3)\zeta_1 - (-W_1^3 + iW_2^3)\zeta_2 \end{pmatrix} + \frac{m}{\boldsymbol{\rho}} (\mathbf{c}^* \xi_1 - \mathbf{b} \eta_2^*).
\end{aligned} \tag{B.84}$$

The Lagrange equation $\frac{\partial \mathcal{L}}{\partial \zeta_2^*} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \zeta_2^*)} \right)$ gives

$$\begin{aligned}
0 = & -i[(-\partial_1 - i\partial_2)\zeta_1 + (\partial_0 + \partial_3)\zeta_2] + \frac{g_1}{2} [(-B_1 - iB_2)\zeta_1 + (B_0 + B_3)\zeta_2] \\
& + \frac{g_2}{2} \begin{pmatrix} -[(-W_1^1 - iW_2^1)\eta_1 + (W_0^1 + W_3^1)\eta_2] \\ +i[(-W_1^2 - iW_2^2)\eta_1 + (W_0^2 + W_3^2)\eta_2] \\ -(-W_1^3 - iW_2^3)\zeta_1 - (W_0^3 + W_3^3)\zeta_2 \end{pmatrix} + \frac{m}{\boldsymbol{\rho}} (\mathbf{c}^* \xi_2 + \mathbf{b} \eta_1^*).
\end{aligned} \tag{B.85}$$

Together these equations read

$$\begin{aligned}
0 = & -i \begin{pmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{pmatrix} \begin{pmatrix} \zeta_1 & 0 \\ \zeta_2 & 0 \end{pmatrix} \\
& + \frac{g_1}{2} \begin{pmatrix} B_0 - B_3 & -B_1 + iB_2 \\ -B_1 - iB_2 & B_0 + B_3 \end{pmatrix} \begin{pmatrix} \zeta_1 & 0 \\ \zeta_2 & 0 \end{pmatrix} \\
& + \frac{g_2}{2} \left[- \begin{pmatrix} W_0^1 - W_3^1 & -W_1^1 + iW_2^1 \\ -W_1^1 - iW_2^1 & W_0^1 + W_3^1 \end{pmatrix} \begin{pmatrix} \eta_1 & 0 \\ \eta_2 & 0 \end{pmatrix} \right. \\
& + i \begin{pmatrix} W_0^2 - W_3^2 & -W_1^2 + iW_2^2 \\ -W_1^2 - iW_2^2 & W_0^2 + W_3^2 \end{pmatrix} \begin{pmatrix} \eta_1 & 0 \\ \eta_2 & 0 \end{pmatrix} \\
& \left. - \begin{pmatrix} W_0^3 - W_3^3 & -W_1^3 + iW_2^3 \\ -W_1^3 - iW_2^3 & W_0^3 + W_3^3 \end{pmatrix} \begin{pmatrix} \zeta_1 & 0 \\ \zeta_2 & 0 \end{pmatrix} \right] \\
& + \frac{m}{\boldsymbol{\rho}} \left[\mathbf{b} \begin{pmatrix} -\eta_2^* & 0 \\ \eta_1^* & 0 \end{pmatrix} + \mathbf{b}^* \begin{pmatrix} \xi_1 & 0 \\ \xi_2 & 0 \end{pmatrix} \right].
\end{aligned} \tag{B.86}$$

Multiplying by $\sqrt{2}$ this reads

$$\begin{aligned}
0 = & \nabla \widehat{\phi}_n \sigma_{21} + \frac{g_1}{2} B \widehat{\phi}_n + \frac{g_2}{2} [(-W^1 + iW^2) \widehat{\phi}_L - W^3 \widehat{\phi}_n] \\
& + \frac{m}{\boldsymbol{\rho}} (\mathbf{c}^* \phi_R + \mathbf{b} \phi_L \sigma_1).
\end{aligned} \tag{B.87}$$

This equation contains the covariant derivative (6.58) of the electronic neutrino. The system made of the two wave equations (B.83) - (B.87) is equivalent to the equation:

$$\mathbf{D}\Psi \gamma_{012} + m\boldsymbol{\rho}\chi = 0 \tag{B.88}$$

where

$$\chi = \frac{1}{\boldsymbol{\rho}^2} \begin{pmatrix} \mathbf{a}^* \phi_e - \mathbf{b} \phi_n \sigma_1 + \mathbf{c}^* \phi_n & \mathbf{b} \phi_L \sigma_1 + \mathbf{c}^* \phi_R \\ -\mathbf{b}^* \widehat{\phi}_L \sigma_1 + \mathbf{c} \widehat{\phi}_R & \mathbf{a} \widehat{\phi}_e + \mathbf{b}^* \widehat{\phi}_n \sigma_1 + \mathbf{c} \widehat{\phi}_n \end{pmatrix} \tag{B.89}$$

or to the invariant equation

$$\widetilde{\Psi} \mathbf{D}\Psi \gamma_{012} + m\boldsymbol{\rho}\widetilde{\Psi}\chi = 0. \tag{B.90}$$

Now we explain how this wave equation has exactly the cancellation of the Lagrangian density (B.72) $\mathcal{L} = 0$ as real part. Then we shall prove the form invariance and the gauge invariance of this equation under the gauge group of electro-weak interactions.

B.3 Invariances

With (B.89) we have

$$\begin{aligned} \boldsymbol{\rho}^2 \tilde{\Psi} \chi &= \begin{pmatrix} \bar{\phi}_e & \phi_n^\dagger \\ \bar{\phi}_n & \phi_e^\dagger \end{pmatrix} \begin{pmatrix} \mathbf{a}^* \phi_e - \mathbf{b} \phi_n \sigma_1 + \mathbf{c}^* \phi_n & \mathbf{b} \phi_L \sigma_1 + \mathbf{c}^* \phi_R \\ -\mathbf{b}^* \hat{\phi}_L \sigma_1 + \mathbf{c} \hat{\phi}_R & \mathbf{a} \hat{\phi}_e + \mathbf{b}^* \hat{\phi}_n \sigma_1 + \mathbf{c} \hat{\phi}_n \end{pmatrix} \\ &= \begin{pmatrix} U & V \\ \hat{V} & \hat{U} \end{pmatrix} \end{aligned} \quad (\text{B.91})$$

$$\begin{aligned} U &= \mathbf{a}^* \bar{\phi}_e \phi_e - \mathbf{b} \bar{\phi}_e \phi_n \sigma_1 + \mathbf{c}^* \bar{\phi}_e \phi_n - \mathbf{b}^* \phi_n^\dagger \hat{\phi}_L \sigma_1 + \mathbf{c} \phi_n^\dagger \hat{\phi}_R \\ V &= \mathbf{b} \bar{\phi}_e \phi_L \sigma_1 + \mathbf{c}^* \bar{\phi}_e \phi_R + \mathbf{a} \phi_n^\dagger \hat{\phi}_e + \mathbf{b}^* \phi_n^\dagger \hat{\phi}_n \sigma_1 + \mathbf{c} \phi_n^\dagger \hat{\phi}_n. \end{aligned} \quad (\text{B.92})$$

We have

$$\mathbf{a}^* \bar{\phi}_e \phi_e = \mathbf{a} \mathbf{a}^* \quad (\text{B.93})$$

$$-\mathbf{b} \bar{\phi}_e \phi_n \sigma_1 - \mathbf{b}^* \phi_n^\dagger \hat{\phi}_L \sigma_1 = \begin{pmatrix} \mathbf{b} \mathbf{b}^* & 0 \\ -\mathbf{b} \mathbf{c} & \mathbf{b} \mathbf{b}^* \end{pmatrix} = \mathbf{b} \mathbf{b}^* + \frac{\mathbf{b} \mathbf{c}}{2} (-\sigma_1 + i \sigma_2) \quad (\text{B.94})$$

$$\mathbf{c}^* \bar{\phi}_e \phi_n + \mathbf{c} \phi_n^\dagger \hat{\phi}_R = \begin{pmatrix} 0 & -\mathbf{b}^* \mathbf{c}^* \\ 0 & 2 \mathbf{c} \mathbf{c}^* \end{pmatrix} = \mathbf{c} \mathbf{c}^* (1 - \sigma_3) + \frac{\mathbf{b}^* \mathbf{c}^*}{2} (-\sigma_1 - i \sigma_2) \quad (\text{B.95})$$

$$U = \rho^2 - \Re(\mathbf{b} \mathbf{c}) \sigma_1 - \Im(\mathbf{b} \mathbf{c}) \sigma_2 - \mathbf{c} \mathbf{c}^* \sigma_3 \quad (\text{B.96})$$

$$\Re(m \boldsymbol{\rho} \tilde{\Psi} \chi) = m \boldsymbol{\rho}. \quad (\text{B.97})$$

Therefore the Lagrangian density (B.72) is also the real part of the invariant equation (B.90). The value of V is more simple because a similar calculation gives

$$V = \mathbf{a} \mathbf{c}^*. \quad (\text{B.98})$$

B.3.1 Form invariance

If M is any invertible matrix defining (1.42) we have

$$\nabla = \bar{M} \nabla' \widehat{M}; \quad \Psi' = N \Psi \quad (\text{B.99})$$

$$\mathbf{D} = \tilde{N} \mathbf{D}' N; \quad N = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix}; \quad \tilde{N} = \begin{pmatrix} \bar{M} & 0 \\ 0 & M^\dagger \end{pmatrix} \quad (\text{B.100})$$

$$\begin{aligned} 0 &= \tilde{\Psi} \mathbf{D} \Psi \gamma_{012} + m \boldsymbol{\rho} \tilde{\Psi} \chi = \tilde{\Psi} \tilde{N} \mathbf{D}' N \Psi \gamma_{012} + m \boldsymbol{\rho} \tilde{\Psi} \chi \\ &= \tilde{\Psi}' \mathbf{D}' \Psi' \gamma_{012} + m' \boldsymbol{\rho}' \tilde{\Psi}' \chi. \end{aligned} \quad (\text{B.101})$$

We shall get the form invariance of the wave equation if and only if

$$\tilde{\Psi} \chi = \tilde{\Psi}' \chi' = \tilde{\Psi} \tilde{N} \chi' \quad (\text{B.102})$$

$$\chi' = \tilde{N}^{-1} \chi. \quad (\text{B.103})$$

But we have

$$\begin{aligned}
& \frac{1}{\boldsymbol{\rho}'^2} (\mathbf{a}'^* \phi'_e - \mathbf{b}' \phi'_n \sigma_1 + \mathbf{c}'^* \phi'_n) \\
&= \frac{1}{r^2 \boldsymbol{\rho}^2} (r e^{-i\theta} \mathbf{a}^* M \phi_e - r e^{-i\theta} \mathbf{b} M \phi_n \sigma_1 + r e^{-i\theta} \mathbf{c}^* M \phi_n) \\
&= \frac{M}{r e^{i\theta}} \frac{1}{\boldsymbol{\rho}^2} (\mathbf{a}^* \phi_e - \mathbf{b} \phi_n \sigma_1 + \mathbf{c}^* \phi_n) = \overline{M}^{-1} \frac{1}{\boldsymbol{\rho}^2} (\mathbf{a}^* \phi_e - \mathbf{b} \phi_n \sigma_1 + \mathbf{c}^* \phi_n). \quad (\text{B.104})
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{\boldsymbol{\rho}'^2} (\mathbf{b}' \phi'_L \sigma_1 + \mathbf{c}'^* \phi'_n) \\
&= \frac{1}{r^2 \boldsymbol{\rho}^2} (r e^{-i\theta} \mathbf{b} M \phi_L \sigma_1 + r e^{-i\theta} \mathbf{c}^* M \phi_n) \\
&= \frac{M}{r e^{i\theta}} \frac{1}{\boldsymbol{\rho}^2} (\mathbf{b} \phi_L \sigma_1 + \mathbf{c}^* \phi_n) = \overline{M}^{-1} \frac{1}{\boldsymbol{\rho}^2} (\mathbf{b} \phi_L \sigma_1 + \mathbf{c}^* \phi_n). \quad (\text{B.105})
\end{aligned}$$

This gives (B.103) because

$$\chi' = \begin{pmatrix} \overline{M}^{-1} & 0 \\ 0 & M^{\dagger-1} \end{pmatrix} \chi = \tilde{N}^{-1} \chi. \quad (\text{B.106})$$

Then the wave equation is form invariant under Cl_3^* , therefore it is relativistic invariant.

B.3.2 Gauge invariance - group generated by P_0

We use the following form of P_0

$$P_0(\Psi) = \frac{\mathbf{i}}{2} \Psi + \frac{3}{2} \Psi \gamma_{21} \quad (\text{B.107})$$

$$\Psi' = [\exp(\theta P_0)](\Psi) = e^{\frac{\theta}{2} \mathbf{i}} \Psi e^{\frac{3\theta}{2} \gamma_{21}}. \quad (\text{B.108})$$

From (B.14) with $a = b = 1$ we deduce

$$\mathbf{D}' \Psi' = e^{-\frac{\theta}{2} \mathbf{i}} (\mathbf{D} \Psi) e^{\frac{3\theta}{2} \gamma_{21}} \quad (\text{B.109})$$

$$\mathbf{D}' \Psi' \gamma_{012} = e^{-\frac{\theta}{2} \mathbf{i}} (\mathbf{D} \Psi \gamma_{012}) e^{\frac{3\theta}{2} \gamma_{21}}. \quad (\text{B.110})$$

(B.108) reads

$$\begin{aligned}
\begin{pmatrix} \widehat{\phi}'_e & \widehat{\phi}'_n \\ \widehat{\phi}'_n & \widehat{\phi}'_e \end{pmatrix} &= \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \begin{pmatrix} \widehat{\phi}_e & \widehat{\phi}_n \\ \widehat{\phi}_n & \widehat{\phi}_e \end{pmatrix} \begin{pmatrix} e^{\frac{3\theta}{2} i\sigma_3} & 0 \\ 0 & e^{\frac{3\theta}{2} i\sigma_3} \end{pmatrix} \\
&= \begin{pmatrix} e^{i\frac{\theta}{2}} \widehat{\phi}_e e^{\frac{3\theta}{2} i\sigma_3} & e^{i\frac{\theta}{2}} \widehat{\phi}_n e^{\frac{3\theta}{2} i\sigma_3} \\ e^{-i\frac{\theta}{2}} \widehat{\phi}_n e^{\frac{3\theta}{2} i\sigma_3} & e^{-i\frac{\theta}{2}} \widehat{\phi}_e e^{\frac{3\theta}{2} i\sigma_3} \end{pmatrix}. \quad (\text{B.111})
\end{aligned}$$

This gives

$$\begin{pmatrix} \xi'_1 & -\eta'^*_2 \\ \xi'_2 & \eta'^*_1 \end{pmatrix} = \begin{pmatrix} e^{2i\theta}\xi_1 & -e^{-i\theta}\eta_2^* \\ e^{2i\theta}\xi_2 & e^{-i\theta}\eta_1^* \end{pmatrix}; \quad \begin{pmatrix} 0 & -\zeta'^*_2 \\ 0 & \zeta'^*_1 \end{pmatrix} = \begin{pmatrix} 0 & -e^{-i\theta}\zeta_2^* \\ 0 & e^{-i\theta}\zeta_1^* \end{pmatrix} \quad (\text{B.112})$$

and we get

$$\begin{pmatrix} \xi'_1 \\ \xi'_2 \end{pmatrix} = e^{2i\theta} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}; \quad \begin{pmatrix} \eta'_1 \\ \eta'_2 \end{pmatrix} = e^{i\theta} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}; \quad \begin{pmatrix} \zeta'_1 \\ \zeta'_2 \end{pmatrix} = e^{i\theta} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}. \quad (\text{B.113})$$

We then have for \mathbf{a} , \mathbf{b} , \mathbf{c} :

$$\mathbf{a}' = e^{i\theta} \mathbf{a}; \quad \mathbf{a}' \mathbf{a}'^* = \mathbf{a} \mathbf{a}^* \quad (\text{B.114})$$

$$\mathbf{b}' = e^{2i\theta} \mathbf{b}; \quad \mathbf{b}' \mathbf{b}'^* = \mathbf{b} \mathbf{b}^* \quad (\text{B.115})$$

$$\mathbf{c}' = e^{i\theta} \mathbf{c}; \quad \mathbf{c}' \mathbf{c}'^* = \mathbf{c} \mathbf{c}^* \quad (\text{B.116})$$

$$\boldsymbol{\rho}' = \boldsymbol{\rho}. \quad (\text{B.117})$$

We must study

$$\chi' = \frac{1}{\boldsymbol{\rho}'^2} \times \begin{pmatrix} \mathbf{a}'^* \phi'_e - \mathbf{b}' \phi'_n \sigma_1 + \mathbf{c}'^* \phi'_n & \mathbf{b}' \phi'_L \sigma_1 + \mathbf{c}'^* \phi'_R \\ -\mathbf{b}'^* \widehat{\phi}'_L \sigma_1 + \mathbf{c}' \widehat{\phi}'_R & \mathbf{a}' \widehat{\phi}'_e + \mathbf{b}'^* \widehat{\phi}'_n \sigma_1 + \mathbf{c}' \widehat{\phi}'_n \end{pmatrix}. \quad (\text{B.118})$$

We get

$$\begin{aligned} & \mathbf{a}'^* \phi'_e - \mathbf{b}' \phi'_n \sigma_1 + \mathbf{c}'^* \phi'_n \\ &= e^{-i\theta} \mathbf{a}^* e^{i\frac{\theta}{2}} \phi_e e^{\frac{3\theta}{2}i\sigma_3} - e^{2i\theta} \mathbf{b} e^{i\frac{\theta}{2}} \phi_n e^{\frac{3\theta}{2}i\sigma_3} + e^{-i\theta} \mathbf{c}^* e^{i\frac{\theta}{2}} \phi_n e^{\frac{3\theta}{2}i\sigma_3} \end{aligned} \quad (\text{B.119})$$

$$\begin{aligned} &= e^{-i\frac{\theta}{2}} (\mathbf{a}^* \phi_e - e^{3i\theta} \mathbf{b} \phi_n e^{3i\theta\sigma_3} + \mathbf{c}^* \phi_n) e^{\frac{3\theta}{2}i\sigma_3} \\ &= e^{-i\frac{\theta}{2}} (\mathbf{a}^* \phi_e - e^{3i\theta} \mathbf{b} \phi_n e^{-3i\theta} \sigma_1 + \mathbf{c}^* \phi_n) e^{\frac{3\theta}{2}i\sigma_3} \\ &= e^{-i\frac{\theta}{2}} (\mathbf{a}^* \phi_e - \mathbf{b} \phi_n \sigma_1 + \mathbf{c}^* \phi_n) e^{\frac{3\theta}{2}i\sigma_3} \end{aligned} \quad (\text{B.120})$$

ans similarly

$$\begin{aligned} \mathbf{b}' \phi'_L \sigma_1 + \mathbf{c}'^* \phi'_R &= e^{2i\theta} \mathbf{b} e^{i\frac{\theta}{2}} \phi_L e^{\frac{3\theta}{2}i\sigma_3} \sigma_1 + e^{-i\theta} \mathbf{c}^* e^{i\frac{\theta}{2}} \phi_R e^{\frac{3\theta}{2}i\sigma_3} \\ &= e^{i\frac{5\theta}{2}} \mathbf{b} \phi_L e^{3i\theta\sigma_3} \sigma_1 e^{\frac{3\theta}{2}i\sigma_3} + e^{-i\frac{\theta}{2}} \mathbf{c}^* \phi_R e^{\frac{3\theta}{2}i\sigma_3} \\ &= (e^{i\frac{5\theta}{2}} \mathbf{b} \phi_L e^{-3i\theta} \sigma_1 + e^{-i\frac{\theta}{2}} \mathbf{c}^* \phi_R) e^{\frac{3\theta}{2}i\sigma_3} \\ &= e^{-i\frac{\theta}{2}} (\mathbf{b} \phi_L \sigma_1 + \mathbf{c}^* \phi_R) e^{\frac{3\theta}{2}i\sigma_3}. \end{aligned} \quad (\text{B.121})$$

This gives

$$\chi' = \frac{1}{\boldsymbol{\rho}'^2} \begin{pmatrix} e^{-i\frac{\theta}{2}} (\mathbf{a}^* \phi_e - \mathbf{b} \phi_n \sigma_1 + \mathbf{c}^* \phi_n) e^{\frac{3\theta}{2}i\sigma_3} & e^{-i\frac{\theta}{2}} (\mathbf{b} \phi_L \sigma_1 + \mathbf{c}^* \phi_R) e^{\frac{3\theta}{2}i\sigma_3} \\ e^{i\frac{\theta}{2}} (-\mathbf{b}'^* \widehat{\phi}'_L \sigma_1 + \mathbf{c}' \widehat{\phi}'_R) e^{\frac{3\theta}{2}i\sigma_3} & e^{i\frac{\theta}{2}} (\mathbf{a}' \widehat{\phi}'_e + \mathbf{b}'^* \widehat{\phi}'_n \sigma_1 + \mathbf{c}' \widehat{\phi}'_n) e^{\frac{3\theta}{2}i\sigma_3} \end{pmatrix} \quad (\text{B.122})$$

which reads

$$\chi' = e^{-\frac{\theta}{2}} \mathbf{i} \chi e^{\frac{3\theta}{2} \gamma_{21}} \quad (\text{B.123})$$

and we finally get

$$\mathbf{D}'\Psi'\gamma_{012} + m\rho'\chi' = e^{-\frac{\theta}{2}\mathbf{i}}(\mathbf{D}\Psi\gamma_{012} + m\rho\chi)e^{\frac{3\theta}{2}\gamma_{21}} = 0. \quad (\text{B.124})$$

This proves that the wave equation for the pair electron-neutrino is gauge invariant under the gauge group generated by P_0 .

B.3.3 Gauge invariance - group generated by P_3

This generator acts only on left waves: we have

$$\xi'_1 = \xi_1 ; \quad \xi'_2 = \xi_2. \quad (\text{B.125})$$

And with left waves we get

$$\Psi'_L = \Psi_L e^{-\theta\mathbf{i}} \quad (\text{B.126})$$

$$\begin{pmatrix} \phi'_L & \phi'_n \\ \widehat{\phi}'_n & \widehat{\phi}'_L \end{pmatrix} = \begin{pmatrix} \phi_L & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_L \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \quad (\text{B.127})$$

$$\phi'_L = e^{-i\theta}\phi_L \quad (\text{B.128})$$

$$\phi'_n = e^{i\theta}\phi_n \quad (\text{B.129})$$

which reads

$$\eta'_1 = e^{i\theta}\eta_1 ; \quad \eta'_2 = e^{i\theta}\eta_2 ; \quad \zeta'_1 = e^{-i\theta}\zeta_1 ; \quad \zeta'_2 = e^{-i\theta}\zeta_2. \quad (\text{B.130})$$

We then get for \mathbf{a} , \mathbf{b} , \mathbf{c} :

$$\mathbf{a}' = e^{-i\theta}\mathbf{a} ; \quad \mathbf{a}'\mathbf{a}'^* = \mathbf{a}\mathbf{a}^* \quad (\text{B.131})$$

$$\mathbf{b}' = \mathbf{b} ; \quad \mathbf{b}'\mathbf{b}'^* = \mathbf{b}\mathbf{b}^* \quad (\text{B.132})$$

$$\mathbf{c}' = e^{i\theta}\mathbf{c} ; \quad \mathbf{c}'\mathbf{c}'^* = \mathbf{c}\mathbf{c}^* \quad (\text{B.133})$$

$$\rho' = \rho. \quad (\text{B.134})$$

The covariant derivative is here reduced to

$$\mathbf{D} = \boldsymbol{\partial} + \frac{g_2}{2}\mathbf{W}^3 P_3. \quad (\text{B.135})$$

We let

$$[\boldsymbol{\partial}\Psi + \frac{g_2}{2}\mathbf{W}^3 P_3(\Psi)]\gamma_{012} + m\rho\chi = \begin{pmatrix} A & B \\ \widehat{B} & \widehat{A} \end{pmatrix}. \quad (\text{B.136})$$

We have

$$A = (\nabla\widehat{\phi}_e + i\frac{g_2}{2}W^3\widehat{\phi}_L)\sigma_{21} + \frac{m}{\rho}(\mathbf{a}^*\phi_e - \mathbf{b}\phi_n\sigma_1 + \mathbf{c}^*\phi_n) \quad (\text{B.137})$$

$$B = (\nabla\widehat{\phi}_n - i\frac{g_2}{2}W^3\widehat{\phi}_n)\sigma_{21} + \frac{m}{\rho}(\mathbf{b}\phi_L\sigma_1 + \mathbf{c}^*\phi_R). \quad (\text{B.138})$$

Only the left column of B is not zero, this gives the simple result:

$$B' = e^{-i\theta} B. \quad (\text{B.139})$$

For A which has a right column and a left column, we note

$$A = A_L + A_R \quad (\text{B.140})$$

$$A_L = (\nabla \widehat{\phi}_L + i \frac{g_2}{2} W^3 \widehat{\phi}_L) \sigma_{21} + \frac{m}{\rho} (\mathbf{a}^* \phi_R - \mathbf{b} \phi_n \sigma_1) \quad (\text{B.141})$$

$$A_R = \nabla \widehat{\phi}_R \sigma_{21} + \frac{m}{\rho} (\mathbf{a}^* \phi_L + \mathbf{c}^* \phi_n). \quad (\text{B.142})$$

We then get

$$A'_L = e^{i\theta} A_L ; \quad A'_R = A_R \quad (\text{B.143})$$

$$A' = A \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{B.144})$$

Since the same matrix multiplies the differential term and the mass term of the equation, we have proved that the equation is invariant under the gauge group generated by P_3 .

B.3.4 Gauge invariance - group generated by P_1

This generator acts also only on left waves, we have

$$\xi'_1 = \xi_1 ; \quad \xi'_2 = \xi_2. \quad (\text{B.145})$$

And with these left waves we have

$$\Psi'_L = \begin{pmatrix} \phi'_L & \phi'_n \\ \widehat{\phi}'_L & \widehat{\phi}'_n \end{pmatrix} = \Psi_L e^{\theta \gamma_3 \mathbf{i}} = \begin{pmatrix} \phi_L & \phi_n \\ \widehat{\phi}_L & \widehat{\phi}_n \end{pmatrix} \begin{pmatrix} \cos(\theta) & -i \sin(\theta) \sigma_3 \\ -i \sin(\theta) \sigma_3 & \cos(\theta) \end{pmatrix} \quad (\text{B.146})$$

$$\phi'_L = \cos(\theta) \phi_L - i \sin(\theta) \phi_n \sigma_3 ; \quad \phi'_n = \cos(\theta) \phi_n - i \sin(\theta) \phi_L \sigma_3 \quad (\text{B.147})$$

which reads with $C = \cos(\theta)$ et $S = \sin(\theta)$:

$$\phi'_L = C \phi_L + i S \phi_n ; \quad \widehat{\phi}'_L = C \widehat{\phi}_L - i S \widehat{\phi}_n \quad (\text{B.148})$$

$$\phi'_n = C \phi_n + i S \phi_L ; \quad \widehat{\phi}'_n = C \widehat{\phi}_n - i S \widehat{\phi}_L \quad (\text{B.149})$$

$$\eta'_1 = C \eta_1 - i S \zeta_1 ; \quad \eta'_2 = C \eta_1 - i S \zeta_2 \quad (\text{B.150})$$

$$\zeta'_1 = C \zeta_1 - i S \eta_1 ; \quad \zeta'_2 = C \zeta_2 - i S \eta_2. \quad (\text{B.151})$$

We then get for \mathbf{a} , \mathbf{b} , \mathbf{c} :

$$\mathbf{a}' = C \mathbf{a} + i S \mathbf{c} \quad (\text{B.152})$$

$$\mathbf{b}' = \mathbf{b} \quad (\text{B.153})$$

$$\mathbf{c}' = C \mathbf{c} + i S \mathbf{a} \quad (\text{B.154})$$

$$\begin{aligned} \rho'^2 &= \mathbf{a}' \mathbf{a}'^* + \mathbf{b}' \mathbf{b}'^* + \mathbf{c}' \mathbf{c}'^* = \mathbf{a} \mathbf{a}^* + \mathbf{b} \mathbf{b}^* + \mathbf{c} \mathbf{c}^* \\ &= \rho^2. \end{aligned} \quad (\text{B.155})$$

The covariant derivative is now reduced to

$$\mathbf{D} = \boldsymbol{\partial} + \frac{g_2}{2} \mathbf{W}^1 P_1. \quad (\text{B.156})$$

We let

$$[\boldsymbol{\partial}\Psi + \frac{g_2}{2} \mathbf{W}^1 P_1(\Psi)]\gamma_{012} + m\rho\chi = \begin{pmatrix} A & B \\ \widehat{B} & \widehat{A} \end{pmatrix}. \quad (\text{B.157})$$

We have

$$A = \nabla\widehat{\phi}_e\sigma_{21} - \frac{g_2}{2} W^1\widehat{\phi}_n + \frac{m}{\boldsymbol{\rho}}(\mathbf{a}^*\phi_e - \mathbf{b}\phi_n\sigma_1 + \mathbf{c}^*\phi_n) \quad (\text{B.158})$$

$$B = \nabla\widehat{\phi}_n\sigma_{21} - \frac{g_2}{2} W^1\widehat{\phi}_L + \frac{m}{\boldsymbol{\rho}}(\mathbf{b}\phi_L\sigma_1 + \mathbf{c}^*\phi_R). \quad (\text{B.159})$$

As previously for A which has a left column and a right column we note:

$$A = A_L + A_R \quad (\text{B.160})$$

$$A_L = (\nabla\widehat{\phi}_L + i\frac{g_2}{2} W^1\widehat{\phi}_L)\sigma_{21} + \frac{m}{\boldsymbol{\rho}}(\mathbf{a}^*\phi_R - \mathbf{b}\phi_n\sigma_1) \quad (\text{B.161})$$

$$A_R = \nabla\widehat{\phi}_R\sigma_{21} + \frac{m}{\boldsymbol{\rho}}(\mathbf{a}^*\phi_L + \mathbf{c}^*\phi_n). \quad (\text{B.162})$$

We then get

$$A'_L = CA_L - iSB \quad (\text{B.163})$$

$$A'_R = A_R \quad (\text{B.164})$$

$$B' = CB - iSA_L. \quad (\text{B.165})$$

Since the mass term changes exactly in the same way as the derivative term we have proved that the wave equation is gauge invariant under the group generated by P_1 . Now it is not necessary to study the group generated by P_2 , because $P_2 = P_3P_1$. We have then proved both the form invariance and the gauge invariance of the equation (B.90) under the Lie group $U(1) \times SU(2)$ generated by P_0, P_1, P_2, P_3 .

C The hydrogen atom

We study the resolution of the homogeneous nonlinear equation for the hydrogen atom. Our resolution uses a method separating the variables in spherical coordinates. The solutions are very near particular solutions of the Dirac equation which are not the usual ones, and which have a Yvon-Takabayasi angle everywhere defined and small.

The hydrogen atom is the jewel of the Dirac theory. The solutions calculated by C. G. Darwin [6], that we may find into newer reports [49], are proper values of an ad hoc operator, coming from the non-relativistic theory, operator that is not the total angular momentum operator. These solutions have as

only physical explanation that they give the expected number of states, the true formula for the energy levels, and to have the expected non-relativistic approximations. This was considered very satisfying. The Darwin's solutions suffer the disadvantage, for most of them, to have a Yvon-Takabayasi angle that is not everywhere defined and small. Therefore they cannot be linear approximations of the solutions for our homogeneous nonlinear equation.

We got previously [9] other solutions of the Dirac equation, that have a Yvon-Takabayasi angle everywhere defined and small, and so that may be the linear approximations of the solutions for our nonlinear equation.

C.1 Separating variables

To solve the Dirac equation (2.42) or the homogeneous nonlinear equation (3.10), in the case of the hydrogen atom, two methods exist. We shall use here, not the initial method based on the non-relativistic wave equations, but the new method invented more recently by H. Krüger [37], classical method on the mathematical point of view for an equation with partial derivatives, separating the variables in spherical coordinates:

$$x^1 = r \sin \theta \cos \varphi \ ; \ x^2 = r \sin \theta \sin \varphi \ ; \ x^3 = r \cos \theta \quad (\text{C.1})$$

We use the following notations⁶⁴:

$$i_1 = \sigma_{23} = i\sigma_1 \ ; \ i_2 = \sigma_{31} = i\sigma_2 \ ; \ i_3 = \sigma_{12} = i\sigma_3 \quad (\text{C.2})$$

$$S = e^{-\frac{\varphi}{2}i_3} e^{-\frac{\theta}{2}i_2} \ ; \ \Omega = r^{-1}(\sin \theta)^{-\frac{1}{2}}S \quad (\text{C.3})$$

$$\vec{\partial}' = \sigma_3 \partial_r + \frac{1}{r} \sigma_1 \partial_\theta + \frac{1}{r \sin \theta} \sigma_2 \partial_\varphi. \quad (\text{C.4})$$

H. Krüger got the remarkable identity:

$$\vec{\partial} = \Omega \vec{\partial}' \Omega^{-1} \quad (\text{C.5})$$

that, with:

$$\nabla' = \partial_0 - \vec{\partial}' = \partial_0 - (\sigma_3 \partial_r + \frac{1}{r} \sigma_1 \partial_\theta + \frac{1}{r \sin \theta} \sigma_2 \partial_\varphi) \quad (\text{C.6})$$

also gives

$$\Omega^{-1} \nabla = \nabla' \Omega^{-1}. \quad (\text{C.7})$$

Into the wave equations (2.42) or (3.10), to separate the temporal variable $x^0 = ct$ and the angular variable φ from the radial variable r and the angular variable θ we let:

$$\phi = \Omega X e^{(\lambda\varphi - Ex^0 + \delta)i_3} \quad (\text{C.8})$$

⁶⁴ S has nothing to do with the tensor S_3 and Ω must not be confused with the relativistic invariants Ω_1 and Ω_2 studied in section 2.

where X is a function, with value into the Pauli algebra, of r and θ alone, $\hbar c E$ is the electron's energy, δ is an arbitrary phase which plays no role as the equations (2.42) and (3.10) are electric gauge invariant. λ is a real constant. We get then

$$\Omega^{-1}\phi = X e^{(\lambda\varphi - Ex^0 + \delta)i_3} \quad (\text{C.9})$$

$$\Omega^{-1}\widehat{\phi} = \widehat{X} e^{(\lambda\varphi - Ex^0 + \delta)i_3}. \quad (\text{C.10})$$

We also have:

$$\begin{aligned} \rho e^{i\beta} &= \det(\phi) = \det(\Omega) \det(X) \det[e^{(\lambda\varphi - Ex^0 + \delta)i_3}] \\ \det(\Omega) &= r^{-2}(\sin\theta)^{-1}; \quad \det[e^{(\lambda\varphi - Ex^0 + \delta)i_3}] = 1 \\ \rho e^{i\beta} &= \frac{\det(X)}{r^2 \sin\theta}. \end{aligned} \quad (\text{C.11})$$

So, if we let:

$$\rho_X e^{i\beta_X} = \det(X) \quad (\text{C.12})$$

we get:

$$\rho = \frac{\rho_X}{r^2 \sin\theta}; \quad \beta = \beta_X. \quad (\text{C.13})$$

Thus with (C.8) for the wave, the Yvon-Takabayasi angle does not depend on time nor on the φ angle, only on r and θ . It is why the separation of variables, in the linear case or in the nonlinear case, may begin in the same way. We have

$$\nabla' \Omega^{-1} \widehat{\phi} = (\partial_0 - \sigma_3 \partial_r - \frac{1}{r} \sigma_1 \partial_\theta - \frac{1}{r \sin\theta} \sigma_2 \partial_\varphi) [\widehat{X} e^{(\lambda\varphi - Ex^0 + \delta)i_3}] \quad (\text{C.14})$$

$$\partial_0(\widehat{X} e^{(\lambda\varphi - Ex^0 + \delta)i_3}) = -E \widehat{X} i_3 e^{(\lambda\varphi - Ex^0 + \delta)i_3} \quad (\text{C.15})$$

$$\partial_r(\widehat{X} e^{(\lambda\varphi - Ex^0 + \delta)i_3}) = (\partial_r \widehat{X}) e^{(\lambda\varphi - Ex^0 + \delta)i_3} \quad (\text{C.16})$$

$$\partial_\theta(\widehat{X} e^{(\lambda\varphi - Ex^0 + \delta)i_3}) = (\partial_\theta \widehat{X}) e^{(\lambda\varphi - Ex^0 + \delta)i_3} \quad (\text{C.17})$$

$$\partial_\varphi(\widehat{X} e^{(\lambda\varphi - Ex^0 + \delta)i_3}) = \lambda \widehat{X} i_3 e^{(\lambda\varphi - Ex^0 + \delta)i_3}. \quad (\text{C.18})$$

We get then:

$$\nabla \widehat{\phi} = \Omega(-E \widehat{X} i_3 - \sigma_3 \partial_r \widehat{X} - \frac{1}{r} \sigma_1 \partial_\theta \widehat{X} - \frac{\lambda}{r \sin\theta} \sigma_2 \widehat{X} i_3) e^{(\lambda\varphi - Ex^0 + \delta)i_3}. \quad (\text{C.19})$$

For the hydrogen atom, we have:

$$qA = qA^0 = -\frac{\alpha}{r}; \quad \alpha = \frac{e^2}{\hbar c} \quad (\text{C.20})$$

where α is the fine structure constant. We have:

$$\begin{aligned} qA \widehat{\phi} \sigma_{12} &= -\frac{\alpha}{r} \widehat{\phi} i_3 = -\frac{\alpha}{r} \Omega \widehat{X} e^{(\lambda\varphi - Ex^0 + \delta)i_3} i_3 \\ &= \Omega(-\frac{\alpha}{r} \widehat{X} i_3) e^{(\lambda\varphi - Ex^0 + \delta)i_3}. \end{aligned} \quad (\text{C.21})$$

So the homogeneous nonlinear equation (3.10) becomes

$$-E\widehat{X}i_3 - \sigma_3\partial_r\widehat{X} - \frac{1}{r}\sigma_1\partial_\theta\widehat{X} - \frac{\lambda}{r\sin\theta}\sigma_2\widehat{X}i_3 - \frac{\alpha}{r}\widehat{X}i_3 + me^{-i\beta}Xi_3 = 0 \quad (\text{C.22})$$

that is to say:

$$(E + \frac{\alpha}{r})\widehat{X}i_3 + \sigma_3\partial_r\widehat{X} + \frac{1}{r}\sigma_1\partial_\theta\widehat{X} + \frac{\lambda}{r\sin\theta}\sigma_2\widehat{X}i_3 = me^{-i\beta}Xi_3 \quad (\text{C.23})$$

while the Dirac equation gives:

$$(E + \frac{\alpha}{r})\widehat{X}i_3 + \sigma_3\partial_r\widehat{X} + \frac{1}{r}\sigma_1\partial_\theta\widehat{X} + \frac{\lambda}{r\sin\theta}\sigma_2\widehat{X}i_3 = mXi_3. \quad (\text{C.24})$$

We let now:

$$X = \begin{pmatrix} \mathbf{a} & -\mathbf{b}^* \\ \mathbf{c} & \mathbf{d}^* \end{pmatrix} \quad (\text{C.25})$$

where \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} are functions with complex value of the real variables r and θ . We get then:

$$\widehat{X} = \begin{pmatrix} \mathbf{d} & -\mathbf{c}^* \\ \mathbf{b} & \mathbf{a}^* \end{pmatrix} \quad (\text{C.26})$$

We get then:

$$e^{-i\beta}Xi_3 = ie^{-i\beta}X\sigma_3 = ie^{-i\beta} \begin{pmatrix} \mathbf{a} & \mathbf{b}^* \\ \mathbf{c} & -\mathbf{d}^* \end{pmatrix} \quad (\text{C.27})$$

$$\widehat{X}i_3 = \begin{pmatrix} \mathbf{d} & -\mathbf{c}^* \\ \mathbf{b} & \mathbf{a}^* \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} i\mathbf{d} & i\mathbf{c}^* \\ i\mathbf{b} & -i\mathbf{a}^* \end{pmatrix} \quad (\text{C.28})$$

$$\sigma_3\partial_r\widehat{X} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial_r\mathbf{d} & -\partial_r\mathbf{c}^* \\ \partial_r\mathbf{b} & \partial_r\mathbf{a}^* \end{pmatrix} = \begin{pmatrix} \partial_r\mathbf{d} & -\partial_r\mathbf{c}^* \\ -\partial_r\mathbf{b} & -\partial_r\mathbf{a}^* \end{pmatrix} \quad (\text{C.29})$$

$$\sigma_1\partial_\theta\widehat{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_\theta\mathbf{d} & -\partial_\theta\mathbf{c}^* \\ \partial_\theta\mathbf{b} & \partial_\theta\mathbf{a}^* \end{pmatrix} = \begin{pmatrix} \partial_\theta\mathbf{b} & \partial_\theta\mathbf{a}^* \\ \partial_\theta\mathbf{d} & -\partial_\theta\mathbf{c}^* \end{pmatrix} \quad (\text{C.30})$$

$$\sigma_2\widehat{X}i_3 = i_2\widehat{X}\sigma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{d} & -\mathbf{c}^* \\ \mathbf{b} & \mathbf{a}^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \mathbf{b} & -\mathbf{a}^* \\ -\mathbf{d} & -\mathbf{c}^* \end{pmatrix}. \quad (\text{C.31})$$

Consequently the nonlinear equation (3.10) becomes:

$$\begin{aligned} & (E + \frac{\alpha}{r}) \begin{pmatrix} i\mathbf{d} & i\mathbf{c}^* \\ i\mathbf{b} & -i\mathbf{a}^* \end{pmatrix} + \begin{pmatrix} \partial_r\mathbf{d} & -\partial_r\mathbf{c}^* \\ -\partial_r\mathbf{b} & -\partial_r\mathbf{a}^* \end{pmatrix} + \frac{1}{r} \begin{pmatrix} \partial_\theta\mathbf{b} & \partial_\theta\mathbf{a}^* \\ \partial_\theta\mathbf{d} & -\partial_\theta\mathbf{c}^* \end{pmatrix} \\ & + \frac{\lambda}{r\sin\theta} \begin{pmatrix} \mathbf{b} & -\mathbf{a}^* \\ -\mathbf{d} & -\mathbf{c}^* \end{pmatrix} = ime^{-i\beta} \begin{pmatrix} \mathbf{a} & \mathbf{b}^* \\ \mathbf{c} & -\mathbf{d}^* \end{pmatrix}. \end{aligned} \quad (\text{C.32})$$

Conjugating equations with $*$, we get the system:

$$\begin{aligned} & i(E + \frac{\alpha}{r})\mathbf{d} + \partial_r\mathbf{d} + \frac{1}{r}(\partial_\theta + \frac{\lambda}{\sin\theta})\mathbf{b} = ime^{-i\beta}\mathbf{a} \\ & -i(E + \frac{\alpha}{r})\mathbf{c} - \partial_r\mathbf{c} + \frac{1}{r}(\partial_\theta - \frac{\lambda}{\sin\theta})\mathbf{a} = -ime^{i\beta}\mathbf{b} \\ & i(E + \frac{\alpha}{r})\mathbf{b} - \partial_r\mathbf{b} + \frac{1}{r}(\partial_\theta - \frac{\lambda}{\sin\theta})\mathbf{d} = ime^{-i\beta}\mathbf{c} \\ & -i(E + \frac{\alpha}{r})\mathbf{a} + \partial_r\mathbf{a} + \frac{1}{r}(\partial_\theta + \frac{\lambda}{\sin\theta})\mathbf{c} = -ime^{i\beta}\mathbf{d}. \end{aligned} \quad (\text{C.33})$$

In addition we have:

$$\rho e^{i\beta} = \det(\phi) = \frac{\det(X)}{r^2 \sin \theta} = \frac{\mathbf{ad}^* + \mathbf{cb}^*}{r^2 \sin \theta} \quad (\text{C.34})$$

so we get:

$$e^{i\beta} = \frac{\mathbf{ad}^* + \mathbf{cb}^*}{|\mathbf{ad}^* + \mathbf{cb}^*|} \quad (\text{C.35})$$

For the four equations (C.33) there are only two angular operators, so we let:

$$\mathbf{a} = AU ; \quad \mathbf{b} = BV ; \quad \mathbf{c} = CV ; \quad \mathbf{d} = DU \quad (\text{C.36})$$

where A, B, C and D are functions of r whilst U and V are functions of θ . The system (C.33) becomes:

$$\begin{aligned} i(E + \frac{\alpha}{r})DU + D'U + \frac{1}{r}(V' + \frac{\lambda}{\sin \theta}V)B &= ime^{-i\beta}AU \\ -i(E + \frac{\alpha}{r})CV - C'V + \frac{1}{r}(U' - \frac{\lambda}{\sin \theta}U)A &= -ime^{i\beta}BV \\ i(E + \frac{\alpha}{r})BV - B'V + \frac{1}{r}(U' - \frac{\lambda}{\sin \theta}U)D &= ime^{-i\beta}CV \\ -i(E + \frac{\alpha}{r})AU + A'U + \frac{1}{r}(V' + \frac{\lambda}{\sin \theta}V)C &= -ime^{i\beta}DU. \end{aligned} \quad (\text{C.37})$$

So if a κ constant exists such as:

$$U' - \frac{\lambda}{\sin \theta}U = -\kappa V ; \quad V' + \frac{\lambda}{\sin \theta}V = \kappa U \quad (\text{C.38})$$

the system (C.37) becomes:

$$\begin{aligned} i(E + \frac{\alpha}{r})D + D' + \frac{\kappa}{r}B &= ime^{-i\beta}A \\ -i(E + \frac{\alpha}{r})C - C' - \frac{\kappa}{r}A &= -ime^{i\beta}B \\ i(E + \frac{\alpha}{r})B - B' - \frac{\kappa}{r}D &= ime^{-i\beta}C \\ -i(E + \frac{\alpha}{r})A + A' + \frac{\kappa}{r}C &= -ime^{i\beta}D \end{aligned} \quad (\text{C.39})$$

To get the system equivalent to the Dirac equation, from the same process, it is enough to replace β by 0, this does not change the angular system (C.38), while in the place of (C.39) we get the system:

$$\begin{aligned} i(E + \frac{\alpha}{r})D + D' + \frac{\kappa}{r}B &= imA \\ -i(E + \frac{\alpha}{r})C - C' - \frac{\kappa}{r}A &= -imB \\ i(E + \frac{\alpha}{r})B - B' - \frac{\kappa}{r}D &= imC \\ -i(E + \frac{\alpha}{r})A + A' + \frac{\kappa}{r}C &= -imD \end{aligned} \quad (\text{C.40})$$

C.2 Angular momentum operators

We established in [9] the form that, in space-time algebra, the angular momentum operators take. With the Pauli algebra, we have (a detailed calculation is in [16] A.3):

$$J_1\phi = (d_1 + \frac{1}{2}\sigma_{23})\phi\sigma_{21}; \quad d_1 = x^2\partial_3 - x^3\partial_2 = -\sin\varphi\partial_\theta - \frac{\cos\varphi}{\tan\theta}\partial_\varphi \quad (\text{C.41})$$

$$J_2\phi = (d_2 + \frac{1}{2}\sigma_{31})\phi\sigma_{21}; \quad d_2 = x^3\partial_1 - x^1\partial_3 = \cos\varphi\partial_\theta - \frac{\sin\varphi}{\tan\theta}\partial_\varphi \quad (\text{C.42})$$

$$J_3\phi = (d_3 + \frac{1}{2}\sigma_{12})\phi\sigma_{21}; \quad d_3 = x^1\partial_2 - x^2\partial_1 = \partial_\varphi. \quad (\text{C.43})$$

Of course we also have

$$J^2 = J_1^2 + J_2^2 + J_3^2. \quad (\text{C.44})$$

We get then

$$J_3\phi = \lambda\phi \iff \phi = \phi(x^0, r, \theta)e^{\lambda\varphi i_3}. \quad (\text{C.45})$$

So the wave ϕ satisfying (C.8) is a proper vector of J_3 and λ is the magnetic quantum number. More, always for a ϕ wave satisfying (C.8), we have:

$$J^2\phi = j(j+1)\phi \quad (\text{C.46})$$

if and only if

$$\partial_{\theta\theta}^2 X + [(j + \frac{1}{2})^2 - \frac{\lambda^2}{\sin^2\theta}]X - \lambda \frac{\cos\theta}{\sin^2\theta} \sigma_{12} X \sigma_{12} = 0. \quad (\text{C.47})$$

But (C.38) implies at the second order

$$0 = U'' + (\kappa^2 - \frac{\lambda^2}{\sin^2\theta})U + \lambda \frac{\cos\theta}{\sin^2\theta} U \quad (\text{C.48})$$

$$0 = V'' + (\kappa^2 - \frac{\lambda^2}{\sin^2\theta})V - \lambda \frac{\cos\theta}{\sin^2\theta} V \quad (\text{C.49})$$

$$0 = \partial_{\theta\theta}^2 X + (\kappa^2 - \frac{\lambda^2}{\sin^2\theta})X - \lambda \frac{\cos\theta}{\sin^2\theta} \sigma_{12} X \sigma_{12} \quad (\text{C.50})$$

Consequently ϕ is a proper vector of J^2 , with the proper value $j(j+1)$, if and only if

$$\kappa^2 = (j + \frac{1}{2})^2; \quad |\kappa| = j + \frac{1}{2}; \quad j = |\kappa| - \frac{1}{2}. \quad (\text{C.51})$$

With (C.3) and (C.8) we can see that the change of φ into $\varphi + 2\pi$ conserves the value of the wave if and only if λ has a half-odd value. General results on angular momentum operators imply then:

$$j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots; \quad \kappa = \pm 1, \pm 2, \pm 3, \dots; \quad \lambda = -j, -j+1, \dots, j-1, j. \quad (\text{C.52})$$

To solve the angular system, if $\lambda > 0$ we let, with $C = C(\theta)$:

$$\begin{aligned} U &= \sin^\lambda \theta [\sin(\frac{\theta}{2})C' - (\kappa + \frac{1}{2} - \lambda) \cos(\frac{\theta}{2})C] \\ V &= \sin^\lambda \theta [\cos(\frac{\theta}{2})C' + (\kappa + \frac{1}{2} - \lambda) \sin(\frac{\theta}{2})C]. \end{aligned} \quad (\text{C.53})$$

If $\lambda < 0$ we let:

$$\begin{aligned} U &= \sin^{-\lambda} \theta [\cos(\frac{\theta}{2})C' + (\kappa + \frac{1}{2} + \lambda) \sin(\frac{\theta}{2})C] \\ V &= \sin^{-\lambda} \theta [-\sin(\frac{\theta}{2})C' + (\kappa + \frac{1}{2} + \lambda) \cos(\frac{\theta}{2})C] \end{aligned} \quad (\text{C.54})$$

The angular system (C.38) is then equivalent [7] to the differential equation :

$$0 = C'' + \frac{2|\lambda|}{\tan \theta} C' + [(\kappa + \frac{1}{2})^2 - \lambda^2] C \quad (\text{C.55})$$

The change of variable:

$$z = \cos \theta ; \quad f(z) = C[\theta(z)] \quad (\text{C.56})$$

gives then the differential equation of the Gegenbauer's polynomials:⁶⁵

$$0 = f''(z) - \frac{1+2|\lambda|}{1-z^2} z f'(z) + \frac{(\kappa + \frac{1}{2})^2 - \lambda^2}{1-z^2} f(z). \quad (\text{C.57})$$

And we get, as only integrable solution:

$$\frac{C(\theta)}{C(0)} = \sum_{n=0}^{\infty} \frac{(|\lambda| - \kappa - \frac{1}{2})_n (|\lambda| + \kappa + \frac{1}{2})_n}{(\frac{1}{2} + |\lambda|)_n n!} \sin^{2n}(\frac{\theta}{2}) \quad (\text{C.58})$$

with:

$$(a)_0 = 1, \quad (a)_n = a(a+1) \dots (a+n-1). \quad (\text{C.59})$$

The $C(0)$ factor is a factor of U and V , its phase may be absorbed by the δ in (C.8), and its amplitude may be transferred on the radial functions. We can take therefore $C(0) = 1$, this gives:

$$C(\theta) = \sum_{n=0}^{\infty} \frac{(|\lambda| - \kappa - \frac{1}{2})_n (|\lambda| + \kappa + \frac{1}{2})_n}{(\frac{1}{2} + |\lambda|)_n n!} \sin^{2n}(\frac{\theta}{2}). \quad (\text{C.60})$$

Since we have the conditions (C.52) on λ and κ , an integer n always exists such as

$$|\lambda| + n = |\kappa + \frac{1}{2}| \quad (\text{C.61})$$

⁶⁵When we solve the Dirac equation with the Darwin's method, that is to say with the ad-hoc operators, we get Legendre's polynomials and spherical harmonics. Here, working with ϕ , that is to say with the Weyl spinors ξ and η , we get the Gegenbauer's polynomials, and it is the degree of the Gegenbauer's polynomial which is the needed quantum number.

and this forces the (C.60) series to be a finite sum, so U and V are integrable. And since U and V have real values, we have:

$$e^{i\beta} = \frac{AD^*U^2 + CB^*V^2}{|AD^*U^2 + CB^*V^2|} \quad (\text{C.62})$$

C.3 Resolution of the linear radial system

We make the change of radial variable:

$$\begin{aligned} x = mr ; \quad \epsilon = \frac{E}{m} ; \quad a(x) = A(r) = A\left(\frac{x}{m}\right) \\ b(x) = B(r) ; \quad c(x) = C(r) ; \quad d(x) = D(r). \end{aligned} \quad (\text{C.63})$$

The radial system (C.40) becomes:

$$\begin{aligned} i\left(\epsilon + \frac{\alpha}{x}\right)d + d' + \frac{\kappa}{x}b &= ia \\ -i\left(\epsilon + \frac{\alpha}{x}\right)c - c' - \frac{\kappa}{x}a &= -ib \\ i\left(\epsilon + \frac{\alpha}{x}\right)b - b' - \frac{\kappa}{x}d &= ic \\ -i\left(\epsilon + \frac{\alpha}{x}\right)a + a' + \frac{\kappa}{x}c &= -id. \end{aligned} \quad (\text{C.64})$$

Adding and subtracting, we get:

$$\begin{aligned} i\left(\epsilon + \frac{\alpha}{x}\right)(d - c) + (d - c)' - \frac{\kappa}{x}(a - b) &= i(a - b) \\ -i\left(\epsilon + \frac{\alpha}{x}\right)(a - b) + (a - b)' - \frac{\kappa}{x}(d - c) &= -i(d - c) \\ i\left(\epsilon + \frac{\alpha}{x}\right)(c + d) + (c + d)' + \frac{\kappa}{x}(a + b) &= i(a + b) \\ i\left(\epsilon + \frac{\alpha}{x}\right)(a + b) - (a + b)' - \frac{\kappa}{x}(c + d) &= i(c + d). \end{aligned} \quad (\text{C.65})$$

We let then:

$$\begin{aligned} a - b = F_- + iG_- ; \quad a + b = F_+ + iG_+ \\ d - c = F_- - iG_- ; \quad c + d = F_+ - iG_+ \end{aligned} \quad (\text{C.66})$$

and the radial system becomes:

$$\begin{aligned} i\left(\epsilon + \frac{\alpha}{x}\right)(F_- - iG_-) + (F_- - iG_-)' - \frac{\kappa}{x}(F_- + iG_-) &= i(F_- + iG_-) \\ -i\left(\epsilon + \frac{\alpha}{x}\right)(F_- + iG_-) + (F_- + iG_-)' - \frac{\kappa}{x}(F_- - iG_-) &= -i(F_- - iG_-) \\ i\left(\epsilon + \frac{\alpha}{x}\right)(F_+ - iG_+) + (F_+ - iG_+)' + \frac{\kappa}{x}(F_+ + iG_+) &= i(F_+ + iG_+) \\ i\left(\epsilon + \frac{\alpha}{x}\right)(F_+ + iG_+) - (F_+ + iG_+)' - \frac{\kappa}{x}(F_+ - iG_+) &= i(F_+ - iG_+). \end{aligned} \quad (\text{C.67})$$

Adding and subtracting equations (C.67), then dividing by i these equations where i is a factor, we get the two separated systems:

$$\begin{aligned} (-1 + \epsilon + \frac{\alpha}{x})F_- - G'_- - \frac{\kappa}{x}G_- &= 0 \\ (1 + \epsilon + \frac{\alpha}{x})G_- + F'_- - \frac{\kappa}{x}F_- &= 0 \end{aligned} \quad (C.68)$$

$$\begin{aligned} (-1 + \epsilon + \frac{\alpha}{x})F_+ - G'_+ + \frac{\kappa}{x}G_+ &= 0 \\ (1 + \epsilon + \frac{\alpha}{x})G_+ + F'_+ + \frac{\kappa}{x}F_+ &= 0. \end{aligned} \quad (C.69)$$

These two systems are exchanged by replacing $-$ indexes by $+$ indexes and vice versa, and by changing κ to $-\kappa$, so it is enough to study one of the two systems. We let now:

$$\begin{aligned} F_- &= \sqrt{1 + \epsilon} e^{-\Lambda x} (\varphi_1 + \varphi_2) ; \quad \Lambda = \sqrt{1 - \epsilon^2} \\ G_- &= \sqrt{1 - \epsilon} e^{-\Lambda x} (\varphi_1 - \varphi_2) \end{aligned} \quad (C.70)$$

Dividing the first of the two equations (C.68) by $\sqrt{1 - \epsilon} e^{-\Lambda x}$ and the second by $\sqrt{1 + \epsilon} e^{-\Lambda x}$, we get:

$$\begin{aligned} -\Lambda(\varphi_1 + \varphi_2) + \frac{\alpha}{x} \sqrt{\frac{1 + \epsilon}{1 - \epsilon}} (\varphi_1 + \varphi_2) + \Lambda(\varphi_1 - \varphi_2) \\ - \varphi'_1 + \varphi'_2 - \frac{\kappa}{x} (\varphi_1 - \varphi_2) &= 0 \\ \Lambda(\varphi_1 - \varphi_2) + \frac{\alpha}{x} \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} (\varphi_1 - \varphi_2) - \Lambda(\varphi_1 + \varphi_2) \\ + \varphi'_1 + \varphi'_2 - \frac{\kappa}{x} (\varphi_1 + \varphi_2) &= 0 \end{aligned} \quad (C.71)$$

But we have:

$$\sqrt{\frac{1 + \epsilon}{1 - \epsilon}} = \frac{1 + \epsilon}{\Lambda} ; \quad \sqrt{\frac{1 - \epsilon}{1 + \epsilon}} = \frac{1 - \epsilon}{\Lambda} \quad (C.72)$$

and we let:

$$c_1 = \frac{\alpha}{\Lambda} ; \quad c_2 = \frac{\alpha \epsilon}{\Lambda}. \quad (C.73)$$

We get then, adding and subtracting the equations (C.71):

$$\begin{aligned} -2\Lambda\varphi_2 + \frac{c_1 - \kappa}{x} \varphi_1 + \frac{c_2}{x} \varphi_2 + \varphi'_2 &= 0 \\ \frac{c_1 + \kappa}{x} \varphi_2 + \frac{c_2}{x} \varphi_1 - \varphi'_1 &= 0. \end{aligned} \quad (C.74)$$

We make then the change of variable:

$$z = 2\Lambda x ; \quad f_1(z) = \varphi_1(x) ; \quad f_2(z) = \varphi_2(x) \quad (C.75)$$

this puts the system (C.74) on the form:

$$\begin{aligned} -f_2 + \frac{c_1 - \kappa}{z} f_1 + \frac{c_2}{z} f_2 + f_2' &= 0 \\ \frac{c_1 + \kappa}{z} f_2 + \frac{c_2}{z} f_1 - f_1' &= 0. \end{aligned} \quad (\text{C.76})$$

We develop now in series:

$$f_1(z) = \sum_{m=0}^{\infty} a_m z^{s+m}; \quad f_2(z) = \sum_{m=0}^{\infty} b_m z^{s+m}. \quad (\text{C.77})$$

The system (C.76) gives, for the coefficients of z^{s-1} :

$$\begin{aligned} (c_1 - \kappa)a_0 + (c_2 + s)b_0 &= 0 \\ (c_2 - s)a_0 + (c_1 + \kappa)b_0 &= 0 \end{aligned} \quad (\text{C.78})$$

A not null solution exists only if the determinant of this system is null:

$$0 = \begin{vmatrix} c_1 - \kappa & c_2 + s \\ c_2 - s & c_1 + \kappa \end{vmatrix} = c_1^2 - \kappa^2 - c_2^2 + s^2 \quad (\text{C.79})$$

But we have, with (C.70) and (C.73):

$$c_1^2 - c_2^2 = \alpha^2 \quad (\text{C.80})$$

So we get:

$$0 = \alpha^2 + s^2 - \kappa^2; \quad s^2 = \kappa^2 - \alpha^2 \quad (\text{C.81})$$

We must take:

$$s = \sqrt{\kappa^2 - \alpha^2} \quad (\text{C.82})$$

to make the wave integrable at the origin. In this case the system (C.78) is reduced to

$$b_0 = \frac{\kappa - c_1}{c_2 + s} a_0 = \frac{s - c_2}{c_1 + \kappa} a_0. \quad (\text{C.83})$$

The system (C.76) gives, for the coefficients of z^{s+m-1} , the system:

$$\begin{aligned} -b_{m-1} + (c_1 - \kappa)a_m + (c_2 + s + m)b_m &= 0 \\ (c_1 + \kappa)b_m + (c_2 - s - m)a_m &= 0 \end{aligned} \quad (\text{C.84})$$

This last equation gives:

$$a_m = \frac{c_1 + \kappa}{-c_2 + s + m} b_m \quad (\text{C.85})$$

and the first one becomes:

$$\begin{aligned} -b_{m-1} + (c_1 - \kappa) \frac{c_1 + \kappa}{-c_2 + s + m} b_m + (c_2 + s + m)b_m &= 0 \\ [(c_1 - \kappa)(c_1 + \kappa) + (s + m)^2 - c_2^2] b_m = (-c_2 + s + m)b_{m-1} \end{aligned} \quad (\text{C.86})$$

which, with (C.79), gives:

$$b_m = \frac{-c_2 + s + m}{(2s + m)m} b_{m-1} = \frac{(-c_2 + s + 1)_m}{(2s + 1)_m m!} b_0. \quad (\text{C.87})$$

And so we have:

$$f_2(z) = b_0 z^s \sum_{m=0}^{\infty} \frac{(-c_2 + s + 1)_m}{(2s + 1)_m m!} z^m = b_0 z^s F(1 + s - c_2, 2s + 1, z) \quad (\text{C.88})$$

where F is the hypergeometric function. We have also:

$$b_m = \frac{-c_2 + s + m}{c_1 + \kappa} a_m ; \quad b_{m-1} = \frac{-c_2 + s + m - 1}{c_1 + \kappa} a_{m-1}. \quad (\text{C.89})$$

The first of the two equations (C.84) becomes:

$$-\frac{-c_2 + s + m - 1}{c_1 + \kappa} a_{m-1} + (c_1 - \kappa) a_m + (c_2 + s + m) \frac{-c_2 + s + m}{c_1 + \kappa} a_m = 0 \quad (\text{C.90})$$

which implies:

$$a_m = \frac{-c_2 + s - 1 + m}{(2s + m)m} a_{m-1} = \frac{(-c_2 + s)_m}{(2s + 1)_m m!} a_0 \quad (\text{C.91})$$

And so we have:

$$f_1(z) = a_0 z^s \sum_{m=0}^{\infty} \frac{(-c_2 + s)_m}{(2s + 1)_m m!} z^m = a_0 z^s F(s - c_2, 2s + 1, z) \quad (\text{C.92})$$

This hypergeometric function is integrable only if the series are polynomial,⁶⁶ (up a coefficient, it is a Laguerre's polynomial) with degree n , that is to say if an integer n exists such as

$$-c_2 + s + n = 0 \quad (\text{C.93})$$

$$s + n = \frac{\epsilon \alpha}{\Lambda} \quad (\text{C.94})$$

which gives, by taking the square ;

$$\begin{aligned} (s + n)^2 (1 - \epsilon^2) &= \epsilon^2 \alpha^2 \\ (s + n)^2 &= [(s + n)^2 + \alpha^2] \epsilon^2 \\ \epsilon^2 &= \frac{1}{1 + \frac{\alpha^2}{(s + n)^2}}. \end{aligned} \quad (\text{C.95})$$

⁶⁶The integrability of the wave functions is not optional, but compulsory, since we have seen in 3.4 that the normalization of the wave comes from the physical fact that the energy of the electron is the energy of its wave.

And we get the Sommerfeld's formula for the energy levels:

$$\epsilon = \frac{1}{\sqrt{1 + \frac{\alpha^2}{(s+n)^2}}}; \quad s = \sqrt{\kappa^2 - \alpha^2}; \quad |\kappa| = j + \frac{1}{2}. \quad (\text{C.96})$$

With (C.75), (C.83), (C.88) and (C.92), we get now:

$$\varphi_1(x) = a_0(2\Lambda x)^s F(-n, 2s+1, 2\Lambda x) \quad (\text{C.97})$$

$$\varphi_2(x) = \frac{-na_0}{c_1 + \kappa} (2\Lambda x)^s F(1-n, 2s+1, 2\Lambda x). \quad (\text{C.98})$$

We let, if $n > 0$:

$$P_1 = F(1-n, 2s+1, 2\Lambda x); \quad P_2 = F(-n, 2s+1, 2\Lambda x). \quad (\text{C.99})$$

And we get:

$$F_- = \frac{\sqrt{1+\epsilon}}{c_1 + \kappa} a_0 e^{-\Lambda x} (2\Lambda x)^s [(c_1 + \kappa)P_2 - nP_1] \quad (\text{C.100})$$

$$G_- = \frac{\sqrt{1-\epsilon}}{c_1 + \kappa} a_0 e^{-\Lambda x} (2\Lambda x)^s [(c_1 + \kappa)P_2 + nP_1]. \quad (\text{C.101})$$

We let then:

$$a_1 = \frac{\sqrt{1+\epsilon}}{c_1 + \kappa} a_0 (2\Lambda)^s. \quad (\text{C.102})$$

We get finally:

$$F_- = a_1 e^{-\Lambda x} x^s [(c_1 + \kappa)P_2 - nP_1] \quad (\text{C.103})$$

$$G_- = \sqrt{\frac{1-\epsilon}{1+\epsilon}} a_1 e^{-\Lambda x} x^s [(c_1 + \kappa)P_2 + nP_1]. \quad (\text{C.104})$$

Since we go from F_- , G_- to F_+ , G_+ by replacing κ by $-\kappa$, we have also:

$$F_+ = a_2 e^{-\Lambda x} x^s [(c_1 - \kappa)P_2 - nP_1] \quad (\text{C.105})$$

$$G_+ = \sqrt{\frac{1-\epsilon}{1+\epsilon}} a_2 e^{-\Lambda x} x^s [(c_1 - \kappa)P_2 + nP_1] \quad (\text{C.106})$$

where a_2 is, as a_1 , a complex constant. We get the same condition on the energy when we say that functions F_+ and G_+ must be polynomials to get integrability of the wave, because (C.96) contains only κ^2 . Therefore if the formula (C.96) is satisfied we get polynomials for the four radial functions and the wave is integrable.

C.4 Calculation of the Yvon-Takabayasi angle

We have with (C.62) and (C.63):

$$e^{i\beta} = \frac{ad^*U^2 + cb^*V^2}{|ad^*U^2 + cb^*V^2|}. \quad (\text{C.107})$$

With (C.66) we get:

$$\begin{aligned} 2a &= F_+ + F_- + i(G_+ + G_-) ; & 2b &= F_+ - F_- + i(G_+ - G_-) \\ 2d &= F_+ + F_- - i(G_+ + G_-) ; & 2c &= F_+ - F_- - i(G_+ - G_-). \end{aligned} \quad (\text{C.108})$$

And we get:

$$\begin{aligned} &4(ad^*U^2 + cb^*V^2) \\ &= (F_+F_+^* + F_-F_-^* - G_+G_+^* - G_-G_-^*)(U^2 + V^2) \\ &+ (F_+F_-^* + F_-F_+^* - G_+G_-^* - G_-G_+^*)(U^2 - V^2) \\ &+ i(F_+G_+^* + F_-G_-^* + G_+F_+^* + G_-F_-^*)(U^2 - V^2) \\ &+ i(F_+G_-^* + F_-G_+^* + G_+F_-^* + G_-F_+^*)(U^2 + V^2) \end{aligned} \quad (\text{C.109})$$

With (C.103) to (C.106), we get then, if $n > 0$:

$$\begin{aligned} &F_+F_+^* + F_-F_-^* - G_+G_+^* - G_-G_-^* \\ &= \frac{2}{1+\epsilon} e^{-2\Lambda x} x^{2s} \left((|a_1|^2 + |a_2|^2)[\epsilon(c_1^2 + \kappa^2)P_2^2 + \epsilon n^2 P_1^2 - 2nc_1 P_1 P_2] \right. \\ &\quad \left. + (|a_2|^2 - |a_1|^2)2\kappa P_2(-\epsilon c_1 P_2 + nP_1) \right) \end{aligned} \quad (\text{C.110})$$

$$\begin{aligned} &F_+F_-^* + F_-F_+^* - G_+G_-^* - G_-G_+^* \\ &= e^{-2\Lambda x} x^{2s} (a_1 a_2^* + a_2 a_1^*) \left(\begin{aligned} &[(c_1 - \kappa)P_2 - nP_1][(c_1 + \kappa)P_2 - nP_1] \\ &- \frac{1-\epsilon}{1+\epsilon} [(c_1 - \kappa)P_2 + nP_1][(c_1 + \kappa)P_2 + nP_1] \end{aligned} \right) \end{aligned} \quad (\text{C.111})$$

$$\begin{aligned} &F_+G_+^* + F_-G_-^* + G_+F_+^* + G_-F_-^* \\ &= 2\sqrt{\frac{1-\epsilon}{1+\epsilon}} e^{-2\Lambda x} x^{2s} \left(\begin{aligned} &|a_2|^2 [(c_1 - \kappa)^2 P_2^2 - n^2 P_1^2] \\ &+ |a_1|^2 [(c_1 + \kappa)^2 P_2^2 - n^2 P_1^2] \end{aligned} \right) \end{aligned} \quad (\text{C.112})$$

$$\begin{aligned} &F_+G_-^* + F_-G_+^* + G_+F_-^* + G_-F_+^* \\ &= e^{-2\Lambda x} x^{2s} (a_1 a_2^* + a_2 a_1^*) \sqrt{\frac{1-\epsilon}{1+\epsilon}} \left(\begin{aligned} &[(c_1 - \kappa)P_2 - nP_1][(c_1 + \kappa)P_2 + nP_1] \\ &+ [(c_1 + \kappa)P_2 - nP_1][(c_1 - \kappa)P_2 + nP_1] \end{aligned} \right). \end{aligned} \quad (\text{C.113})$$

There is a great reduction, that we will let now, if:

$$a_1 a_2^* + a_2 a_1^* = 0. \quad (\text{C.114})$$

In addition, we have:

$$c_2 = s + n = \frac{\alpha\epsilon}{\Lambda} ; \quad c_1 = \frac{\alpha}{\Lambda} = \frac{s+n}{\epsilon} = \sqrt{(s+n)^2 + \alpha^2}. \quad (\text{C.115})$$

We have: $s \geq 0$, $n \geq 0$, therefore $(s+n)^2 \geq s^2$, and

$$c_1 = \sqrt{(s+n)^2 + \alpha^2} \geq \sqrt{s^2 + \alpha^2} = \sqrt{\kappa^2} = |\kappa| \geq \pm\kappa. \quad (\text{C.116})$$

So we always have:

$$c_1 - \kappa \geq 0 ; \quad c_1 + \kappa \geq 0. \quad (\text{C.117})$$

If we choose to let:

$$|a_1|^2 = (c_1 - \kappa)k ; \quad |a_2|^2 = (c_1 + \kappa)k \quad (\text{C.118})$$

where k is a real positive constant, we get:

$$\begin{aligned} & F_+ F_+^* + F_- F_-^* - G_+ G_+^* - G_- G_-^* \\ &= \frac{2k}{1+\epsilon} e^{-2\Lambda x} x^{2s} [2\epsilon c_1 (c_1^2 - \kappa^2) P_2^2 + 2\epsilon c_1 n^2 P_1^2 - 4n(c_1^2 - \kappa^2) P_1 P_2] \end{aligned} \quad (\text{C.119})$$

and since:

$$c_1^2 - \kappa^2 = n(n+2s) ; \quad \epsilon c_1 = s+n \quad (\text{C.120})$$

we get:

$$\begin{aligned} & F_+ F_+^* + F_- F_-^* - G_+ G_+^* - G_- G_-^* \\ &= \frac{4nk}{1+\epsilon} e^{-2\Lambda x} x^{2s} \left((n+2s)[(s+n)P_2^2 - 2nP_1 P_2] + n(s+n)P_1^2 \right) \\ &= \frac{4nk}{1+\epsilon} e^{-2\Lambda x} x^{2s} \left[(n+2s) \left(\sqrt{s+n} P_2 - \frac{n}{\sqrt{s+n}} P_1 \right)^2 + \frac{ns^2}{s+n} P_1^2 \right]. \end{aligned} \quad (\text{C.121})$$

And this term, which is the sum of two squares, is always positive, two successive Laguerre's polynomials having no common zero. Then we get

$$\begin{aligned} & F_+ G_+^* + F_- G_-^* + G_+ F_+^* + G_- F_-^* \\ &= \frac{2\sqrt{1-\epsilon^2}}{1+\epsilon} e^{-2\Lambda x} x^{2s} \left(|a_2|^2 [(c_1 - \kappa)^2 P_2^2 - n^2 P_1^2] \right. \\ &\quad \left. + |a_1|^2 [(c_1 + \kappa)^2 P_2^2 - n^2 P_1^2] \right) \\ &= \frac{4c_1 \Lambda k}{1+\epsilon} e^{-2\Lambda x} x^{2s} [(c_1^2 - \kappa^2) P_2^2 - n^2 P_1^2] \\ &= \frac{4\alpha nk}{1+\epsilon} e^{-2\Lambda x} x^{2s} [(n+2s) P_2^2 - n P_1^2] \end{aligned} \quad (\text{C.122})$$

This allows us to write the Yvon-Takabayasi angle such as:

$$\tan \beta = \frac{\alpha[(2s+n)P_2^2 - nP_1^2]}{(n+2s) \left(\sqrt{s+n} P_2 - \frac{n}{\sqrt{s+n}} P_1 \right)^2 + \frac{ns^2}{s+n} P_1^2} \times \frac{U^2 - V^2}{U^2 + V^2} \quad (\text{C.123})$$

The denominator contains only sums of squares, which cannot be together null. Consequently, for all the states with a $n > 0$ quantum number, a solution exists such that the Yvon-Takabayasi β angle is everywhere defined. In addition the presence of the fine structure constant, which is small, implies that the β angle

is everywhere small. Moreover we now explain why $U^2 - V^2$ is exactly null, for any value of κ and λ , in the plane $x^1 O x^2$. We start here from the differential equation (C.57). If f is a solution, then g defined by $g(z) = f(-z)$ is also a solution. Since there is only one polynomial solution with degree n , up to a real factor, we get necessarily $f(-z) = \pm f(z)$ and f is either an even polynomial or an odd polynomial. Therefore C is an even or an odd polynomial of $\cos \theta$. Now from (C.53) we get

$$U^2 + V^2 = \sin^2 \lambda \theta [C'^2 + (\kappa + \frac{1}{2} - \lambda)^2 C^2] \quad (\text{C.124})$$

$$U^2 - V^2 = \sin^2 \lambda \theta [-\cos \theta C'^2 - 2(\kappa + \frac{1}{2} - \lambda) \sin \theta C' C + (\kappa + \frac{1}{2} - \lambda)^2 \cos \theta C^2]$$

From (C.54) we get

$$U^2 + V^2 = \sin^{-2\lambda} \theta [C'^2 + (\kappa + \frac{1}{2} + \lambda)^2 C^2] \quad (\text{C.125})$$

$$U^2 - V^2 = \sin^{-2\lambda} \theta [\cos \theta C'^2 + 2(\kappa + \frac{1}{2} + \lambda) \sin \theta C' C - (\kappa + \frac{1}{2} + \lambda)^2 \cos \theta C^2]$$

This gives

$$(U^2 - V^2)(\frac{\pi}{2}) = -\frac{\lambda}{|\lambda|} (\kappa + \frac{1}{2} - |\lambda|) (C' C)(\frac{\pi}{2}) \quad (\text{C.126})$$

If C is a constant $C' = 0$. Otherwise either C is an even polynomial of $\cos \theta$ and then C' is odd and the product $C' C$ contains a $\cos \theta$ factor, or C is an odd polynomial of $\cos \theta$ and the product $C' C$ contains also a $\cos \theta$ factor. This factor is null if $\theta = \frac{\pi}{2}$. This proves that

$$\beta(\frac{\pi}{2}) = 0 \quad (\text{C.127})$$

The solutions of the linear Dirac equation satisfying (C.114) and (C.118) may therefore be the linear approximations of solutions for the homogeneous non-linear equation. Now if we have a solution ϕ_0 of the nonlinear homogeneous equation (3.10) with a not small value β_0 of the Yvon-Takabayasi angle at a point M_0 with coordinates $(x_0, y_0, 0)$, since the nonlinear homogeneous equation is globally gauge invariant under the chiral gauge (3.31), we let

$$\phi = e^{-i\frac{\beta_0}{2}} \phi_0 \quad (\text{C.128})$$

And we get

$$(\phi \bar{\phi})(M_0) = e^{-i\frac{\beta_0}{2}} \phi_0 e^{-i\frac{\beta_0}{2}} \bar{\phi}_0 = e^{-i\beta_0} \rho_0 e^{i\beta_0} = \rho_0. \quad (\text{C.129})$$

And ϕ has at this point M_0 a null β angle, so the equation (3.10) at this point is exactly the Dirac equation, we get the separation of variables at this point, we get the angular system (C.38) and the radial system (C.39) which is identical to the radial system (C.40) of the linear equation. Then the β angle is null in all the $z = 0$ plane and the radial system (3.39) is identical to (C.40) in all the $z = 0$ plane. Then the necessity of integrability imposes the existence of radial polynomials and we get the quantification of the energy levels and the Sommerfeld's formula (C.96)

C.5 Radial polynomials with degree 0

To get absolutely all results of the Dirac equation, we have a last thing to explain: why do we get $2\mathbf{n}^2$ different states with a principal quantum number $\mathbf{n} = |\kappa| + n$, and we must return to the particular case where radial polynomials are constants. We start directly from (C.64), and we let:

$$a = a_0 e^{-\Lambda x} x^s ; \quad b = b_0 e^{-\Lambda x} x^s ; \quad c = c_0 e^{-\Lambda x} x^s ; \quad d = d_0 e^{-\Lambda x} x^s. \quad (\text{C.130})$$

We get from (C.39):

$$\begin{aligned} e^{-\Lambda x} (i\epsilon d_0 x^s + i\alpha d_0 x^{s-1} - \Lambda d_0 x^s + s d_0 x^{s-1} + \kappa b_0 x^{s-1}) &= i a_0 e^{-\Lambda x} x^s \\ e^{-\Lambda x} (-i\epsilon c_0 x^s - i\alpha c_0 x^{s-1} + \Lambda c_0 x^s - s c_0 x^{s-1} - \kappa a_0 x^{s-1}) &= -i b_0 e^{-\Lambda x} x^s \\ e^{-\Lambda x} (i\epsilon b_0 x^s + i\alpha b_0 x^{s-1} + \Lambda b_0 x^s - s b_0 x^{s-1} - \kappa d_0 x^{s-1}) &= i c_0 e^{-\Lambda x} x^s \\ e^{-\Lambda x} (-i\epsilon a_0 x^s - i\alpha a_0 x^{s-1} - \Lambda a_0 x^s + s a_0 x^{s-1} + \kappa c_0 x^{s-1}) &= -i d_0 e^{-\Lambda x} x^s. \end{aligned} \quad (\text{C.131})$$

This is equivalent to the set formed by the four following systems:

$$\begin{aligned} \kappa b_0 + (i\alpha + s)d_0 &= 0 \\ (i\alpha - s)b_0 - \kappa d_0 &= 0 \end{aligned} \quad (\text{C.132})$$

$$\begin{aligned} -\kappa a_0 - (i\alpha + s)c_0 &= 0 \\ -(i\alpha - s)a_0 + \kappa c_0 &= 0 \end{aligned} \quad (\text{C.133})$$

$$\begin{aligned} -i a_0 + (i\epsilon - \Lambda)d_0 &= 0 \\ -(i\epsilon + \Lambda)a_0 + i d_0 &= 0 \end{aligned} \quad (\text{C.134})$$

$$\begin{aligned} i b_0 - (i\epsilon - \Lambda)c_0 &= 0 \\ (i\epsilon + \Lambda)b_0 - i c_0 &= 0. \end{aligned} \quad (\text{C.135})$$

The cancellation of the determinant in (C.132) and (C.133) gives again (C.81) and (C.82). The cancellation of the determinant in (C.134) and (C.135) is simply equivalent to $\Lambda^2 = 1 - \epsilon^2$, which comes from the definition of Λ . Each system (C.132) to (C.135) is then reduced into one equation:

$$\begin{aligned} \kappa d_0 &= (i\alpha - s)b_0 \\ \kappa c_0 &= (i\alpha - s)a_0 \\ d_0 &= (\epsilon - i\Lambda)a_0 \\ b_0 &= (\epsilon + i\Lambda)c_0. \end{aligned} \quad (\text{C.136})$$

We get then:

$$\kappa d_0 = \kappa(\epsilon - i\Lambda)a_0 = (i\alpha - s)b_0 = (i\alpha - s)(\epsilon + i\Lambda)c_0 = \frac{(i\alpha - s)^2(\epsilon + i\Lambda)}{\kappa} a_0. \quad (\text{C.137})$$

We have a not null solution only if:

$$\begin{aligned}\kappa(\epsilon - i\Lambda) &= \frac{(i\alpha - s)^2(\epsilon + i\Lambda)}{\kappa} \\ \kappa^2(\epsilon - i\Lambda)^2 &= (s - i\alpha)^2 \\ \kappa(\epsilon - i\Lambda) &= \pm(s - i\alpha).\end{aligned}\tag{C.138}$$

Since ϵ , s , Λ and α are positive, we finally get

$$|\kappa| = \frac{s}{\epsilon} = \frac{\alpha}{\Lambda}.\tag{C.139}$$

This last equality gives again the formula of energy levels (C.96) with $n = 0$. Since κ comes with its absolute value, we can equally have $\kappa < 0$ or $\kappa > 0$. But the calculation of solutions by C. G. Darwin, who works with real constants, not with complex constants at this stage of his computation, forbids to κ to be negative, and it is that thing that allows, for a given principal quantum number $\mathbf{n} = n + |\kappa|$, to get $\mathbf{n}(\mathbf{n}+1) + \mathbf{n}(\mathbf{n}-1) = 2\mathbf{n}^2$ states. Whatever really happens is that to change sign in κ comes to change V into $-V$. And if we change the sign of κ and V , then \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} are invariant if $n = 0$, and the wave is unchanged. To change the sign of κ brings no more solutions and we can use only solutions with $\kappa > 0$, in the case $n = 0$. And this allows to get the true number of states.

The formula obtained for the energy levels does not account for the Lamb effect, which gives, if $n > 0$, a very small split between energy levels with same other quantum numbers but with opposite signs of κ . If the formula (C.96) was not the same for two opposite values of κ we should not be able to get four polynomial radial functions with only one condition which gives the quantification of the energy levels. Here also the standard model has already an answer, with the polarization of the void. But the calculation must be revised, both to avoid divergent integrals and to use our solutions instead of Darwin's solutions coming from the non-relativistic Pauli equation.

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