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### Central limit theorems for a supercritical branching process in a random environment

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#### Abstract

For a supercritical branching process  $(Z_n)$  in a stationary and ergodic environment  $\xi$ , we study the rate of convergence of the normalized population  $W_n = Z_n/E[Z_n|\xi]$  to its limit  $W_\infty$ : we show a central limit theorem for  $W_\infty - W_n$ with suitable normalization and derive a Berry-Esseen bound for the rate of convergence in the central limit theorem when the environment is independent and identically distributed. Similar results are also shown for  $W_{n+k} - W_n$  for each fixed  $k \in \mathbb{N}^*$ .

*Keywords:* Branching processes, random environment, central limit theorem, martingale, rate of convergence. 2000 MSC: 60J80, 60F05

#### 1. Introduction

Galton-Watson processes have been studied by many authors, due to a wide range of applications. See for example the books by Harris (1963) and Athreya and Ney (1972). In a Galton-Watson process  $\{Z_n, n = 0, 1, ...\}$ , particles behave independently, each gives birth to a random number of particles of the next generation with a fixed distribution  $\{p_k : k = 0, 1, ...\}$ .

A branching process in a random environment is a natural and important extension of the Galton-Watson process. It is a class of non-homogeneous Galton-Watson processes indexed by a time-environment  $\xi = (\xi_0, \xi_1, \xi_2, ...)$ , which is

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supposed to be stationary and ergodic; given the environment  $\xi$ , the particles of *n*-th generation have offspring distribution  $\{p_k(\xi_n) : k \in \mathbb{N}\}$  depending on  $\xi_n$ . For first important works on the subject, see Smith and Wilkinson (1969) and Athreya and Karlin (1971a,b).

For a Galton-Waston process with  $Z_0 = 1$  and  $m = EZ_1 \in (0, \infty)$ , it is well known that  $\{W_n = Z_n/m^n : n = 0, 1, ...\}$  forms a non-negative martingale, and converges almost surely to a random variable  $W_{\infty}$ . For the convergence rate of the martingale, Heyde (1971) and Bühler (1969) obtained respectively that if  $\operatorname{Var}(Z_1) = \sigma^2 < \infty$ , then conditioned on  $Z_n > 0$ , the conditional laws of

$$(m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n^{-\frac{1}{2}} m^n (W_\infty - W_n)$$

and

$$\left(m^{k}/(m^{k}-1)\right)^{\frac{1}{2}}(m^{2}-m)^{\frac{1}{2}}\sigma^{-1}Z_{n}^{-\frac{1}{2}}m^{n}(W_{n+k}-W_{n}) \qquad k \in \mathbb{N}^{*}$$

converge to the normal law  $\mathcal{N}(0, 1)$ ; Heyde and Brown (1971) gave an estimation of its convergence rate under a third moment condition.

The object of this paper is to extend the theorems of Bühler (1969), Heyde (1971) and Heyde and Brown (1971) to a branching process in a random environment. The main results are Theorems 2.1 and 2.2.

#### 2. Main Results

As usual, we write  $\mathbb{N} = \{0, 1, 2, \dots\}$ ,  $\mathbb{N}^* = \{1, 2, \dots\}$  and  $\mathbb{R}$  for the set of real numbers.

Let us first recall the definition of a branching process in a random environment. For reference on the subject, see for example Athreya and Karlin (1971a,b), and Athreya and Ney (1972).

A random environment  $\xi = (\xi_n)$  is formulated as a stationary and ergodic sequence of random variables taking values in some measurable space  $(\Theta, \mathcal{F})$ . Each realization of  $\xi_n$  corresponds to a probability distribution  $p(\xi_n) = \{p_i(\xi_n) : i \in \mathbb{N}\}$  where

$$p_i(\xi_n) \ge 0, \quad \sum_{i=0}^{\infty} p_i(\xi_n) = 1, \quad 0 < \sum_{i=0}^{\infty} i p_i(\xi_n) < \infty.$$
 (1)

Without loss of generality, we can take  $\xi_n$  as coordinate functions defined on the product space  $(\Theta^{\mathbb{N}}, \mathcal{F}^{\otimes \mathbb{N}})$ , equipped with a probability law  $\tau$ , which is invariant and ergodic under the usual shift transformation  $\theta$  on  $\Theta^{\mathbb{N}}$ :  $\theta(\xi_0, \xi_1, \cdots) = (\xi_1, \xi_2, \cdots)$ . A branching process  $(Z_n)_{n\geq 0}$  in the random environment  $\xi$  is a class of non-homogeneous branching processes indexed by  $\xi$ . By definition,

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \ n \ge 0,$$
 (2)

where given  $\xi$ ,  $\{X_{n,i} : n \ge 0, i \ge 1\}$  is a family of (conditionally) independent random variables, each  $X_{n,i}$  has the common law  $p(\xi_n)$ . Notice that when all  $\xi_n$  are the same constant,  $(Z_n)$  reduces to the classical Galton-Waston process.

Let  $(\Gamma, P_{\xi})$  be the probability space under which the process is defined when the environment  $\xi$  is given. As usual,  $P_{\xi}$  is called *quenched law*. The total probability space can be formulated as the product space  $(\Gamma \times \Theta^{\mathbb{N}}, P)$ , where  $P = P_{\xi} \otimes \tau$  in the sense that for all measurable and positive function g, we have

$$\int g dP = \int \int g(\xi, y) dP_{\xi}(y) d\tau(\xi), \qquad (3)$$

(recall that  $\tau$  is the law of the environment  $\xi$ ). The total probability P is usually called *annealed law*. The quenched law  $P_{\xi}$  may be considered to be the conditional probability of the annealed law P given  $\xi$ . The expectation with respect to  $P_{\xi}$  (resp. P) will be denoted  $E_{\xi}$  (resp. E).

For  $n \ge 0$ , define

$$m_n(a) = m(\xi_n, a) = \sum_{i=1}^{\infty} i^a p_i(\xi_n), \quad a \in \mathbb{R},$$
(4)

$$m_n = m_n(1), \quad \sigma_n^2 = m_n(2) - m_n^2,$$
 (5)

 $\pi_0 = 1 \quad and \quad \pi_n = \pi_n(\xi) = m_0 \cdots m_{n-1} \quad \text{for} \quad n \ge 1.$  (6)

Then  $\pi_n = E_{\xi} Z_n$  for  $n \ge 0$ . It is well known that

$$W_n = Z_n / \pi_n \tag{7}$$

is a martingale with respect to the filtration

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_n = \sigma\{\xi, X_{j,i} : j \le n - 1, i \ge 1\} \quad (n \ge 1), \tag{8}$$

so that the limit

$$W_{\infty} = \lim_{n \to \infty} W_n \tag{9}$$

exists almost surely (a.s.) with  $EW \leq 1$  by Fatou's lemma.

Throughout the paper, we always assume that

$$E\ln m_0 > 0$$
 and  $E\left(\frac{Z_1}{m_0}\ln^+ Z_1\right) < \infty.$  (10)

The first assumption ensures that the process is *supercritical* (cf. Athreya and Karlin (1971a)); the second one together with the first implies that  $EW_{\infty} = 1$ ; moreover,

$$P_{\xi}(W_{\infty} > 0) = P_{\xi}(Z_n \to \infty) = \lim_{n \to \infty} P_{\xi}(Z_n > 0) = 1 - q(\xi) > 0 \quad \text{a.s.}, \quad (11)$$

where  $q(\xi) = \lim_{n \to \infty} P_{\xi}(Z_n = 0)$  is the extinct probability.

In this paper, we search for central limit theorems on  $W_{\infty} - W_n$  and  $W_{n+k} - W_n$  for fixed  $k \ge 1$  with an appropriate normalization. Assum that  $m_0(2) < \infty$  a.s., and let

$$\Delta_k^2 = \Delta_k^2(\xi) = \sum_{0 \le i < k} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2} \quad \text{for} \quad k \in \mathbb{N}^* \cup \{\infty\}.$$

Then for  $k \in \mathbb{N}^*$ ,  $\Delta_k^2(\xi)$  is the variance of  $W_k$  under  $P_{\xi}$ ;  $\Delta_{\infty}^2(\xi)$  is the variance of  $W_{\infty}$  if the series converges (i.e.  $\Delta_{\infty}^2(\xi) < \infty$ ): see Lemma 3.2.

We can now formulate our first main result.

**Theorem 2.1.** Suppose that (10) holds and that  $m_0(2) < \infty$  a.s.. In the case where  $k = \infty$ , assume additionally that  $E \ln^+(\sigma_0^2/m_0^2) < \infty$ . Write

$$U_{n,k} = \frac{\pi_n(W_{n+k} - W_n)}{\sqrt{Z_n}\Delta_k(\theta^n\xi)} \quad for \quad k \in \mathbb{N}^* \cup \{\infty\},$$

where by convention  $W_{n+k} = W_{\infty}$  if  $k = \infty$ . Then for each  $k \in \mathbb{N}^* \cup \{\infty\}$ , as  $n \to \infty$ ,

$$\sup_{x \in \mathbb{R}} |P_{\xi}(U_{n,k} \le x | Z_n > 0) - \Phi(x)| \to 0 \quad in \quad L^1,$$
(12)

and

$$\sup_{x \in \mathbb{R}} |P(U_{n,k} \le x | Z_n > 0) - \Phi(x)| \to 0.$$
(13)

We believe that for each  $k \in \mathbb{N} \cup \{\infty\}$ 

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |P_{\xi}(U_{n,k} \le x | Z_n > 0) - \Phi(x)| = 0 \quad \text{a.s.}.$$

We notice that in the classical Galton-Waston process, (13) reduces to the results of Bühler (1969) and Heyde (1971). Our second main result concerns the rate of convergence in the above central limit theorem for a branching process with an independent and identically distributed environment.

**Theorem 2.2.** Let the environment  $\{\xi_n\}$  be independent and identically distributed. Assume that (10) holds and that  $m_0(2) < \infty$  a.s.. In the case where  $k = \infty$ , assume additionally that  $E \ln^+(\sigma_0^2/m_0^2) < \infty$ . For each  $k \in \mathbb{N}^* \cup \{\infty\}$ , if  $E |\frac{W_k - 1}{\Delta_k}|^{2+\delta} < \infty$  for some  $\delta \in (0, 1]$ , then

$$\sup_{x \in \mathbb{R}} |P(U_{n,k} \le x | Z_n > 0) - \Phi(x)| \le \frac{C_{\delta} \left( Em_0(-\frac{\delta}{2}) \right)^n E \left| \frac{W_k - 1}{\Delta_k} \right|^{2+\delta}}{P(Z_n > 0)}, \quad (14)$$

where  $U_{n,k}$  is defined in Theorem 2.1 and  $C_{\delta}$  is the Berry-Esseen constant.

Remark 2.3. It maybe useful to notice that if

$$E(Z_1/m_0)^{2+\delta} < \infty$$
,  $Em_0^{-(1+\delta)} < 1$  and  $m_0(2)/m_0^2 \ge A$ 

for some constant A > 1, then  $E \left| \frac{W_{\infty} - 1}{\Delta_{\infty}} \right|^{2+\delta} < +\infty$ . In fact by Theorem 3 of Guivarc'h and Liu (2001), the first two conditions imply that  $E|W_{\infty} - 1|^{2+\delta} < \infty$ , while the last one implies that  $\Delta_{\infty}^2 \ge A - 1 > 0$ .

For the classical Galton-Watson process with  $\delta = 1$ , Theorem 2.2 reduces to Theorem 2 of (Heyde and Brown, 1971, p.272).

#### 3. Proof of Theorem 2.1

In this section, we consider a central limit theorem under a second moment condition in proving Theorem 2.1. We first give some lemmas.

**Lemma 3.1** (Grincevičjus (1974)). Let  $\{(\alpha_n, \beta_n), n = 0, 1, 2, \dots\}$  be a stationary and ergodic sequence of random variables with values in  $\mathbb{R}^2$ . If

$$E\ln|\alpha_0| < 0 \quad and \quad E\ln^+|\beta_0| < \infty,$$

then

$$\sum_{n=0}^{\infty} |\alpha_0 \alpha_1 \cdots \alpha_{n-1} \beta_n| < \infty \quad a.s.$$

In fact, the result is a direct consequence of the ergodic theorem and Cauchy's criterion for the convergence of series.

Using the above lemma, we can easily obtain the following result.

**Lemma 3.2.** Under the assumptions in Theorem 2.1, for each  $k \in \mathbb{N}^* \cup \{\infty\}$ ,

$$\operatorname{Var}_{\xi}(W_k) = \Delta_k^2(\xi) = \sum_{0 \le i < k} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}.$$
 (15)

This has been known for branching processes in varying environment, see e.g. (Jagers, 1974, p.175) in a slightly different form. For reader's convenience, we present a proof in the following.

Proof of Lemma 3.2. By (2) and the definition of  $W_n$ , we have

$$W_{n+1} - W_n = \frac{1}{\pi_n} \sum_{j=1}^{Z_n} (\frac{X_{n,j}}{m_n} - 1).$$

Recall that under  $P_{\xi}$ , the random variables  $\{X_{n,j}\}$  are independent of each other and have the common distribution  $p(\xi_n)$  with expectation  $m_n$ . Hence a direct calculation shows that

$$E_{\xi}((W_{n+1} - W_n)^2) = E_{\xi}\left(E_{\xi}((W_{n+1} - W_n)^2 | \mathcal{F}_n)\right)$$
$$= E_{\xi}\left(\frac{Z_n}{\pi_n^2} \frac{\sigma_n^2}{m_n^2}\right) = \frac{1}{\pi_n} \frac{\sigma_n^2}{m_n^2}.$$

As  $\{W_n\}$  is a martingale, it follows that

$$E_{\xi}W_k^2 = E_{\xi}W_0^2 + \sum_{i=0}^{k-1} E((W_{i+1} - W_i)^2) = 1 + \sum_{i=0}^{k-1} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}.$$

Therefore for each fixed integer k,

$$\operatorname{Var}_{\xi}(W_k) = E_{\xi}(W_k^2) - 1 = \sum_{i=0}^{k-1} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}$$

Now we turn to the calculation of  $\operatorname{Var}_{\xi}(W_{\infty})$ . By Lemma 3.1, when  $E \ln m_0 > 0$ and  $E \ln^+ \frac{\sigma_i^2}{m_i^2} < \infty$ ,

$$\sup_{n} E_{\xi}(W_{n}^{2}) = 1 + \sum_{i=0}^{\infty} \frac{1}{\pi_{i}} \frac{\sigma_{i}^{2}}{m_{i}^{2}} < \infty \quad \text{a.s.}$$

So  $W_n$  converges to  $W_\infty$  in  $L^2$  under  $P_\xi$  and

$$E_{\xi}(W_{\infty}^{2}) = \lim_{k \to \infty} E_{\xi}(W_{k}^{2}) = 1 + \sum_{i=0}^{\infty} \frac{1}{\pi_{i}} \frac{\sigma_{i}^{2}}{m_{i}^{2}}.$$

It follows that

$$\operatorname{Var}_{\xi}(W_{\infty}) = E_{\xi}(W_{\infty}^2) - 1 = \Delta_{\infty}^2(\xi) = \sum_{i=0}^{\infty} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2} < \infty$$
 a.s.

To give our next lemma, we will need some notations, which will also be used in the proof of the main theorems. By definition,

$$Z_{n+k} = \sum_{j=1}^{Z_n} Z_k(n,j),$$
(16)

where  $Z_k(n, j)$  denotes the number of descendants in the (n + k)-th generation

of the *j*-th particle among the  $Z_n$  particles in *n*-th generation. Writing  $W_k(n,j) = \frac{Z_k(n,j)}{\pi_k(\theta^n \xi)}$  and using (16), we obtain the following decomposition:

$$\pi_n(W_{n+k} - W_n) = \sum_{j=1}^{Z_n} (W_k(n, j) - 1).$$
(17)

Letting  $k \to \infty$ , it follows that

$$\pi_n(W_\infty - W_n) = \sum_{j=1}^{Z_n} (W_\infty(n, j) - 1),$$
(18)

where under  $P_{\xi}$ , the random variables  $\{W_{\infty}(n, j)\}_{j}$  are independent of each other and have the common conditional distribution

$$P_{\xi}(W_{\infty}(n,j)\in \cdot) = P_{\theta^n\xi}(W_{\infty}\in \cdot)$$

**Lemma 3.3.** Suppose that the assumptions of Theorem 2.1 hold. Let  $r_n \in \mathbb{N}$  with  $r_n \to \infty$ . For  $k \in \mathbb{N}^* \cup \{\infty\}$ , define

$$Y_{k,n} = \frac{1}{\sqrt{r_n}} \sum_{j=1}^{r_n} \frac{W_k(n,j) - 1}{\Delta_k(\theta^n \xi)}$$

Fix  $k \in \mathbb{N}^* \cup \{\infty\}$ . Then for each subsequence  $\{n'\}$  of  $\mathbb{N}$  with  $n' \to \infty$ , there is a subsequence  $\{n''\}$  of  $\{n'\}$  with  $n'' \to \infty$  such that for a.e.  $\xi$  and all  $x \in \mathbb{R}$ , as  $n'' \to \infty$ ,

$$P_{\xi}(Y_{k,n''} \le x) \to \Phi(x).$$

*Proof.* Fix  $k \in \mathbb{N}^* \cup \{\infty\}$ . In order to use Lindeberg's theorem, for  $n \in \mathbb{N}$  and  $\epsilon > 0$ , we consider the quantity

$$L_k(\xi,\epsilon,n) = \frac{1}{r_n} \sum_{j=1}^{r_n} E_{\xi} \left( \left( \frac{W_k(n,j) - 1}{\Delta_k(\theta^n \xi)} \right)^2; \left| \frac{W_k(n,j) - 1}{\Delta_k(\theta^n \xi)\sqrt{r_n}} \right| > \epsilon \right),$$

where for a set A, we write  $E_{\xi}(x; A)$  for  $E_{\xi}(X\mathbf{1}_A)$ ,  $\mathbf{1}_A$  denoting the indicator function of A. By the stationarity and ergodicity of the environment, for all  $\epsilon > 0$ , as  $n \to \infty$ ,

$$EL_k(\xi,\epsilon,n) = E\left[\left(\frac{W_k - 1}{\Delta_k}\right)^2; \left|\frac{W_k - 1}{\Delta_k}\right| > \sqrt{r_n}\epsilon\right] \to 0.$$
(19)

Let  $\{n'\}$  be a subsequence of  $\mathbb{N}$ . Notice that from (19), we can choose a subsequence  $\{n''\}$  for which  $L_k(\xi, \varepsilon, n'') \to 0$  a.s., but this sequence may depend of  $\epsilon$ . We will use a diagonal argument to select a subsequence  $\{n''\}$  of  $\{n'\}$  such that a.s.  $L_k(\xi, \varepsilon, n'') \xrightarrow{n'' \to \infty} 0$  for all  $\epsilon > 0$ . Set

$$\epsilon_m = 1/m \quad \text{for} \quad m \ge 1.$$

Let  $\{n_{0,i}\} = \{n'\}$ . Because of (19), there is a subsequence  $\{n_{1,i}\}$  of  $\{n_{0,i}\}$  and a set  $\Lambda_1$  with  $\tau(\Lambda_1) = 1$  such that  $\forall \xi \in \Lambda_1$ ,

$$\lim_{i \to \infty} L_k(\xi, \epsilon_1, n_{1,i}) = 0.$$

Inductively for  $m \geq 1$ , when  $\Lambda_m$  and  $\{n_{m,i}\}$  are defined such that  $\tau(\Lambda_m) = 1$ and  $\forall \xi \in \Lambda_m$ ,  $L_k(\xi, \epsilon_m, n_{m,i}) \to 0$ , there is a subsequence  $\{n_{m+1,i}\} \subset \{n_{m,i}\}$ and a set  $\Lambda_{m+1}$  with  $\tau(\Lambda_{m+1}) = 1$  such that  $\forall \xi \in \Lambda_{m+1}$ ,

$$\lim_{i \to \infty} L_k(\xi, \epsilon_{m+1}, n_{m+1,i}) = 0.$$

We now consider the diagonal sequence  $\{n_{i,i}\}_{i\geq 1}$  and  $\Lambda = \bigcap_{j=1}^{\infty} \Lambda_j$ . For each fixed  $\epsilon > 0$ , let  $m \geq \frac{1}{\epsilon}$ . Then  $\epsilon_m \leq \epsilon$  and by the monoticity of  $L_k(\xi, \epsilon, n)$  in  $\epsilon$ , we see that  $\forall \xi \in \Lambda$ ,

$$L_k(\xi, \epsilon, n_{m,i}) \le L_k(\xi, \epsilon_m, n_{m,i}) \to 0 \text{ as } i \to \infty.$$

As  $\{n_{i,i}\}$  is a subsequence of  $\{n_{m,i}\}$  whenever i > m, this implies that

$$\lim_{i \to \infty} L_k(\xi, \epsilon, n_{i,i}) = 0.$$
<sup>(20)</sup>

Since  $\tau(\Lambda) = 1$ , we have shown that for all  $\epsilon > 0$ , (20) holds a.s.. It follows that a.s. (20) holds for all rational  $\epsilon > 0$ , and therefore for all real  $\epsilon > 0$  by the monoticity of  $L_k(\xi, \epsilon, n_{ii})$  in  $\epsilon$ . So by Lindeberg's theorem, it is a.s. that for all  $x \in \mathbb{R}$ , as  $i \to \infty$ ,

$$P_{\xi}(Y_{k,n_{i,i}} \le x) \to \Phi(x).$$

Thus the lemma has been proved with  $\{n''\} = \{n_{i,i}\}.$ 

Proof of Theorem 2.1. We shall only deal with the case where  $k = \infty$ , as the case where  $k \in \mathbb{N}^*$  can be treated similarly.

We first prove the following assertion: for each sequence  $\{n'\}$  of  $\mathbb{N}$  with  $n' \to \infty$ , there exist a subsequence  $\{n''\}$  of  $\{n'\}$  with  $n'' \to \infty$  such that for a.e.  $\xi$  and all x, as  $n'' \to \infty$ ,

$$P_{\xi}(U_{n'',\infty} \le x | Z_{n''} > 0) \to \Phi(x).$$

$$\tag{21}$$

By the definition of  $U_{n,\infty}$  and the relation (18), we get :

$$U_{n,\infty} = \frac{\pi_n (W_\infty - W_n)}{\sqrt{Z_n} \Delta_\infty(\theta^n \xi)} = \frac{1}{\sqrt{Z_n}} \sum_{j=1}^{Z_n} \frac{W_\infty(n, j) - 1}{\Delta_\infty(\theta^n \xi)},$$

where we recall that under  $P_{\xi}$ ,  $\{W_{\infty}(n, j), j \geq 1\}$  is a family of random variables independent of each other and independent of  $Z_n$ , each has the same law as  $W_{\infty}$ under  $P_{\theta^n \xi}$ . Set

$$u_n(r,x) = P_{\xi}\left(\frac{1}{\sqrt{r}}\sum_{j=1}^r \frac{W_{\infty}(n,j)-1}{\Delta_{\infty}(\theta^n\xi)} \le x\right), \quad r \in \mathbb{N}^*, \quad x \in \mathbb{R}.$$

Then

$$P_{\xi}(U_{n,\infty} \le x | Z_n > 0) = [P_{\xi}(Z_n > 0)]^{-1} \sum_{r=1}^{\infty} P_{\xi}(U_{n,\infty} \le x, Z_n = r)$$
$$= \sum_{r=1}^{\infty} u_n(r, x) \frac{P_{\xi}(Z_n = r)}{P_{\xi}(Z_n > 0)}.$$
(22)

To show the main idea, let us first consider the special case where  $q(\xi) = 0$ a.s., i.e. for a.e.  $\xi$ ,

$$Z_n \to \infty$$
  $P_{\xi}^*$ -a.s..

In this case, the relation (22) becomes

$$P_{\xi}(U_{n,\infty} \le x) = \sum_{r=1}^{\infty} u_n(r,x) P_{\xi}(Z_n = r) = E_{\xi} u_n(Z_n,x).$$

By Lemma 3.3, for each subsequence  $\{n'\}$  of  $\mathbb{N}$  with  $n' \to \infty$ , there exist a subsequence  $\{n''\}$  of  $\{n'\}$  with  $n'' \to \infty$  such that for a.e.  $\xi$  and all x, as  $n'' \to \infty$ ,

$$u_{n''}(Z_{n''}, x) \to \Phi(x)$$

By the dominated convergence theorem, for a.e.  $\xi$  and all x, as  $n'' \to \infty$ ,

$$P_{\xi}(U_{n^{\prime\prime},\infty} \leq x) = E_{\xi}[u_{n^{\prime\prime}}(Z_{n^{\prime\prime}},x)] \to \Phi(x).$$

So we have proved (21) when  $q(\xi) = 0$  a.s..

We now consider the general case where  $0 \le q(\xi) < 1$  a.s..

For each  $\xi \in \Theta^{\mathbb{N}}$ , let  $Z_n^*$  be random variables defined on some probability space  $(\Gamma^*, \mathbb{P}^*_{\varepsilon})$  with law

$$P_{\xi}^{*}(Z_{n}^{*}=r) = \frac{P_{\xi}(Z_{n}=r)}{P_{\xi}(Z_{n}>0)}, \qquad r \in \mathbb{N}^{*}$$

Then

$$P_{\xi}(U_{n,\infty} \le x | Z_n > 0) = E_{\xi}^* u_n(Z_n^*, x),$$

where  $E_{\xi}^{*}$  denotes the expectation with respect to  $P_{\xi}^{*}$ . Let  $\{n'\}$  be a sequence of  $\mathbb{N}$  with  $n' \to \infty$ . If for a.e.  $\xi$ ,

$$Z_{n'}^* \to \infty$$
  $P_{\xi}^*$ -a.s.,

then as above we can use Lemma 3.3 and the dominated convergence theorem to show that there is a sequence  $\{n''\}$  of  $\{n'\}$  with  $n'' \to \infty$  such that for all x, as  $n'' \to \infty$ ,

$$E_{\varepsilon}^* u_{n^{\prime\prime}}(Z_{n^{\prime\prime}}^*, x) \to \Phi(x).$$

By the fact that  $Z_n^* \to \infty$  in probability under  $P_{\xi}^*$ , we can choose a subsequence for which  $Z_n^* \to \infty P_{\xi}^*$ -a.s.. But to apply Lemma 3.3, we need that the sequence does not depend on  $\xi$ . We therefore pass to the probability  $P^*$  to overcome this difficulty, where  $P^* = P_{\xi}^* \otimes \tau$  is defined on the product space  $\Gamma^* \times \Theta^{\mathbb{N}}$  just as P was defined on  $\Gamma \times \Theta^{\mathbb{N}}$ .

Notice that for each  $r \in \mathbb{N}^*$ , as  $n \to \infty$ ,

$$P_{\xi}^{*}(Z_{n}^{*}=r) = \frac{P_{\xi}(Z_{n}=r)}{P_{\xi}(Z_{n}>0)} \to 0,$$

where the last step holds as  $Z_n \to \infty$  a.s. on the survival event  $S = \{Z_n > 0, \forall n \ge 1\}$  (see (11) or Tanny (1977) for this fact). Then  $Z_n^* \to +\infty$  in probability under  $P_{\xi}^*$ . By the dominated convergence theorem, this implies that  $Z_n^* \to +\infty$  in probability under  $P^*$ . Therefore for each subsequence  $\{n'\}$  of  $\mathbb{N}$  with  $n' \to \infty$ , there is a subsequence  $\{\tilde{n}\} \subset \{n'\}$  with  $\tilde{n} \to \infty$  such that  $Z_{\tilde{n}}^* \to +\infty$  a.s. under  $P^*$ . This implies that for a.e.  $\xi$ , as  $\tilde{n} \to \infty$ ,

$$Z^*_{\tilde{n}} \to +\infty \quad P^*_{\mathcal{E}}$$
-a.s.

Now by Lemma 3.3, there exists a subsequence  $\{n''\}$  of  $\{\tilde{n}\}$  such that for a.e.  $\xi$  and all x, as  $n'' \to \infty$ ,

$$u_{n^{\prime\prime}}(Z_{n^{\prime\prime}}^*,x)\to \Phi(x)$$

By the dominated convergence theorem, for almost every  $\xi$  and each x, as  $n'' \to \infty$ ,

$$P_{\xi}(U_{n'',\infty} \le x | Z_{n''} > 0) = E_{\xi} u_{n''}(Z_{n''}^*, x) \to \Phi(x).$$
(23)

So combining the above two cases, we have proved (21).

Since  $P_{\xi}(U_{n'',\infty} \leq x | Z_{n''} > 0)$  are distribution functions and  $\Phi(x)$  is a continuous distribution function, by Dini's Theorem we see that for a.e.  $\xi$ , as  $n'' \to \infty$ ,

$$\sup_{x} |P_{\xi}(U_{n'',\infty} \le x | Z_{n''} > 0) - \Phi(x)| \to 0.$$
(24)

By the dominated convergence theorem, (24) implies that as  $n'' \to \infty$ ,

$$E \sup_{x} |P_{\xi}(U_{n'',\infty} \le x | Z_{n''} > 0) - \Phi(x)| \to 0.$$
(25)

Therefore we have proved that for each sequence  $\{n'\}$  of  $\mathbb{N}$  with  $n' \to \infty$ , there is a subsequence  $\{n''\}$  of  $\{n'\}$  with  $n'' \to \infty$  such that (25) holds. Hence

$$E \sup_{x} |P_{\xi}(U_{n,\infty} \le x | Z_n > 0) - \Phi(x)| \to 0.$$

This gives (12) for  $k = \infty$ . The proof for  $k \in \mathbb{N}^*$  is similar.

We now begin to prove (13).

As we have proved that for each subsequence  $\{n'\}$  of  $\mathbb{N}$ , there is a subsequence  $\{n''\}$  so that (24) holds, which implies: for a.e.  $\xi$  and all  $x \in \mathbb{R}$ , as  $n'' \to \infty$ ,

$$|P_{\xi}(U_{n'',\infty} \le x | Z_{n''} > 0) - \Phi(x)| \to 0.$$

It follows that for a.e.  $\xi$  and all  $x \in \mathbb{R}$ ,

$$|P_{\xi}(U_{n'',\infty} \le x, Z_{n''} > 0) - P_{\xi}(Z_{n''} > 0)\Phi(x)| \to 0.$$

So by the dominated convergence theorem, we see that for each  $x \in \mathbb{R}$ , as  $n'' \to \infty$ ,

$$|P(U_{n'',\infty} \le x, Z_{n''} > 0) - P(Z_{n''} > 0)\Phi(x)| \to 0,$$

and hence

$$P(U_{n'',\infty} \le x | Z_{n''} > 0) \to \Phi(x).$$

By Dini's Theorem, it follows that

$$\sup_{x} |P(U_{n'',\infty} \le x | Z_{n''} > 0) - \Phi(x)| \to 0.$$
(26)

Therefore we have proved that for each sequence  $\{n'\}$  of  $\mathbb{N}$ , there is a subsequence  $\{n''\}$  of  $\{n'\}$  with  $n'' \to \infty$  such that (26) holds. Hence

$$\sup_{x} |P(U_{n,\infty} \le x | Z_n > 0) - \Phi(x)| \to 0$$

Thus the proof is completed.

#### 4. Proof of Theorem 2.2

In this section, we consider the rate of convergence in the central limit theorem under a moment condition of order  $2 + \delta$ , in proving Theorem 2.2.

Notice that by the definition (4) of  $m_n(a)$ , we have

$$m_n(a) = E_{\xi} X^a_{n,i}$$
 if  $a > 0$ ,  $m_n(a) = E_{\xi} X^a_{n,i} \mathbf{1}_{\{X_{n,i} > 0\}}$  if  $a \le 0$ , (27)

where  $X_{n,i}$  is as in (2). For a > 0, define

$$R_n = [m_0(-a)\cdots m_{n-1}(-a)]^{-1} Z_n^{-a} \mathbf{1}_{\{Z_n > 0\}}, \quad n \ge 0.$$

**Lemma 4.1.**  $(R_n, \mathcal{F}_n)_{n\geq 0}$  is a supermartingale, where  $\mathcal{F}_n$  were defined in (8).

*Proof.* Using the decomposition (2) of  $Z_{n+1}$ , we have

$$Z_{n+1}^{-a} \mathbf{1}_{\{Z_{n+1}>0\}} = \left[\sum_{i=1}^{Z_n} X_{n,i}\right]^{-a} \mathbf{1}_{\{Z_n>0\}} \mathbf{1}_{\{Z_{n+1}>0\}}$$
$$= Z_n^{-a} \left[\frac{1}{Z_n} \sum_{i=1}^{Z_n} X_{n,i} \mathbf{1}_{\{X_{n,i}\}>0}\right]^{-a} \mathbf{1}_{\{Z_n>0\}} \mathbf{1}_{\{Z_{n+1}>0\}}$$
$$\leq Z_n^{-a} \frac{1}{Z_n} \sum_{i=1}^{Z_n} (X_{n,i} \mathbf{1}_{\{X_{n,i}\}>0})^{-a} \mathbf{1}_{\{Z_n>0\}} \mathbf{1}_{\{Z_{n+1}>0\}},$$

where the last inequality is due to the convexity property of the function  $x^{-a}(a > 0)$ .

Taking conditional expectation with respect to  $\mathcal{F}_n$  and  $P_{\xi}$  on both sides of the above inequality, we obtain that

$$E_{\xi}(Z_{n+1}^{-a}\mathbf{1}_{\{Z_{n+1}>0\}}|\mathcal{F}_n) \le Z_n^{-a}\mathbf{1}_{\{Z_n>0\}}m_n(-a),$$
(28)

which gives the desired result.

Since  $Z_0 = 1$ , by (28), we immediate obtain the following

**Lemma 4.2.** For a > 0, we have

$$E_{\xi} Z_n^{-a} \mathbf{1}_{\{Z_n > 0\}} \le m_0(-a) \cdots m_{n-1}(-a)$$
(29)

If the environment sequence  $\{\xi_n\}$  is independent and identically distributed, then

$$EZ_n^{-a} \mathbf{1}_{\{Z_n > 0\}} \le (Em_0(-a))^n.$$
(30)

Now we give the proof of Theorem 2.2.

Proof of Theorem 2.2. We shall only deal with the case  $k = \infty$ , as the case where  $k \in \mathbb{N}^*$  can be treated similarly.

Consider the probability space  $(\Gamma^* \times \Theta^{\mathbb{N}}, P^*)$  and define random variables  $Z_n^*$  as in the proof of Theorem 2.1. By definition,

$$u_n(Z_n^*, x) = P_{\xi}\left(\frac{1}{\sqrt{Z_n^*}}\sum_{j=1}^{Z_n^*}\frac{W_{\infty}(n, j) - 1}{\Delta_{\infty}(\theta^n \xi)} \le x\right)$$

By our hypothesis and the Berry-Esseen theorem (see e.g. Theorem 6 of (Petrov, 1995, p.115)), we have

$$\begin{aligned} |u_n(Z_n^*, x) - \Phi(x)| &\leq \frac{C_{\delta}}{(Z_n^*)^{1+\frac{\delta}{2}}} \sum_{j=1}^{Z_n^*} E_{\xi} \left| \frac{W_{\infty}(n, j) - 1}{\Delta_{\infty}(\theta^n \xi)} \right|^{2+\delta} \\ &= C_{\delta}(Z_n^*)^{-\frac{\delta}{2}} E_{\theta^n \xi} \left| \frac{W_{\infty} - 1}{\Delta_{\infty}} \right|^{2+\delta}, \end{aligned}$$

where  $C_{\delta}$  is the Berry-Esseen constant. Using this evaluation, we can derive that

$$\begin{aligned} |P_{\xi}(U_{n,\infty} \le x | Z_n > 0) - \Phi(x)| &\le E_{\xi}^* |u_n(Z_n^*, x) - \Phi(x)| \\ &\le C_{\delta} E_{\xi}^*(Z_n^*)^{-\frac{\delta}{2}} E_{\theta^n \xi} \left| \frac{W_{\infty} - 1}{\Delta_{\infty}} \right|^{2+\delta}. \end{aligned}$$

By the definition of  $Z_n^*$ , this implies that

$$|P_{\xi}(U_{n,\infty} \le x, Z_n > 0) - P_{\xi}(Z_n > 0)\Phi(x)|$$
  
$$\leq C_{\delta}E_{\xi}\left(Z_n^{-\frac{\delta}{2}}I_{\{Z_n > 0\}}\right)E_{\theta^n\xi}\left|\frac{W_{\infty} - 1}{\Delta_{\infty}}\right|^{2+\delta}.$$
 (31)

Using (31) and the fact that the sequence  $\{\xi_n\}$  is independent and identically distributed, we get

$$|P(U_{n,\infty} \le x, Z_n > 0) - P(Z_n > 0)\Phi(x)| \\ \le E|P_{\xi}(U_{n,\infty} \le x, Z_n > 0) - P_{\xi}(Z_n > 0)\Phi(x) \\ \le C_{\delta}E\left(Z_n^{-\frac{\delta}{2}}I_{\{Z_n > 0\}}\right)E\left|\frac{W_{\infty} - 1}{\Delta_{\infty}}\right|^{2+\delta}.$$

Together with (30), we obtain that

$$|P(U_{n,\infty} \le x | Z_n > 0) - \Phi(x)| \le \frac{C_{\delta}(Em_0(-\frac{\delta}{2}))^n E |\frac{W_{\infty} - 1}{\Delta_{\infty}}|^{2+\delta}}{P(Z_n > 0)}.$$

Then the proof is completed.

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