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Central limit theorems for a supercritical branching process in a random environment

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Abstract

For a supercritical branching process (Z_n) in a stationary and ergodic environment ξ , we study the rate of convergence of the normalized population $W_n = Z_n/E[Z_n|\xi]$ to its limit W_∞ : we show a central limit theorem for $W_\infty - W_n$ with suitable normalization and derive a Berry-Esseen bound for the rate of convergence in the central limit theorem when the environment is independent and identically distributed. Similar results are also shown for $W_{n+k} - W_n$ for each fixed $k \in \mathbb{N}^*$.

Keywords: Branching processes, random environment, central limit theorem, martingale, rate of convergence.

2000 MSC: 60J80, 60F05

1. Introduction

Galton-Watson processes have been studied by many authors, due to a wide range of applications. See for example the books by Harris (1963) and Athreya and Ney (1972). In a Galton-Watson process $\{Z_n, n = 0, 1, \dots\}$, particles behave independently, each gives birth to a random number of particles of the next generation with a fixed distribution $\{p_k : k = 0, 1, \dots\}$.

A branching process in a random environment is a natural and important extension of the Galton-Watson process. It is a class of non-homogeneous Galton-Watson processes indexed by a time-environment $\xi = (\xi_0, \xi_1, \xi_2, \dots)$, which is

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supposed to be stationary and ergodic; given the environment ξ , the particles of n -th generation have offspring distribution $\{p_k(\xi_n) : k \in \mathbb{N}\}$ depending on ξ_n . For first important works on the subject, see Smith and Wilkinson (1969) and Athreya and Karlin (1971a,b).

For a Galton-Watson process with $Z_0 = 1$ and $m = EZ_1 \in (0, \infty)$, it is well known that $\{W_n = Z_n/m^n : n = 0, 1, \dots\}$ forms a non-negative martingale, and converges almost surely to a random variable W_∞ . For the convergence rate of the martingale, Heyde (1971) and Bühler (1969) obtained respectively that if $\text{Var}(Z_1) = \sigma^2 < \infty$, then conditioned on $Z_n > 0$, the conditional laws of

$$(m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n^{-\frac{1}{2}} m^n (W_\infty - W_n)$$

and

$$(m^k / (m^k - 1))^{\frac{1}{2}} (m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n^{-\frac{1}{2}} m^n (W_{n+k} - W_n) \quad k \in \mathbb{N}^*$$

converge to the normal law $\mathcal{N}(0, 1)$; Heyde and Brown (1971) gave an estimation of its convergence rate under a third moment condition.

The object of this paper is to extend the theorems of Bühler (1969), Heyde (1971) and Heyde and Brown (1971) to a branching process in a random environment. The main results are Theorems 2.1 and 2.2.

2. Main Results

As usual, we write $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}^* = \{1, 2, \dots\}$ and \mathbb{R} for the set of real numbers.

Let us first recall the definition of a branching process in a random environment. For reference on the subject, see for example Athreya and Karlin (1971a,b), and Athreya and Ney (1972).

A *random environment* $\xi = (\xi_n)$ is formulated as a stationary and ergodic sequence of random variables taking values in some measurable space (Θ, \mathcal{F}) . Each realization of ξ_n corresponds to a probability distribution $p(\xi_n) = \{p_i(\xi_n) : i \in \mathbb{N}\}$ where

$$p_i(\xi_n) \geq 0, \quad \sum_{i=0}^{\infty} p_i(\xi_n) = 1, \quad 0 < \sum_{i=0}^{\infty} i p_i(\xi_n) < \infty. \quad (1)$$

Without loss of generality, we can take ξ_n as coordinate functions defined on the product space $(\Theta^{\mathbb{N}}, \mathcal{F}^{\otimes \mathbb{N}})$, equipped with a probability law τ , which is invariant and ergodic under the usual shift transformation θ on $\Theta^{\mathbb{N}}$: $\theta(\xi_0, \xi_1, \dots) = (\xi_1, \xi_2, \dots)$. A branching process $(Z_n)_{n \geq 0}$ in the random environment ξ is a class of non-homogeneous branching processes indexed by ξ . By definition,

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \quad n \geq 0, \quad (2)$$

where given ξ , $\{X_{n,i} : n \geq 0, i \geq 1\}$ is a family of (conditionally) independent random variables, each $X_{n,i}$ has the common law $p(\xi_n)$. Notice that when all ξ_n are the same constant, (Z_n) reduces to the classical Galton-Waston process.

Let (Γ, P_ξ) be the probability space under which the process is defined when the environment ξ is given. As usual, P_ξ is called *quenched law*. The total probability space can be formulated as the product space $(\Gamma \times \Theta^\mathbb{N}, P)$, where $P = P_\xi \otimes \tau$ in the sense that for all measurable and positive function g , we have

$$\int g dP = \int \int g(\xi, y) dP_\xi(y) d\tau(\xi), \quad (3)$$

(recall that τ is the law of the environment ξ). The total probability P is usually called *annealed law*. The quenched law P_ξ may be considered to be the conditional probability of the annealed law P given ξ . The expectation with respect to P_ξ (resp. P) will be denoted E_ξ (resp. E).

For $n \geq 0$, define

$$m_n(a) = m(\xi_n, a) = \sum_{i=1}^{\infty} i^a p_i(\xi_n), \quad a \in \mathbb{R}, \quad (4)$$

$$m_n = m_n(1), \quad \sigma_n^2 = m_n(2) - m_n^2, \quad (5)$$

$$\pi_0 = 1 \quad \text{and} \quad \pi_n = \pi_n(\xi) = m_0 \cdots m_{n-1} \quad \text{for } n \geq 1. \quad (6)$$

Then $\pi_n = E_\xi Z_n$ for $n \geq 0$. It is well known that

$$W_n = Z_n / \pi_n \quad (7)$$

is a martingale with respect to the filtration

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_n = \sigma\{\xi, X_{j,i} : j \leq n-1, i \geq 1\} \quad (n \geq 1), \quad (8)$$

so that the limit

$$W_\infty = \lim_{n \rightarrow \infty} W_n \quad (9)$$

exists almost surely (a.s.) with $EW \leq 1$ by Fatou's lemma.

Throughout the paper, we always assume that

$$E \ln m_0 > 0 \quad \text{and} \quad E \left(\frac{Z_1}{m_0} \ln^+ Z_1 \right) < \infty. \quad (10)$$

The first assumption ensures that the process is *supercritical* (cf. Athreya and Karlin (1971a)); the second one together with the first implies that $EW_\infty = 1$; moreover,

$$P_\xi(W_\infty > 0) = P_\xi(Z_n \rightarrow \infty) = \lim_{n \rightarrow \infty} P_\xi(Z_n > 0) = 1 - q(\xi) > 0 \quad \text{a.s.}, \quad (11)$$

where $q(\xi) = \lim_{n \rightarrow \infty} P_\xi(Z_n = 0)$ is the extinct probability.

In this paper, we search for central limit theorems on $W_\infty - W_n$ and $W_{n+k} - W_n$ for fixed $k \geq 1$ with an appropriate normalization. Assume that $m_0(2) < \infty$ a.s., and let

$$\Delta_k^2 = \Delta_k^2(\xi) = \sum_{0 \leq i < k} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2} \quad \text{for } k \in \mathbb{N}^* \cup \{\infty\}.$$

Then for $k \in \mathbb{N}^*$, $\Delta_k^2(\xi)$ is the variance of W_k under P_ξ ; $\Delta_\infty^2(\xi)$ is the variance of W_∞ if the series converges (i.e. $\Delta_\infty^2(\xi) < \infty$): see Lemma 3.2.

We can now formulate our first main result.

Theorem 2.1. *Suppose that (10) holds and that $m_0(2) < \infty$ a.s.. In the case where $k = \infty$, assume additionally that $E \ln^+(\sigma_0^2/m_0^2) < \infty$. Write*

$$U_{n,k} = \frac{\pi_n(W_{n+k} - W_n)}{\sqrt{Z_n} \Delta_k(\theta^n \xi)} \quad \text{for } k \in \mathbb{N}^* \cup \{\infty\},$$

where by convention $W_{n+k} = W_\infty$ if $k = \infty$. Then for each $k \in \mathbb{N}^* \cup \{\infty\}$, as $n \rightarrow \infty$,

$$\sup_{x \in \mathbb{R}} |P_\xi(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| \rightarrow 0 \quad \text{in } L^1, \quad (12)$$

and

$$\sup_{x \in \mathbb{R}} |P(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| \rightarrow 0. \quad (13)$$

We believe that for each $k \in \mathbb{N} \cup \{\infty\}$

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |P_\xi(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| = 0 \quad \text{a.s..}$$

We notice that in the classical Galton-Waston process, (13) reduces to the results of Bühler (1969) and Heyde (1971). Our second main result concerns the rate of convergence in the above central limit theorem for a branching process with an independent and identically distributed environment.

Theorem 2.2. *Let the environment $\{\xi_n\}$ be independent and identically distributed. Assume that (10) holds and that $m_0(2) < \infty$ a.s.. In the case where $k = \infty$, assume additionally that $E \ln^+(\sigma_0^2/m_0^2) < \infty$. For each $k \in \mathbb{N}^* \cup \{\infty\}$, if $E \left| \frac{W_k - 1}{\Delta_k} \right|^{2+\delta} < \infty$ for some $\delta \in (0, 1]$, then*

$$\sup_{x \in \mathbb{R}} |P(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| \leq \frac{C_\delta \left(E m_0(-\frac{\delta}{2}) \right)^n E \left| \frac{W_k - 1}{\Delta_k} \right|^{2+\delta}}{P(Z_n > 0)}, \quad (14)$$

where $U_{n,k}$ is defined in Theorem 2.1 and C_δ is the Berry-Esseen constant.

Remark 2.3. It maybe useful to notice that if

$$E(Z_1/m_0)^{2+\delta} < \infty, \quad E m_0^{-(1+\delta)} < 1 \quad \text{and} \quad m_0(2)/m_0^2 \geq A$$

for some constant $A > 1$, then $E \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta} < +\infty$. In fact by Theorem 3 of Guivarc'h and Liu (2001), the first two conditions imply that $E|W_\infty - 1|^{2+\delta} < \infty$, while the last one implies that $\Delta_\infty^2 \geq A - 1 > 0$.

For the classical Galton-Watson process with $\delta = 1$, Theorem 2.2 reduces to Theorem 2 of (Heyde and Brown, 1971, p.272).

3. Proof of Theorem 2.1

In this section, we consider a central limit theorem under a second moment condition in proving Theorem 2.1. We first give some lemmas.

Lemma 3.1 (Grincevičius (1974)). *Let $\{(\alpha_n, \beta_n), n = 0, 1, 2, \dots\}$ be a stationary and ergodic sequence of random variables with values in \mathbb{R}^2 . If*

$$E \ln |\alpha_0| < 0 \quad \text{and} \quad E \ln^+ |\beta_0| < \infty,$$

then

$$\sum_{n=0}^{\infty} |\alpha_0 \alpha_1 \cdots \alpha_{n-1} \beta_n| < \infty \quad a.s.$$

In fact, the result is a direct consequence of the ergodic theorem and Cauchy's criterion for the convergence of series.

Using the above lemma, we can easily obtain the following result.

Lemma 3.2. *Under the assumptions in Theorem 2.1, for each $k \in \mathbb{N}^* \cup \{\infty\}$,*

$$\text{Var}_\xi(W_k) = \Delta_k^2(\xi) = \sum_{0 \leq i < k} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}. \quad (15)$$

This has been known for branching processes in varying environment, see e.g. (Jagers, 1974, p.175) in a slightly different form. For reader's convenience, we present a proof in the following.

Proof of Lemma 3.2. By (2) and the definition of W_n , we have

$$W_{n+1} - W_n = \frac{1}{\pi_n} \sum_{j=1}^{Z_n} \left(\frac{X_{n,j}}{m_n} - 1 \right).$$

Recall that under P_ξ , the random variables $\{X_{n,j}\}$ are independent of each other and have the common distribution $p(\xi_n)$ with expectation m_n . Hence a direct calculation shows that

$$\begin{aligned} E_\xi((W_{n+1} - W_n)^2) &= E_\xi \left(E_\xi((W_{n+1} - W_n)^2 | \mathcal{F}_n) \right) \\ &= E_\xi \left(\frac{Z_n}{\pi_n^2} \frac{\sigma_n^2}{m_n^2} \right) = \frac{1}{\pi_n} \frac{\sigma_n^2}{m_n^2}. \end{aligned}$$

As $\{W_n\}$ is a martingale, it follows that

$$E_\xi W_k^2 = E_\xi W_0^2 + \sum_{i=0}^{k-1} E((W_{i+1} - W_i)^2) = 1 + \sum_{i=0}^{k-1} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}.$$

Therefore for each fixed integer k ,

$$\text{Var}_\xi(W_k) = E_\xi(W_k^2) - 1 = \sum_{i=0}^{k-1} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}.$$

Now we turn to the calculation of $\text{Var}_\xi(W_\infty)$. By Lemma 3.1, when $E \ln m_0 > 0$ and $E \ln^+ \frac{\sigma_i^2}{m_i^2} < \infty$,

$$\sup_n E_\xi(W_n^2) = 1 + \sum_{i=0}^{\infty} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2} < \infty \quad \text{a.s.}$$

So W_n converges to W_∞ in L^2 under P_ξ and

$$E_\xi(W_\infty^2) = \lim_{k \rightarrow \infty} E_\xi(W_k^2) = 1 + \sum_{i=0}^{\infty} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}.$$

It follows that

$$\text{Var}_\xi(W_\infty) = E_\xi(W_\infty^2) - 1 = \Delta_\infty^2(\xi) = \sum_{i=0}^{\infty} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2} < \infty \quad \text{a.s.}$$

□

To give our next lemma, we will need some notations, which will also be used in the proof of the main theorems. By definition,

$$Z_{n+k} = \sum_{j=1}^{Z_n} Z_k(n, j), \quad (16)$$

where $Z_k(n, j)$ denotes the number of descendants in the $(n+k)$ -th generation of the j -th particle among the Z_n particles in n -th generation.

Writing $W_k(n, j) = \frac{Z_k(n, j)}{\pi_k(\theta^n \xi)}$ and using (16), we obtain the following decomposition:

$$\pi_n(W_{n+k} - W_n) = \sum_{j=1}^{Z_n} (W_k(n, j) - 1). \quad (17)$$

Letting $k \rightarrow \infty$, it follows that

$$\pi_n(W_\infty - W_n) = \sum_{j=1}^{Z_n} (W_\infty(n, j) - 1), \quad (18)$$

where under P_ξ , the random variables $\{W_\infty(n, j)\}_j$ are independent of each other and have the common conditional distribution

$$P_\xi(W_\infty(n, j) \in \cdot) = P_{\theta^n \xi}(W_\infty \in \cdot).$$

Lemma 3.3. *Suppose that the assumptions of Theorem 2.1 hold. Let $r_n \in \mathbb{N}$ with $r_n \rightarrow \infty$. For $k \in \mathbb{N}^* \cup \{\infty\}$, define*

$$Y_{k,n} = \frac{1}{\sqrt{r_n}} \sum_{j=1}^{r_n} \frac{W_k(n, j) - 1}{\Delta_k(\theta^n \xi)}.$$

Fix $k \in \mathbb{N}^ \cup \{\infty\}$. Then for each subsequence $\{n'\}$ of \mathbb{N} with $n' \rightarrow \infty$, there is a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \rightarrow \infty$ such that for a.e. ξ and all $x \in \mathbb{R}$, as $n'' \rightarrow \infty$,*

$$P_\xi(Y_{k,n''} \leq x) \rightarrow \Phi(x).$$

Proof. Fix $k \in \mathbb{N}^* \cup \{\infty\}$. In order to use Lindeberg's theorem, for $n \in \mathbb{N}$ and $\epsilon > 0$, we consider the quantity

$$L_k(\xi, \epsilon, n) = \frac{1}{r_n} \sum_{j=1}^{r_n} E_\xi \left(\left(\frac{W_k(n, j) - 1}{\Delta_k(\theta^n \xi)} \right)^2 ; \left| \frac{W_k(n, j) - 1}{\Delta_k(\theta^n \xi) \sqrt{r_n}} \right| > \epsilon \right),$$

where for a set A , we write $E_\xi(x; A)$ for $E_\xi(X \mathbf{1}_A)$, $\mathbf{1}_A$ denoting the indicator function of A . By the stationarity and ergodicity of the environment, for all $\epsilon > 0$, as $n \rightarrow \infty$,

$$EL_k(\xi, \epsilon, n) = E \left[\left(\frac{W_k - 1}{\Delta_k} \right)^2 ; \left| \frac{W_k - 1}{\Delta_k} \right| > \sqrt{r_n} \epsilon \right] \rightarrow 0. \quad (19)$$

Let $\{n'\}$ be a subsequence of \mathbb{N} . Notice that from (19), we can choose a subsequence $\{n''\}$ for which $L_k(\xi, \epsilon, n'') \rightarrow 0$ a.s., but this sequence may depend of ϵ . We will use a diagonal argument to select a subsequence $\{n''\}$ of $\{n'\}$ such that a.s. $L_k(\xi, \epsilon, n'') \xrightarrow{n'' \rightarrow \infty} 0$ for all $\epsilon > 0$. Set

$$\epsilon_m = 1/m \quad \text{for } m \geq 1.$$

Let $\{n_{0,i}\} = \{n'\}$. Because of (19), there is a subsequence $\{n_{1,i}\}$ of $\{n_{0,i}\}$ and a set Λ_1 with $\tau(\Lambda_1) = 1$ such that $\forall \xi \in \Lambda_1$,

$$\lim_{i \rightarrow \infty} L_k(\xi, \epsilon_1, n_{1,i}) = 0.$$

Inductively for $m \geq 1$, when Λ_m and $\{n_{m,i}\}$ are defined such that $\tau(\Lambda_m) = 1$ and $\forall \xi \in \Lambda_m$, $L_k(\xi, \epsilon_m, n_{m,i}) \rightarrow 0$, there is a subsequence $\{n_{m+1,i}\} \subset \{n_{m,i}\}$ and a set Λ_{m+1} with $\tau(\Lambda_{m+1}) = 1$ such that $\forall \xi \in \Lambda_{m+1}$,

$$\lim_{i \rightarrow \infty} L_k(\xi, \epsilon_{m+1}, n_{m+1,i}) = 0.$$

We now consider the diagonal sequence $\{n_{i,i}\}_{i \geq 1}$ and $\Lambda = \bigcap_{j=1}^{\infty} \Lambda_j$. For each fixed $\epsilon > 0$, let $m \geq \frac{1}{\epsilon}$. Then $\epsilon_m \leq \epsilon$ and by the monotonicity of $L_k(\xi, \epsilon, n)$ in ϵ , we see that $\forall \xi \in \Lambda$,

$$L_k(\xi, \epsilon, n_{m,i}) \leq L_k(\xi, \epsilon_m, n_{m,i}) \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

As $\{n_{i,i}\}$ is a subsequence of $\{n_{m,i}\}$ whenever $i > m$, this implies that

$$\lim_{i \rightarrow \infty} L_k(\xi, \epsilon, n_{i,i}) = 0. \quad (20)$$

Since $\tau(\Lambda) = 1$, we have shown that for all $\epsilon > 0$, (20) holds a.s.. It follows that a.s. (20) holds for all rational $\epsilon > 0$, and therefore for all real $\epsilon > 0$ by the monotonicity of $L_k(\xi, \epsilon, n_{i,i})$ in ϵ . So by Lindeberg's theorem, it is a.s. that for all $x \in \mathbb{R}$, as $i \rightarrow \infty$,

$$P_{\xi}(Y_{k,n_{i,i}} \leq x) \rightarrow \Phi(x).$$

Thus the lemma has been proved with $\{n''\} = \{n_{i,i}\}$. \square

Proof of Theorem 2.1. We shall only deal with the case where $k = \infty$, as the case where $k \in \mathbb{N}^*$ can be treated similarly.

We first prove the following assertion: *for each sequence $\{n'\}$ of \mathbb{N} with $n' \rightarrow \infty$, there exist a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \rightarrow \infty$ such that for a.e. ξ and all x , as $n'' \rightarrow \infty$,*

$$P_{\xi}(U_{n'',\infty} \leq x | Z_{n''} > 0) \rightarrow \Phi(x). \quad (21)$$

By the definition of $U_{n,\infty}$ and the relation (18), we get :

$$U_{n,\infty} = \frac{\pi_n(W_{\infty} - W_n)}{\sqrt{Z_n} \Delta_{\infty}(\theta^n \xi)} = \frac{1}{\sqrt{Z_n}} \sum_{j=1}^{Z_n} \frac{W_{\infty}(n, j) - 1}{\Delta_{\infty}(\theta^n \xi)},$$

where we recall that under P_{ξ} , $\{W_{\infty}(n, j), j \geq 1\}$ is a family of random variables independent of each other and independent of Z_n , each has the same law as W_{∞} under $P_{\theta^n \xi}$. Set

$$u_n(r, x) = P_{\xi} \left(\frac{1}{\sqrt{r}} \sum_{j=1}^r \frac{W_{\infty}(n, j) - 1}{\Delta_{\infty}(\theta^n \xi)} \leq x \right), \quad r \in \mathbb{N}^*, \quad x \in \mathbb{R}.$$

Then

$$\begin{aligned} P_{\xi}(U_{n,\infty} \leq x | Z_n > 0) &= [P_{\xi}(Z_n > 0)]^{-1} \sum_{r=1}^{\infty} P_{\xi}(U_{n,\infty} \leq x, Z_n = r) \\ &= \sum_{r=1}^{\infty} u_n(r, x) \frac{P_{\xi}(Z_n = r)}{P_{\xi}(Z_n > 0)}. \end{aligned} \quad (22)$$

To show the main idea, let us first consider the special case where $q(\xi) = 0$ a.s., i.e. for a.e. ξ ,

$$Z_n \rightarrow \infty \quad P_\xi^* \text{-a.s.}$$

In this case, the relation (22) becomes

$$P_\xi(U_{n,\infty} \leq x) = \sum_{r=1}^{\infty} u_n(r, x) P_\xi(Z_n = r) = E_\xi u_n(Z_n, x).$$

By Lemma 3.3, for each subsequence $\{n'\}$ of \mathbb{N} with $n' \rightarrow \infty$, there exist a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \rightarrow \infty$ such that for a.e. ξ and all x , as $n'' \rightarrow \infty$,

$$u_{n''}(Z_{n''}, x) \rightarrow \Phi(x).$$

By the dominated convergence theorem, for a.e. ξ and all x , as $n'' \rightarrow \infty$,

$$P_\xi(U_{n'',\infty} \leq x) = E_\xi[u_{n''}(Z_{n''}, x)] \rightarrow \Phi(x).$$

So we have proved (21) when $q(\xi) = 0$ a.s..

We now consider the general case where $0 \leq q(\xi) < 1$ a.s..

For each $\xi \in \Theta^\mathbb{N}$, let Z_n^* be random variables defined on some probability space $(\Gamma^*, \mathbb{P}_\xi^*)$ with law

$$P_\xi^*(Z_n^* = r) = \frac{P_\xi(Z_n = r)}{P_\xi(Z_n > 0)}, \quad r \in \mathbb{N}^*.$$

Then

$$P_\xi(U_{n,\infty} \leq x | Z_n > 0) = E_\xi^* u_n(Z_n^*, x),$$

where E_ξ^* denotes the expectation with respect to P_ξ^* .

Let $\{n'\}$ be a sequence of \mathbb{N} with $n' \rightarrow \infty$. If for a.e. ξ ,

$$Z_{n'}^* \rightarrow \infty \quad P_\xi^* \text{-a.s.},$$

then as above we can use Lemma 3.3 and the dominated convergence theorem to show that there is a sequence $\{n''\}$ of $\{n'\}$ with $n'' \rightarrow \infty$ such that for all x , as $n'' \rightarrow \infty$,

$$E_\xi^* u_{n''}(Z_{n''}^*, x) \rightarrow \Phi(x).$$

By the fact that $Z_n^* \rightarrow \infty$ in probability under P_ξ^* , we can choose a subsequence for which $Z_n^* \rightarrow \infty$ P_ξ^* -a.s.. But to apply Lemma 3.3, we need that the sequence does not depend on ξ . We therefore pass to the probability P^* to overcome this difficulty, where $P^* = P_\xi^* \otimes \tau$ is defined on the product space $\Gamma^* \times \Theta^\mathbb{N}$ just as P was defined on $\Gamma \times \Theta^\mathbb{N}$.

Notice that for each $r \in \mathbb{N}^*$, as $n \rightarrow \infty$,

$$P_\xi^*(Z_n^* = r) = \frac{P_\xi(Z_n = r)}{P_\xi(Z_n > 0)} \rightarrow 0,$$

where the last step holds as $Z_n \rightarrow \infty$ a.s. on the survival event $S = \{Z_n > 0, \forall n \geq 1\}$ (see (11) or Tanny (1977) for this fact). Then $Z_n^* \rightarrow +\infty$ in probability under P_ξ^* . By the dominated convergence theorem, this implies that $Z_n^* \rightarrow +\infty$ in probability under P^* . Therefore for each subsequence $\{n'\}$ of \mathbb{N} with $n' \rightarrow \infty$, there is a subsequence $\{\tilde{n}\} \subset \{n'\}$ with $\tilde{n} \rightarrow \infty$ such that $Z_{\tilde{n}}^* \rightarrow +\infty$ a.s. under P^* . This implies that for a.e. ξ , as $\tilde{n} \rightarrow \infty$,

$$Z_{\tilde{n}}^* \rightarrow +\infty \quad P_\xi^* \text{-a.s.}$$

Now by Lemma 3.3, there exists a subsequence $\{n''\}$ of $\{\tilde{n}\}$ such that for a.e. ξ and all x , as $n'' \rightarrow \infty$,

$$u_{n''}(Z_{n''}^*, x) \rightarrow \Phi(x).$$

By the dominated convergence theorem, for almost every ξ and each x , as $n'' \rightarrow \infty$,

$$P_\xi(U_{n'', \infty} \leq x | Z_{n''} > 0) = E_\xi u_{n''}(Z_{n''}^*, x) \rightarrow \Phi(x). \quad (23)$$

So combining the above two cases, we have proved (21).

Since $P_\xi(U_{n'', \infty} \leq x | Z_{n''} > 0)$ are distribution functions and $\Phi(x)$ is a continuous distribution function, by Dini's Theorem we see that for a.e. ξ , as $n'' \rightarrow \infty$,

$$\sup_x |P_\xi(U_{n'', \infty} \leq x | Z_{n''} > 0) - \Phi(x)| \rightarrow 0. \quad (24)$$

By the dominated convergence theorem, (24) implies that as $n'' \rightarrow \infty$,

$$E \sup_x |P_\xi(U_{n'', \infty} \leq x | Z_{n''} > 0) - \Phi(x)| \rightarrow 0. \quad (25)$$

Therefore we have proved that for each sequence $\{n'\}$ of \mathbb{N} with $n' \rightarrow \infty$, there is a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \rightarrow \infty$ such that (25) holds. Hence

$$E \sup_x |P_\xi(U_{n, \infty} \leq x | Z_n > 0) - \Phi(x)| \rightarrow 0.$$

This gives (12) for $k = \infty$. The proof for $k \in \mathbb{N}^*$ is similar.

We now begin to prove (13).

As we have proved that for each subsequence $\{n'\}$ of \mathbb{N} , there is a subsequence $\{n''\}$ so that (24) holds, which implies: for a.e. ξ and all $x \in \mathbb{R}$, as $n'' \rightarrow \infty$,

$$|P_\xi(U_{n'', \infty} \leq x | Z_{n''} > 0) - \Phi(x)| \rightarrow 0.$$

It follows that for a.e. ξ and all $x \in \mathbb{R}$,

$$|P_\xi(U_{n'', \infty} \leq x, Z_{n''} > 0) - P_\xi(Z_{n''} > 0)\Phi(x)| \rightarrow 0.$$

So by the dominated convergence theorem, we see that for each $x \in \mathbb{R}$, as $n'' \rightarrow \infty$,

$$|P(U_{n'', \infty} \leq x, Z_{n''} > 0) - P(Z_{n''} > 0)\Phi(x)| \rightarrow 0,$$

and hence

$$P(U_{n'',\infty} \leq x | Z_{n''} > 0) \rightarrow \Phi(x).$$

By Dini's Theorem, it follows that

$$\sup_x |P(U_{n'',\infty} \leq x | Z_{n''} > 0) - \Phi(x)| \rightarrow 0. \quad (26)$$

Therefore we have proved that for each sequence $\{n'\}$ of \mathbb{N} , there is a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \rightarrow \infty$ such that (26) holds. Hence

$$\sup_x |P(U_{n,\infty} \leq x | Z_n > 0) - \Phi(x)| \rightarrow 0.$$

Thus the proof is completed. \square

4. Proof of Theorem 2.2

In this section, we consider the rate of convergence in the central limit theorem under a moment condition of order $2 + \delta$, in proving Theorem 2.2.

Notice that by the definition (4) of $m_n(a)$, we have

$$m_n(a) = E_\xi X_{n,i}^a \text{ if } a > 0, \quad m_n(a) = E_\xi X_{n,i}^a \mathbf{1}_{\{X_{n,i} > 0\}} \text{ if } a \leq 0, \quad (27)$$

where $X_{n,i}$ is as in (2). For $a > 0$, define

$$R_n = [m_0(-a) \cdots m_{n-1}(-a)]^{-1} Z_n^{-a} \mathbf{1}_{\{Z_n > 0\}}, \quad n \geq 0.$$

Lemma 4.1. $(R_n, \mathcal{F}_n)_{n \geq 0}$ is a supermartingale, where \mathcal{F}_n were defined in (8).

Proof. Using the decomposition (2) of Z_{n+1} , we have

$$\begin{aligned} Z_{n+1}^{-a} \mathbf{1}_{\{Z_{n+1} > 0\}} &= \left[\sum_{i=1}^{Z_n} X_{n,i} \right]^{-a} \mathbf{1}_{\{Z_n > 0\}} \mathbf{1}_{\{Z_{n+1} > 0\}} \\ &= Z_n^{-a} \left[\frac{1}{Z_n} \sum_{i=1}^{Z_n} X_{n,i} \mathbf{1}_{\{X_{n,i} > 0\}} \right]^{-a} \mathbf{1}_{\{Z_n > 0\}} \mathbf{1}_{\{Z_{n+1} > 0\}} \\ &\leq Z_n^{-a} \frac{1}{Z_n} \sum_{i=1}^{Z_n} (X_{n,i} \mathbf{1}_{\{X_{n,i} > 0\}})^{-a} \mathbf{1}_{\{Z_n > 0\}} \mathbf{1}_{\{Z_{n+1} > 0\}}, \end{aligned}$$

where the last inequality is due to the convexity property of the function x^{-a} ($a > 0$).

Taking conditional expectation with respect to \mathcal{F}_n and P_ξ on both sides of the above inequality, we obtain that

$$E_\xi(Z_{n+1}^{-a} \mathbf{1}_{\{Z_{n+1} > 0\}} | \mathcal{F}_n) \leq Z_n^{-a} \mathbf{1}_{\{Z_n > 0\}} m_n(-a), \quad (28)$$

which gives the desired result. \square

Since $Z_0 = 1$, by (28), we immediately obtain the following

Lemma 4.2. *For $a > 0$, we have*

$$E_\xi Z_n^{-a} \mathbf{1}_{\{Z_n > 0\}} \leq m_0(-a) \cdots m_{n-1}(-a) \quad (29)$$

If the environment sequence $\{\xi_n\}$ is independent and identically distributed, then

$$E Z_n^{-a} \mathbf{1}_{\{Z_n > 0\}} \leq (E m_0(-a))^n. \quad (30)$$

Now we give the proof of Theorem 2.2.

Proof of Theorem 2.2. We shall only deal with the case $k = \infty$, as the case where $k \in \mathbb{N}^*$ can be treated similarly.

Consider the probability space $(\Gamma^* \times \Theta^{\mathbb{N}}, P^*)$ and define random variables Z_n^* as in the proof of Theorem 2.1. By definition,

$$u_n(Z_n^*, x) = P_\xi \left(\frac{1}{\sqrt{Z_n^*}} \sum_{j=1}^{Z_n^*} \frac{W_\infty(n, j) - 1}{\Delta_\infty(\theta^n \xi)} \leq x \right).$$

By our hypothesis and the Berry-Esseen theorem (see e.g. Theorem 6 of (Petrov, 1995, p.115)), we have

$$\begin{aligned} |u_n(Z_n^*, x) - \Phi(x)| &\leq \frac{C_\delta}{(Z_n^*)^{1+\frac{\delta}{2}}} \sum_{j=1}^{Z_n^*} E_\xi \left| \frac{W_\infty(n, j) - 1}{\Delta_\infty(\theta^n \xi)} \right|^{2+\delta} \\ &= C_\delta (Z_n^*)^{-\frac{\delta}{2}} E_{\theta^n \xi} \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}, \end{aligned}$$

where C_δ is the Berry-Esseen constant. Using this evaluation, we can derive that

$$\begin{aligned} |P_\xi(U_{n,\infty} \leq x | Z_n > 0) - \Phi(x)| &\leq E_\xi^* |u_n(Z_n^*, x) - \Phi(x)| \\ &\leq C_\delta E_\xi^* (Z_n^*)^{-\frac{\delta}{2}} E_{\theta^n \xi} \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}. \end{aligned}$$

By the definition of Z_n^* , this implies that

$$\begin{aligned} |P_\xi(U_{n,\infty} \leq x, Z_n > 0) - P_\xi(Z_n > 0)\Phi(x)| \\ \leq C_\delta E_\xi \left(Z_n^{-\frac{\delta}{2}} I_{\{Z_n > 0\}} \right) E_{\theta^n \xi} \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}. \end{aligned} \quad (31)$$

Using (31) and the fact that the sequence $\{\xi_n\}$ is independent and identically distributed, we get

$$\begin{aligned} &|P(U_{n,\infty} \leq x, Z_n > 0) - P(Z_n > 0)\Phi(x)| \\ &\leq E |P_\xi(U_{n,\infty} \leq x, Z_n > 0) - P_\xi(Z_n > 0)\Phi(x)| \\ &\leq C_\delta E \left(Z_n^{-\frac{\delta}{2}} I_{\{Z_n > 0\}} \right) E \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}. \end{aligned}$$

Together with (30), we obtain that

$$|P(U_{n,\infty} \leq x | Z_n > 0) - \Phi(x)| \leq \frac{C_\delta (Em_0(-\frac{\delta}{2}))^n E| \frac{W_\infty - 1}{\Delta_\infty} |^{2+\delta}}{P(Z_n > 0)}.$$

Then the proof is completed. \square

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References

- Athreya, K. B., Karlin, S., 1971a. On branching processes with random environments. I. Extinction probabilities. *Ann. Math. Statist.* 42, 1499–1520.
- Athreya, K. B., Karlin, S., 1971b. Branching processes with random environments. II. Limit theorems. *Ann. Math. Statist.* 42, 1843–1858.
- Athreya, K. B., Ney, P. E., 1972. *Branching processes*. Springer-Verlag, New York, die Grundlehren der mathematischen Wissenschaften, Band 196.
- Bühler, W. J., 1969. Ein zentraler Grenzwertsatz für Verzweigungsprozesse. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 11, 139–141.
- Grincevičius, A. K., 1974. The continuity of the distribution of a certain sum of dependent variables that is connected with independent walks on lines. *Teor. Veroyatnost. i Primenen.* 19, 163–168.
- Guivarc’h, Y., Liu, Q., 2001. Propriétés asymptotiques des processus de branchement en environnement aléatoire. *C. R. Acad. Sci. Paris Sér. I Math.* 332 (4), 339–344.
- Harris, T. E., 1963. *The theory of branching processes*. Die Grundlehren der Mathematischen Wissenschaften, Bd. 119. Springer-Verlag, Berlin.
- Heyde, C. C., 1971. Some central limit analogues for supercritical Galton-Watson processes. *J. Appl. Probability* 8, 52–59.
- Heyde, C. C., Brown, B. M., 1971. An invariance principle and some convergence rate results for branching processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 20, 271–278.

- Jagers, P., 1974. Galton-Watson processes in varying environments. *J. Appl. Probability* 11, 174–178.
- Petrov, V. V., 1995. Limit theorems of probability theory. Vol. 4 of *Oxford Studies in Probability*. The Clarendon Press Oxford University Press, New York, sequences of independent random variables, Oxford Science Publications.
- Smith, W. L., Wilkinson, W. E., 1969. On branching processes in random environments. *Ann. Math. Statist.* 40, 814–827.
- Tanny, D., 1977. Limit theorems for branching processes in a random environment. *Ann. Probability* 5 (1), 100–116.