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Accumulation points of the sets of real parts of zeros of the partial sums of the Riemann zeta function

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ABSTRACT Let $\zeta_n(z) := \sum_{k=1}^n \frac{1}{k^z}$ be the n th partial sum of the Riemann zeta function. In this paper it is shown that for every integer $n > 2$ there exists $\delta_n > 0$ such that $R^{(n)} := \{\Re z : \zeta_n(z) = 0\}$ contains the interval $[-\delta_n, b^{(n)}]$, where $b^{(n)} := \sup \{\Re z : \zeta_n(z) = 0\}$. It is also demonstrated that $b^{(n)}$ is positive for all $n > 2$. Then, for every $n > 2$, $\zeta_n(z)$ possesses infinitely many zeros, with negative and positive real part, arbitrarily close to any line contained in the strip $[-\delta_n, b^{(n)}] \times \mathbb{R}$. Finally, noting $b^{(n)} = 1 + \left(\frac{4}{\pi} - 1 + o(1)\right) \frac{\log \log n}{\log n}$, $n \rightarrow \infty$, it has been deduced that $\bigcap_{n=3}^{\infty} R^{(n)}$ contains a non-degenerated interval containing 0, so 0 is an accumulation point common to all the sets $P_{\zeta_n} := \{\Re z : \zeta_n(z) = 0\}$, $n > 2$, and in particular $0 \in R^{(n)}$ for all $n \geq 2$.

AMS Subject Classification: 30Axx, 30D05.

Key words: Zeros of partial sums of the Riemann zeta function, Almost-periodic functions, Level curves.

1 Introduction

The distribution of the zeros of exponential polynomials was mainly studied in the first third of XX^{th} century linked to the development of the theory of differential equations, as we can see for instance in [8, 23, 24, 28]. On the concrete case of the partial sums of the Riemann zeta function, $\zeta_n(z) := \sum_{k=1}^n \frac{1}{k^z}$, the interest for knowing the distribution of their zeros had its starting point in 1948 as consequence of a paper of Turán [26] where he pointed out an intriguing connexion between a particular distribution of such zeros near the line $x = 1$ and the Riemann Hypothesis (briefly R.H.). Namely, Turán [26, Th. 1] proved that if

$$\zeta_n(z) \neq 0 \text{ for } \Re z \geq 1 + n^{-\frac{1}{2} + \epsilon}, n \geq N_0(\epsilon),$$

then the Riemann Hypothesis is true.

The first step to avoid Turán's dream of proving R.H. via the previous theorem was given by Haselgrove [7] few years later. Indeed, by means of an added note at the end of [7], Haselgrove showed that the suggestion of Turán [26, (25.1)] on the non-negativity for all $x \geq 1$ of the function $T(x) := \sum_{k \leq x} \frac{\lambda(k)}{k}$, was false, where $\lambda(k) := (-1)^{p(k)}$ and $p(k)$ is the number of prime factors of k counting multiplicities. Hence, for some n , one has

$$\sum_{k=1}^n \frac{\lambda(k)}{k} < 0. \tag{1}$$

This n , allows to define the Dirichlet polynomial

$$D_n(z) := \sum_{k=1}^n \frac{\lambda(k)}{k^z}$$

and then the corresponding partial sum

$$\zeta_n(z) = \sum_{k=1}^n \frac{1}{k^z},$$

because the equivalence theorem of Bohr [4], attains the same set of values in any half-plane $\Re z > a$. That is, for arbitrary real a ,

$$\{D_n(z) : \Re z > a\} = \{\zeta_n(z) : \Re z > a\}.$$

Thus, since for sufficiently large real z , $D_n(z)$ is near 1 and, by (1.1), $D_n(1) < 0$, there exists a real zero $x > 1$ of $D_n(z)$. Therefore, there is a zero of $\zeta_n(z)$ in the half-plane $\Re z > 1$.

In 1968 Spira [22] demonstrated the same for $n = 19$. That is, the partial sum $\zeta_{19}(z)$ has a zero in the half-plane $\Re z > 1$. Levinson [9, Theorem 1] in 1973 found an asymptotic formula, for large n , for the location of the zeros of $\zeta_n(z)$ near the point $z = 1$. In particular, he proved that these zeros have real part less than 1. Voronin [27] in 1974 showed that $\zeta_n(z)$ has zeros in $\Re z > 1$ for infinitely many n .

A sharp result on the upper bound

$$b^{(n)} := \sup \{\Re z : \zeta_n(z) = 0\}$$

was demonstrated in 2001 by Montgomery and Vaughan in [12]. Namely, they proved that there exists N_0 such that if $n > N_0$ then $\zeta_n(z) \neq 0$ whenever

$$\Re z \geq 1 + \left(\frac{4}{\pi} - 1\right) \frac{\log \log n}{\log n}. \quad (2)$$

But in the opposite direction, in 1983, Montgomery [11] had already shown that for every $0 < c < \frac{4}{\pi} - 1$ there exists $N_0(c)$ such that $\zeta_n(z)$ has zeros in the half-plane

$$\Re z > 1 + c \frac{\log \log n}{\log n}, \quad n > N_0(c). \quad (3)$$

Since for any positive c , n sufficiently large and any $0 < \epsilon < \frac{1}{2}$, one has

$$c \frac{\log \log n}{\log n} > n^{-\frac{1}{2} + \epsilon},$$

Montgomery's result implies the existence of zeros of $\zeta_n(z)$ having real part much larger than the bound $1 + n^{-\frac{1}{2} + \epsilon}$ considered by Turán in [26, Th. I]. Hence, this old theorem of Turán of 1948 became vacuous and consequently the hope of

proving R.H. via such hypothetical localization of the zeros of the partial sums of the Riemann zeta function died.

The results of Montgomery and Vaughan, (1.2) and (1.3), imply that, fixed $0 < c < \frac{4}{\pi} - 1$, for all $n > N_1 := \max \{N_0, N_0(c)\}$, the upper bound $b^{(n)}$ satisfies the inequalities

$$1 + c \frac{\log \log n}{\log n} < b^{(n)} \leq 1 + \left(\frac{4}{\pi} - 1\right) \frac{\log \log n}{\log n}. \quad (4)$$

Then, since $\lim_{n \rightarrow \infty} \frac{\log \log n}{\log n} = 0$, it follows that

$$\lim_{n \rightarrow \infty} b^{(n)} = 1.$$

On the other hand, Balazard and Velásquez-Castañón [2] demonstrated that the lower bound

$$a^{(n)} := \inf \{\Re z : \zeta_n(z) = 0\}$$

is so that

$$\lim_{n \rightarrow \infty} \frac{a^{(n)}}{n} = -\log 2. \quad (5)$$

Therefore

$$\lim_{n \rightarrow \infty} a^{(n)} = -\infty.$$

Once the estimation of the bounds $a^{(n)}$ and $b^{(n)}$ has been settled, our interest is focused on the distribution of the real parts of the zeros of $\zeta_n(z)$ in the interior of its critical interval $[a^{(n)}, b^{(n)}]$ and, in this sense, we suggest to see [6, 13, 14, 15, 16]. A way of knowing the distribution of the zeros of $\zeta_n(z)$ is by means of the study of the set of their real projections

$$P_{\zeta_n} := \{\Re z : \zeta_n(z) = 0\}$$

and its closure set, denoted by $R^{(n)}$. Indeed, the difference between $R^{(n)}$ and the critical interval $[a^{(n)}, b^{(n)}]$ gives us a certain measure of the uniformity of the distribution of the real part of the zeros of $\zeta_n(z)$ and, consequently, the position of those zeros with respect to vertical lines contained in the critical strip $[a^{(n)}, b^{(n)}] \times \mathbb{R}$ where are located all the zeros of $\zeta_n(z)$.

The zeros of $\zeta_2(z) = 1 + \frac{1}{2^z}$, or those of the function $G_2(z) := 1 + 2^z$, are of the form

$$\frac{\pi(2k+1)i}{\log 2}, \quad k \in \mathbb{Z}. \quad (6)$$

Then $a^{(2)} = b^{(2)} = 0$, so $R^{(2)} = \{0\}$ and trivially $R^{(n)} = [a^{(n)}, b^{(n)}]$ for $n = 2$.

Since $R^{(n)} \subset [a^{(n)}, b^{(n)}]$ for all $n \geq 2$, the best possible relation between both sets one reaches when they are equal. Thus, whenever $R^{(n)} = [a^{(n)}, b^{(n)}]$, we will say that the distribution of the real parts of the zeros of $\zeta_n(z)$ is completely uniform on its critical interval. It means that, given any vertical line contained in $[a^{(n)}, b^{(n)}] \times \mathbb{R}$, $\zeta_n(z)$ possesses infinitely many zeros arbitrarily close to that

line. We have proved [15, Th.12] that this is exactly the case of the partial sums $\zeta_n(z)$ for sufficiently large n . That is, we have demonstrated in [15, Th.12] the existence of a positive integer N such that

$$R^{(n)} = [a^{(n)}, b^{(n)}] \text{ for all } n > N. \quad (7)$$

This result points out that the distribution of the real part of the zeros of $\zeta_n(z)$, for all $n > N$, is completely uniform on its corresponding critical interval. Consequently, the partial sums $\zeta_n(z)$, $n > N$, in spite of the set of their frequencies $\{-\log k : k = 2, \dots, n\}$ is not linearly independent (briefly l.i.) over the rationals for any $n > 3$, share with the Dirichlet polynomials, with l.i. frequencies and coefficients of modulus 1, the property of having the real parts of their zeros uniformly distributed on their critical intervals (see [16, Th. 10]).

In the present paper, by means of our Theorem 2, we have given a positive answer to an old question, on the negativity of the real part of the zeros of the partial sums of the Riemann zeta function, posed by Bellman and Hooke in 1963 [3, p. 439, Question 25]. It has been also proved in our Theorem 17 that for every integer $n > 2$ there exists a number $\delta_n > 0$ such that $[-\delta_n, b^{(n)}] \subset R^{(n)}$. Then, whenever $n > N$, by (1.7), the Theorem 17 is improved of an optimal form because the δ_n 's can be taken equal to $-a^{(n)} = n \log 2 + o(n)$, noting (1.5). Some relevant consequences are derived from Theorem 17:

a) By (1.4) and Corollary 3 of this paper, the set $\bigcap_{n=3}^{\infty} R^{(n)}$ contains an interval of the form $[-\delta, b]$ for some $\delta \geq 0$ and $b := \min \{b^{(n)} : n > 2\}$, which is a positive number. Then the strip $[-\delta, b] \times \mathbb{R}$ is common to all the n th partial sums $\zeta_n(z)$, $n > 2$. That is, given any vertical line contained in $[-\delta, b] \times \mathbb{R}$, there are infinitely many zeros of every $\zeta_n(z)$, for all $n > 2$, arbitrarily close to that line.

b) In particular, we deduce that 0 is an accumulation point common to all the sets $P_{\zeta_n} = \{\Re z : \zeta_n(z) = 0\}$, $n > 2$. Moreover, because of Lemma 1, for every $n > 2$, there exist infinitely many zeros of each $\zeta_n(z)$, with negative and positive real parts, which are arbitrarily close to the imaginay axis. Hence, the Theorem 17 and its consequences have applications to the stability of the solutions of neutral functional differential equations with delays (see for instance [10]).

c) By regarding the partial sum of Riemann zeta function $\zeta_2(z) := 1 + \frac{1}{2^z}$, noting (1.6), one has

$$\bigcap_{n=2}^{\infty} R^{(n)} = \{0\},$$

which proves Sepulcre and Vidal's conjecture [20, Conjecture 15].

2 Distribution of the zeros of $\zeta_n(z)$ on the half-planes $\Re z > 0$ and $\Re z < 0$

Lemma 1 *Let $f(z)$ be an almost-periodic function not identically null on a strip $S_{a,b} := \{z : a < \Re z < b\}$. If $f(z)$ has a zero at a point $z_0 \in S_{a,b}$ then it has infinitely many zeros in $S_{\epsilon} := \{z : \Re z_0 - \epsilon < \Re z < \Re z_0 + \epsilon\}$ for arbitrary $\epsilon > 0$.*

Proof. We firstly note that, since the almost-periodicity implies analyticity [5, p.101], $f(z)$ is analytic on $S_{a,b}$. Secondly, by [5, p.102], $f(z)$ is bounded on any closed sub-strip $S_{\alpha,\beta} := \{z : a < \alpha \leq \Re z \leq \beta < b\}$ of $S_{a,b}$. Now, let z_1 be a point on the line $x = \Re z_0$ such that $f(z_1) \neq 0$. Thus we claim that there exist positive numbers δ and l such that on any segment of length l of the line $x = \Re z_0$ there is a point $z_1 + iT$ verifying

$$|f(z_1 + iT)| \geq \delta. \quad (8)$$

Indeed, by taking $\delta = \frac{|f(z_1)|}{2} > 0$, the almost-periodicity of $f(z)$ involves the existence of a real number $l = l(\delta)$ such that every interval of length l on the imaginary axis contains at least a number $T_1 = T_1(\delta)$, called translation number, satisfying

$$|f(z + iT_1) - f(z)| \leq \delta \text{ for all } z \in S_{a,b}. \quad (9)$$

Then, by setting $z = z_1$ in (2.2) and according to the choice of δ , the inequality (2.1) follows, as claimed. By reiterating this process for every $\delta_n = \frac{\delta}{2^{n-1}}$, $n \geq 1$, the almost-periodicity of $f(z)$ assures the existence of a real number $l_n = l_n(\delta_n)$ such that every interval of length l_n contains at least a translation number $T_n = T_n(\delta_n)$ which, without loss of generality, can be chosen in such a way that $(T_n)_{n=1,2,\dots}$ be an increasing sequence of positive numbers with $T_{n+1} - T_n > 1$, for all n , satisfying

$$|f(z + iT_n) - f(z)| \leq \delta_n \text{ for all } z \in S_{a,b}. \quad (10)$$

Then, by just making $z = z_0$ in (2.3), we have

$$\lim_{n \rightarrow \infty} f(z_0 + iT_n) = 0. \quad (11)$$

We now consider the sequence of analytic functions on $S_{a,b}$

$$g_n(z) := f(z + iT_n), \quad z \in S_{a,b}, \quad n = 1, 2, \dots, \quad (12)$$

which are uniformly bounded on the compacts of $S_{a,b}$ by virtue of the boundedness of $f(z)$ on any substrip of $S_{a,b}$. Hence, by Montel's theorem [1, p. 165], there exists a uniformly convergent subsequence $\{g_{n_k}(z) : k = 1, 2, \dots\}$ to an analytic function, say $g(z)$, on $S_{a,b}$. The function $g(z)$, by (2.1), is not identically 0 on $S_{a,b}$ and, because of (2.5) and (2.4), has a zero at the point z_0 which, by analyticity, is not isolated. Then we can determine a positive number

$$r < \min \left\{ x_0 - a, b - x_0, \epsilon, \frac{1}{2} \right\}$$

such that $|g(z)| \neq 0$ for all z on the circle $|z - z_0| = r$, so the number

$$m := \min \{|g(z)| : |z - z_0| = r\} > 0.$$

Noticing $|z - z_0| = r$ is a compact set in the strip $S_{a,b}$ and $\lim_{k \rightarrow \infty} g_{n_k}(z) = g(z)$ is uniform on the compacts of $S_{a,b}$, there exists a positive integer k_0 such that

$$|g(z) - g_{n_k}(z)| < m \leq |g(z)| \text{ for all } |z - z_0| = r \text{ and for all } k > k_0.$$

This means, by Rouché's theorem [1, p. 88], that every function $g_{n_k}(z)$, for all $k > k_0$, has in the disk $|z - z_0| < r$ as many zeros as $g(z)$ does, so at least one, say z_{n_k} . Then, from (2.5), one has

$$f(z_{n_k} + iT_{n_k}) = 0 \text{ for all } k > k_0.$$

Finally, since $|z_{n_k} - z_0| < r < \frac{1}{2}$ for all $k > k_0$ and $T_{n_{k+1}} - T_{n_k} > 1$ for all k , the points $z_{n_k} + iT_{n_k}$ are distinct for all $k > k_0$. Hence, the function $f(z)$ has a zero at each point $z_{n_k} + iT_{n_k}$ for any $k > k_0$. Consequently, $f(z)$ has infinitely many zeros in the strip S_ϵ . This completes the proof. ■

Since the partial sums of the Riemann zeta function are almost-periodic functions, from the previous lemma, we deduce a relevant property on the distribution of their zeros with respect to the imaginary axis.

Theorem 2 *Every partial sum of the Riemann zeta function, $\zeta_n(z) := \sum_{k=1}^n \frac{1}{k^z}$, $n \geq 2$, possesses infinitely many zeros in each half-plane $\{z : \Re z < 0\}$ and $\{z : \Re z > 0\}$ except for $n = 2$ whose zeros are all on the imaginary axis.*

Proof. For any $n \geq 2$, by regarding the functions

$$G_n(z) := \zeta_n(-z), \quad n \geq 2,$$

the sets of zeros of $G_n(z)$ and $\zeta_n(z)$, denoted by $Z_{G_n(z)}$ and $Z_{\zeta_n(z)}$ respectively, are related by $Z_{G_n(z)} = -Z_{\zeta_n(z)}$. Then the bounds

$$a_n := \inf \{\Re z : G_n(z) = 0\}, \quad b_n := \sup \{\Re z : G_n(z) = 0\}$$

and

$$a^{(n)} := \inf \{\Re z : \zeta_n(z) = 0\}, \quad b^{(n)} := \sup \{\Re z : \zeta_n(z) = 0\},$$

satisfy

$$a^{(n)} = -b_n, \quad b^{(n)} = -a_n \text{ for all } n \geq 2. \quad (13)$$

Concerning the sets $Z_{G_n(z)}$ it is needed to remind that in [13, Propositions 1,2,3] was proved:

- a) For any $n \geq 2$, $Z_{G_n(z)}$ is infinite.
- b) For every $n \geq 2$, there exist real numbers r_n, s_n , such that $Z_{G_n(z)}$ is contained in the vertical strip $\{z \in \mathbb{C} : r_n \leq \Re z \leq s_n\}$.
- c) For any $n \geq 2$, $Z_{G_n(z)}$ is not contained in the imaginary axis, except for $n = 2$.

As we have seen in (1.6) the zeros of $\zeta_2(z)$ are all imaginary, then the last part of the theorem follows. Hence, from now on, we assume that $n > 2$. Let suppose that there exists an integer $m > 2$ such that the corresponding function $G_m(z)$ has its zeros, say $(\alpha_{m,l})_{l \geq 1}$, of the form

$$\Re \alpha_{m,l} \neq 0 \text{ for } l = 1, \dots, p, \quad \Re \alpha_{m,l} = 0 \text{ if } l > p, \text{ for some integer } p \geq 1. \quad (14)$$

Noticing $G_n(\bar{z}) = \overline{G_n(z)}$ for all $z \in \mathbb{C}$, $(\alpha_{m,l})_{l \geq 1}$ are conjugate. Then p is necessarily even, so $p = 2q \geq 2$, and the zeros $\alpha_{m,l}$ with $l > p$ are of the form

$\pm iy_j$ with $y_j > 0$. Let denote by $P(z)$ the polynomial with leader coefficient 1 and zeros $(\alpha_{m,l})_{l=1,2,\dots,p}$. Then

$$P(z) = (z^2 - 2a_1z + |\alpha_{m,1}|^2) \cdots (z^2 - 2a_qz + |\alpha_{m,q}|^2), \quad (15)$$

where $a_l := \Re\alpha_{m,l}$, $l = 1, \dots, q$. Therefore the function

$$H_m(z) := \frac{G_m(z)}{P(z)}$$

is entire of order 1 with zeros $(\pm iy_j)_{j=1,2,\dots}$. By Hadamard's factorization theorem [1, Th. 4.4.3] we can write

$$H_m(z) = e^{Az+B} \prod_{j=1}^{\infty} \left(1 + \frac{z^2}{y_j^2}\right), \quad (16)$$

where the constants B and A , because of (2.8), satisfy

$$e^B = H_m(0) = \frac{m}{|\alpha_{m,1} \cdots \alpha_{m,q}|^2},$$

$$A = \frac{H'_m(0)}{e^B} = \frac{\log(m!)}{m} + 2 \left(\frac{a_1}{|\alpha_{m,1}|^2} + \cdots + \frac{a_q}{|\alpha_{m,q}|^2} \right).$$

By (2.9) the function $H_m(z)e^{-Az}$ is even, then

$$H_m(z)e^{-Az} = H_m(-z)e^{Az} \text{ for all } z \in \mathbb{C}.$$

Now, substituting $H_m(z)$ by $\frac{G_m(z)}{P(z)}$, we get

$$\frac{P(-z)}{P(z)} = \frac{G_m(-z)e^{2Az}}{G_m(z)} \text{ for all } z \in \mathbb{C}. \quad (17)$$

If we assume that any zero of $P(z)$ is a zero of $P(-z)$, the polynomials $P(z)$ and $P(-z)$ are equal and their zeros $(\alpha_{m,l})_{l=1,2,\dots,p}$ are conjugate and opposite. Hence $q = \frac{p}{2}$ is even,

$$\frac{a_1}{|\alpha_{m,1}|^2} + \cdots + \frac{a_q}{|\alpha_{m,q}|^2} = 0$$

and then $A = \frac{\log(m!)}{m}$. But this leads us to the following contradiction: noticing $(n!)^2 > n^n$ for all $n > 2$, the right side of (2.10) tends to $+\infty$ as $z \in \mathbb{R}$ does, whereas its left side is identically 1. Consequently, at least one zero of $P(z)$ cannot be a zero of $P(-z)$. Thus the left side of (2.10) is a meromorphic function having at least a pole. On the other hand, the right side of (2.10) is a quotient of exponential polynomials with a finite number of poles (the zeros of $P(z)$ that are not cancelled with the zeros of $P(-z)$). Then Shields's theorem [21] asserts that $\frac{G_m(-z)e^{2Az}}{G_m(z)}$ is an exponential polynomial, which represents a

new contradiction because $\frac{P(-z)}{P(z)}$ has at least a pole. As consequence, there is no function $G_m(z)$, $m > 2$, having its zeros of the form (2.7). Therefore, taking into account c), every function $G_n(z)$, $n > 2$, has infinitely many zeros in at least one of the half-planes $\{z : \Re z < 0\}$ and $\{z : \Re z > 0\}$. However $G_n(z)$, $n > 2$, cannot have all its zeros, say $(z_{n,l})_{l=1,2,\dots}$, with real part of the same sign. Indeed, by expressing $G_n(z) = 1 + 2^z + \dots + n^z$ of the form

$$1 + e^{z \log 2} + \dots + e^{z \log n},$$

by Ritt's formula [19, formula (9)], we have

$$\sum_{l=1}^{\infty} \Re z_{n,l} = O(1). \quad (18)$$

Now, by assuming, for instance, $\Re z_{n,l} \geq 0$ for all $l \geq 1$, from c), there exists at least a zero of $G_n(z)$, say z_{n,l_0} , such that $\Re z_{n,l_0} > 0$. By taking $\epsilon = \Re z_{n,l_0}$ and noting that $G_n(z)$ is an almost-periodic function, because of Lemma 1, $G_n(z)$ has infinitely many zeros in the strip

$$S_{\frac{\epsilon}{2}} := \left\{ z : \frac{\epsilon}{2} < \Re z < \frac{3\epsilon}{2} \right\}.$$

This implies that

$$\sum_{l=1}^{\infty} \Re z_{n,l} = +\infty,$$

contradicting (2.11). Hence, $G_n(z)$ has at least a zero in $\{z : \Re z < 0\}$ and again by using Lemma 1, $G_n(z)$ has infinitely many zeros with negative real part. Then the theorem follows for the functions $G_n(z)$ and, taking into account (2.6), it is also true for the partial sums $\zeta_n(z)$. ■

The next result is an easy consequence of the above theorem.

Corollary 3 *The bounds a_n , b_n and $a^{(n)}$, $b^{(n)}$ corresponding to the functions $G_n(z)$ and $\zeta_n(z)$, respectively, satisfy*

$$a_n = b_n = a^{(n)} = b^{(n)} = 0 \text{ if } n = 2$$

and

$$a_n < 0, b_n > 0, a^{(n)} < 0, b^{(n)} > 0 \text{ for all } n > 2.$$

3 The level curves $|G_n^*(z)| = p_{k_n}^{x_0}$, $n > 2$, $x_0 \in \mathbb{R}$

The curves $|f(z)| = \text{constant}$, where $f(z)$ is an analytic function on a given region of the complex plane, are defined as level curves (see for instance Titchmarsh's book [25, p.121]). This concept will be crucial for attaining the main objective of our paper, namely, to prove the existence of non-degenerate intervals contained in the sets

$$R_n := \overline{\{\Re z : G_n(z) = 0\}} \quad (19)$$

and

$$R^{(n)} := \overline{\{\Re z : \zeta_n(z) = 0\}}, \quad (20)$$

for all integer $n > 2$.

From the definition of the bounds corresponding to the functions $G_n(z)$, $\zeta_n(z)$, and (3.1), (3.2), it follows

$$a_n, b_n \in R_n \subset [a_n, b_n]$$

and

$$a^{(n)}, b^{(n)} \in R^{(n)} \subset [a^{(n)}, b^{(n)}],$$

for all $n \geq 2$. By Corollary 3, the set $R_n = R^{(n)} = \{0\}$ if and only if $n = 2$. Therefore $n = 2$ is the trivial case and whether $n > 2$, the sets $R_n, R^{(n)}$ have at least two distinct points, namely, the bounds a_n, b_n and $a^{(n)}, b^{(n)}$, respectively.

For every $n > 2$ we define the function

$$G_n^*(z) := G_n(z) - p_{k_n}^z,$$

where $G_n(z) := 1 + 2^z + \dots + n^z$ and p_{k_n} is the last prime not exceeding n . The goal is to study properties of the level curves

$$|G_n^*(z)| = p_{k_n}^{x_0}, \quad (21)$$

for each integer $n > 2$ and a given real number x_0 .

In order to emphasize the importance of the level curves, we recall that the sets R_n , defined in (3.1), were characterized in [6] in terms of such curves as follows.

Theorem 4 (*Dubon, Mora, Sepulcre, Ubeda, Vidal [6, Th. 2]*) *A real number x_0 belongs to R_n , $n > 2$, if and only if the vertical line $x = x_0$ intersects the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$.*

By defining the real functions

$$A_n(x, y) := |G_n^*(x + iy)| - p_{k_n}^x; \quad x, y \in \mathbb{R}, \quad n > 2,$$

the above result was completed in [15] under the form:

Theorem 5 (*Mora [15, Th. 2]*) *For every integer $n > 2$, a real number x belongs to R_n if and only if there exists some $y \in \mathbb{R}$ such that $A_n(x, y) = 0$. Furthermore, $A_n(x, 0) \geq 0$ for all $x \in [a_n, b_n]$.*

By squaring (3.3) we obtain the cartesian equation of the level curve for a given real number x_0 ,

$$1 + 2^{2x} + \dots + n^{2x} + 2 \cdot 1^x \sum_{m=2}^n m^x \cos\left(y \log\left(\frac{m}{1}\right)\right) + 2 \cdot 2^x \sum_{m=3}^n m^x \cos\left(y \log\left(\frac{m}{2}\right)\right) +$$

$$+\dots + 2(n-1)^x \sum_{m=n}^n m^x \cos\left(y \log\left(\frac{m}{n-1}\right)\right) = p_{k_n}^{2x_0}, \quad (22)$$

where in the left side of (3.4) the integer variables do not take the value p_{k_n} .

Since, for any value of y , the limit of the left side of (3.4) is $+\infty$ when $x \rightarrow +\infty$ and the right side of (3.4) is fixed, the range of x is always upper bounded for every x_0 . However, as the limit of the left side of (3.4) is 1 when $x \rightarrow -\infty$, if $x_0 \neq 0$, the range of the variable x must be lower bounded. Finally, from equation (3.4), it is immediate that the level curve neither contains a vertical segment nor has a vertical asymptote. Thus, fixed $n > 2$ and $x_0 \in \mathbb{R}$, the variable x in the equation (3.4) either runs on a finite interval

$$[b_{n,x_0}^-, b_{n,x_0}], \text{ if } x_0 \neq 0,$$

or on the infinite interval

$$(-\infty, b_{n,x_0}], \text{ if } x_0 = 0,$$

where the extremes b_{n,x_0}^- , b_{n,x_0} are defined as follows.

Definition 6 Fixed an integer $n > 2$ and a real number x_0 , we define b_{n,x_0} , called upper extreme associated to x_0 , as the maximum value that we can assign to the variable x to have, for at least a value of y , a point (b_{n,x_0}, y) of the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$. When $x_0 \neq 0$, we define b_{n,x_0}^- , called lower extreme associated to x_0 , as the minimum value that we can assign to the variable x to have, for at least a value of y , a point (b_{n,x_0}^-, y) of the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$.

To illustrate the above definition we determine in the next example the points extremes corresponding to the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$ for $n = 3$.

Example 7 The level curve $|G_3^*(z)| = p_{k_3}^{x_0}$, $x_0 \in \mathbb{R}$.

For $n = 3$ one has $p_{k_3} = 3$, then $G_3^*(z) := G_3(z) - p_{k_3}^z = 1 + 2^z$. Hence the level curve is $|1 + 2^z| = 3^{x_0}$. By squaring we obtain its cartesian equation,

$$1 + 2^{2x} + 2^{x+1} \cos(y \log 2) = 3^{2x_0}. \quad (23)$$

The lower extreme associated to $x_0 \neq 0$ is

$$b_{3,x_0}^- = \begin{cases} \frac{\log(1-3^{x_0})}{\log 2} & \text{if } x_0 < 0 \\ \frac{\log(3^{x_0}-1)}{\log 2} & \text{if } x_0 > 0 \end{cases}$$

and the upper extreme associated to any x_0 is

$$b_{3,x_0} = \frac{\log(1+3^{x_0})}{\log 2}.$$

If $x_0 = 0$, by dividing (3.5) by 2^{x+1} , the equation of the level curve is

$$2^{x-1} + \cos(y \log 2) = 0$$

and then x runs on $(-\infty, 1]$.

Thus, for $x_0 < 0$, (3.5) represents an infinite family of closed curves containing each of them one zero of $G_3^*(z)$, so the level curve has infinitely many arc-connected components. For $x_0 = 0$, the above equation represents an infinite family of open curves having horizontal asymptotes oriented to $-\infty$ of equations $y = (2k + 1) \frac{\pi}{2 \log 2}$, $k \in \mathbb{Z}$, so the level curve has infinitely many arc-connected components. Finally, for $x_0 > 0$, (3.5) is the equation of a unique open curve, that is the level curve has only one arc-connected component. The curve meets to the real axis at the point $b_{3,x_0}^- = \frac{\log(3^{x_0}-1)}{\log 2}$, and the variable y takes all real numbers.

The next result proves that the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$ for $n > 3$ is essentially the same as the case $n = 3$.

Proposition 8 *Fixed an integer $n > 2$ and a real number x_0 , let us consider the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$. Then, if $x_0 < 0$, the level curve has infinitely many arc-connected components which are closed curves; if $x_0 = 0$, the level curve has infinitely many arc-connected components which are open curves with horizontal asymptotes, as $x \rightarrow -\infty$, of equations $y = (2k + 1) \frac{\pi}{2 \log 2}$, $k \in \mathbb{Z}$. Moreover, the level curve only meets to the real axis at a point, say c_{n,x_0} , if and only if $x_0 > 0$; in this case the lower extreme $b_{n,x_0}^- = c_{n,x_0}$ and the level curve has only one arc-connected component which is an open curve where the variable y takes all real values.*

Proof. Because of the function of the left side of (3.4) is even with respect to the variable y , we will assume that $y \geq 0$. For a given a real number $x_0 \leq 0$ we write (3.4) as

$$\begin{aligned} 2^{2x} + \dots + n^{2x} + 2.1^x \sum_{m=2}^n m^x \cos\left(y \log\left(\frac{m}{1}\right)\right) + 2.2^x \sum_{m=3}^n m^x \cos\left(y \log\left(\frac{m}{2}\right)\right) + \\ + \dots + 2(n-1)^x \sum_{m=n}^n m^x \cos\left(y \log\left(\frac{m}{n-1}\right)\right) = p_{k_n}^{2x_0} - 1 \end{aligned} \quad (24)$$

and, since $p_{k_n}^{2x_0} - 1 \leq 0$, the variable y satisfies either

$$\frac{\pi}{2} < y \log n, \text{ if } n \text{ is a composite number,} \quad (25)$$

or

$$\frac{\pi}{2} < y \log(n-1), \text{ if } n \text{ is prime.} \quad (26)$$

(Otherwise the left side of (3.6) would be positive). Hence, by (3.7), or (3.8), the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$, for a given a real number $x_0 \leq 0$, has no point in the strip $\left\{ (x, y) : 0 \leq y \leq \frac{\pi}{2 \log n} \right\}$, when n is composite, and it has no point in the strip $\left\{ (x, y) : 0 \leq y \leq \frac{\pi}{2 \log(n-1)} \right\}$, if n is prime.

Now assume that $x_0 < 0$. By writing $|G_n^*(z)| = p_{k_n}^{x_0}$ as $|1 + w| = r_{x_0}$, where $w := 2^z + \dots + n^z$ (this expression does not contain the term $p_{k_n}^z$) and $r_{x_0} := p_{k_n}^{x_0}$,

the equation (3.6) describes the continuous motion of each zero $z = (x, y)$ of the function

$$F_{n,w}^*(z) := G_n^*(z) - 1 - w$$

when w runs on the circle $|1 + w| = r_{x_0}$. Indeed, fixed $n > 2$, for each w of the circle $|1 + w| = r_{x_0}$ the function $F_{n,w}^*(z)$ has infinitely many zeros on the half-plane $\{z : \Im z \geq 0\}$ situated on a vertical strip. These zeros will be supposed lexicographically ordered. That is, given a point w of the circle $|1 + w| = r_{x_0}$, a zero $z_w = (x, y)$ of $F_{n,w}^*(z)$ precedes to other zero $z'_w = (x', y')$ if and only if either $y < y'$ or $x < x'$, if $y = y'$. Then, by taking the interval $(0, 2\pi]$ as a determination of the argument (the domain corresponding to the log function is $\Omega := \mathbb{C} \setminus [0, +\infty)$), let us define the map

$$\varphi_1(w) := z_{1,w},$$

that assigns to each w of the circle $|1 + w| = r_{x_0}$ the first zero, $z_{1,w}$, of $F_{n,w}^*(z)$ with respect to the order defined. Because of $0 < r_{x_0} < 1$, the circle $|1 + w| = r_{x_0}$ is in Ω and the argument of w varies on a certain interval $[\pi - \alpha_{x_0}, \pi + \alpha_{x_0}]$ for some α_{x_0} such that $0 < \alpha_{x_0} < \frac{\pi}{4}$, so $[\pi - \alpha_{x_0}, \pi + \alpha_{x_0}] \subset (0, 2\pi]$. Thus, the function φ_1 transforms the circle $|1 + w| = r_{x_0}$ on a closed curve whose equation is given by (3.6) and it constitutes the first arc-connected component of the level curve. By analogy, the function

$$\varphi_j(w) := z_{j,w}, \quad j > 1,$$

where $z_{j,w}$ is the j th zero of $F_{n,w}^*(z)$ with respect to the order defined, transforms the circle $|1 + w| = r_{x_0}$ on a closed curve, whose equation is given by (3.6), and it is the j th arc-connected component of the level curve, and so on. Finally, as the function of the left side of equation (3.6) is even with respect to the variable y , we obtain by symmetry with respect to the real axis the closed arc-connected components of the level curve in the half-plane $\{z : \Im z < 0\}$.

If $x_0 = 0$, the right side of (3.6) is 0. Then dividing (3.6) by 2^{x+1} , the equation of the level curve is

$$\begin{aligned} 2^{x-1} + \dots + \left(\frac{n}{2}\right)^{2x} 2^{x-1} + \cos(y \log 2) + 1^x \sum_{m=3}^n \left(\frac{m}{2}\right)^x \cos\left(y \log\left(\frac{m}{1}\right)\right) + \\ + 2^x \sum_{m=3}^n \left(\frac{m}{2}\right)^x \cos\left(y \log\left(\frac{m}{2}\right)\right) + \dots + \\ + (n-1)^x \sum_{m=n}^n \left(\frac{m}{2}\right)^x \cos\left(y \log\left(\frac{m}{n-1}\right)\right) = 0. \end{aligned} \quad (27)$$

By taking the limit as $x \rightarrow -\infty$, all the terms of the left side of (3.9), except $\cos(y \log 2)$, tend to 0. Then the equation $\cos(y \log 2) = 0$ gives us the horizontal asymptotes of the level curve. Consequently, the level curve has infinitely many arc-connected components which are open curves (closed on the right) having asymptotes of equations $y = (2k + 1) \frac{\pi}{2 \log 2}$, $k \in \mathbb{Z}$, oriented to $-\infty$.

For studying the case $x_0 > 0$, let y_0 be a fixed real number. By substituting it in (3.4), we have an expression of the form

$$1 + a_2 2^x + a_3 3^x + \dots + a_{n(n-1)} (n(n-1))^x + a_{n^2} n^{2x} = p_{k_n}^{2x_0}, \quad (28)$$

where the a 's are real numbers (the left side of (3.10) does not contain the term $a_{p_{k_n}} p_{k_n}^x$). Because the left side of (3.10) has been obtained from (3.3) as the sum of the squares of the real and imaginary parts of $G_n^*(z)$, for $z = x + iy_0$, the number of changes of the sign of its coefficients is a nonnegative even integer. Hence, since $p_{k_n}^{2x_0} > 1$, the number, say W , of changes of the sign of the coefficients of the equation, equivalent to (3.10),

$$1 - p_{k_n}^{2x_0} + a_2 2^x + a_3 3^x + \dots + a_{n(n-1)} (n(n-1))^x + a_{n^2} n^{2x} = 0,$$

is odd. Then, by Pólya and Szëgo's formula [18, p. 46], $W - N$ is an even nonnegative integer, where N denotes the number of real zeros (counting multiplicities) of the preceding equation. Therefore, N must be positive, which implies that equation (3.10) has solution. Consequently, as y_0 is arbitrary, the variable y takes all the real values. On the other hand, by making $y = 0$ in (3.4), one has

$$1 + 2^x + \dots + n^x = p_{k_n}^{x_0}$$

(the left side of this equation does not contain the term $p_{k_n}^x$) and we then claim that the above equation has real solution if and only if $x_0 > 0$. Indeed, clearly the equation has no real solution if $x_0 \leq 0$. Then if the equation has solution, x_0 must be positive. Conversely, by assuming $x_0 > 0$, the number $p_{k_n}^{x_0}$ is greater than 1 and W , the number of changes of the sign of the equation

$$1 - p_{k_n}^{x_0} + 2^x + \dots + n^x = 0,$$

is 1. Thus, by Pólya and Szëgo's formula [18], the number, N , of real solutions of the above equation is necessarily 1. Hence the above equation has only a real solution, say c_{n,x_0} , and the claim follows. Furthermore, since $(c_{n,x_0}, 0)$ is a point of the level curve, from Definition 6, one has

$$b_{n,x_0}^- \leq c_{n,x_0}.$$

But, if we suppose that $b_{n,x_0}^- < c_{n,x_0}$, again from Definition 6, for some y , the point (b_{n,x_0}^-, y) belongs to the level curve and then $|G_n^*(b_{n,x_0}^- + iy)| = p_{k_n}^{x_0}$. Now, because $G_n^*(z)$ is strictly increasing on the real axis, we obtain

$$p_{k_n}^{x_0} = |G_n^*(b_{n,x_0}^- + iy)| \leq |G_n^*(b_{n,x_0}^-)| < |G_n^*(c_{n,x_0})| = p_{k_n}^{x_0},$$

a contradiction. Therefore,

$$b_{n,x_0}^- = c_{n,x_0}.$$

Finally, since the variable y takes all real numbers, the unique point that the level curve meets to the real axis is $(b_{n,x_0}^-, 0)$ and there is no vertical asymptote, the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$ has only one arc-connected component. The proof is now completed. ■

From Example 7 and Proposition 8 we obtain the following consequence.

Corollary 9 For a given integer $n > 2$ and a real number x_0 , let $|G_n^*(z)| = p_{k_n}^{x_0}$ be the corresponding level curve. If a point $z' = (x', y')$ satisfies $|G_n^*(z')| < p_{k_n}^{x_0}$, then there exists a point $z = (x, y)$ of the level curve such that $x > x'$.

Proof. By Example 7 and Proposition 8, if $x_0 < 0$, the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$ has infinitely many arc-connected components which are closed curves. Thus, the level curve divides to the complex plane in two regions $\{w : |G_n^*(w)| < p_{k_n}^{x_0}\}$ and $\{w : |G_n^*(w)| > p_{k_n}^{x_0}\}$, defined by the interior and the exterior points of its arc-connected components, respectively. Thus, if $z' = (x', y')$ is a point satisfying $|G_n^*(z')| < p_{k_n}^{x_0}$, z' is in the interior of a closed arc-connected component and consequently it must have a point $z = (x, y)$ of the level curve situated on the right of z' , so $x > x'$. The case $x_0 = 0$ is analogous to the previous one because the arc-connected components (open curves) are closed on the right side and open on the left side with asymptotes oriented to $-\infty$. Finally, if $x_0 > 0$, the variable y take all real values and x runs on the interval $[b_{n,x_0}^-, b_{n,x_0}]$. Then, since for any $n > 2$,

$$\max \{|G_n^*(w)| : \Re w \leq b_{n,x_0}^-\} = G_n^*(b_{n,x_0}^-),$$

the set $\{w : |G_n^*(w)| < p_{k_n}^{x_0}\}$ coincides exactly with the region situated on the left of the level curve. Hence, if $z' = (x', y')$ satisfies $|G_n^*(z')| < p_{k_n}^{x_0}$, there exists a point $z = (x, y)$ of the level curve on the right of z' and, consequently, $x > x'$. The proof is now completed. ■

The next result shows that the set $R_n := \overline{\{\Re z : G_n(z) = 0\}}$ contains intervals.

Theorem 10 Let $[a_n, b_n]$ be the critical interval of $G_n(z)$, $n > 2$, $x_0 \in R_n$ and b_{n,x_0} its upper extreme associated with respect to the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$. Then

$$[x_0, b_{n,x_0}] \cap [a_n, b_n] \subset R_n.$$

Proof. Since $x_0 \in R_n$, because of Theorem 4, there exists a value y_0 such that the point (x_0, y_0) lies on the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$. From Definition 6, if $x_0 \neq 0$, then $x_0 \in [b_{n,x_0}^-, b_{n,x_0}]$ and $x_0 \in (-\infty, b_{n,x_0}]$, if $x_0 = 0$. Therefore, for any $x_0 \in R_n$, one has $x_0 \leq b_{n,x_0}$. Thus, if $x_0 = b_{n,x_0}$, since $R_n \subset [a_n, b_n]$, the theorem trivially follows. Let us assume that $x_0 < b_{n,x_0}$ and let $x_1 \in [a_n, b_n]$ be a point such that $x_0 < x_1 < b_{n,x_0}$. Then $x_1 \in [b_{n,x_0}^-, b_{n,x_0}]$, or $x_1 \in (-\infty, b_{n,x_0}]$, so the line of equation $x = x_1$ intersects to the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$ at least at a point, say $z_1 = x_1 + iy_1$. Consequently $|G_n^*(z_1)| = p_{k_n}^{x_0}$. Now, by the definition of $A_n(x, y)$, we obtain

$$A_n(x_1, y_1) = |G_n^*(z_1)| - p_{k_n}^{x_1} = p_{k_n}^{x_0} - p_{k_n}^{x_1} < 0.$$

On the other hand, since $x_1 \in [a_n, b_n]$, by Theorem 5, one has $A_n(x_1, 0) \geq 0$. Then, the continuity of $A_n(x_1, y)$ as function of y implies the existence of a value, say y_2 , such that $A_n(x_1, y_2) = 0$. Now, because of Theorem 5, it follows that $x_1 \in R_n$ and then

$$[x_0, b_{n,x_0}] \cap [a_n, b_n] \subset R_n. \quad (29)$$

By supposing $b_{n,x_0} > b_n$, from (3.11), we get

$$[x_0, b_{n,x_0}] \cap [a_n, b_n] = [x_0, b_{n,x_0}) \cap [a_n, b_n] \subset R_n$$

and then the theorem follows. If $b_{n,x_0} \leq b_n$, since $x_0 \in R_n \subset [a_n, b_n]$, again by (3.11), one has

$$[x_0, b_{n,x_0}) \cap [a_n, b_n] = [x_0, b_{n,x_0}) \subset R_n,$$

but, as R_n is closed, the point b_{n,x_0} also belongs to R_n . Then, $[x_0, b_{n,x_0}] \subset R_n$ and hence, in this case, we also get

$$[x_0, b_{n,x_0}] \cap [a_n, b_n] = [x_0, b_{n,x_0}] \subset R_n,$$

which proves completely the theorem. ■

An important conclusion on the sets R_3 and $R^{(3)}$ is deduced from Theorem 10.

Corollary 11 *The sets R_3 and $R^{(3)}$ are equal to $[a_3, b_3]$ and $[a^{(3)}, b^{(3)}]$, respectively.*

Proof. The bound $a_3 := \inf \{\Re z : G_3(z) = 0\}$ belongs to R_3 and, by Corollary 3, a_3 is negative. Its upper extreme associated, with respect to the level curve $|G_3^*(z)| = p_{k_3}^{x_0}$, determined in Example 7, is

$$b_{3,a_3} = \frac{\log(1 + 3^{a_3})}{\log 2} > 0.$$

Then, by taking $x_0 = a_3$ in Theorem 10, we have

$$\left[a_3, \frac{\log(1 + 3^{a_3})}{\log 2} \right] \cap [a_3, b_3] \subset R_3. \quad (30)$$

Hence, the corollary follows, if $\frac{\log(1+3^{a_3})}{\log 2} \geq b_3$, because $R_3 \subset [a_3, b_3]$. Let us assume $\frac{\log(1+3^{a_3})}{\log 2} < b_3$. Then, since $a_3 < 0$ and $\frac{\log(1+3^{a_3})}{\log 2} > 0$, from (3.12), it deduces in particular that $0 \in R_3$. But, as we have just seen in Example 7, the upper extreme associated to 0 is 1, so, again from Theorem 10, we get

$$[0, 1] \cap [a_3, b_3] \subset R_3. \quad (31)$$

From Corollary 3 and [6, Lemma 1], we know that

$$0 < b_3 \leq \sup \{x : 1 + 2^x \geq 3^x\} = 1.$$

Then, by (3.13), one has

$$[0, 1] \cap [a_3, b_3] = [0, b_3] \subset R_3,$$

which, jointly with (3.12), involves that

$$R_3 = [a_3, b_3].$$

Now, taking into account (2.6), one has $R^{(3)} = [-b_3, -a_3] = [a^{(3)}, b^{(3)}]$. Then the proof is completed. ■

From the above corollary it follows that the functions $G_3(z)$ and $\zeta_3(z)$ have infinitely many zeros arbitrarily near any vertical line contained in $[a_3, b_3] \times \mathbb{R}$ and $[a^{(3)}, b^{(3)}] \times \mathbb{R}$, respectively. Therefore, the distribution of the real parts of the zeros of $G_3(z)$ and $\zeta_3(z)$ is completely uniform on their critical intervals.

Lemma 12 *For every $n > 2$, let us denote by f_n the function that assigns to each real number x_0 its upper extreme associated b_{n,x_0} with respect to the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$. Then f_n is a strictly increasing function.*

Proof. Let us consider two real numbers $x_1 < x_0$. Then, since $p_{k_n} \geq 2$ for all $n \geq 2$, one has $p_{k_n}^{x_1} < p_{k_n}^{x_0}$. From Definition 6, given $f_n(x_1) := b_{n,x_1}$, there exists some y_1 such that the point $z_1 = (b_{n,x_1}, y_1)$ belongs to the level curve $|G_n^*(z)| = p_{k_n}^{x_1}$. Then, since $p_{k_n}^{x_1} < p_{k_n}^{x_0}$, the point z_1 satisfies $|G_n^*(z_1)| < p_{k_n}^{x_0}$. By applying Corollary 9, there exists a point $z_2 = (x_2, y_2)$ of the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$ such that $b_{n,x_1} < x_2$. Now, as z_2 belongs to the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$, again from Definition 6, we have $x_2 \leq b_{n,x_0} = f_n(x_0)$. Thus

$$f_n(x_1) = b_{n,x_1} < x_2 \leq b_{n,x_0} = f_n(x_0),$$

so f_n is strictly increasing. ■

In the next result we find, for each integer $n > 2$, a lower bound of the set of upper extremes b_{n,x_0} when $x_0 \in \mathbb{R}$.

Lemma 13 *For every $n > 2$, let us denote $b_n^* = \sup \{\Re z : G_n^*(z) = 0\}$. Then*

$$b_n^* < b_{n,x_0} \text{ for all } x_0 \in \mathbb{R}.$$

Proof. Let us assume that $G_n^*(z)$ has a zero, say w , on the vertical line $x = b_n^*$. Then $\Re w = b_n^*$. Now, let x_0 be a real number. Then, since $G_n^*(w) = 0$, one has $|G_n^*(w)| < p_{k_n}^{x_0}$. From Corollary 9, there exists a point $z_1 = (x_1, y_1)$ of the level curve $|G_n^*(z)| = p_{k_n}^{x_0}$ such that $b_n^* < x_1$. By Definition 6 it follows that $x_1 \leq b_{n,x_0}$ and then

$$b_n^* < x_1 \leq b_{n,x_0},$$

which proves the lemma under the above assumption.

If $G_n^*(z)$ has no zero on the line $x = b_n^*$, according to the definition of b_n^* , there exists a sequence of zeros $(z_m)_m$ of $G_n^*(z)$ with $\Re z_m < b_n^*$ such that $\lim_m \Re z_m = b_n^*$. Let us consider a real number $x_1 < x_0$. Then, as $G_n^*(z_m) = 0$, we get $|G_n^*(z_m)| < p_{k_n}^{x_1}$ for all m . This implies, from Corollary 9, the existence of a sequence of points of the level curve $|G_n^*(z)| = p_{k_n}^{x_1}$, say $(w_m)_m$, such that $\Re z_m < \Re w_m$ for all m . On the other hand, since every w_m is on the level curve $|G_n^*(z)| = p_{k_n}^{x_1}$, from Definition 6, one has $\Re w_m \leq b_{n,x_1}$ for all m . Hence, we have $\Re z_m < \Re w_m \leq b_{n,x_1}$ for all m and, taking the limit when $m \rightarrow \infty$, we obtain

$$b_n^* \leq b_{n,x_1}. \tag{32}$$

Now, by Lemma 12, since $x_1 < x_0$, it follows that $b_{n,x_1} < b_{n,x_0}$ and, noting inequality (3.14), we get

$$b_n^* \leq b_{n,x_1} < b_{n,x_0},$$

which completes the proof. ■

The above lemma, jointly with Theorem 10, allows us to determine the sets R_n and $R^{(n)}$ in the case $n = 4$.

Theorem 14 *The sets R_4 and $R^{(4)}$ are equal to $[a_4, b_4]$ and $[a^{(4)}, b^{(4)}]$, respectively.*

Proof. The last prime not exceeding 4 is 3, so $p_{k_4} = 3$. Then $G_4^*(z) = 1 + 2^z + 4^z$, and its zeros are

$$\frac{2\pi(3k+1)i}{3\log 2}, \frac{2\pi(3k+2)i}{3\log 2}, k \in \mathbb{Z}.$$

Hence, $b_4^* := \sup \{\Re z : G_4^*(z) = 0\} = 0$. By applying Theorem 10 to $x_0 = a_4$, which is negative by Corollary 3, one has

$$[a_4, 0] \subset R_4, \quad (33)$$

because $b_4 > 0$ by virtue of Corollary 3 and $b_{4,x_0} > b_4^* = 0$ by Lemma 13. From (3.4), the equation of the level curve $|G_4^*(z)| = 3^{x_0}$ is

$$\begin{aligned} 1 + 2^{2x} + 4^{2x} + 2 \cdot 2^x \cos(y \log 2) + 2 \cdot 4^x \cos(y \log 4) + \\ + 2 \cdot 2^x \cdot 4^x \cos(y \log 2) = 3^{2x_0}. \end{aligned} \quad (34)$$

Since $\cos(y \log 4) = 2 \cos^2(y \log 2) - 1$, from (3.16), we obtain

$$\cos(y \log 2) = \frac{-(1 + 4^x) \pm \sqrt{(2 \cdot 3^{x_0})^2 - (\sqrt{3}(4^x - 1))^2}}{4 \cdot 2^x} \quad (35)$$

and then the variable x must satisfy $(\sqrt{3}(4^x - 1))^2 \leq (2 \cdot 3^{x_0})^2$, which is equivalent to say that

$$4^x \in \left[1 - 2 \cdot 3^{x_0 - \frac{1}{2}}, 1 + 2 \cdot 3^{x_0 - \frac{1}{2}} \right]. \quad (36)$$

Now, for $x_0 \geq 0$, is $1 - 2 \cdot 3^{x_0 - \frac{1}{2}} < 0$. Thus, noting that $4^x > 0$ for any x , from (3.18) it follows that the variable x satisfies

$$x \leq \frac{\log \left(1 + 2 \cdot 3^{x_0 - \frac{1}{2}} \right)}{\log 4}.$$

Then, by taking $x = x_1 := \frac{\log(1 + 2 \cdot 3^{x_0 - \frac{1}{2}})}{\log 4}$ in (3.17) (observe that $(2 \cdot 3^{x_0})^2 - (\sqrt{3}(4^{x_1} - 1))^2 = 0$), there exists a value for y , given by

$$\cos(y \log 2) = -\frac{1 + 3^{x_0 - \frac{1}{2}}}{2 \left(1 + 2 \cdot 3^{x_0 - \frac{1}{2}} \right)^{\frac{1}{2}}}, \quad (37)$$

provided that $\left| -\frac{1+3^{x_0-\frac{1}{2}}}{2(1+2\cdot 3^{x_0-\frac{1}{2}})^{\frac{1}{2}}} \right| \leq 1$. But $0 < \frac{1+3^{x_0-\frac{1}{2}}}{2(1+2\cdot 3^{x_0-\frac{1}{2}})^{\frac{1}{2}}} \leq 1$ if and only if $x_0 \leq \frac{\log(6+3\sqrt{3})}{\log 3}$. Then, we claim that any $x_0 \in [0, b_4)$, where $b_4 := \sup \{ \Re z : G_4(z) = 0 \}$, satisfies the above condition. Indeed, because of [6, Lemma 1], one has $b_4 \leq \sup \{ x : 1 + 2^x + 3^x \geq 4^x \}$ and, on the other hand, it is immediate to check that

$$\sup \{ x : 1 + 2^x + 3^x \geq 4^x \} < 2 < \frac{\log(6+3\sqrt{3})}{\log 3},$$

so $x_0 < b_4 < 2 < \frac{\log(6+3\sqrt{3})}{\log 3}$, and the claim follows. Consequently, for each $x_0 \in [0, b_4)$, by making $x = x_1$ in (3.17), there exists a value of y , say y_1 , given by (3.19), determining a point (x_1, y_1) of the level curve $|G_4^*(z)| = 3^{x_0}$. Thus, by Definition 6, $b_{4,x_0}^- \leq x_1 \leq b_{4,x_0}$. On the other hand, from the definition of x_1 , one has

$$x_1 > x_0 \text{ if and only if } 4^{x_0} < 1 + 2\cdot 3^{x_0-\frac{1}{2}}. \quad (38)$$

We define $\beta_4 := \sup \{ x : 4^x \leq 1 + 2\cdot 3^{x-\frac{1}{2}} \}$. Then, from Pólya and Szégo's formula [18, p. 46], the equation $4^x = 1 + 2\cdot 3^{x-\frac{1}{2}}$ has a unique real solution, so equal to β_4 . Now we claim that

$$b_4 \leq \beta_4. \quad (39)$$

Indeed, if $z = x + iy$ a zero of $G_4(z)$, one has $1 + 2^z + 4^z = -3^z$. Then the real and imaginary parts are equal,

$$\begin{aligned} 1 + 2^x \cos(y \log 2) + 4^x \cos(y \log 4) &= -3^x \cos(y \log 3), \\ 2^x \sin(y \log 2) + 4^x \sin(y \log 4) &= -3^x \sin(y \log 3). \end{aligned}$$

By squaring and adding, we obtain the same expression that in (3.9), but the right side term which is equal to 3^{2x} . That means that $x = \Re z$, $y = \Im z$ satisfy equation (3.16), with x_0 replaced by x , so necessarily $4^x \leq 1 + 2\cdot 3^{x-\frac{1}{2}}$. Therefore (3.21) follows, as claimed. In consequence, if $x_0 \in [0, b_4)$, by (3.21) one has $x_0 < \beta_4$ which implies that $4^{x_0} < 1 + 2\cdot 3^{x_0-\frac{1}{2}}$. That is, x_0 satisfies (3.20) and then we have

$$x_0 < x_1 \leq b_{4,x_0}, \text{ for all } x_0 \in [0, b_4). \quad (40)$$

Let x_0 be an arbitrary point such that $0 < x_0 < b_4$. The point b_{4,x_0}^- , by Proposition 8, is obtained by making $y = 0$ in equation (3.16), so b_{4,x_0}^- is the real solution of the equation

$$1 + 2^x + 4^x = 3^{x_0}. \quad (41)$$

Then we claim that $b_{4,x_0}^- < x_0$. Indeed, if $b_{4,x_0}^- \geq x_0$, from (3.23), we would have

$$3^{x_0} = 1 + 2^{b_{4,x_0}^-} + 4^{b_{4,x_0}^-} \geq 1 + 2^{x_0} + 4^{x_0} > 3^{x_0},$$

a contradiction. Then, it follows, according to (3.22) that $b_{4,x_0}^- < x_0 < b_{4,x_0}$, which means that the line $x = x_0$ meets to the level curve $|G_4^*(z)| = 3^{x_0}$, and, by Theorem 4, $x_0 \in R_4$. As consequence, $(0, b_4) \subset R_4$. Now, noting that R_4 is closed, one deduces that $[0, b_4] \subset R_4$. Finally, taking into account (3.15), one has $[a_4, b_4] \subset R_4$ and since $R_4 \subset [a_4, b_4]$, we definitively get $[a_4, b_4] = R_4$. Now, by using (2.6), the set $R^{(4)}$ corresponding to the partial sum $\zeta_4(z)$ also satisfies

$$R^{(4)} = [-b_4, -a_4] = [a^{(4)}, b^{(4)}].$$

The proof is now completed. ■

4 Distribution of the zeros of $G_n^*(z)$ on the half-planes $\Re z > 0$ and $\Re z < 0$

Now it is needed to prove a similar result to that of Theorem 2 for the functions $G_n^*(z)$.

Theorem 15 *Every function $G_n^*(z)$, $n > 2$, possesses infinitely many zeros in each half-plane $\{z : \Re z < 0\}$ and $\{z : \Re z > 0\}$ except for $n = 3, 4$, whose zeros are all imaginary.*

Proof. For $n = 3$ one has $p_{k_3} = 3$. Then $G_3^*(z) = 1 + 2^z$, whose zeros are

$$\frac{i\pi(2k+1)}{\log 2}, k \in \mathbb{Z}.$$

For $n = 4$, $p_{k_4} = 3$, so $G_4^*(z) = 1 + 2^z + 4^z$ and then its zeros are

$$\frac{i2\pi(3k+1)}{3\log 2}, \frac{i2\pi(3k+2)}{3\log 2}, k \in \mathbb{Z}.$$

Hence, for $n = 3, 4$, the zeros of $G_n^*(z)$ are all imaginary and, consequently, the theorem follows in both cases.

Assume that, for some $m > 4$, there exists a function $G_m^*(z)$ whose zeros $(\alpha_{m,l})_{l=1,2,\dots}$ are of the form

$$\Re \alpha_{m,l} \neq 0 \text{ if } l \leq p, \Re \alpha_{m,l} = 0 \text{ if } l > p, \quad (42)$$

for some positive integer p . By noting that $G_m^*(\bar{z}) = \overline{G_m^*(z)}$ for all $z \in \mathbb{C}$, the zeros of $G_m^*(z)$ are conjugate and then necessarily p is even. Hence, for some integer q , the number $p = 2q \geq 2$ and the zeros $\alpha_{m,l}$, if $l > p$, are of the form $\pm iy_j$ with $y_j > 0$ for all $j \geq 1$. Let us denote by $P(z)$ the polynomial with leader coefficient 1 and zeros $(\alpha_{m,l})_{l=1,2,\dots,p}$. Then

$$P(z) = (z^2 - 2a_1z + |\alpha_{m,1}|^2) \cdots (z^2 - 2a_qz + |\alpha_{m,q}|^2), \quad (43)$$

where $a_l := \Re \alpha_{m,l}$, $l = 1, \dots, q$. By considering the function

$$H_m^*(z) := \frac{G_m^*(z)}{P(z)},$$

entire of order 1 with zeros $\alpha_{m,l}$ for $l > p$, by Hadamard's factorization theorem [1, Th. 4.4.3], we have

$$H_m^*(z) = e^{A_m z + B_m} \prod_{j=1}^{\infty} \left(1 + \frac{z^2}{y_j^2}\right). \quad (44)$$

The constants B_m and A_m are given by the relations

$$e^{B_m} = H_m^*(0) = \frac{m-1}{|\alpha_{m,1} \dots \alpha_{m,q}|^2},$$

$$A_m = \frac{(H_m^*)'(0)}{e^{B_m}} = \frac{\log\left(\frac{m!}{p_{k_m}!}\right)}{m-1} + 2 \left(\frac{a_1}{|\alpha_{m,1}|^2} + \dots + \frac{a_q}{|\alpha_{m,q}|^2} \right),$$

where p_{k_m} denotes, as usual, the last prime not exceeding m . Noticing (4.3), $H_m^*(z)e^{-A_m z}$ is an even function, then

$$H_m^*(z)e^{-A_m z} = H_m^*(-z)e^{A_m z} \text{ for all } z \in \mathbb{C}.$$

Hence, substituting $H_m^*(z)$ by $\frac{G_m^*(z)}{P(z)}$, we get

$$\frac{P(-z)}{P(z)G_m^*(-z)} = \frac{e^{2A_m z}}{G_m^*(z)}, \text{ for all } z \in \mathbb{C}. \quad (45)$$

Now, if we assume that any zero of $P(z)$ is a zero of $P(-z)$, then $(\alpha_{m,l})_{l=1,2,\dots,p}$ are conjugate and opposite, so q is even, and $P(z)$ coincides with $P(-z)$. Thus $\frac{a_1}{|\alpha_{m,1}|^2} + \dots + \frac{a_q}{|\alpha_{m,q}|^2} = 0$, so

$$A_m = \frac{\log\left(\frac{m!}{p_{k_m}!}\right)}{m-1}. \quad (46)$$

By writing $G_m^*(z) = 1 + 2^z + \dots + (m^*)^z$ (it does not have the term $p_{k_m}^z$), where

$$m^* := \begin{cases} m-1, & \text{if } m \text{ is prime} \\ m, & \text{otherwise.} \end{cases},$$

we claim that $2A_m - \log m^* > 0$ for all $m > 4$. Indeed, by assuming that m is prime, $m = p_{k_m}$, so $m^* = m-1$ and, by (4.5), one has

$$2A_m - \log m^* = 2 \frac{\log((m-1)!)}{m-1} - \log(m-1). \quad (47)$$

But, $2\frac{\log((m-1)!)}{m-1} - \log(m-1)$ is positive by virtue of the inequality

$$(k!)^2 > k^k \text{ for all integer } k > 2, \quad (48)$$

which can be easily proved by induction and by the mean value theorem applied on the log function. Hence the claim follows provided that m be a prime number. Let us suppose that m is a composite number greater than 5. From (4.5) and the definition of m^* , we get

$$2A_m - \log m^* = 2\frac{\log(\frac{m!}{p_{k_m}})}{m-1} - \log m.$$

Then $2A_m - \log m^*$ is positive if and only if $(m!)^2 > m^{m-1}(p_{k_m})^2$. But this last inequality, taking into account that $p_{k_m} \leq m-1$, can be immediately deduced from the inequality, sharpest than (4.7),

$$(k!)^2 > k^{k-1}(k-1)^2 \text{ for all integer } k > 5,$$

which can be also demonstrated by using induction and by the mean value theorem applied on the log function. Hence,

$$2A_m - \log m^* > 0 \text{ for any } m > 4, \quad (49)$$

as we claimed. Now, by making $z = x > 0$ in (4.4) and by taking the limit when $x \rightarrow +\infty$, because of (4.8), the left side of (4.4) tends to 1 whereas the right side tends to $+\infty$, a contradiction. Therefore, there is at least one zero of $P(z)$ which is not a zero of $P(-z)$. Then, by writing (4.4) of the form

$$\frac{P(-z)}{P(z)} = \frac{e^{2A_m z} G_m^*(-z)}{G_m^*(z)}, \text{ for all } z \in \mathbb{C}, \quad (50)$$

the left side of (4.9) is a meromorphic function with at least a pole, whereas the right side is a quotient of exponential polynomials with a finite number of poles. It means, by Shields's theorem [21], that the right side of (4.9) is an exponential polynomial and then we are led to a new contradiction. Therefore, there is no integer $m > 4$ for which the function $G_m^*(z)$ have its zeros of the form (4.1) and, in particular, are not all of them situated on the imaginary axis. Indeed, if this were so, by defining the polynomial $P(z)$ as identically 1, we would be led to the following contradiction: by taking the limit when $z = x \rightarrow +\infty$ in (4.4), the left side of (4.4) tends to 1 whereas the right side one tends to $+\infty$. Consequently, all the functions $G_n^*(z)$, $n > 4$, have infinitely many zeros in at least one of the half-planes $\{z : \Re z < 0\}$, $\{z : \Re z > 0\}$. Let $(z_{n,l}^*)_{l=1,2,\dots}$ be all the zeros of $G_n^*(z)$ for a fixed $n > 4$. Suppose, for example, that $\Re z_{n,l}^* \geq 0$ for all $l \geq 1$ and let $z_{n,j}^*$ be a zero of $G_n^*(z)$ such that $\Re z_{n,j}^* > 0$. Then if we take $\epsilon = \Re z_{n,j}^*$, noticing $G_n^*(z)$ is an almost-periodic function, because Lemma 1, $G_n^*(z)$ has infinitely many zeros in the strip

$$S_{\frac{\epsilon}{2}} := \left\{ z : \frac{\epsilon}{2} < \Re z < \frac{3\epsilon}{2} \right\},$$

which implies that

$$\sum_{l=1}^{\infty} \Re z_{n,l}^* = +\infty. \quad (51)$$

On the other hand, since $G_n^*(z)$ is an exponential polynomial with all its coefficients equal to 1, by Ritt's formula [19, formula (9)], we have

$$\sum_{l=1}^{\infty} \Re z_{n,l}^* = O(1),$$

contradicting (4.10). Then $G_n^*(z)$ has at least a zero in the half-plane $\{z : \Re z < 0\}$ and again, by Lemma 1, $G_n^*(z)$ has infinitely many zeros with negative real part. This completes the proof. ■

The next result is obtained directly from the above theorem.

Corollary 16 *The bounds corresponding to the functions $G_n^*(z)$*

$$a_n^* := \inf \{\Re z : G_n^*(z) = 0\}, \quad b_n^* := \sup \{\Re z : G_n^*(z) = 0\}$$

satisfy

$$a_n^* = b_n^* = 0 \text{ if } n = 3, 4$$

and

$$a_n^* < 0, \quad b_n^* > 0 \text{ for all } n > 4.$$

Now we are ready to prove the main result of this paper.

Theorem 17 *For each $n > 2$, there exists $\delta_n > 0$ such that $[a_n, \delta_n] \subset R_n$ and $[-\delta_n, b^{(n)}] \subset R^{(n)}$.*

Proof. From Corollary 3 we have $a_n < 0$ and $b_n > 0$, for all $n > 2$. By Lemma 13, $b_n^* < b_{n,x_0}$ for every integer $n > 2$ and for arbitrary real x_0 . Then, in particular, for $x_0 = a_n$, we have

$$b_n^* < b_{n,a_n} \text{ for every } n > 2. \quad (52)$$

Since $a_n \in R_n$, by Theorem 10,

$$[a_n, b_{n,a_n}] \cap [a_n, b_n] \subset R_n. \quad (53)$$

Because of Corollary 16, $0 \leq b_n^*$ for every $n > 2$. Then by (4.11), $0 < b_{n,a_n}$ for every $n > 2$. Hence,

$$\delta_n := \min \{b_{n,a_n}, b_n\}$$

is a positive number for every $n > 2$. Thus, by (4.12), we get

$$[a_n, b_{n,a_n}] \cap [a_n, b_n] = [a_n, \delta_n] \subset R_n \text{ for every } n > 2.$$

Finally, from (2.6), we obtain

$$[-\delta_n, -a_n] = [-\delta_n, b^{(n)}] \subset R^{(n)} \text{ for every } n > 2,$$

and then the theorem follows. ■

Remark 18 Observe that the proof of Theorem 17, valid for all $n > 2$, is essentially based on the old notion of level curve. Nevertheless (1.7) follows for all $n > N$, for some positive integer N whose existence depends of the Prime Number Theorem (see [15, Lemma 7]). Then [15, Th.12] improves the Theorem 17 of an optimal form, provided that $n > N$, because the δ_n 's are equal to $b_n = -a^{(n)} = n \log 2 + o(n)$, $n \rightarrow \infty$, by virtue of (1.5). The same occurs when $n = 3, 4$, by virtue of Corollary 11 and Theorem 14, respectively.

An important conclusion is deduced from Theorem 17, namely, each set $\cap_{n=3}^{\infty} R_n, \cap_{n=3}^{\infty} R^{(n)}$ contains a non-degenerated closed interval.

Corollary 19 There exist $\delta \geq 0$ and $a < 0$ such that $\cap_{n=3}^{\infty} R_n \supset [a, \delta]$ and $\cap_{n=3}^{\infty} R^{(n)} \supset [-\delta, -a]$.

Proof. By virtue of Theorem 17, we define $\delta := \inf \{\delta_n : n > 2\}$ and then, since $\delta_n > 0$ for all $n > 2$, necessarily $\delta \geq 0$. By (1.4), all the terms of the sequence $(b^{(n)})_{n>2}$, except at most a finite quantity of them, are arbitrarily close to 1. By Corollary 3, $b^{(n)} > 0$ for all $n > 2$, then $b := \min \{b^{(n)} : n > 2\}$ is a positive number. On the other hand, by (2.6), one has $-a_n = b^{(n)}$ for all $n > 2$, so $a := \max \{a_n : n > 2\} = -b < 0$. Now the proof is completed. ■

Corollary 20 For all $n > 2$, there exist infinitely many zeros, having positive and negative real part, of $G_n(z)$ and $\zeta_n(z)$ arbitrarily close to the imaginary axis. In particular, the point 0 is an accumulation point common to the sets $P_{G_n} = \{\Re z : G_n(z) = 0\}$ and $P_{\zeta_n} = \{\Re z : \zeta_n(z) = 0\}$ for all $n > 2$.

Proof. From Theorem 17, it is enough to take into account that 0 is an interior point to each set $R_n := \overline{P_{G_n}}$, $R^{(n)} := \overline{P_{\zeta_n}}$, for all $n > 2$. ■

In [20, Conjecture 15] Sepulcre and Vidal conjectured that the point $0 \in R_n$ for all $n \geq 2$. Now, as an easy consequence of Theorem 17, we can give a positive answer to such question.

Corollary 21 The conjecture of Sepulcre and Vidal is true.

Proof. By Corollary 19, $\cap_{n=3}^{\infty} R_n \supset [a, \delta]$, where $a < 0$ and $\delta \geq 0$. Then, in particular, $0 \in R_n$ for all $n > 2$. Noting (1.6), the set $R_2 = \{0\}$, therefore it follows that $0 \in R_n$ for all $n \geq 2$. Consequently Sepulcre and Vidal's conjecture is true. ■

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