Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting

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BOLTZMANN EQUATION FOR GRANULAR MEDIA WITH THERMAL FORCE IN A WEAKLY INHOMOGENEOUS SETTING

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Abstract. In this paper, we consider the spatially inhomogeneous diffusively driven inelastic Boltzmann equation in different cases: the restitution coefficient can be constant or can depend on the impact velocity (which is a more physically relevant case), including in particular the case of viscoelastic hard-spheres. In the weak thermalization regime, i.e when the diffusion parameter is sufficiently small, we prove existence of global solutions considering both the close-to-equilibrium and the weakly inhomogeneous regimes. We also study the long-time behavior of these solutions and prove a convergence to equilibrium with an exponential rate. The basis of the proof is the study of the linearized equation. We obtain a new result on it, we prove existence of a spectral gap in weighted (stretched exponential and polynomial) Sobolev spaces, more precisely, there is a one-dimensional eigenvalue which is negative and which tends to 0 when the diffusion parameter tends to 0. To do that, we take advantage of the recent paper [18] where the spatially inhomogeneous equation for elastic hard spheres has been considered. As far as the case of a constant coefficient is concerned, the present paper improves similar results obtained in [24] in a spatially homogeneous framework. Concerning the case of a non-constant coefficient, this kind of results is new and we use results on steady states of the linearized equation from [5].

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Contents

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1. Introduction

1.1. **Model and main result.** We investigate in the present paper the Cauchy theory associated to the spatially inhomogeneous diffusively driven inelastic Boltzmann equation for hard spheres interactions and constant or non-constant restitution coefficient. More precisely, we consider hard-spheres particles described by their distribution density $f = f(t,x,v) \geq 0$ undergoing inelastic collisions in the torus in dimension $d = 3$. The spatial coordinates are $x \in \mathbb{T}^3$ (3-dimensional flat torus) and the velocities are $v \in \mathbb{R}^3$. The distribution f satisfies the following equation:

$$\partial_t f = Q_{e_\lambda}(f,f) + \lambda^\gamma\, \Delta_v f - v \cdot \nabla_x f. \tag{1.1}$$

Let us point out that in the case of a constant restitution coefficient, $e_\lambda(\cdot)$ is constant equal to $1-\lambda$ and γ is equal to 1, the equation hence becomes:

$$\partial_t f = Q_{1-\lambda}(f,f) + \lambda\, \Delta_v f - v \cdot \nabla_x f, \tag{1.2}$$

The term $\lambda^\gamma\, \Delta_v f$ represents a constant heat bath which models particles uncorrelated random accelerations between collisions. The quadratic collision operator Q_{e_λ} models the interactions of hard-spheres by inelastic binary collisions where the inelasticity is characterized by the so-called normal restitution coefficient $e_\lambda(\cdot)$ which can be, in contrast with previous contributions on the subject, constant or non-constant. In the non-constant case, this restitution coefficient quantifies the loss of relative normal velocity of a pair of colliding particles after the collision with respect to the impact velocity. Namely, if v and v_* denote the velocities of two particles before collision, their respective velocities v' and v'_* after collision are such that

$$(u' \cdot \widehat{n}) = -(u \cdot \widehat{n})\, e_\lambda(u \cdot \widehat{n}), \tag{1.3}$$

where $e_\lambda(\cdot) := e(\lambda\,\cdot)$ and $e := e(|u \cdot \widehat{n}|)$ is such that $0 \leq e \leq 1$. The unitary vector $\widehat{n} \in \mathbb{S}^2$ determines the impact direction, that is, $\widehat{n}$ stands for the unit vector that points from the v-particle center to the v_*-particle center at the instant of impact. Here above,

$$u = v - v_*, \quad u' = v' - v'_*,$$

denote respectively the relative velocity before and after collision. Assuming the granular particles to be perfectly smooth hard-spheres of mass $m = 1$, the velocities after collision

v' and v'_* are given, in virtue of (1.3) and the conservation of momentum, by

$$v' = v - \frac{1+e_\lambda}{2}\,(u\cdot\widehat{n})\,\widehat{n}, \quad v'_* = v_* + \frac{1+e_\lambda}{2}\,(u\cdot\widehat{n})\,\widehat{n}. \tag{1.4}$$

The main assumption on $e(\cdot)$ we shall need is listed in the following (see [1] for more details).

Assumptions 1.1.

(1) *The mapping* $r \to e(r)$ *from* $\mathbb{R}^+$ *to* $(0,1]$ *is absolutely continuous and non-increasing.*
(2) *The mapping* $r \to r\,e(r)$ *is strictly increasing on* $\mathbb{R}^+$.
(3) *There exist* $a, b > 0$ *and* $\overline{\gamma} > \gamma > 0$ *such that*

$$\forall\, r \geq 0, \quad |e(r) - 1 + a\,r^\gamma| \leq b\,r^{\overline{\gamma}}.$$

The assumptions (1) and (2) are trivially satisfied in the constant case which is enough to apply most of the results from [5]. The assumption (3) is crucial to do a fine study of spectrum of the linearized operator close to 0 in the non-constant case (see step 4 of proof of Theorem 2.14). Let us also emphasize that the three assumptions are met by the visco-elastic hard-spheres model which is the most physically relevant model for applications (see [12] and Subsection 1.2). In the remaining part of the paper, we suppose that the restitution coefficient $e(\cdot)$ is constant or satisfies Assumptions 1.1.

In the sequel, it shall be more convenient to deal with a second, and equivalent, parametrization of the post-collisional velocities. Fix v and v_* with $v \neq v_*$ and let $\widehat{u} = u/|u|$. Performing in (1.4) the change of unknown $\sigma = \widehat{u} - 2\,(\widehat{u}\cdot\widehat{n})\,\widehat{n} \in \mathbb{S}^2$ provides an alternative parametrization of the unit sphere $\mathbb{S}^2$. In this case, the impact velocity reads $|u\cdot\widehat{n}| = |u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}$ and the post-collisional velocities v' and v'_* are then given by

$$v' = v - \frac{1+e_\lambda}{2}\,\frac{u-|u|\sigma}{2}, \quad v'_* = v_* + \frac{1+e_\lambda}{2}\,\frac{u-|u|\sigma}{2}. \tag{1.5}$$

This representation allows us to give a precise definition of the Boltzmann collision operator in weak form by

$$\int_{\mathbb{R}^3} Q_{e_\lambda}(g,f)\,\psi\,dv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)\left[\psi(v') - \psi(v)\right]|v-v_*|\,d\sigma\,dv_*\,dv, \tag{1.6}$$

for any $\psi = \psi(v)$ a suitable regular test function. Here, the post-collisional velocities v' and v'_* are defined by (1.5). Notice that

$$|v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -|u|^2\,\frac{1-\widehat{u}\cdot\sigma}{4}\left(1 - e\left(|u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)^2\right). \tag{1.7}$$

The operator Q_{e_λ} defined by (1.6) preserves mass and momentum, and since the Laplacian also does so, the equation preserves mass and momentum. However, energy is not preserved either by the collisional operator (which tends to cool down the gas because of (1.7)) or by the diffusive operator (which warms it up).

The formula (1.6) suggests the natural splitting $Q_{e_\lambda} = Q^+_{e_\lambda} - Q^-_{e_\lambda}$ between gain and loss parts. The loss part $Q^-_{e_\lambda}$ can easily be defined in strong form noticing that

$$\langle Q^-_{e_\lambda}(g,f),\psi\rangle = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)f(v)\psi(v)|v-v_*|\,d\sigma\,dv_*\,dv =: \langle f\,L(g),\psi\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in L^2 and L is the convolution operator

$$L(g)(v) = 4\pi(|\cdot| * g)(v).$$

In particular, we can notice that L and $Q^-_{e_\lambda}$ are independent of the normal restitution coefficient.

We also define the symmetrized (or polar form of the) bilinear collision operator $\widetilde{Q}_{e_\lambda}$ by setting

$$\int_{\mathbb{R}^3} \widetilde{Q}_{e_\lambda}(g,h)\psi \, dv = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)h(v)|v-v_*| \left[\psi(v') + \psi(v'_*)\right] d\sigma \, dv_* \, dv - \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} g(v_*)h(v)|v-v_*| \left[\psi(v) + \psi(v_*)\right] d\sigma \, dv_* \, dv. \tag{1.8}$$

In other words, $\widetilde{Q}_{e_\lambda}(g,h) = (Q_{e_\lambda}(g,h) + Q_{e_\lambda}(h,g))/2$. The formula (1.8) also suggests a splitting $\widetilde{Q}_{e_\lambda} = \widetilde{Q}^+_{e_\lambda} - \widetilde{Q}^-_{e_\lambda}$ between gain and loss parts. We can notice that we have $\widetilde{Q}^+_{e_\lambda}(g,h) = (Q^+_{e_\lambda}(g,h) + Q^+_{e_\lambda}(h,g))/2$ and $\widetilde{Q}^-_{e_\lambda}(g,h) = (Q^-_{e_\lambda}(g,h) + Q^-_{e_\lambda}(h,g))/2$.

In the elastic case ($\lambda = 0$), we can easily define the collision operator in strong form using the pre-post collisional change of variables:

$$Q_1(g,f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \left[f(v')g(v'_*) - f(v)g(v_*)\right] |v-v_*| \, dv_* \, d\sigma.$$

Our main result is the proof of existence of solutions for the non-linear problem (1.1) as well as stability and relaxation to equilibrium for these solutions. This work stands out from others because it deals with the spatially inhomogeneous case. We know from [5] that there exists $G_\lambda = G_\lambda(v)$ a space homogeneous solution of the stationary equation

$$Q_{e_\lambda}(f,f) + \lambda^\gamma \Delta_v f = 0$$

with mass 1 and vanishing momentum. Moreover, G_λ is unique for λ close enough to 0. We refer to Subsection 2.2 for more details.

Theorem 1.2. *Let us consider λ in $[0,\lambda_0]$ where $\lambda_0 \in (0,1)$ will be defined in Lemma 2.8. Consider the functional space $\mathcal{E}_0 = W^{s,1}_x W^{2,1}_v\left(\langle v\rangle e^{b\langle v\rangle^\beta}\right)$ where $b > 0$, $\beta \in (0,1)$ and $s > 6$. Consider a spatially homogeneous distribution $g_{in} = g_{in}(v) \in W^{2,1}_v\left(\langle v\rangle^5 e^{b\langle v\rangle^\beta}\right)$ with the same global mass and momentum as G_λ. Let us denote $M := \|g_{in}\|_{W^{2,1}_v\left(\langle v\rangle^5 e^{b\langle v\rangle^\beta}\right)}$.*

There is some constructive constant $\varepsilon(M) > 0$ such that for any initial data $f_{in} \in \mathcal{E}_0$ satisfying

$$\|f_{in} - g_{in}\|_{\mathcal{E}_0} \le \varepsilon(M),$$

and f_{in} has the same mass and momentum as G_λ and g_{in}, there exists a global solution $f \in L^\infty_t(\mathcal{E}_0)$ to (1.1).

Moreover, for any $\widetilde{\alpha} \in (0, -\mu_\lambda)$ (where $\mu_\lambda < 0$ will be defined in Theorem 2.14), this solution satisfies

$$\|f_t - G_\lambda\|_{\mathcal{E}_0} \le C e^{-\widetilde{\alpha} t}$$

for some constructive constant $C > 0$.

1.2. **Physical and mathematical motivation.** For a detailed physical introduction to granular gases we refer to [12, 14]. As can be seen from the references included in the latter, granular flows have become a subject of physical research on their own in the last decades, and for certain regimes of dilute and rapid flows, these studies are based on kinetic theory. By contrast, the mathematical kinetic theory of granular gas is rather young and began in the late 1990 decade. We refer to [25, 22] for some (short) mathematical introduction to this theory and a (non exhaustive) list of references. As explained in these papers, granular gases are composed of grains of macroscopic size with contact collisional interactions, when one does not consider other additional possible self-interaction mechanisms such as gravitation – for cosmic clouds for instance – or electromagnetism – for "dusty plasmas" for instance –. Therefore the natural assumption about the binary interaction between grains is that of inelastic hard spheres, with no loss of "tangential relative velocity" (according to the impact direction) and a loss in "normal relative velocity". This loss is quantified in some (normal) restitution coefficient. The latter is either assumed to be constant as a first approximation or can be more intricate: for instance it is a function of the modulus $|v' - v|$ of the normal relative velocity in the case of "visco-elastic hard spheres" (see [3], [4], [5] and [12]). In this paper, we consider both constant and non-constant restitution coefficients.

We restrict to the case of a small diffusion parameter (weak thermalization regime), which corresponds to small inelasticity. There are several motivations from mathematics and physics for such a choice:

- the first reason is related to the regime of validity of kinetic theory: as explained in [12, Chapter 6] for instance, the more inelasticity, the more correlations between grains are created during the binary collisions, and therefore the molecular chaos assumption, which is the core of the validity of Boltzmann's theory, suggests weak inelasticity to be the most effective;
- second, as emphasized in [12] again, the case of small inelasticity has been widely considered in physics or mathematical physics since it allows to use expansions around the elastic case, and since conversely it is an interesting question to understand the connection of the inelastic case (dissipative at the microscopic level) to the elastic case ("Hamiltonian" at the microscopic level);
- finally, this case of a small inelasticity is reasonable from the viewpoint of applications, since it applies to interstellar dust clouds in astrophysics, or sands and dusts in earth-bound experiments, and more generally to visco-elastic hard spheres whose restitution coefficient is not constant but close to 1 on the average.

Let us now describe the most physically relevant model, the one corresponding to viscoelastic hard spheres for which the restitution coefficient has been derived in [30]. For this peculiar model, $e(\cdot)$ admits the following representation as an infinite expansion series

$$e(|u\cdot\widehat{n}|) = 1 + \sum_{k=1}^{\infty} (-1)^k\, a_k |u\cdot\widehat{n}|^{k/5}, \quad u\in\mathbb{R}^3, \quad \widehat{n}\in\mathbb{S}^2 \tag{1.9}$$

where $a_k > 0$ for any $k \in \mathbb{N}$ are parameters depending on the material viscosity. We can see that in this case, $e(\cdot)$ satisfies Assumptions 1.1. More precisely, the assumption (3) is satisfied with $\gamma = 1/5$ and $\overline{\gamma} = 2/5$. In the case of a non-constant restitution coefficient, this is the principal example of application of the results in the paper, though, as we shall see, our results will cover more general cases.

1.3. **Function spaces.** For some given Borel weight function $m > 0$ on $\mathbb{R}^3$, let us define $L^q_v L^p_x(m)$, $1 \le p, q \le +\infty$, as the Lebesgue space associated to the norm

$$\|h\|_{L^q_v L^p_x(m)} = \| \|h(\cdot, v)\|_{L^p_x} \, m(v)\|_{L^q_v}.$$

We also consider the standard higher-order Sobolev generalizations $W^{\sigma,q}_v W^{s,p}_x(m)$ for $\sigma, s \in \mathbb{N}$ defined by the norm

$$\|h\|_{W^{\sigma,q}_v W^{s,p}_x(m)} = \sum_{0 \le s' \le s,\, 0 \le \sigma' \le \sigma,\, s'+\sigma' \le \max(s,\sigma)} \| \|\nabla^{s'}_x \nabla^{\sigma'}_v h(\cdot, v)\|_{L^p_x} \, m(v)\|_{L^q_v}.$$

This definition reduces to the usual weighted Sobolev space $W^{s,p}_{x,v}(m)$ when $q = p$ and $\sigma = s$, and we recall the shorthand notation $H^s = W^{s,2}$.

1.4. **Known results.** Let us briefly review the existing results concerning inelastic hard spheres Boltzmann models. We shall mention that most of them are established in an homogeneous framework and that the major part of the investigation has been devoted to the particular case of a constant restitution coefficient.

For the inhomogeneous inelastic Boltzmann equation, the literature is more scarce; in this respect we mention the work [1] that treats the Cauchy problem in the case of near-vacuum data. It is worthwhile mentioning that the scarcity of results regarding existence of solutions for the inhomogeneous case is explained by the lack of entropy estimates for the inelastic Boltzmann equation; thus, well-known theories like the DiPerna-Lions renormalized solutions are no longer available. Let us now give an overview of papers dealing with homogeneous equations.

Let us begin by papers considering constant restitution coefficient and dealing with existence, uniqueness or properties of self-similar profiles (resp. stationary solutions) for freely cooling (resp. driven by a thermal bath) inelastic hard spheres. In the paper [11], existence of self-similar profiles or stationary solutions is assumed and *a priori* polynomial and exponential moments bounds are shown. The paper [15] completes the previous one showing existence of stationary solutions for inelastic hard spheres driven by a thermal bath, and improving the estimates on their tails of [11] into pointwise ones. The paper [22] shows, for freely cooling inelastic hard spheres, existence of self-similar profile(s) as well as propagation of regularity and damping of singularities with time. The paper [23] proves uniqueness of the stationary solution in the physical regime of a small inelasticity and provides various results on the linear stability and nonlinear stability of this stationary solution. Finally, the paper [24] gives similar answers as in [23] adding a thermal bath term. We can also mention the paper [29] which investigates the long-time behavior of the solutions for an "anomalous" gas. Existence and uniqueness of blow up profiles for this model are studied, together with the trend to equilibrium and the cooling law associated.

Let us now mention the papers dealing with inelastic hard spheres models with more general restitution coefficient. The paper [25] provides a Cauchy theory for freely cooling inelastic hard spheres with a broad family of collision kernels (including in particular restitution coefficients possibly depending on the relative velocity and/or the temperature), and studies whether the gas cools down in finite time or asymptotically, depending on the collision kernel. The paper [3] shows the generalized Haff's law yielding the optimal algebraic cooling rate of the temperature of a granular gas described by the homogeneous Boltzmann equation for inelastic interactions with non constant restitution coefficient. The paper [4] improves the previous one giving two simpler proofs of the Haff's law. The paper [5] studies uniqueness and regularity of the steady states of the diffusively driven Boltzmann equation in the physically relevant case where the restitution coefficient depends on the impact velocity including, in particular, the case of viscoelastic hard-spheres.

Our results are established in an inhomogeneous setting in a small inelasticity regime (close to the elastic one). To obtain them, we use results on the linearized elastic equation. We hence give an overview on the results already obtained on the linearized elastic equation. Let us denote $\mu := G_0$ the elastic equilibrium which is a Maxwellian distribution.

In the spatially homogeneous case, the study of the linearized collision operator $\overline{\mathcal{L}}_0$ goes back to Hilbert [19, 20] who computed the collisional invariant, the linearized operator and its kernel in the hard spheres case, and showed the boundedness and complete continuity of the non-local part of $\overline{\mathcal{L}}_0$. The operator is self-adjoint non-positive and generates a strongly continuous semigroup in in the space $L^2_v(\mu^{-1/2})$. Carleman [13] then proved the existence of a spectral gap by using Weyl's theorem and the compactness of the non-local part of $\overline{\mathcal{L}}_0$. Grad [16, 17] then extended these results to the so-called hard potentials with cutoff. All these arguments are based on non-constructive arguments. The first constructive estimates were obtained only recently in [8]. These spectral gap estimates can easily be extended to $H^s_v(\mu^{-1/2})$, $s \in \mathbb{N}^*$, see for instance the introduction of derivatives in the proof of Lemma 2.8.

Let us also briefly mention the works [33, 9, 10] for the different setting of Maxwell molecules where the eigenbasis and eigenvalues can be explicitely computed by means of Fourier transform methods. Although these techniques do not apply here, the explicit formula computed are an important source of inspiration for dealing with more general physical models.

The full linearized operator $\mathcal{L}_0$ is the sum of the self-adjoint non-positive operator $\overline{\mathcal{L}}_0$ and the skew-symmetric transport operator $-v\cdot\nabla_x$. It was first established in [31, Theorem 1.1] that it has a spectral gap in the Hilbert space $H^s_x L^2_v(\mu^{-1/2})$, $s \in \mathbb{N}$ by non-constructive arguments. Ukai [31] also showed the spectral property in $H^s_x L^\infty_v((1+|v|^k)\mu^{-1/2})$, $k > 3/2$, using an argument initially due to Grad [17] for constructing local-in-time solutions. In [28, Theorems 1.1 & 3.1], quantitative spectral gap estimates are established in $H^s_{x,v}(\mu^{-1/2})$, $s \in \mathbb{N}^*$.

All the studies mentioned above are done in weighted spaces with a Maxwellian weight prescribed by the equilibrium. Let us now talk about the improvements made to weights. For the spatially homogeneous case, in [6] a first extension of the decay estimate to L^1 with polynomial weight was obtained by an intricate nonconstructive approach based on decomposition of the solution and some dyadic decomposition of the velocity variable. This argument was then extended to L^p spaces in [34, 35]. In [27], another improvement was made, a spectral gap estimate on the space homogeneous semigroup $e^{t\overline{\mathcal{L}}_0}$ was extended to the space $L^1_v(m)$ for a stretched exponential weight m, by constructive means, with optimal rate. Let us also mention that in [7], some non-constructive decay estimates were obtained in a Sobolev space in position combined with a polynomially weighted L^∞ space in velocity. Finally, let us underline the theory of enlargement of spectral gap developed in [18] which gives explicit spectral gap estimates on the semigroup associated to the linearized non homogeneous operator $\mathcal{L}_0$ in $W^{s,p}_x W^{\sigma,q}_v(m)$ with polynomial or stretched exponential weight m.

1.5. **Method of proof.** The main outcome of this paper is a new Cauchy theory for the non-homogeneous Boltzmann equation for inelastic hard spheres (1.1). We prove existence and stability of solutions for this equation. In order to do so, we first establish the asymptotic stability of the linearized equation by a perturbation argument which uses the spectral analysis of the linearized elastic Boltzmann equation.

Let us explain in more details how we deal with the linearized problem, our method is in the spirit of the one in [24]. However, our study largely improves the one done in [24] in three aspects:

- we are able to deal with the spatial dependency in the torus;
- we are able to deal with non-constant restitution coefficients;
- we are able to obtain a decay estimate on the semigroup using the localization of the spectrum.

The perturbative argument around the elastic operator allows us to obtain results on the localization of the spectrum of the inelastic operator. It is based on the following facts:

- the inelastic operator can be written as the sum of a regularizing part and a dissipative part (these operators are defined through an appropriate mollification-truncation process, described later on);
- the inelastic operator is a small perturbation of the elastic one for a diffusion parameter sufficiently small;
- we know that the spectrum of the elastic operator is well localized.

To prove the first two points, we get estimates on the difference between the elastic and the inelastic collision operators which is small when taking λ close enough to 0. We establish these estimates in an inhomogeneous setting; this kind of estimates was only known to hold in an homogeneous setting (see [23] for the case of a constant restitution coefficient and [5] for the non-constant case).

About the third point, let us emphasize that equilibriums in the inelastic case do not decrease enough to belong to spaces with Maxwellian weights. Therefore, a perturbative theory close to the elastic equation is not possible in spaces of this type. But the results obtained in [18] via the theory of enlargement of spectral gap allows us to apply a perturbative theory. Indeed, estimates on the elastic collision operator are proved in spaces of type $W^{s,p}_x W^{\sigma,q}_v(m)$ where m is a polynomial or stretched exponential weight.

Using these facts, we prove our main result on the linearized inelastic operator. Its spectrum $\Sigma(\mathcal{L}_\lambda)$ is well localized: there is a constructive constant $\alpha > 0$ such that

$$\Sigma(\mathcal{L}_\lambda) \cap \{z \in \mathbb{C}, \, \Re e \, z > -\alpha\} = \{\mu_\lambda, 0\},$$

0 is a four-dimensional eigenvalue (due to the conservation of mass and momentum) and $\mu_\lambda \in \mathbb{R}$, the "energy" eigenvalue, is a one-dimensional eigenvalue. We also obtain an estimate on μ_λ which is negative for λ close enough to 0. The behavior of μ_λ is linked with the fact that the energy is not preserved by the operator. Let us finally emphasize that we prove that these spectral properties imply the decay of the semigroup associated with an exponential rate.

Let us now explain how we go back to the nonlinear problem. We construct perturbative solutions close to the equilibrium or close to the spatially homogeneous case. To do so, we use the two following points:

- we introduce a dissipative Banach norm for the fully linearized operator which provides the key a priori estimate to get the "linearization trap";
- we prove bilinear estimates to control the nonlinear remainder in the equation.

As far as the close-to-equilibirum regime is concerned, the idea of the proof is to gather these two points; we can then prove that taking a sufficiently small initial data, the solution is trapped close to the equilibrium.

To deal with the weakly inhomogeneous regime, we also prove a local in time stability. We can then capture a general solution around the subset of spatially homogeneous solutions and then the general solution is driven towards equilibrium thanks to the relaxation estimates known for the spatially homogeneous solutions. Finally, we use the previous case once the stability neighborhood is entered by the solution.

1.6. **Outline of the paper.** In Section 2, we introduce the splitting of the inelastic linearized Boltzmann operator as the sum of a regularizing part and a dissipative part. We show that our inelastic operator is a small perturbation of the elastic one. We also make a fine study of spectrum close to 0, which allows us to prove existence of a spectral gap. We then obtain a property of semigroup decay in $W^{s,1}_x W^{2,1}_v(\langle v \rangle m)$ for a stretched exponential weight m. This section ends by the introduction of a new norm which is dissipative for the full linearized operator.

In Section 3, we go back to the nonlinear problem. We consider first the close-to-equilibrium regime and we state our main theorem concerning the weakly inhomogeneous regime.

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2. Properties of the linearized operator

2.1. **Notations and definitions.** For a given real number $a \in \mathbb{R}$, we define the half complex plane

$$\Delta_a := \{z \in \mathbb{C},\, \Re e\, z > a\}.$$

For some given Banach spaces $(E, \|\cdot\|_E)$ and $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, we denote by $\mathscr{B}(E, \mathcal{E})$ the space of bounded linear operators from E to $\mathcal{E}$ and we denote by $\|\cdot\|_{\mathscr{B}(E,\mathcal{E})}$ or $\|\cdot\|_{E\to\mathcal{E}}$ the associated operator norm. We write $\mathscr{B}(E) = \mathscr{B}(E, E)$ when $E = \mathcal{E}$. We denote by $\mathscr{C}(E, \mathcal{E})$ the space of closed unbounded linear operators from E to $\mathcal{E}$ with dense domain, and $\mathscr{C}(E) = \mathscr{C}(E, E)$ in the case $E = \mathcal{E}$.

For a Banach space X and $\Lambda \in \mathscr{C}(X)$ we denote by $S_\Lambda(t)$ or $e^{\Lambda t}$, $t \geq 0$, its semigroup, by $\mathrm{D}(\Lambda)$ its domain, by $\mathrm{N}(\Lambda)$ its null space and by $\mathrm{R}(\Lambda)$ its range. We introduce the $\mathrm{D}(\Lambda)$-norm defined as $\|f\|_{\mathrm{D}(\Lambda)} = \|f\|_X + \|\Lambda f\|_X$ for $f \in \mathrm{D}(\Lambda)$. More generally, for $k \in \mathbb{N}$, we define

$$\|f\|_{\mathrm{D}(\Lambda^k)} = \sum_{j=0}^{k} \|\Lambda^j f\|_X, \quad f \in \mathrm{D}(\Lambda^k).$$

We also denote by $\Sigma(\Lambda)$ its spectrum, so that for any z belonging to the resolvent set $\rho(\Lambda) := \mathbb{C}\backslash\Sigma(\Lambda)$, the operator $\Lambda - z$ is invertible and the resolvent operator

$$\mathcal{R}_\Lambda(z) := (\Lambda - z)^{-1}$$

is well-defined, belongs to $\mathscr{B}(X)$ and has range equal to $\mathrm{D}(\Lambda)$. We recall that $\xi \in \Sigma(\Lambda)$ is said to be an eigenvalue if $N(\Lambda - \xi) \neq \{0\}$. Moreover an eigenvalue $\xi \in \Sigma(\Lambda)$ is said to be isolated if there exists $r > 0$ such that

$$\Sigma(\Lambda) \cap \{z \in \mathbb{C},\ |z - \xi| \leq r\} = \{\xi\}.$$

In the case when ξ is an isolated eigenvalue we may define $\Pi_{\Lambda,\xi} \in \mathscr{B}(X)$ the associated spectral projector by

$$\Pi_{\Lambda,\eta} := -\frac{1}{2i\pi} \int_{|z-\xi|=r'} (\Lambda - z)^{-1}\, dz$$

with $0 < r' < r$. Note that this definition is independent of the value of r' as the application $\mathbb{C} \setminus \Sigma(\Lambda) \to \mathscr{B}(X)$, $z \to \mathcal{R}_\Lambda(z)$ is holomorphic. For any $\xi \in \Sigma(\Lambda)$ isolated, it is well-known (see [21, Paragraph III-6.19]) that $\Pi^2_{\Lambda,\xi} = \Pi_{\Lambda,\xi}$, so that $\Pi_{\Lambda,\xi}$ is indeed a projector, and that the "associated projected semigroup"

$$S_{\Lambda,\xi}(t) := -\frac{1}{2i\pi} \int_{|z-\xi|=r'} e^{zt} \mathcal{R}_\Lambda(z)\, dz, \quad t > 0,$$

satisfies

$$\forall\, t > 0, \quad S_{\Lambda,\xi}(t) = \Pi_{\Lambda,\xi} S_\Lambda(t) = S_\Lambda(t) \Pi_{\Lambda,\xi}.$$

When moreover the so-called "algebraic eigenspace" $\mathrm{R}(\Pi_{\Lambda,\xi})$ is finite dimensional we say that ξ is a discrete eigenvalue, written as $\xi \in \Sigma_d(\Lambda)$. In that case, $\mathcal{R}_\Lambda$ is a meromorphic function on a neighborhood of ξ, with non-removable finite-order pole ξ, and there exists $\alpha_0 \in \mathbb{N}^*$ such that

$$\mathrm{R}(\Pi_{\Lambda,\xi}) = N(\Lambda - \xi)^{\alpha_0} = N(\Lambda - \xi)^{\alpha} \ \text{ for any } \ \alpha \geq \alpha_0.$$

On the other hand, for any $\xi \in \mathbb{C}$ we may also define the "classical algebraic eigenspace"

$$M(\Lambda - \xi) := \lim_{\alpha \to \infty} N(\Lambda - \xi)^\alpha.$$

We have then $M(\Lambda - \xi) \neq \{0\}$ if $\xi \in \Sigma(\Lambda)$ is an eigenvalue and $M(\Lambda - \xi) = \mathrm{R}(\Pi_{\Lambda,\xi})$ if $\xi \in \Sigma_d(\Lambda)$.

Finally for any $a \in \mathbb{R}$ such that

$$\Sigma(\Lambda) \cap \Delta_a = \{\xi_1, \dots, \xi_k\}$$

where $\xi_1, \dots, \xi_k$ are distinct discrete eigenvalues, we define without ambiguity

$$\Pi_{\Lambda,a} := \Pi_{\Lambda,\xi_1} + \dots \Pi_{\Lambda,\xi_k}.$$

We shall need the following definition on the convolution of semigroup (corresponding to composition at the level of the resolvent operators). If one consider some Banach spaces X_1, X_2, X_3, for two given functions

$$S_1 \in L^1(\mathbb{R}_+; \mathscr{B}(X_1, X_2)) \ \text{ and } \ S_2 \in L^1(\mathbb{R}_+; \mathscr{B}(X_2, X_3)),$$

the convolution $S_2 * S_1 \in L^1(\mathbb{R}_+; \mathscr{B}(X_1, X_3))$ is defined as

$$\forall\, t \geq 0, \quad (S_2 * S_1)(t) := \int_0^t S_2(s)\, S_1(t-s)\, ds.$$

When $S_1 = S_2$ and $X_1 = X_2 = X_3$, $S^{(*\ell)}$ is defined recursively by $S^{(*1)} = S$ and for any $\ell \geq 2$, $S^{(*\ell)} = S * S^{(*(\ell-1))}$.

One can immediately see that if S_i satisfies $\|S_i(t)\|_{\mathscr{B}(X_i, X_{i+1})} \leq C_i\, t^{\alpha_i}\, e^{a_i\, t}$ for any $t \geq 0$ and some $a_i \in \mathbb{R}$, $\alpha_i \in \mathbb{N}$, $C_i \in (0, \infty)$, then

$$\forall\, t \geq 0, \quad \|S_1 * S_2(t)\|_{\mathscr{B}(X_1, X_2)} \leq C_1\, C_2\, \frac{\alpha_1!\, \alpha_2!}{(\alpha_1 + \alpha_2 + 1)!}\, t^{\alpha_1 + \alpha_2 + 1}\, e^{\max(a_1, a_2)\, t}.$$

This implies that if S satisfies $\|S(t)\|_{\mathscr{B}(X)} \leq Ce^{a\,t}$ for any $t \geq 0$ and some $a \in \mathbb{R}$, $C \in (0, \infty)$, then

$$\forall\, t \geq 0, \quad \|S^{(*n)}(t)\|_{\mathscr{B}(X)} \leq C^n \frac{1}{(n-1)!}\, t^{n-1}\, e^{a\,t}.$$

Let us now introduce the notion of hypodissipative operators. If one consider a Banach space $(X, \|\cdot\|_X)$ and some operator $\Lambda \in \mathscr{C}(X)$, $(\Lambda - a)$ is said to be hypodissipative on X if there exists some norm $|||\cdot|||_X$ on X equivalent to the initial norm $\|\cdot\|_X$ such that

$$\forall\, f \in \mathrm{D}(\Lambda), \quad \exists\, \phi \in F(f) \quad \text{s.t} \quad \Re e\langle \phi, (\Lambda - a) f\rangle \leq 0,$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket for the duality in X and X^* and $F(f) \subset X^*$ is the dual set of f defined by

$$F(f) = F_{|||\cdot|||_X}(f) := \left\{ \phi \in X^*,\, \langle \phi, f\rangle = |||f|||_X^2 = |||\phi|||_{X^*}^2 \right\}.$$

One classically sees (cf [18]) that if X is a Banach space and Λ is the generator of a semigroup S_Λ, for given constants $a \in \mathbb{R}$, $M > 0$ the following assertions are equivalent:
(a) $\Lambda - a$ is hypodissipative;
(b) the semigroup satisfies the growth estimate $\|S_\Lambda(t)\|_{\mathscr{B}(X)} \leq M\, e^{a\,t}$, $t \geq 0$;
(c) there exists some norm $|||\cdot|||$ on X equivalent to the initial norm, and more precisely satisfying

$$\forall\, f \in X, \quad \|f\| \leq |||f||| \leq M\, \|f\|,$$

such that $\rho(\Lambda) \supset]a, \infty[$ and

$$\forall\, \lambda > a, \ \forall\, f \in \mathrm{D}(\Lambda), \quad |||(\Lambda - \lambda)\, f||| \geq (\lambda - a)\, |||f|||.$$

We refer to [18, Subsection 2.3] for further details on this subject.

2.2. **Preliminaries on the steady states.** Let us first recall results about the stationary equation

$$Q_{e_\lambda}(f, f) + \lambda^\gamma\, \Delta_v f = 0. \tag{2.1}$$

The main references for this subsection are [24] for the constant case and [5] for the non-constant case. We introduce the following notation: we shall say that a restitution coefficient $e(\cdot)$ satisfying Assumptions 1.1 is belonging to the class $\mathbb{E}_m$ for some integer $m \geq 1$ if $e(\cdot) \in \mathcal{C}^m(0, \infty)$ and

$$\forall\, k = 1, \ldots, m, \quad \sup_{r \geq 0} r e^{(k)}(r) < \infty,$$

where $e^{(k)}(\cdot)$ denotes the k-th order derivative of $e(\cdot)$.

Remark 2.1. *For the physically relevant case of visco-elastic hard-spheres, the restitution coefficient $e(\cdot)$ is given by (1.9) but admits also the following implicit representation (see [12]):*

$$\forall\, r > 0, \quad e(r) + a r^{\frac{1}{5}} e^{\frac{3}{5}}(r) = 1$$

for some $a > 0$. Then, it is possible to deduce from such representation that $e(\cdot)$ belongs to the class $\mathbb{E}_m$ for any integer $m \geq 1$.

In [5, Theorem 4.5], the authors state that if $e(\cdot)$ belongs to the class $\mathbb{E}_m$ for some integer $m \geq 4$, there exists $\lambda^\dagger \in (0,1]$ such that for any $\lambda \in [0,\lambda^\dagger)$, there exists a unique solution in L^1_2 of (2.1) of mass 1 and vanishing momentum. We denote G_λ this solution.

It is also proved in [5, Proposition 3.3] that there exist $A>0$, $M>0$ such that for any $\lambda \in (0,\lambda^\dagger]$, G_λ satisfies

$$\int_{\mathbb{R}^3} G_\lambda(v)\, e^{A\,|v|^{3/2}}\, dv \leq M. \tag{2.2}$$

Let us point out that in the case of a constant coefficient, these results were already established. In [11, Theorem 1] and [15, Theorem 5.2 & Lemma 7.2], existence of solutions and regularity estimates are proved. In [24, Section 2.1], it is proved that these estimates are uniform in terms of the coefficient of inelasticity and in [24, Theorem 1.2], uniqueness of steady states is proved for a sufficiently small coefficient of inelasticity.

We denote $m(v) = e^{b\langle v\rangle^\beta}$, $b>0$ and $\beta \in (0,1)$. We now state several lemmas on steady states G_λ which are straightforward consequences of results from [24] and [5]. We shall use them several times in what follows. First, we recall a result of interpolation [23, Lemma B.1] which is going to be very useful.

Lemma 2.2. *For any k, $q \in \mathbb{N}$, there exists $C>0$ such that for any $h \in H^{k'}_v \cap L^1_v(m^{12})$ with $k' = 8k + 7(1+3/2)$*

$$\|h\|_{W^{k,1}_v(\langle v\rangle^q m)} \leq C \|h\|^{1/8}_{H^{k'}_v} \|h\|^{1/8}_{L^1_v(m^{12})} \|h\|^{3/4}_{L^1_v(m)}.$$

Let us now prove estimate on Sobolev norm of G_λ.

Lemma 2.3. *Let k, $q \in \mathbb{N}$. We denote $k' = 8k+7(1+3/2)$. If $e(\cdot)$ belongs to the space $\mathbb{E}_{k'+1}$, then there exists $C>0$ such that*

$$\forall\, \lambda \in (0,\lambda^\dagger], \quad \|G_\lambda\|_{W^{k,1}_v(\langle v\rangle^q m)} \leq C.$$

Proof. We deduce from (2.2) that there exists $C>0$ such that for any $\lambda \in (0,\lambda^\dagger]$, $\|G_\lambda\|_{L^1_v(m)} \leq C$ and $\|G_\lambda\|_{L^1_v(m^{12})} \leq C$. We now use [5, Theorem 3.6], it gives us the following:

$$\forall\, q \in \mathbb{N},\ \forall\, \ell \in [0,k'], \quad \sup_{\lambda \in (0,\lambda^\dagger]} \|G_\lambda\|_{H^\ell_v(\langle v\rangle^q)} < \infty.$$

Gathering the previous estimates and using Lemma 2.2, we obtain the result. Let us mention that in the case of a constant coefficient, we can prove this result using [24, Proposition 2.1]. □

Let us now give an estimate on the difference between G_λ and G_0, the elastic equilibrium which is a Maxwelian distribution.

Lemma 2.4. *Let $k \in \mathbb{N}$, $q \in \mathbb{N}$. We denote $k' = 8k+7(1+3/2)$. If $e(\cdot)$ belongs to the space $\mathbb{E}_{k'+1}$, then there exists a function $\varepsilon_1(\lambda)$ such that for any $\lambda \in (0,\lambda^\dagger]$*

$$\|G_\lambda - G_0\|_{W^{k,1}_v(\langle v\rangle^q m)} \leq \varepsilon_1(\lambda) \quad \textit{with} \quad \varepsilon_1(\lambda) \xrightarrow[\lambda\to 0]{} 0.$$

Proof. Theorem 4.1 from [5] implies that

$$\|G_\lambda - G_0\|_{H^{k'}_v} \xrightarrow[\lambda\to 0]{} 0.$$

Using this estimate with Lemma 2.2 and Lemma 2.3, it yields the result. We here mention that in the case of a constant coefficient, we can conclude using [24, Lemma 4.3]. □

2.3. **The linearized operator and its splitting.** Considering the linearization $f = G_\lambda + h$, we obtain at first order the linearized equation around the equilibrium G_λ

$$\partial_t h = \mathcal{L}_\lambda h := Q_{e_\lambda}(G_\lambda, h) + Q_{e_\lambda}(h, G_\lambda) + \lambda^\gamma \Delta_v h - v \cdot \nabla_x h, \tag{2.3}$$

for $h = h(t, x, v)$, $x \in \mathbb{T}^3$, $v \in \mathbb{R}^3$.

We define the operator $\widehat{Q}_{e_\lambda}$ by

$$\widehat{Q}_{e_\lambda}(h) = Q_{e_\lambda}(G_\lambda, h) + Q_{e_\lambda}(h, G_\lambda) = 2\,\widetilde{Q}_{e_\lambda}(h, G_\lambda),$$

where $\widetilde{Q}_{e_\lambda}$ is defined in (1.8). Using the weak formulation, we have

$$\int_{\mathbb{R}^3} \widehat{Q}_{e_\lambda}(h)\psi\, dv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} G_\lambda(v) h(v_*)|v - v_*| \left[\psi(v') + \psi(v'_*) - \psi(v) - \psi(v_*)\right] d\sigma\, dv_*\, dv$$

for any test function ψ.

2.3.1. *Decomposition of the linearized operator.* Let us introduce the decomposition of the linearized operator $\mathcal{L}_\lambda$. For any $\delta \in (0, 1)$, we consider $\Theta_\delta \in \mathcal{C}^\infty$ bounded by one, which equals one on

$$\left\{|v| \le \delta^{-1} \text{ and } 2\delta \le |v - v_*| \le \delta^{-1} \text{ and } |cos\,\theta| \le 1 - 2\delta\right\}$$

and whose support is included in

$$\left\{|v| \le 2\delta^{-1} \text{ and } \delta \le |v - v_*| \le 2\delta^{-1} \text{ and } |cos\,\theta| \le 1 - \delta\right\}.$$

We introduce the following splitting of the linearized elastic collisional operator $\widehat{Q}_1$ defined as $\widehat{Q}_1(h) = Q_1(G_0, h) + Q_1(h, G_0)$:

$$\widehat{Q}_1 = \widehat{Q}^{+,*}_{1,S} + \widehat{Q}^{+,*}_{1,R} - L(G_0)$$

with the truncated operator

$$\widehat{Q}^{+,*}_{1,S}(h) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \Theta_\delta\left[G_0(v'_*)h(v') + G_0(v')h(v'_*) - G_0(v)h(v_*)\right] |v - v_*|\, dv_*\, d\sigma,$$

the corresponding remainder operator

$$\widehat{Q}^{+,*}_{1,R}(h) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} (1 - \Theta_\delta)\left[G_0(v'_*)h(v') + G_0(v')h(v'_*) - G_0(v)h(v_*)\right] |v - v_*|\, dv_*\, d\sigma$$

and

$$L(G_0) = 4\pi\ (G_0 * |\cdot|).$$

We can then write a decomposition for the full linearized operator $\mathcal{L}_\lambda$:

$$\begin{aligned} \mathcal{L}_\lambda h &= \widehat{Q}_{e_\lambda}(h) - \widehat{Q}_1(h) + \widehat{Q}_1(h) + \lambda^\gamma \Delta_v h - v \cdot \nabla_x h \\ &= \widehat{Q}_{e_\lambda}(h) - \widehat{Q}_1(h) + \widehat{Q}^{*,+}_{1,S}(h) + \widehat{Q}^{+,*}_{1,R}(h) - L(G_0)\, h + \lambda^\gamma \Delta_v h - v \cdot \nabla_x h. \end{aligned}$$

Let us denote

$$\mathcal{A}_\delta h := \widehat{Q}^{*,+}_{1,S}(h)$$

and

$$\mathcal{B}_{\lambda,\delta} h := \widehat{Q}_{e_\lambda}(h) - \widehat{Q}_1(h) + \widehat{Q}^{+,*}_{1,R}(h) + \lambda^\gamma \Delta_v h - v \cdot \nabla_x h - L(G_0)\, h.$$

Thanks to the truncation, we can use the so-called Carleman representation (see [32, Chapter 1, Section 4.4]) and write the truncated operator $\mathcal{A}_\delta$ as an integral operator

$$\mathcal{A}_\delta(h)(v) = \int_{\mathbb{R}^3} k_\delta(v, v_*)\, h(v_*)\, dv_* \tag{2.4}$$

for some smooth kernel $k_\delta \in C^\infty_c\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$.

We also introduce the collision frequency $\nu := L(G_0)$ which satisfies $\nu(v) \approx \langle v \rangle$ i.e there exist some constants ν_0, $\nu_1 > 0$ such that:

$$\forall\, v \in \mathbb{R}^3, \quad 0 < \nu_0 \leq \nu_0 \langle v \rangle \leq \nu(v) \leq \nu_1 \langle v \rangle. \tag{2.5}$$

2.3.2. *Spaces at stake.* Let us consider the three Banach spaces

$$\begin{aligned} \mathcal{E}_1 &= W^{s+2,1}_x W^{4,1}_v(\langle v \rangle^2 m), \\ \mathcal{E}_0 &= W^{s,1}_x W^{2,1}_v(\langle v \rangle m), \\ \mathcal{E}_{-1} &= W^{s-1,1}_x L^1_v(m) \end{aligned}$$

for some $s \in \mathbb{N}$ such that $s/2 > 3$.

In the remaining part of the paper, we suppose that the following assumption on $e(\cdot)$ holds:

Assumption 2.5. *The coefficient of restitution $e(\cdot)$ belongs to $\mathbb{E}_{k^\dagger+1}$ where $k^\dagger := 32 + 7(1 + 3/2)$.*

It allows us to get uniform bounds on the $\mathcal{E}_j$-norms of G_λ and uniform estimates on the $\mathcal{E}_j$-norms of the difference $G_\lambda - G_0$ for $j = -1, 0, 1$ (thanks to Lemmas 2.3 and 2.4).

The operator $\mathcal{L}_\lambda$ is bounded from $\mathcal{E}_j$ to $\mathcal{E}_{j-1}$ for $j = 0, 1$. The operators Δ_v and $v \cdot \nabla_x$ are clearly bounded from $\mathcal{E}_j$ to $\mathcal{E}_{j-1}$. As far as $\widehat{Q}_{e_\lambda}$ is concerned, we are going to use the result of interpolation Lemma 2.2.

Lemma 2.6. *Let us consider k, $q \in \mathbb{N}$. If $e(\cdot)$ is regular enough, $\widehat{Q}_{e_\lambda}$ is bounded from $W^{s,1}_x W^{k,1}_v(\langle v \rangle^{q+1} m)$ to $W^{s,1}_x W^{k,1}_v(\langle v \rangle^q m)$.*

Proof. As far as the case of a constant coefficient is concerned, Proposition 3.1 from [23] gives us

$$\|\widehat{Q}_{e_\lambda}(h)\|_{L^1_v(\langle v \rangle^q m)} \leq C \|G_\lambda\|_{L^1_v(\langle v \rangle^{q+1} m)} \|h\|_{L^1_v(\langle v \rangle^{q+1} m)} \leq C \|h\|_{L^1_v(\langle v \rangle^{q+1} m)},$$

where the last inequality comes from Lemma 2.3. Concerning the case of a non-constant coefficient, we use both Lemma 2.3 and [2, Theorem 1] and we get:

$$\|\widehat{Q}_{e_\lambda}(h)\|_{L^1_v(\langle v \rangle^q m)} \leq C \|G_\lambda\|_{L^1_v(\langle v \rangle^{q+1} m)} \|h\|_{L^1_v(\langle v \rangle^{q+1} m)} \leq C \|h\|_{L^1_v(\langle v \rangle^{q+1} m)}.$$

The x-derivatives commute with the operator $\widehat{Q}_{e_\lambda}$, therefore we can do the proof with $s = 0$ without loss of generality. We first look at the case $L^1_x L^1_v(\langle v \rangle^q m)$ before treating the v-derivatives. Using Fubini theorem and the previous inequalities, we obtain

$$\|\widehat{Q}_{e_\lambda} h\|_{L^1_x L^1_v(\langle v \rangle^q m)} \leq C \|h\|_{L^1_x L^1_v(\langle v \rangle^{q+1} m)}.$$

We now treat the case $L^1_x W^{1,1}_v(\langle v \rangle^q m)$. We use the property

$$\partial_v Q^{\pm}_{e_\lambda}(f, g) = Q^{\pm}_{e_\lambda}(\partial_v f, g) + Q^{\pm}_{e_\lambda}(f, \partial_v g). \tag{2.6}$$

We then compute

$$\partial_v \widehat{Q}_{e_\lambda} h = Q_{e_\lambda}(\partial_v G_\lambda, h) + Q_{e_\lambda}(G_\lambda, \partial_v h) + Q_{e_\lambda}(\partial_v h, G_\lambda) + Q_{e_\lambda}(h, \partial_v G_\lambda).$$

Using Lemma 2.3, [23, Proposition 3.1] in the constant case and [2, Theorem 1] in the non-constant case, the $L^1_v(\langle v \rangle^q m)$-norm of each term can be bounded by $C \|h\|_{W^{1,1}_v(\langle v \rangle^{q+1} m)}$. Again using Fubini theorem, we deduce that

$$\|\partial_v \widehat{Q}_{e_\lambda} h\|_{L^1_x L^1_v(\langle v \rangle^q m)} \leq C \|h\|_{L^1_x W^{1,1}_v(\langle v \rangle^{q+1} m)}.$$

The higher-order terms are dealt with in a similar manner, which concludes the proof. ☐

Under the assumptions made on $e(\cdot)$, using the previous lemma, we can conclude that $\widehat{Q}_{e_\lambda}$ is bounded from $\mathcal{E}_j$ to $\mathcal{E}_{j-1}$ for $j = 0, 1$.

2.4. Hypodissipativity of $\mathcal{B}_{\lambda,\delta}$ and boundedness of $\mathcal{A}_\delta$.

Lemma 2.7. *Let us consider $s \geq 0$, $k \geq 0$ and $q \geq 0$. If $e(\cdot)$ is regular enough, then there exist $\lambda_0 \in (0, \lambda^\dagger)$, $\delta > 0$ and $\alpha_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$, $\mathcal{B}_{\lambda,\delta} + \alpha_0$ is hypodissipative in $W^{s,1}_x W^{k,1}_v(\langle v\rangle^q m)$.*

Proof. Observe first that the x-derivatives commute with the operator $\mathcal{B}_{\lambda,\delta}$, therefore we can do the proof for $s = 0$ without loss of generality.

We consider a solution h_t to the linear equation $\partial_t h_t = \mathcal{B}_{\lambda,\delta}(h_t)$ with given initial datum h_0. We first look at the case $L^1_x L^1_v(\langle v\rangle^q m)$ before treating the v-derivatives. We compute

$$\begin{aligned}
\frac{d}{dt}\|h_t\|_{L^1_x L^1_v((\langle v\rangle^q m)} =& \frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} |h_t|\, dx\, \langle v\rangle^q m(v)\, dv \\
=& \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \partial_t h_t \operatorname{sign}(h_t)\, dx\, \langle v\rangle^q m(v)\, dv \\
=& \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \mathcal{B}_{\lambda,\delta}(h_t) \operatorname{sign}(h_t)\, dx\, \langle v\rangle^q m(v)\, dv \\
=& \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} (\widehat{Q}_{e_\lambda}(h_t) - \widehat{Q}_1(h_t)) \operatorname{sign}(h_t)\, dx\, \langle v\rangle^q m(v)\, dv \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \widehat{Q}^{+,*}_{1,R}(h_t) \operatorname{sign}(h_t)\, dx\, \langle v\rangle^q m(v)\, dv \\
&+ \lambda^\gamma \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \Delta_v h_t \operatorname{sign}(h_t)\, dx\, \langle v\rangle^q m(v)\, dv \\
&- \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} v\cdot\nabla_x h_t \operatorname{sign}(h_t)\, dx\, \langle v\rangle^q m(v)\, dv \\
&- \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \nu\, h_t \operatorname{sign}(h_t)\, dx\, \langle v\rangle^q m(v)\, dv \\
=:& I_1(h_t) + I_2(h_t) + I_3(h_t) + I_4(h_t) + I_5(h_t).
\end{aligned}$$

We first deal with I_1 splitting the difference $\widehat{Q}_{e_\lambda} - \widehat{Q}_1$ into several parts and using that $Q^-_{e_\lambda} = Q^-_1$:

$$\begin{aligned}
\widehat{Q}_{e_\lambda} h - \widehat{Q}_1 h =& Q^+_{e_\lambda}(h, G_\lambda) - Q^+_1(h, G_\lambda) + Q^+_1(h, G_\lambda - G_0) \\
&+ Q^+_{e_\lambda}(G_\lambda, h) - Q^+_1(G_\lambda, h) + Q^+_1(G_\lambda - G_0, h) \\
&- Q^-_1(h, G_\lambda - G_0) - Q^-_1(G_\lambda - G_0, h) \\
=& 2\left[\widetilde{Q}^+_{e_\lambda}(h, G_\lambda) - \widetilde{Q}^+_1(h, G_\lambda) + \widetilde{Q}^+_1(h, G_\lambda - G_0) - \widetilde{Q}^-_1(h, G_\lambda - G_0)\right].
\end{aligned}$$

We now use a result given by [23, Proposition 3.1] which can be easily extended to others weights of type $\langle v\rangle^q m$. We can treat together the terms $\widetilde{Q}^+_1(h, G_\lambda - G_0)$ and $\widetilde{Q}^-_1(h, G_\lambda - G_0)$. Because of [23, Proposition 3.1], their $L^1_v(\langle v\rangle^q m)$-norm are bounded from above by $C\,\|G_\lambda - G_0\|_{L^1_v(\langle v\rangle^{q+1} m)} \|h\|_{L^1_v(\langle v\rangle^{q+1} m)}$. Then, using Lemma 2.4, we obtain

$$\|\widetilde{Q}^\pm_1(h, G_\lambda - G_0)\|_{L^1_v(\langle v\rangle^q m)} \leq C\,\varepsilon_1(\lambda) \|h\|_{L^1_v(\langle v\rangle^{q+1} m)} \tag{2.7}$$

with $\varepsilon_1(\lambda) \xrightarrow[\lambda\to 0]{} 0$. Concerning the term $\widetilde{Q}^+_{e_\lambda}(h_t, G_\lambda) - \widetilde{Q}^+_1(h_t, G_\lambda)$, we use [5, Theorem 3.11] (we can use [23, Proposition 3.2] for the constant case) and Lemma 2.3. It gives us that there exists $\lambda_1 \in (0, \lambda^\dagger]$ such that for any $\lambda \in (0, \lambda_1)$:

$$\begin{aligned} \|\widetilde{Q}^+_{e_\lambda}(h, G_\lambda) - \widetilde{Q}^+_1(h, G_\lambda)\|_{L^1_v(\langle v\rangle^q m)} &\leq C\lambda^{\frac{\gamma}{8+3\gamma}} \|G_\lambda\|_{W^{1,1}_v(\langle v\rangle^{q+1}m)} \|h\|_{L^1_v(\langle v\rangle^{q+1}m)} \\ &\leq C\varepsilon_2(\lambda) \|h\|_{L^1_v(\langle v\rangle^{q+1}m)} \end{aligned} \tag{2.8}$$

with $\varepsilon_2(\lambda) \xrightarrow[\lambda\to 0]{} 0$. Gathering (2.7) and (2.8), we thus obtain

$$I_1(h) \leq \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \left|\widehat{Q}_{e_\lambda}(h) - \widehat{Q}_1(h)\right| \, dx \, \langle v\rangle^q m(v) \, dv \leq \varepsilon(\lambda)\|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1}m)} \tag{2.9}$$

with $\varepsilon(\lambda) \xrightarrow[\lambda\to 0]{} 0$.

As far as I_2 is concerned, we are going to use the same method as in the proof of [18, Lemma 4.14]. Recall that [27, Proposition 2.1] establishes that there holds

$$\forall\, h \in L^1_v(\langle v\rangle m), \quad \|\widehat{Q}^{+,*}_{1,R}(h)\|_{L^1_v(m)} \leq \Lambda(\delta)\|h\|_{L^1_v(\langle v\rangle m)} \quad \text{with} \quad \Lambda(\delta) \xrightarrow[\delta\to 0]{} 0,$$

where however the definition of Θ_δ is slightly different and only the case $q = 0$ is treated. But it is straightforward to extend the proof to the present situation. We hence have

$$I_2(h) \leq \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} |\widehat{Q}^{+,*}_{1,R}(h_t)| \, dx \, \langle v\rangle^q m(v) \, dv \leq \Lambda(\delta)\|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1}m)}, \tag{2.10}$$

with $\Lambda(\delta) \xrightarrow[\delta\to 0]{} 0$.

Concerning the term with the Laplacian, we write performing two integrations by parts

$$\begin{aligned} \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \Delta_v h_t \,\mathrm{sign}(h_t)\, \langle v\rangle^q m \, dv \, dx &= -\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |\nabla_v h|^2 \, \mathrm{sign}'(h)\, \langle v\rangle^q m \, dv \, dx \\ &\quad - \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \nabla_v h \, \mathrm{sign}(h) \cdot \nabla_v(\langle v\rangle^q m(v)) \, dv \, dx \\ &\leq \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \nabla_v|h| \cdot \nabla_v(\langle v\rangle^q m) \, dv \, dx \\ &= \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |h| \, \Delta_v(\langle v\rangle^q m) \, dv \, dx \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} |h| \, \langle v\rangle^q m \, \frac{\Delta_v(\langle v\rangle^q m)}{\langle v\rangle^q m} \, dx \, dv. \end{aligned}$$

Since $\Delta_v(\langle v\rangle^q m)/(\langle v\rangle^q m)$ is bounded in $\mathbb{R}^3$, we can write

$$I_3(h) \leq C\,\lambda^\gamma \,\|h\|_{L^1_x L^1_v(\langle v\rangle^q m)} \leq C\,\lambda^\gamma\,\|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1}m)}. \tag{2.11}$$

We notice that

$$I_4(h) = 0 \tag{2.12}$$

because the term $v \cdot \nabla_x h$ has a divergence structure in x.

Finally, let us deal with I_5. We use property (2.5), more precisely the fact that $\nu(v)$ is bounded below by $\nu_0\langle v\rangle$:

$$I_5(h) = -\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} |h| \, dx \, \nu \, \langle v\rangle^q m(v) \, dv \leq -\nu_0 \,\|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1}m)}. \tag{2.13}$$

Gathering (2.9), (2.10), (2.11), (2.12) and (2.13), we obtain that for any $\lambda \in (0, \lambda_1)$

$$\int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \mathcal{B}_{\lambda,\delta} h \, \mathrm{sign}(h) \, dx \, \langle v \rangle^q m(v) \, dv \leq (\Lambda(\delta) + \varepsilon(\lambda) + C\lambda^\gamma - \nu_0) \|h\|_{L^1_x L^1_v(\langle v \rangle^{q+1} m)}.$$

We choose $\lambda_0 \in (0, \lambda_1]$ small enough so that for any $\lambda \in [0, \lambda_0]$, $\varepsilon(\lambda) + C\lambda^\gamma < \nu_0$. Then, we choose δ close enough to 0 in order to have

$$\alpha_0 := -\left(\Lambda(\delta) + \max_{\lambda \in [0,\lambda_0]} [\varepsilon(\lambda) + C\lambda^\gamma] - \nu_0 \right) > 0. \tag{2.14}$$

We hence have

$$\int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \mathcal{B}_{\lambda,\delta} h \, \mathrm{sign}(h) \, dx \, \langle v \rangle^q m(v) \, dv \leq -\alpha_0 \, \|h\|_{L^1_x L^1_v(\langle v \rangle^{q+1} m)}.$$

In particular, we deduce that for any $\lambda \in [0, \lambda_0]$, $\mathcal{B}_{\lambda,\delta} + \alpha_0$ is dissipative in $L^1_x L^1_v(\langle v \rangle^q m)$.

Let us now treat the v-derivatives. We are going to deal with the case $L^1_x W^{1,1}_v(\langle v \rangle^q m)$, the higher-order cases are similar. Thanks to (2.6), we compute the evolution of the v-derivatives:

$$\partial_t \partial_v h_t = \partial_v \left(\widehat{Q}^{+,*}_{1,R}(h_t) - \nu h_t \right) + \partial_v \left((\widehat{Q}_{e_\lambda} - \widehat{Q}_1)(h_t) \right) + \lambda^\gamma \Delta_v \partial_v h_t - \partial_x h_t - v \cdot \nabla_x \partial_v h_t.$$

Let us treat the first term:

$$\partial_v \left(\widehat{Q}^{+,*}_{1,R}(h) - \nu h \right) = \widehat{Q}^{+,*}_{1,R}(\partial_v h) - \nu \, \partial_v h + \mathcal{R} h$$

with

$$\mathcal{R} h = Q_1(h, \partial_v G_0) + Q_1(\partial_v G_0, h) - (\partial_v \mathcal{A}_\delta)(h) + \mathcal{A}_\delta(\partial_v h).$$

We obtain because of [23, Proposition 3.1] and the regularization property of $\mathcal{A}_\delta$ that

$$\|\mathcal{R} h\|_{L^1_x L^1_v(\langle v \rangle^q m)} \leq C_\delta \, \|h\|_{L^1_x L^1_v(\langle v \rangle^{q+1} m)}$$

for some constant $C_\delta > 0$.

Let us now deal with the second term coming from the difference $\widehat{Q}_{e_\lambda} - \widehat{Q}_1$:

$$\begin{aligned} \partial_v \left((\widehat{Q}_{e_\lambda} - \widehat{Q}_1) h \right) = & \, (\widehat{Q}_{e_\lambda} - \widehat{Q}_1)(\partial_v h) \\ & + 2 \left[\widetilde{Q}^+_{e_\lambda}(h, \partial_v G_\lambda) - \widetilde{Q}^+_1(h, \partial_v G_\lambda) \right] \\ & + 2 \left[\widetilde{Q}^+_1(h, \partial_v(G_\lambda - G_0)) - \widetilde{Q}^-_1(h, \partial_v(G_\lambda - G_0)) \right]. \end{aligned}$$

Arguing as before, we obtain

$$\|\widetilde{Q}^+_{e_\lambda}(h, \partial_v G_\lambda) - \widetilde{Q}^+_1(h, \partial_v G_\lambda)\|_{L^1_x L^1_v(\langle v \rangle^q m)} \leq \varepsilon(\lambda) \|h\|_{L^1_x L^1_v(\langle v \rangle^{q+1} m)}$$

and

$$\|\widetilde{Q}^+_1(h, \partial_v(G_\lambda - G_0)) - \widetilde{Q}^-_1(h, \partial_v(G_\lambda - G_0))\|_{L^1_x L^1_v(\langle v \rangle^q m)} \leq \varepsilon(\lambda) \|h\|_{L^1_x L^1_v(\langle v \rangle^{q+1} m)}$$

with $\varepsilon(\lambda) \xrightarrow[\alpha \to 0]{} 0$.

All together, we deduce that

$$\partial_t \partial_v h_t = \mathcal{B}_{\lambda,\delta} h_t + \mathcal{R}'(h_t)$$

with

$$\|\mathcal{R}' h\|_{L^1_x L^1_v(\langle v \rangle^q m)} \leq C_\delta \, \|h\|_{L^1_x L^1_v(\langle v \rangle^{q+1} m)} + \varepsilon(\lambda) \, \|h\|_{L^1_x L^1_v(\langle v \rangle^{q+1} m)} + \|\nabla_x h\|_{L^1_x L^1_v(\langle v \rangle^q m)}.$$

We now use the proof of the previous case to finally deduce the following estimate:

$$\begin{aligned}\frac{d}{dt}\|\nabla_v h_t\|_{L^1_x L^1_v(\langle v\rangle^q m)} \le & -\alpha_0 \|\nabla_v h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + C_\delta \, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} \\ & + \varepsilon(\lambda)\, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \|\nabla_x h\|_{L^1_x L^1_v(\langle v\rangle^q m)},\end{aligned}$$

where α_0 is defined in (2.14).

Again using the proof of the previous case, we also have:

$$\begin{aligned}&\frac{d}{dt}\left(\|h_t\|_{L^1_x L^1_v(\langle v\rangle^q m)} + \|\nabla_x h_t\|_{L^1_x L^1_v(\langle v\rangle^q m)}\right) \\ &\le -\alpha_0 \left(\|h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \|\nabla_x h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}\right).\end{aligned}$$

We now introduce the norm

$$\|h\|_* := \|h\|_{L^1_x L^1_v(\langle v\rangle^q m)} + \|\nabla_x h\|_{L^1_x L^1_v(\langle v\rangle^q m)} + \eta \|\nabla_v h\|_{L^1_x L^1_v(\langle v\rangle^q m)}$$

for some $\eta > 0$ to be fixed later. We deduce

$$\begin{aligned}\frac{d}{dt}\|h_t\|_* \le & -\alpha_0 \left(\|h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \|\nabla_x h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \eta \|\nabla_v h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}\right) \\ & + \eta \left(C_\delta \, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \varepsilon(\lambda)\, \|h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \|\nabla_x h\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}\right) \\ \le & (-\alpha_0 + o(\eta)) \left(\|h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \|\nabla_x h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \eta \|\nabla_v h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}\right)\end{aligned}$$

with $o(\eta) \xrightarrow[\eta \to 0]{} 0$. We choose η close enough to 0 so that $\alpha_1 := \alpha_0 - o(\eta) > 0$. We thus obtain

$$\begin{aligned}\frac{d}{dt}\|h_t\|_* &\le -\alpha_1 \left(\|h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \|\nabla_x h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)} + \eta \|\nabla_v h_t\|_{L^1_x L^1_v(\langle v\rangle^{q+1} m)}\right) \\ &\le -\alpha_1 \|h_t\|_*,\end{aligned}$$

with $\alpha_1 > 0$, which concludes the proof. □

Let us clarify what implies the previous lemma. We can deduce the following result:

Lemma 2.8. *Under the assumptions made on $e(\cdot)$, there exist $\lambda_0 \in (0, \lambda^\dagger]$, $\delta > 0$ and $\alpha_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$, $\mathcal{B}_{\lambda,\delta} + \alpha_0$ is hypodissipative in $\mathcal{E}_j$, $j = -1, 0, 1$.*

The boundedness of $\mathcal{A}_\delta$ is treated in [18]. Let us recall Lemma 4.16 of [18].

Lemma 2.9. *For any $s \in \mathbb{N}$, the operator $\mathcal{A}_\delta$ maps $L^1_v(\langle v\rangle)$ into H^s_v functions with compact support, with explicit bounds (depending on δ) on the $L^1_v(\langle v\rangle) \to H^s_v$ norm and on the size of the support.*

More precisely, there are two constants $C_{s,\delta}$ and R_δ such that for any $h \in L^1_v(\langle v\rangle)$

$$K := \operatorname{supp} \mathcal{A}_\delta h \subset B(0, R_\delta), \quad \|\mathcal{A}_\delta h\|_{H^s_v(K)} \le C_{s,\delta} \|h\|_{L^1_v(\langle v\rangle)}.$$

In particular, we deduce that $\mathcal{A}_\delta$ is in $\mathscr{B}(\mathcal{E}_j)$ for $j = -1, 0, 1$.

2.5. **Regularization properties of $T_n := \left(\mathcal{A}_\delta\, S_{\mathcal{B}_{\lambda,\delta}}\right)^{(*n)}$.** Let us consider λ_0 and α_0 provided by Lemma 2.8.

Lemma 2.10. *Let λ be in $(0, \lambda_0)$. The time indexed family T_n of operators satisfies the following: for any $\alpha_0' \in (0, \alpha_0)$, there are some constructive constants $C_\delta > 0$ and R_δ such that for any $t \ge 0$*

$$supp\, T_n(t) h \subset K := B(0, R_\delta),$$

and

$$\|T_1(t)\|_{W^{s+1,1}_{x,v}(K)} \le C\frac{e^{-\alpha_0' t}}{t}\|h\|_{W^{s,1}_{x,v}(\langle v\rangle m)}, \quad \textit{if } s \ge 1; \tag{2.15}$$

$$\|T_2(t)\|_{W^{s+1/2,1}_{x,v}(K)} \le C e^{-\alpha_0' t}\|h\|_{W^{s,1}_{x,v}(\langle v\rangle m)}, \quad \textit{if } s \ge 0. \tag{2.16}$$

The proof is straightforwardly adapted from [18, Lemma 4.19]. We have here the same operator $\mathcal{A}_\delta$ and we also have the dissipativity of $\mathcal{B}_{\lambda,\delta}+\alpha_0$, which allows us to do a similar proof.

Proof. We first consider $h_0 \in W^{s,1}_{x,v}(\langle v\rangle m)$. Using Lemma 2.9, the fact that the x-derivatives commute with $T_1(t)$ and the fact that $\mathcal{B}_{\lambda,\delta}+\alpha_0$ is dissipative in $W^{s,1}_{x,v}(\langle v\rangle m)$ (Lemma 2.8), we get

$$\|T_1(t)h_0\|_{W^{s,1}_x W^{s+1,1}_v(K)} \le C e^{-\alpha_0 t}\|h_0\|_{W^{s,1}_{x,v}}(\langle v\rangle m). \tag{2.17}$$

Assume now $h_0 \in W^{s,1}_x W^{s+1,1}_v(\langle v\rangle m)$ and consider $g_t = e^{\mathcal{B}_{\lambda,\delta}t}(\partial^\beta_x h_0)$, for any $|\beta| \le s$, which satisfies (using the fact that the x-derivatives commute with the semigroup)

$$\partial_t g_t + v\cdot\nabla_x g_t = Q_1(G_0,g_t)+Q_1(g_t,G_0)+Q_{e_\lambda}(G_\lambda,g_t)+Q_{e_\lambda}(g_t,G_\lambda)+\lambda^\gamma\Delta_v g_t-\mathcal{A}_\delta g_t.$$

Let us define $D_t := t\nabla_x+\nabla_v$. D_t commute with the free transport equation and the Laplacian Δ_v. Using these properties of commutativity and the property (2.6) of the collision operator, we have

$$\begin{aligned}\partial_t(D_t g_t)+v\cdot\nabla_x(D_t g_t) = {} & Q_{e_\lambda}(\nabla_v G_\lambda,g_t)+Q_{e_\lambda}(g_t,\nabla_v G_\lambda)+Q_{e_\lambda}(G_\lambda,D_t g_t)\\ & + Q_{e_\lambda}(D_t g_t,G_\lambda)+\lambda^\gamma\Delta_v g_t - D_t(\mathcal{A}_\delta g_t).\end{aligned}$$

With the notations of (2.4), we rewrite the last term as

$$\begin{aligned}D_t(\mathcal{A}_\delta g_t)(v) &= D_t\int_{\mathbb{R}^3} k_\delta(v,v_*)\,g_t(v_*)\,dv_*\\ &= \int_{\mathbb{R}^3}\nabla_v k_\delta(v,v_*)\,g_t(v_*)\,dv_* - \int_{\mathbb{R}^3} k_\delta(v,v_*)\,\nabla_{v_*}g_t(v_*)\,dv_*\\ &\quad + \int_{\mathbb{R}^3} k_\delta(v,v_*)\,(D_t g_t)(v_*)\,dv_*\\ &= \mathcal{A}^1_\delta g_t + \mathcal{A}^2_\delta g_t + \mathcal{A}_\delta(D_t g_t),\end{aligned}$$

where $\mathcal{A}^1_\delta$ stands for the integral operator associated to the kernel $\nabla_v k_\delta$ and $\mathcal{A}^2_\delta$ stands for the integral operator associated to the kernel $\nabla_{v_*}k_\delta$. All together, we may write

$$\partial_t(D_t g_t) = \mathcal{B}_{\lambda,\delta}(D_t g_t)+\mathcal{I}_\delta(g_t)$$

with

$$\mathcal{I}_\delta f = Q_{e_\lambda}(\nabla_v G_\lambda,f)+Q_{e_\lambda}(f,\nabla_v G_\lambda)-\mathcal{A}^1_\delta f-\mathcal{A}^2_\delta f,$$

which satisfies

$$\|\mathcal{I}_\delta f\|_{L^1_v(\langle v\rangle m)} \le C_\delta\|f\|_{L^1_v(\langle v\rangle^2 m)}.$$

Then arguing as in the proof of Lemma 2.8, we obtain, for any $\alpha_0'' \in (0,\alpha_0)$ and for η small enough

$$\frac{d}{dt}\left(e^{\alpha_0'' t}\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}(\eta|D_t g_t|+|g_t|)\langle v\rangle m\,dx\,dv\right) \le 0,$$

which implies

$$\forall\, t\ge 0, \quad \|D_t g_t\|_{L^1(\langle v\rangle m)}+\|g_t\|_{L^1(\langle v\rangle m)} \le \eta^{-1}e^{-\alpha_0'' t}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}. \tag{2.18}$$

Using (2.18), we get

$$t\,\|\nabla_x T_1(t)(\partial_x^\beta h_0)\|_{L^1(K)} \le C\,\eta^{-1} e^{-\alpha_0'' t}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}.$$

Together with estimate (2.17) and Lemma 2.9, for $s \ge 0$, we conclude that

$$\|T_1(t)(\partial_x^\beta h_0)\|_{W^{s+1,1}_x W^{1,1}_v(K)} \le \frac{Ce^{-\alpha_0'' t}}{t}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)},$$

which in turn implies (2.15).

Now interpolating the last inequality and (2.17), for $s \ge 0$, we have

$$\|T_1(t)h_0\|_{W^{s+1/2,1}_{x,v}(K)} \le \frac{Ce^{-\alpha_0'' t}}{\sqrt{t}}\|h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}. \tag{2.19}$$

Putting together (2.15) and (2.19), for $s \ge 0$, we obtain

$$\begin{aligned}\|T_2(t)h_0\|_{W^{s+1/2,1}_{x,v}(K)} &\le \int_0^t \|T_1(t-s)T_1(s)h_0\|_{W^{s+1/2,1}_{x,v}(K)}\,ds \\ &\le C\int_0^t \frac{e^{-\alpha_0''(t-s)}}{(t-s)^{1/2}}\|T_1(s)h_0\|_{W^{s,1}_x W^{1,1}_v(\langle v\rangle m)}\,ds \\ &\le C\left(\int_0^t \frac{e^{-\alpha_0''(t-s)}}{(t-s)^{1/2}}e^{-\alpha_0 s}\,ds\right)\|h_0\|_{W^{s,1}_{x,v}(\langle v\rangle m)} \\ &\le C\sqrt{t}e^{-\alpha_0'' t}\|h_0\|_{W^{s,1}_{x,v}(\langle v\rangle m)},\end{aligned}$$

which concludes the proof. □

Let us now recall [18, Lemma 2.17] which yields an estimate on the norms $\|T_n\|_{\mathscr{B}(\mathcal{E}_j,\mathcal{E}_{j+1})}$ for $j = -1, 0$.

Lemma 2.11. *Let E, $\mathcal{E}$ be two Banach space with $E \subset \mathcal{E}$ dense with continuous embedding, and consider $\mathcal{L} \in \mathcal{E}$ and $a \in \mathbb{R}$. We assume that there exist some intermediate spaces*

$$E = \mathcal{E}_J \subset \mathcal{E}_{J-1} \subset ... \subset \mathcal{E}_2 \subset \mathcal{E}_1 = \mathcal{E}, \quad J \ge 2$$

such that, denoting $\mathcal{A}_j := \mathcal{A}_{|\mathcal{E}_j}$ and $\mathcal{B}_j := \mathcal{B}_{|\mathcal{E}_j}$

- **(i)** *$(\mathcal{B}_j - a)$ is hypodissipative and $\mathcal{A}_j$ is bounded on $\mathcal{E}_j$ for $1 \le j \le J$;*
- **(ii)** *there are some constants $\ell_0 \in \mathbb{N}^*$, $C \ge 1$, $K \in \mathbb{R}$, $\gamma \in [0,1)$ such that*

$$\forall\, t \ge 0, \quad \|T_{\ell_0}(t)\|_{\mathscr{B}(\mathcal{E}_j,\mathcal{E}_{j+1})} \le C\frac{e^{Kt}}{t^\gamma},$$

for $1 \le j \le J-1$, with the notation $T_\ell := (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}$.

Then for any $a' > a$, there exist some constructive constants $n \in \mathbb{N}$, $C_{a'} \ge 1$ such that

$$\forall\, t \ge 0, \quad \|T_n(t)\|_{\mathscr{B}(\mathcal{E},E)} \le C_{a'}e^{a't}.$$

Combining Lemmas 2.8 and 2.10, we can apply Lemma 2.11 and deduce the following result:

Lemma 2.12. *Let λ be in $(0,\lambda_0)$. For any $\alpha_0' \in (0,\alpha_0)$, there exist some constructive constants $n \in \mathbb{N}$ and $C_{\alpha_0'} \ge 1$ such that*

$$\forall\, t \ge 0, \quad \|T_n(t)\|_{\mathscr{B}(\mathcal{E}_j,\mathcal{E}_{j+1})} \le C_{\alpha_0'}e^{-\alpha_0' t}, \quad j = -1, 0.$$

2.6. **Estimate on $\mathcal{L}_\lambda - \mathcal{L}_0$.** Using estimates from the proof of Lemma 2.8, we can prove the following result:

Lemma 2.13. *There exists a function $\eta_1(\lambda)$ such that $\eta_1(\lambda) \xrightarrow[\lambda\to 0]{} 0$ and the difference $\mathcal{L}_\lambda - \mathcal{L}_0$ satisfies*

$$\|\mathcal{L}_\lambda - \mathcal{L}_0\|_{\mathscr{B}(\mathcal{E}_j,\mathcal{E}_{j-1})} \le \eta_1(\lambda), \;\; j = 0, 1.$$

Proof. We have

$$\mathcal{L}_\lambda - \mathcal{L}_0 = \lambda^\gamma \Delta_v + \widehat{Q}_{e_\lambda} - \widehat{Q}_1.$$

First, we have the following inequality:

$$\|\lambda^\gamma \Delta_v(h)\|_{\mathcal{E}_{j-1}} \le \lambda^\gamma \|h\|_{\mathcal{E}_j}, \quad j = 0, 1. \tag{2.20}$$

Concerning the term $\widehat{Q}_{e_\lambda} - \widehat{Q}_1$, we have obtained in the proof of Lemma 2.8

$$\|(\widehat{Q}_{e_\lambda} - \widehat{Q}_1)h\|_{L^1_v(\langle v\rangle m)} \le C\,\varepsilon(\lambda)\|h\|_{L^1_v(\langle v\rangle^2 m)}$$

with $\varepsilon(\lambda) \xrightarrow[\lambda\to 0]{} 0$. Again arguing as in the proof of Lemma 2.8, we obtain

$$\|\partial_v(\widehat{Q}_{e_\lambda} - \widehat{Q}_1)h\|_{L^1_v(\langle v\rangle m)} \le C\,\varepsilon(\lambda)\|h\|_{W^{1,1}_v(\langle v\rangle^2 m)}.$$

We obtain the higher-order derivatives in the same way and we can conclude that

$$\|(\widehat{Q}_{e_\lambda} - \widehat{Q}_1)h\|_{\mathcal{E}_0} \le C\,\varepsilon(\lambda)\|h\|_{\mathcal{E}_1}. \tag{2.21}$$

Gathering (2.20) and (2.21), we deduce that

$$\|(\mathcal{L}_\lambda - \mathcal{L}_0)h\|_{\mathcal{E}_0} \le \eta_1(\lambda)\|h\|_{\mathcal{E}_1}.$$

Using the same method, we obtain:

$$\|(\mathcal{L}_\lambda - \mathcal{L}_0)h\|_{\mathcal{E}_{-1}} \le \eta_1(\lambda)\|h\|_{\mathcal{E}_0}.$$

□

In the remaining part of the paper, δ is fixed (given by Lemma 2.8), we hence denote $\mathcal{A} = \mathcal{A}_\delta$ and $\mathcal{B}_\lambda = \mathcal{B}_{\lambda,\delta}$.

2.7. **Semigroup spectral analysis of the linearized operator.** In this section we shall state some results on the geometry of the spectrum of the linearized diffusive inelastic collision operator for a diffusion parameter.

Theorem 2.14. *There exists $\lambda' \in [0, 1)$ such that for any $\lambda \in [0, \lambda']$, $\mathcal{L}_\lambda$ satisfies the following properties in $\mathcal{E}_0$:*

- **(i)** *There exists $\mu_\lambda \in \mathbb{R}$ such that $\Sigma(\mathcal{L}_\lambda) \cap \Delta_{-\alpha} = \{\mu_\lambda, 0\}$ where α is given by Theorem 2.15. Moreover, 0 is a four-dimensional eigenvalue and μ_λ is a one-dimensional eigenvalue.*
- **(ii)** *μ_λ satisfies the following estimate*

$$\mu_\lambda = -C\lambda^\gamma + o(\lambda^\gamma) \tag{2.22}$$

for some $C > 0$.
- **(iii)** *For any $\alpha' \in (0, \min(\alpha, \alpha_0)) \setminus \{-\mu_\lambda\}$ (where α_0 is provided by Lemma 2.8), the semigroup generated by $\mathcal{L}_\lambda$ has the following decay property*

$$\forall\, t \ge 0, \quad \|e^{\mathcal{L}_\lambda t} - e^{\mathcal{L}_\lambda t}\Pi_{\mathcal{L}_\lambda,0} - e^{\mathcal{L}_\lambda t}\Pi_{\mathcal{L}_\lambda,\mu_\lambda}\|_{\mathscr{B}(\mathcal{E}_0)} \le Ce^{-\alpha' t} \tag{2.23}$$

for some $C > 0$.

The proof is divided into several steps.

2.7.1. *Step 1 of the proof: the linearized elastic operator.* We recall hypodissipativity results for the semigroup associated to the linearized elastic Boltzmann equation which are proved in [18]. Among other things, the following is proved in this paper (Theorem 4.2):

Theorem 2.15. *There are constructive constants $C \geq 1$, $\alpha > 0$, such that the operator $\mathcal{L}_0$ satisfies in $\mathcal{E}_0$:*

$$\Sigma(\mathcal{L}_0) \cap \Delta_{-\alpha} = \{0\} \quad \text{and} \quad N(\mathcal{L}_0) = Span\{G_0, v_1 G_0, v_2 G_0, v_3 G_0, |v|^2 G_0\}.$$

Moreover, $\mathcal{L}_0$ is the generator of a strongly continuous semigroup $h(t) = S_{\mathcal{L}_0}(t) h_{in}$ in $\mathcal{E}_0$, solution to the initial value problem (2.3) with $\lambda = 0$, which satisfies:

$$\forall\, t \geq 0, \quad \|h(t) - \Pi_{\mathcal{L}_0,0} h_{in}\|_{\mathcal{E}_0} \leq C e^{-\alpha t} \|h_{in} - \Pi_{\mathcal{L}_0,0} h_{in}\|_{\mathcal{E}_0}.$$

2.7.2. *Step 2 of the proof: localization of spectrum of $\mathcal{L}_\lambda$.*

Lemma 2.16. *Let us define $K_\lambda(z)$ for any $z \in \Omega := \Delta_{-\alpha} \setminus \{0\}$ (where α is given by Theorem 2.15) by*

$$K_\lambda(z) = (-1)^n \, (\mathcal{L}_\lambda - \mathcal{L}_0) \, \mathcal{R}_{\mathcal{L}_0}(z) \, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^n.$$

Then, there exists $\eta_2(\lambda)$ with $\eta_2(\lambda) \xrightarrow[\lambda \to 0]{} 0$ such that

$$\forall\, z \in \Omega_\lambda := \Delta_{-\alpha} \setminus \bar{B}(0, \eta_2(\lambda)), \quad \|K_\lambda(z)\|_{\mathscr{B}(\mathcal{E}_0)} \leq \eta_2(\lambda).$$

Moreover, there exists $\lambda' \in (0, \lambda_0]$ (where λ_0 is given by Lemma 2.8) such that for any $\lambda \in [0, \lambda']$, we have

- **(i)** *$I + K_\lambda(z)$ is invertible for any $z \in \Omega_\lambda$*
- **(ii)** *$\mathcal{L}_\lambda - z$ is also invertible for any $z \in \Omega_\lambda$ and*

$$\forall\, z \in \Omega_\lambda, \quad \mathcal{R}_{\mathcal{L}_\lambda}(z) = \mathcal{U}_\lambda(z) \, (I + K_\lambda(z))^{-1}$$

where

$$\mathcal{U}_\lambda(z) = \mathcal{R}_{\mathcal{B}_\lambda}(z) + ... + (-1)^{n-1} \, \mathcal{R}_{\mathcal{B}_\lambda}(z) \, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^{n-1} + (-1)^n \, \mathcal{R}_{\mathcal{L}_0} \, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^n.$$

We thus deduce that

$$\Sigma(\mathcal{L}_\lambda) \cap \Delta_{-\alpha} \subset B(0, \eta_2(\lambda)).$$

Proof. Step 1. We first notice that $(\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^n \in \mathscr{B}(\mathcal{E}_0, \mathcal{E}_1)$, $\mathcal{R}_{\mathcal{L}_0}(z) \in \mathscr{B}(\mathcal{E}_1)$ and $\mathcal{L}_\lambda - \mathcal{L}_0 \in \mathscr{B}(\mathcal{E}_1, \mathcal{E}_0)$ for any $z \in \Omega$ because of Lemma 2.12, Theorem 2.15 and Lemma 2.13. Moreover, there exist $n \in \mathbb{N}$ and $C_0 > 0$ such that $\|\mathcal{R}_{\mathcal{L}_0}(z)\|_{\mathscr{B}(\mathcal{E}_1)} \leq C_0/|z|^n$ for any z in Ω. Indeed, we know from [21, paragraph I.5.3] that in $\mathcal{E}_1$, the following Laurent series

$$\mathcal{R}_{\mathcal{L}_0}(z) = \sum_{k=-n}^{+\infty} z^k \, \mathcal{C}_k$$

where $\mathcal{C}_k$ are some bounded operators in $\mathscr{B}(\mathcal{E}_1)$, converges for z close to 0. We thus deduce the previous estimate on $\|\mathcal{R}_{\mathcal{L}_0}(z)\|_{\mathscr{B}(\mathcal{E}_1)}$. Let us finally define $\eta_2(\lambda) := \left(C_0 C_{\lambda'_0} \eta_1(\lambda)\right)^{\frac{1}{n+1}}$ where λ'_0 is fixed in $(0, \lambda_0)$ and $C_{\lambda'_0}$ is given by Lemma 2.12. We deduce that

$$\forall\, z \in \Omega_\lambda, \quad \|K_\lambda(z)\|_{\mathscr{B}(\mathcal{E}_0)} \leq \eta_1(\lambda) \frac{C_0}{\eta_2(\lambda)^n} C_{\lambda'_0} = \eta_2(\lambda).$$

We then choose $\lambda' \in (0, \lambda_0]$ such that for any $\lambda \in (0, \lambda']$, $\eta_2(\lambda) < 1$. We hence obtain that $I + K_\lambda(z)$ is an invertible operator for any $\lambda \in (0, \lambda']$. Let us now consider $\lambda \in (0, \lambda']$.

Step 2. $\mathcal{U}_\lambda(z)\,(I + K_\lambda(z))^{-1}$ is a right-inverse of $\mathcal{L}_\lambda - z$ on Ω_λ. For any $z \in \Omega_\lambda$, we compute

$$\begin{aligned}(\mathcal{L}_\lambda - z)\,\mathcal{U}_\lambda(z) &= (\mathcal{B}_\lambda - z + \mathcal{A})\,\{\mathcal{R}_{\mathcal{B}_\lambda}(z) + ... + (-1)^{n-1}\,\mathcal{R}_{\mathcal{B}_\lambda}(z)\,(\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\lambda})^{n-1}(z)\} \\ &+ \quad (-1)^n\,(\mathcal{L}_\lambda - \mathcal{L}_0 + \mathcal{L}_0 - z)\,\mathcal{R}_{\mathcal{L}_0}(z)\,(\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\lambda})^n(z) \\ &= Id + K_\lambda(z).\end{aligned}$$

Because of the previous step, we deduce that for $z \in \Omega_\lambda$, $\mathcal{U}_\lambda(z)\,(I + K_\lambda(z))^{-1}$ is a right-inverse of $\mathcal{L}_\lambda - z$.

Step 3. There exists $z_0 \in \Omega_\lambda$ such that $\mathcal{L}_\lambda - z_0$ is invertible on Ω_λ. Indeed, we write

$$\mathcal{L}_\lambda - z_0 = (\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\lambda}(z_0) + I)\,(\mathcal{B}_\lambda - z_0)$$

where $(\mathcal{A}\,\mathcal{R}_{\mathcal{B}_\lambda}(z_0) + I)$ is invertible for $\Re e\, z_0$ large enough because of Lemma 2.8. As a consequence, $\mathcal{L}_\lambda - z_0$ is the product of two invertible operators, we hence obtain that $\mathcal{L}_\lambda - z_0$ is invertible.

Step 4. $\mathcal{L}_\lambda - z$ is invertible close to z_0. Since $\mathcal{L}_\lambda - z_0$ is invertible on Ω_λ, we have $\mathcal{R}_{\mathcal{L}_\lambda}(z_0) = \mathcal{U}_\lambda(z_0)\,(I + K_\lambda(z_0))^{-1}$. Moreover, if $\|\mathcal{R}_{\mathcal{L}_\lambda}(z_0)\| \le C$ for some $C > 0$, then $\mathcal{L}_\lambda - z$ in invertible on the disc $B(z_0, 1/C)$ with

$$\forall\, z \in B(z_0, 1/C), \quad \mathcal{R}_{\mathcal{L}_\lambda}(z) = \mathcal{R}_{\mathcal{L}_\lambda}(z_0) \sum_{n=0}^{+\infty} (z - z_0)^n\,\mathcal{R}_{\mathcal{L}_\lambda}(z_0)^n, \tag{2.24}$$

and arguing as before, $\mathcal{R}_{\mathcal{L}_\lambda}(z) = \mathcal{U}_\lambda(z)\,(I + K_\lambda(z))^{-1}$ on $B(z_0, 1/C)$ since $\mathcal{U}_\lambda(z)\,(I + K_\lambda(z))^{-1}$ is a right inverse of $\mathcal{L}_\lambda - z$ for any $z \in \Omega_\lambda$.

Step 5. $\mathcal{L}_\lambda - z$ is invertible on Ω_λ. For a given $z_1 \in \Omega_\lambda$, we consider a continuous path Γ from z_0 to z_1 included in Ω_λ, i.e. a continuous function $\Gamma : [0,1] \to \Omega_\lambda$ such that $\Gamma(0) = z_0$, $\Gamma(1) = z_1$. We know that $(\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^\ell$, $1 \le \ell \le n-1$, $\mathcal{R}_{\mathcal{L}_0}(z)(\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^n$ and $(I + K_\lambda(z))^{-1}$ are locally uniformly bounded in $\mathscr{B}(\mathcal{E}_0)$ on Ω_λ, which implies

$$\sup_{z \in \Gamma([0,1])} \|\mathcal{U}_\lambda(z)(I + K_\lambda(z))^{-1}\|_{\mathscr{B}(\mathcal{E}_0)} := K < \infty.$$

Since $(\mathcal{L}_\lambda - z_0)$ is invertible we deduce that $(\mathcal{L}_\lambda - z)$ is invertible with $\mathcal{R}_{\mathcal{L}_\lambda}(z)$ locally bounded around z_0 with a bound K which is uniform along Γ (and a similar series expansion as in (2.24)). By a continuation argument we hence obtain that $(\mathcal{L}_\lambda - z)$ is invertible in $\mathcal{E}_0$ all along the path Γ with

$$\mathcal{R}_{\mathcal{L}_\lambda}(z) = \mathcal{U}_\lambda(z)(I + K_\lambda(z))^{-1} \;\text{ and }\; \|\mathcal{R}_{\mathcal{L}_\lambda}(z)\|_{\mathscr{B}(\mathcal{E}_0)} \le K.$$

Hence we conclude that $(\mathcal{L}_\lambda - z_1)$ is invertible with $\mathcal{R}_{\mathcal{L}_\lambda}(z_1) = \mathcal{U}_\lambda(z_1)(I + K_\lambda(z_1))^{-1}$. □

2.7.3. *Step 3 of the proof: dimension of eigenspaces.*

Lemma 2.17. *There exist a constant $C > 0$ and a function $\eta_3(\lambda)$ such that*

$$\|\Pi_{\mathcal{L}_\lambda, -\alpha}\|_{\mathscr{B}(\mathcal{E}_0, \mathcal{E}_1)} \le C, \tag{2.25}$$

and

$$\|\Pi_{\mathcal{L}_\lambda, -\alpha} - \Pi_{\mathcal{L}_0, -\alpha}\|_{\mathscr{B}(\mathcal{E}_0)} \le \eta_3(\lambda), \quad \eta_3(\lambda) \xrightarrow[\lambda \to 0]{} 0. \tag{2.26}$$

It implies that for λ close enough to 0*, we have*

$$\dim \mathrm{R}(\Pi_{\mathcal{L}_\lambda,-\alpha}) = \dim \mathrm{R}(\Pi_{\mathcal{L}_0,-\alpha}) = 5.$$

The following lemma from [21, paragraph I.4.6] is going to be useful for the proof.

Lemma 2.18. *Let X be a Banach space and P, Q be two projectors in $\mathscr{B}(X)$ such that $\|P-Q\|_{\mathscr{B}(X)} < 1$. Then the ranges of P and Q are isomorphic. In particular,* $\dim(\mathrm{R}(P)) = \dim(\mathrm{R}(Q))$.

Let us now prove Lemma 2.17.

Proof. Let Γ be a simple closed curve which encloses an open set containing $\bigcup_{j=1}^k \bar{B}(\xi_j, \eta_2(\lambda))$. We set $N := 2n$ and we define

$$\mathcal{U}_\lambda^0 := \mathcal{R}_{\mathcal{B}_\lambda} + ... + (-1)^{N-1}\, \mathcal{R}_{\mathcal{B}_\lambda}\, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda})^{N-1} \text{ and } \quad \mathcal{U}_\lambda^1 := (-1)^N\, \mathcal{R}_{\mathcal{L}_0}\, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda})^N,$$

Then we compute:

$$\begin{aligned}
\Pi_{\mathcal{L}_\lambda,-\alpha} &= \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\mathcal{L}_\lambda}(z)\, dz \\
&= \frac{i}{2\pi}\int_\Gamma \mathcal{U}_\lambda(z)\, (I + K_\lambda(z))^{-1}\, dz \\
&= \frac{i}{2\pi}\int_\Gamma \mathcal{U}_\lambda^0(z)\, \{I - K_\lambda(z)\, (I+K_\lambda(z))^{-1}\}\, dz \\
&\quad + \frac{i}{2\pi}\int_\Gamma \mathcal{U}_\lambda^1(z)\, (I + K_\lambda(z))^{-1}\, dz \\
&= \frac{1}{2i\pi}\int_\Gamma \mathcal{U}_\lambda^0(z)\, K_\lambda(z)\, (I + K_\lambda(z))^{-1}\, dz \\
&\quad + (-1)^n\, \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\mathcal{L}_0}(z)\, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^N\, (I + K_\lambda(z))^{-1}\, dz.
\end{aligned}$$

Since $(\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^N$ appears in the two parts of the expression of $\Pi_{\mathcal{L}_\lambda,-\alpha}$, we deduce that (2.25) holds.

Concerning the estimate on $\Pi_{\mathcal{L}_0,-\alpha} - \Pi_{\mathcal{L}_\lambda,-\alpha}$, we begin by writing

$$\mathcal{R}_{\mathcal{L}_0}(z) = \mathcal{R}_{\mathcal{B}_0}(z) + ... + (-1)^{N-1}\, \mathcal{R}_{\mathcal{B}_0}(z)\, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z))^{N-1} + (-1)^N\, \mathcal{R}_{\mathcal{L}_0}(z)\, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z))^N$$

which implies that

$$\begin{aligned}
\Pi_{\mathcal{L}_0,-\alpha} &= \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\mathcal{L}_0}(z)\, dz \\
&= (-1)^n\, \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\mathcal{L}_0}(z)\, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z))^N\, dz.
\end{aligned}$$

Finally, we deduce that

$$\begin{aligned}
&\Pi_{\mathcal{L}_0,-\alpha} - \Pi_{\mathcal{L}_\lambda,-\alpha} \\
&= (-1)^n\, \frac{i}{2\pi}\int_\Gamma \mathcal{R}_{\mathcal{L}_0}(z)\, \{(\mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z))^N - (\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\lambda}(z))^N\, (I + K_\lambda(z))^{-1}\}\, dz \\
&\quad - \frac{1}{2i\pi}\int_\Gamma \mathcal{U}_\lambda^0(z)\, K_\lambda(z)\, (I + K_\lambda(z))^{-1}\, dz.
\end{aligned}$$

Since $K_\lambda(z)$ appears in the second term, we deduce that it is bounded by $\eta_2(\lambda)$. Concerning the first term, we rewrite it as

$$(\mathcal{A}\mathcal{R}_{\mathcal{B}_0}(z))^{2n} - (\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^{2n} + (\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^{2n}(I - (I + K_\lambda(z))^{-1}).$$

The second part of this expression is bounded by $\eta_2(\lambda)/(1 - \eta_2(\lambda))$ because of the bound on the norm of K_λ. The first part can be written as

$$\sum_{k=0}^{2n} (\mathcal{A}\mathcal{R}_{\mathcal{B}_0}(z))^k \, \mathcal{A}\,(\mathcal{R}_{\mathcal{B}_0}(z) - \mathcal{R}_{\mathcal{B}_\lambda}(z))\,(\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^{2n-k-1}.$$

In addition, the bound on the norm of $\mathcal{B}_\lambda - \mathcal{B}_0$ given by Lemma 2.13 gives a bound on the norm of $\mathcal{R}_{\mathcal{B}_\lambda}(z) - \mathcal{R}_{\mathcal{B}_0}(z)$ because

$$\mathcal{R}_{\mathcal{B}_1}(z) - \mathcal{R}_{\mathcal{B}_\lambda}(z) = \mathcal{R}_{\mathcal{B}_\lambda}(z)\,(\mathcal{B}_\lambda - \mathcal{B}_0)\,\mathcal{R}_{\mathcal{B}_0}(z).$$

Since for all k, $0 \le k \le 2n$ we have $k \ge n$ or $2n - k - 1 \ge n$, we can use Lemma 2.12 and conclude that $(\mathcal{A}\mathcal{R}_{\mathcal{B}_0}(z))^{2n} - (\mathcal{A}\mathcal{R}_{\mathcal{B}_\lambda}(z))^{2n}$ is bounded by $C\eta_1(\lambda)$, which concludes the proof of (2.26).

The last part of Lemma 2.17 is nothing but Lemma 2.18 because for λ close enough to 0, $\eta_3(\lambda) < 1$. □

We can now finish the proof of Theorem 2.14-**(i)**. The previous lemma implies that there exist $\xi_1, ..., \xi_5 \in \mathbb{C}$ such that

$$\Sigma(\mathcal{L}_\lambda) \cap \Delta_{-\alpha} = \{\xi_1, ...\xi_5\}.$$

Moreover, we know that 0 is a four-dimensional eigenvalue due to the conservation of mass and momentum. Since the operator is real, we can deduce that there exists $\mu_\lambda \in \mathbb{R}$ such that

$$\Sigma(\mathcal{L}_\lambda) \cap \Delta_{-\alpha} = \{0, \mu_\lambda\}.$$

2.7.4. *Step 4 of the proof: fine study of spectrum close to 0.* Concerning the case of a constant coefficient of inelasticity, we refer to [24, Section 5.2, Step 2] for the proof of Theorem 2.14-**(ii)** (the first order expansion of μ_λ (2.22)). Let us deal with the non-constant case.

We first denote ϕ_0 the energy eigenfunction of the the elastic linearized operator associated to 0 such that $\|\phi_0\|_{L^1_v(\langle v \rangle^2)} = 1$. We also denote Π_0 the projection on $\mathbb{R}\,\phi_0$ and $\pi_0\psi$ the coordinate of $\Pi_0\,\psi$ on $\mathbb{R}\,\phi_0$ i.e $\Pi_0\,\psi = (\pi_0\psi)\,\phi_0$. Finally, we denote ϕ_λ the unique associated eigenfunction such that $\|\phi_\lambda\|_{L^1_v(\langle v \rangle^2)} = 1$ and $\pi_0\phi_\lambda \ge 0$.

By integrating in v the eigenvalue equation related to μ_λ

$$\mathcal{L}_\lambda\,\phi_\lambda = \mu_\lambda\,\phi_\lambda$$

against $|v|^2$, we get

$$2\int_{\mathbb{R}^3} \widetilde{Q}_{e_\lambda}(G_\lambda, \phi_\lambda)\,|v|^2\,dv + \lambda^\gamma \int_{\mathbb{R}^3} \Delta_v\,\phi_\lambda\,|v|^2\,dv = \mu_\lambda\,\mathcal{E}(\phi_\lambda). \tag{2.27}$$

We now compute the left-hand side of (2.27). By a classical computation which uses (1.7), we have:

$$2\int_{\mathbb{R}^3} \widetilde{Q}_{e_\lambda}(G_\lambda, \phi_\lambda)\,|v|^2\,dv = -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |u|^3 G_{\lambda_*}\phi_\lambda\,\frac{1 - \widehat{u}\cdot\sigma}{4}(1 - {e_\lambda}^2)\,d\sigma\,dv_*\,dv$$

and using polar coordinates

$$\int_{\mathbb{S}^2} \frac{1-\widehat{u}\cdot\sigma}{4}\left(1-{e_\lambda}^2\left(|u|\sqrt{\frac{1-\widehat{u}\cdot\sigma}{2}}\right)\right)d\sigma = 4\pi\int_0^1 \left(1-{e_\lambda}^2\left(|u|y\right)\right)y^3\,dy.$$

Let us define

$$\psi_e(r) := 4\pi\, r^{3/2}\int_0^1 (1-e^2(\sqrt{r}z))\,z^3\,dz,$$

we can compute $\psi_{e_\lambda}(r) = \lambda^{-3}\,\psi_e(\lambda^2 r)$. We deduce that

$$2\int_{\mathbb{R}^3}\widetilde{Q}_{e_\lambda}(G_\lambda,\phi_\lambda)\,|v|^2\,dv = -\frac{1}{\lambda^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_{\lambda_*}\phi_\lambda\,\psi_e(\lambda^2|u|^2)\,dv_*\,dv.$$

We also have

$$\int_{\mathbb{R}^3}\Delta_v\,\phi_\lambda\,|v|^2\,dv = 6\int_{\mathbb{R}^3}\phi_\lambda\,dv = 6\,\rho(\phi_\lambda).$$

Dividing (2.27) by λ^γ, we hence obtain

$$-\frac{1}{\lambda^{3+\gamma}}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_{\lambda_*}\phi_\lambda\,\psi_e(\lambda^2|u|^2)\,dv_*\,dv + 6\,\rho(\phi_\lambda) = \frac{1}{\lambda^\gamma}\,\mu_\lambda\,\mathcal{E}(\phi_\lambda). \tag{2.28}$$

We would like to make λ tend to 0 in (2.28). To do that, we introduce the following notations:

$$I_\lambda(f,g) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_* g\,\zeta_\lambda(|u|^2)\,dv_*\,dv \quad \text{with} \quad \zeta_\lambda(r^2) = \frac{1}{\lambda^{3+\gamma}}\,\psi_e(\lambda^2 r^2),$$

and

$$I_0(f,g) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_* g\,\zeta_0(|u|^2)\,dv_*\,dv \quad \text{with} \quad \zeta_0(r^2) = \frac{a}{4+\gamma}\,r^{3+\gamma}.$$

Let us now prove that $I_\lambda(G_\lambda,\phi_\lambda)$ tends to $I_0(G_0,\phi_0)$ as λ tends to 0. We state the following lemma which is going to be useful. We do not prove it here because the proof is the same as the one of [23, Lemma 5.17].

Lemma 2.19. *Let k, $q\in\mathbb{N}$. We have the following result:*

$$\|\phi_\lambda-\phi_0\|_{W^{k,1}_v(\langle v\rangle^q m)} \xrightarrow[\lambda\to 0]{} 0.$$

To prove that $I_\lambda(G_\lambda,\phi_\lambda)$ tends to $I_0(G_0,\phi_0)$ as λ tends to 0, let us write the following inequality:

$$\begin{aligned}|I_\lambda(G_\lambda,\phi_\lambda)-I_0(G_0,\phi_0)| &\le |I_\lambda(G_\lambda,\phi_\lambda)-I_0(G_\lambda,\phi_\lambda)| + |I_0(G_\lambda,\phi_\lambda)-I_0(G_0,\phi_0)|\\ &=: J^1_\lambda + J^2_\lambda.\end{aligned}$$

We first deal with J^2_λ:

$$\begin{aligned}J^2_\lambda &= \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(G_{\lambda_*}\phi_\lambda - G_{0_*}\phi_0)\,\zeta_0(|u|^2)\,dv_*\,dv\right|\\
&\le C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|G_{\lambda_*}-G_{0_*}|\,\phi_0\,\langle v\rangle^{3+\gamma}\,\langle v_*\rangle^{3+\gamma}\,dv_*\,dv\\
&\quad + C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_{\lambda_*}\,|\phi_\lambda-\phi_0|\,\langle v\rangle^{3+\gamma}\,\langle v_*\rangle^{3+\gamma}\,dv_*\,dv\\
&\le C\left(\|G_\lambda-G_0\|_{L^1_v(\langle v\rangle^{3+\gamma})}\,\|\phi_0\|_{L^1_v(\langle v\rangle^{3+\gamma})} + \|G_\lambda\|_{L^1_v(\langle v\rangle^{3+\gamma})}\,\|\phi_\lambda-\phi_0\|_{L^1_v(\langle v\rangle^{3+\gamma})}\right)\\
&\le C\left(\|G_\lambda-G_0\|_{L^1_v(\langle v\rangle^{3+\gamma})} + \|\phi_\lambda-\phi_0\|_{L^1_v(\langle v\rangle^{3+\gamma})}\right) \xrightarrow[\lambda\to 0]{} 0\end{aligned}$$

because of Lemmas 2.4 and 2.19.

Let us now establish an estimate on J^1_λ:

$$J^1_\lambda \leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_{\lambda_*}\phi_\lambda \, |\zeta_\lambda(|u|^2) - \zeta_0(|u|^2)| \, dv_* \, dv =: D_\lambda.$$

We can rewrite the difference $\zeta_\lambda(r^2) - \zeta_0(r^2)$ in the following way:

$$\zeta_\lambda(r^2) - \zeta_0(r^2) = \frac{r^{3+\gamma}}{2}\int_0^1 \left(\frac{1-e^2(\lambda r z)}{(\lambda r z)^\gamma} - 2a\right) z^{3+\gamma}\, dz,$$

which allows us to get an estimate on this difference using Assumption 1.1-(3). There exists a constant $C > 0$ such that

$$\forall\, \lambda \in (0,1], \quad \forall r > 0, \quad |\zeta_\lambda(r^2) - \zeta_0(r^2)| \leq C\left(r^{3+2\gamma}\,\lambda^\gamma + r^{3+\gamma+\overline{\gamma}}\,\lambda^{\overline{\gamma}} + r^{3+\overline{\gamma}}\,\lambda^{\overline{\gamma}-\gamma}\right).$$

Denoting $\widetilde{\gamma} := \min(\gamma, \overline{\gamma} - \gamma)$, we can deduce that

$$\begin{aligned} D_\lambda &\leq \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_{\lambda_*}\phi_\lambda\, \lambda^{\widetilde{\gamma}} |u|^{3+\gamma+\overline{\gamma}}\, dv_*\, dv \\ &\leq C\,\lambda^{\widetilde{\gamma}}\, \|G_\lambda\|_{L^1_v(\langle v\rangle^{3+\gamma+\overline{\gamma}})} \|\phi_\lambda\|_{L^1_v(\langle v\rangle^{3+\gamma+\overline{\gamma}})} \\ &\leq C\,\lambda^{\widetilde{\gamma}}. \end{aligned}$$

It yields the result: $J^1_\lambda \xrightarrow[\lambda\to 0]{} 0$.

We can now make λ tend to 0 in (2.28). Using the previous result $I_\lambda(G_\lambda, \phi_\lambda) \to I_0(G_0, \phi_0)$, the fact that the mass of ϕ_0 is 0 and the convergence of $G_\lambda \to G_0$ and $\phi_\lambda \to \phi_0$ (Lemmas 2.4 and 2.19), we deduce that

$$\frac{\mu_\lambda}{\lambda^\gamma}\mathcal{E}(\phi_0) = -I_0(G_0, \phi_0) + o(1).$$

We finally conclude that there exists a constant $C > 0$ such that

$$\mu_\lambda = -C\lambda^\gamma + o(\lambda^\gamma).$$

2.7.5. *Step 5 of the proof: semigroup decay.* In order to get our semigroup decay, we are going to apply the following quantitative spectral mapping theorem which comes from [26]. Let us remind it.

Proposition 2.20. *Consider a Banach space X and an operator $\Lambda \in \mathscr{C}(X)$ so that $\Lambda = \mathcal{A} + \mathcal{B}$ where $\mathcal{A} \in \mathscr{B}(X)$ and $\mathcal{B} - a$ is hypodissipative on X for some $a \in \mathbb{R}$. We assume furthermore that there exists a family X_j, $1 \leq j \leq m$, $m \geq 2$ of intermediate spaces such that*

$$X_m \subset \mathrm{D}(\Lambda^2) \subset X_{m-1} \subset ... \subset X_2 \subset X_1 = X,$$

and a family of operators $\Lambda_j, \mathcal{A}_j, \mathcal{B}_j \in \mathscr{C}(X_j)$ such that

$$\Lambda_j = \mathcal{A}_j + \mathcal{B}_j, \quad \Lambda_j = \Lambda_{|X_j}, \quad \mathcal{A}_j = \mathcal{A}_{|X_j}, \quad \mathcal{B}_j = \mathcal{B}_{|X_j},$$

and that there holds

- **(i)** *$(\mathcal{B}_j - a)$ is hypodissipative on X_j;*
- **(ii)** *$\mathcal{A}_j \in \mathscr{B}(X_j)$;*
- **(iii)** *there exists $n \in \mathbb{N}$ such that $T_n(t) := (\mathcal{A}S_\mathcal{B}(t))^{(*n)}$ satisfies $\|T_n(t)\|_{\mathscr{B}(X,X_m)} \leq Ce^{at}$.*

Then the following localization of the principal part of the spectrum

(1) *there are some distinct complex numbers* $\xi_1, ..., \xi_k \in \Delta_a$, $k \in \mathbb{N}$ *(with the convention* $\{\xi_1, ...\xi_k\} = \emptyset$ *if* $k = 0$*) such that one has*

$$\Sigma(\Lambda) \cap \Delta_a = \{\xi_1, ..., \xi_k\} \subset \Sigma_d(\Lambda).$$

implies the following quantitative growth estimate on the semigroup

(2) *for any* $a' \in (a, \infty) \setminus \{\Re e\, \xi_j, j = 1, ..., k\}$*, there exists some constructive constant* $C_{a'} > 0$ *such that*

$$\forall t \geq 0, \quad \left\| S_\Lambda(t) - \sum_{j=1}^{k} e^{t\Lambda \Pi_{\Lambda,\xi_j}} \Pi_{\Lambda,\xi_j} \right\|_{\mathscr{B}(X)} \leq C_{a'} e^{a' t}.$$

In particular, the following partial (but principal) spectral mapping theorem holds

$$\forall\, t \geq 0, \quad \forall\, a' > a, \quad \Sigma(e^{\Lambda t}) \cap \Delta_{e^{a' t}} = e^{\Sigma(\Lambda) \cap \Delta_{a'} t}.$$

Proof. We have the following representation formula (see for instance the proof of [18, Theorem 2.13]):

$$S_\Lambda(t) f = \sum_{j=1}^{k} S_{\Lambda,\xi_j}(t) f + \sum_{\ell=0}^{n+1} (-1)^\ell S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}(t) f + \mathcal{Z}(t) f,$$

for any $f \in \mathrm{D}(\Lambda)$ and $t \geq 0$, where

$$\mathcal{Z}(t) f := \lim_{M \to \infty} \frac{(-1)^n}{2i\pi} \int_{a'-iM}^{a'+iM} e^{zt}\, \mathcal{R}_\Lambda(z)\, (\mathcal{A} \mathcal{R}_\mathcal{B}(z))^{n+2} f \, dz.$$

On the one hand, we know from **(i)** and **(ii)** that

$$\forall\, \ell = 0, ..., n+1, \quad \| S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}(t) \|_{\mathscr{B}(X)} \leq C_{a'} e^{a' t}.$$

On the other hand, because of **(iii)**, we have

$$\sup_{z \in a' + i\mathbb{R}} \| (\mathcal{A} \mathcal{R}_\mathcal{B})^n(z) \|_{\mathscr{B}(X, \mathrm{D}(\Lambda^2))} \leq K^1_{a'}$$

and because of **(1)**, since Λ generates a semigroup,

$$\sup_{z \in a' + i\mathbb{R}} \| \mathcal{R}_\Lambda(z) \|_{\mathscr{B}(X)} \leq K^2_{a'}.$$

Then, we are going to use the resolvent identity

$$\forall\, z \notin \Sigma(\mathcal{B}), \quad \mathcal{R}_\mathcal{B}(z) = z^{-1} [\mathcal{R}_\mathcal{B}(z) \mathcal{B} - I] \tag{2.29}$$

to get an estimate on $\| (\mathcal{A} \mathcal{R}_\mathcal{B})^2(z) \|_{\mathscr{B}(\mathrm{D}(\Lambda^2), X)}$ if $|z| \geq 1$. Using twice (2.29), we obtain

$$\forall\, z \in \mathbb{C},\, |z| \geq 1, \quad \| (\mathcal{A} \mathcal{R}_\mathcal{B})^2(z) f \|_X \leq K^3_{a'} |z|^{-2} \| f \|_{\mathrm{D}(\mathcal{B}^2)}$$

and we notice that $\mathrm{D}(\mathcal{B}^2) = \mathrm{D}(\Lambda^2)$ because $\mathcal{A}$ is bounded. We finally obtain

$$\forall\, z \in \mathbb{C},\, |z| \geq 1, \quad \| (\mathcal{A} \mathcal{R}_\mathcal{B})^2(z) f \|_X \leq K^3_{a'} \frac{1}{1 + |z|^2} \| f \|_{\mathrm{D}(\Lambda^2)}.$$

Moreover, we also have

$$\forall\, z \in \mathbb{C},\, |z| \leq 1, \quad \| (\mathcal{A} \mathcal{R}_\mathcal{B})^2(z) f \|_X \leq K^4_{a'} \frac{1}{1 + |z|^2} \| f \|_{\mathrm{D}(\Lambda^2)}.$$

All together, we deduce that

$$\|\mathcal{Z}(t)\|_{\mathscr{B}(X)} \le K_{a'} \frac{e^{a't}}{2\pi} \int_{\mathbb{R}} \frac{dy}{1+y^2},$$

which yields the result. □

We can now prove the estimate on the semigroup decay (2.23). We apply Proposition 2.20 with $a := \max(-\alpha, -\alpha_0) < 0$. We have $\mathcal{E}_1 \subset \mathrm{D}(\mathcal{L}_\lambda^2) \subset \mathcal{E}_0 \subset \mathcal{E}_{-1}$. Assumptions **(i)**, **(ii)** and **(iii)** are nothing but Lemmas 2.8, 2.9 and 2.12. And **(1)** is given by the previous steps of the proof. We hence conclude that we have the decay result (2.23) for any $\alpha' \in (0, \min(\alpha, \alpha_0)) \setminus \{-\mu_\lambda\}$.

Remark 2.21. *Thanks to the first order expansion of μ_λ (2.22), we deduce that $\mu_\lambda < 0$ for λ close enough to* 0. *As a consequence, for any $\alpha_\lambda \in (0, -\mu_\lambda)$, we have*

$$\|e^{\mathcal{L}_\lambda t} - e^{\mathcal{L}_\lambda t}\Pi_{\mathcal{L}_\lambda,0}\|_{\mathscr{B}(\mathcal{E}_0)} \le C e^{-\alpha_\lambda t}. \tag{2.30}$$

2.8. **A dissipative Banach norm for the full linearized operator.** Let us define a new norm on $\mathcal{E}_0$ by

$$|||h|||_{\mathcal{E}_0} := \eta \|h\|_{\mathcal{E}_0} + \int_0^{+\infty} \|S_{\mathcal{L}_\lambda}(\tau)h\|_{\mathcal{E}_0}\, d\tau, \quad \eta > 0. \tag{2.31}$$

Proposition 2.22. *There exist $\eta > 0$ and $\alpha_1 > 0$ such that for any $h_{in} \in \mathcal{E}_0$, $\Pi_{\mathcal{L}_\lambda,0} h_{in} = 0$, the solution $h(t) := S_{\mathcal{L}_\lambda}(t)h_{in}$ to the initial value problem (2.3) satisfies:*

$$\forall\, t \ge 0, \quad \frac{d}{dt}|||h_t|||_{\mathcal{E}_0} \le -\alpha_1 |||h_t|||_{\mathcal{E}_0^1},$$

where $\mathcal{E}_0^1 := W_x^{s,1} W_v^{2,1}(\langle v\rangle^2 m)$ and $|||\cdot|||_{\mathcal{E}_0^1}$ is defined as in (2.31).

The proof is adapted from the proof of [18, Proposition 5.15].

Proof. From the decay property of $\mathcal{L}_\lambda$ provided by (2.30), we have

$$\|S_{\mathcal{L}_\lambda}(\tau)h\|_{\mathcal{E}_0} \le C e^{-\alpha_\lambda \tau} \|h\|_{\mathcal{E}_0}.$$

We thus deduce that the norms $\|\cdot\|_{\mathcal{E}_0}$ and $|||\cdot|||_{\mathcal{E}_0}$ are equivalent for any $\eta > 0$.

Let us now compute the time derivative of the norm $\mathcal{E}_0$ along h_t where h_t solves the linear evolution problem (2.3). Observe that $\Pi_{\mathcal{L}_\lambda,0} h_t = 0$ due to the mass and momentum conservation of the linearized equation. Since the x-derivatives commute with the equation, we can set $s = 0$. We first treat the case $L_x^1 L_v^1(\langle v\rangle m)$. We compute

$$\frac{d}{dt}|||h_t|||_{\mathcal{E}_0} = \eta \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} \mathcal{L}_\lambda(h_t)\,\mathrm{sign}(h_t)\, dx \right) \langle v\rangle m\, dv + \int_0^\infty \frac{\partial}{\partial t}\|h_{t+\tau}\|_{\mathcal{E}_0}\, d\tau =: I_1 + I_2.$$

Concerning the first term, arguing as in the proof of Lemma 2.8, we have from the dissipativity of $\mathcal{B}_\lambda$ and the bounds on $\mathcal{A}$

$$I_1 \le \eta\, (C\|h_t\|_{\mathcal{E}_0} - K\|h_t\|_{\mathcal{E}_0^1})$$

for some constants $C, K > 0$.

The second term is computed exactly:

$$I_2 = \int_0^\infty \frac{\partial}{\partial t}\|h_{t+\tau}\|_{\mathcal{E}_0}\, d\tau = \int_0^\infty \frac{\partial}{\partial \tau}\|h_{t+\tau}\|_{\mathcal{E}_0}\, d\tau = -\|h_t\|_{\mathcal{E}_0}.$$

The combination of the two last equations yields the desired result by choosing η small enough. The case of higher-order v-derivativeq is treated similarly as in Lemma 2.8. □

3. The nonlinear Boltzmann equation

3.1. **The bilinear estimates.**

Let us recall a bilinear estimate on the nonlinear term in equation (1.1).

Lemma 3.1. *In the space* $\mathcal{E}^q := W^{\sigma,1}_v W^{s,1}_x(\langle v\rangle^q m)$ *with* $s, \sigma \in \mathbb{N}$, $s > 6$ *and* $q \in \mathbb{N}$, *the collision operator* Q *satisfies*

$$\|Q_\alpha(g,f)\|_{\mathcal{E}^q} \le C(\|g\|_{\mathcal{E}^{q+1}}\|f\|_{\mathcal{E}^q} + \|g\|_{\mathcal{E}^q}\|f\|_{\mathcal{E}^{q+1}})$$

for some constant $C > 0$, *where* $\mathcal{E}^{q+1}$ *is defined as* $\mathcal{E}^q$.

The proof is similar to the one done in [18, Lemma 5.16]. We shall only mention the main steps.

Proof. Let us first consider the velocity aspect only of the norm with $\sigma = 0$. Concerning the case of a constant coefficient of inelasticity, we use that the elastic collision operator Q_1 satisfies (cf [27])

$$\|Q_1(g,f)\|_{L^1_v(m)} \le C(\|f\|_{L^1_v(m)}\|g\|_{L^1_v(\langle v\rangle m)} + \|f\|_{L^1_v(\langle v\rangle m)}\|g\|_{L^1_v(m)}).$$

First, it can be straightforwardly adapted to the case $L^1(\langle v\rangle^q m)$. Then, if v'_λ and v'_0 denotes the post-collisional velocities in the inelastic case and in the elastic case with obvious notations, using the fact that we both have

$$|v'_\lambda|^2 \le |v|^2 + |v_*|^2$$

and

$$|v'_0|^2 \le |v|^2 + |v_*|^2,$$

the same proof can be done in the inelastic case. We hence obtain that

$$\|Q_{e_\lambda}(g,f)\|_{L^1_v(\langle v\rangle^q m)} \le C\left(\|f\|_{L^1_v(\langle v\rangle^q m)}\|g\|_{L^1_v(\langle v\rangle^{q+1} m)} + \|f\|_{L^1_v(\langle v\rangle^{q+1} m)}\|g\|_{L^1_v(\langle v\rangle^q m)}\right). \tag{3.1}$$

Then, from property (2.6) and inequality (3.1), we deduce that

$$\begin{aligned}\|Q_{e_\lambda}(g,f)\|_{W^{\sigma,1}_v(\langle v\rangle^q m)} &\le C\Big(\|f\|_{W^{\sigma,1}_v(\langle v\rangle^q m)}\|g\|_{W^{\sigma,1}_v(\langle v\rangle^{q+1} m)} + \\ &\qquad \|f\|_{W^{\sigma,1}_v(\langle v\rangle^{q+1} m)}\|g\|_{W^{\sigma,1}_v(\langle v\rangle^q m)}\Big)\end{aligned}$$

as well as similar results from the other estimates.

As a final step, we consider the x aspect of the norm. We use the Sobolev embedding $W^{s/2,1}_x(\mathbb{T}^3) \subset L^\infty_x(\mathbb{T}^3)$ with continuous embedding since $s > 6$ and we conclude as in [18]. □

3.2. **The main results.**

Let us now give some results on the stability and relaxation to equilibrium for solutions to the full non-linear problem (1.1). We consider first the close-to-equilibrium regime (Theorem 3.2), and then the weakly inhomogeneous regime (Theorem 3.3 which is a precise version of Theorem 1.2).

Theorem 3.2 (Perturbative solutions close to equilibrium)**.** *Let us consider* $\lambda \in [0, \lambda']$ *(where* λ' *is given by Theorem 2.14). There is some constructive constant* $\varepsilon > 0$ *such that for any initial data* $f_{in} \in \mathcal{E}_0$ *satisfying*

$$\|f_{in} - G_\lambda\|_{\mathcal{E}_0} \le \varepsilon,$$

and f_{in} *has the same global mass and momentum as the equilibrium* G_λ *defined in subsection 3.1, there exists a unique global solution* $f \in L^\infty_t(\mathcal{E}_0)$ *to (1.1).*

This solution furthermore satisfies that for any $\widetilde{\alpha} \in (0, -\mu_\lambda)$*:*

$$\forall\, t \geq 0, \quad \|f(t) - G_\lambda\|_{\mathcal{E}_0} \leq C e^{-\widetilde{\alpha} t} \|f_{in} - G_\lambda\|_{\mathcal{E}_0}$$

for some constructive constant $C > 0$*.*

Theorem 3.3 (Weakly inhomogeneous solutions)**.** *Let us consider* λ *in* $[0, \lambda']$*. Consider a spatially homogeneous distribution* $g_{in} = g_{in}(v) \in W^{2,1}_v\left(\langle v\rangle^5 e^{b\langle v\rangle^\beta}\right)$ *with the same global mass and momentum as* G_λ*. Let us denote* $M := \|g_{in}\|_{W^{2,1}_v\left(\langle v\rangle^5 e^{b\langle v\rangle^\beta}\right)}$*.*

There is some constructive constant $\varepsilon(M) > 0$ *such that for any initial data* $f_{in} \in \mathcal{E}_0$ *satisfying*

$$\|f_{in} - g_{in}\|_{\mathcal{E}_0} \leq \varepsilon(M),$$

and f_{in} *has the same mass and momentum as* G_λ *and* g_{in}*, there exists a global solution* $f \in L^\infty_t(\mathcal{E}_0)$ *to (1.1).*

Moreover, this solution satisfies

$$\forall\, t \geq 0, \quad \|f_t - g_t\|_{\mathcal{E}_0} \leq C\varepsilon(M)$$

where g_t *is the solution to the spatially homogeneous Boltzmann equation starting from* g_{in} *for some constructive constant* $C > 0$*.*

For any $\widetilde{\alpha} \in (0, -\mu_\lambda)$*, this solution also satisfies*

$$\|f_t - G_\lambda\|_{\mathcal{E}_0} \leq C e^{-\widetilde{\alpha} t}$$

for some constructive constant $C > 0$*.*

3.3. **Proof of the main results.**

3.3.1. *Proof of Theorem 3.2.* Again, the proof is adapted from [18] and we will only mention the main ideas of the proof. We begin by giving the key a priori estimate.

Lemma 3.4. *With the notations of Theorem 3.2, in the space* $\mathcal{E}_0$*, a solution* f_t *to the Boltzmann equation formally writes* $f_t = G_\lambda + h_t$*,* $\Pi_{\mathcal{L}_\lambda,0} h_t = 0$*, and* h_t *satisfies the estimate*

$$\frac{d}{dt} |\!|\!| h_t |\!|\!|_{\mathcal{E}_0} \leq (C |\!|\!| h_t |\!|\!|_{\mathcal{E}_0} - K) |\!|\!| h_t |\!|\!|_{\mathcal{E}_0^1}$$

for some constants C*,* $K > 0$*.*

Proof. We consider the case $L^1_x L^1_v(\langle v\rangle m)$, we will skip the proof of other cases which is similar. We have

$$\frac{d}{dt} |\!|\!| h_t |\!|\!|_{L^1_x L^1_v(\langle v\rangle m)} = I_1 + I_2$$

with

$$\begin{aligned} I_1 :=\, & \eta \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} \mathcal{L}_\lambda h_t \,\mathrm{sign}(h_t)\, dx \right) \langle v\rangle m\, dv \\ & + \int_0^\infty \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} e^{\tau \mathcal{L}_\lambda} (\mathcal{L}_\lambda h_t)\, \mathrm{sign}(e^{\tau\mathcal{L}_\lambda} h_t)\, dx \right) \langle v\rangle m\, dv\, d\tau \end{aligned}$$

and

$$\begin{aligned} I_2 :=\, & \eta \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} Q_{e_\lambda}(h_t, h_t)\, \mathrm{sign}(h_t)\, dx \right) \langle v\rangle m\, dv \\ & + \int_0^\infty \int_{\mathbb{R}^3} \left(\int_{\mathbb{T}^3} e^{\tau \mathcal{L}_\lambda} Q_{e_\lambda}(h_t, h_t)\, \mathrm{sign}(e^{\tau\mathcal{L}_\lambda} h_t)\, dx \right) \langle v\rangle m\, dv\, d\tau \end{aligned}$$

We already know from Proposition 2.22 that by choosing η small enough, we have

$$I_1 \leq -K |||h|||_{L^1_x L^1_v(\langle v\rangle^2 m)}, \quad K > 0.$$

For the second term, we have

$$\begin{aligned} I_2 \leq & \eta \int_{\mathbb{R}^3} \|Q_{e_\lambda}(h_t,h_t)\|_{L^1_x(\langle v\rangle m)}\, dv + \int_0^\infty \int_{\mathbb{R}^3} \|e^{\tau\mathcal{L}_\lambda} Q_{e_\lambda}(h_t,h_t)\|_{L^1_x(\langle v\rangle m)}\, dv\, d\tau \\ \leq & \eta\, \|Q_{e_\lambda}(h_t,h_t)\|_{L^1_x L^1_v(\langle v\rangle m)} + \int_0^\infty \|e^{\tau\mathcal{L}_\lambda} Q_{e_\lambda}(h_t,h_t)\|_{L^1_x L^1_v(\langle v\rangle m)}\, d\tau. \end{aligned}$$

We thus deduce

$$\frac{d}{dt} |||h_t|||_{L^1_x L^1_v(\langle v\rangle m)} \leq -K |||h_t|||_{L^1_x L^1_v(\langle v\rangle^2 m)} + |||Q_{e_\lambda}(h_t,h_t)|||_{L^1_x L^1_v(\langle v\rangle m)}.$$

Now using the bilinear estimate coming from Lemma 3.1, the semigroup decay (2.30) and the fact that $\Pi_{\mathcal{L}_\lambda,0} Q_{e_\lambda}(h_t,h_t) = 0$, we obtain

$$\begin{aligned} |||Q_{e_\lambda}(h_t,h_t)|||_{L^1_x L^1_v(\langle v\rangle m)} \leq & \eta\, \|Q_{e_\lambda}(h_t,h_t)\|_{L^1_x L^1_v(\langle v\rangle m)} \\ & + \int_0^\infty \|S_{\mathcal{L}_\lambda}(\tau) Q_{e_\lambda}(h_t,h_t)\|_{L^1_x L^1_v(\langle v\rangle m)}\, d\tau \\ \leq & \eta\, \|h_t\|_{L^1_x L^1_v(\langle v\rangle m)} \|h_t\|_{L^1_x L^1_v(\langle v\rangle^2 m)} \\ & + C \left(\int_0^\infty e^{-\alpha_\lambda \tau}\, d\tau \right) \|h_t\|_{L^1_x L^1_v(\langle v\rangle m)} \|h_t\|_{L^1_x L^1_v(\langle v\rangle^2 m)} \\ \leq & C\, \|h_t\|_{L^1_x L^1_v(\langle v\rangle m)} \|h_t\|_{L^1_x L^1_v(\langle v\rangle^2 m)} \\ \leq & C\, |||h_t|||_{L^1_x L^1_v(\langle v\rangle m)} |||h_t|||_{L^1_x L^1_v(\langle v\rangle^2 m)}, \end{aligned}$$

which concludes the proof. □

We shall now construct solutions by considering the following iterative scheme

$$\partial_t h^{n+1} = \mathcal{L}_\lambda h^{n+1} + Q_{e_\lambda}(h^n,h^n), \quad n \geq 1,$$

with the initialization

$$\partial_t h^0 = \mathcal{L}_\lambda h^0, \quad h^0_{in} = h_{in}$$

and we assume $|||h_{in}|||_{\mathcal{E}_0} \leq \varepsilon/2$. The functions h^n, $n \geq 0$ are well-defined in $\mathcal{E}_0$ thanks to Theorem 2.14.

The proof is split into three steps.

Step 1. Stability of the scheme. Let us prove by induction the following control

$$\forall\, n \geq 0, \quad \sup_{t \geq 0} \left(|||h^n_t|||_{\mathcal{E}_0} + K \int_0^t |||h^n_\tau|||_{\mathcal{E}^1_0}\, d\tau \right) \leq \varepsilon \tag{3.2}$$

as soon as $\varepsilon \leq K/(2C)$.

The initialization is deduced from Proposition 2.22 and the fact that $\|h_{in}\|_{\mathcal{E}_0} \leq \varepsilon/2$:

$$\sup_{t \geq 0} \left(|||h^0_t|||_{\mathcal{E}_0} + K \int_0^t |||h^0_\tau|||_{\mathcal{E}^1_0}\, d\tau \right) \leq \varepsilon.$$

Let us now assume that (3.2) is satisfied for any $0 \leq n \leq N \in \mathbb{N}^*$ and let us prove it for $n = N+1$. A similar computation as in Lemma 3.4 yields

$$\frac{d}{dt} |||h^{N+1}|||_{\mathcal{E}_0} + K \|h^{N+1}\|_{\mathcal{E}^1_0}\, d\tau \leq C |||Q_{e_\lambda}(h^N,h^N)|||_{\mathcal{E}_0}$$

for some constants $C, K > 0$, which implies

$$\begin{aligned}\|h_t^{N+1}\|_{\mathcal{E}_0} + K \int_0^t \|h_\tau^{N+1}\|_{\mathcal{E}_0^1}\,d\tau &\leq |||h_{in}|||_{\mathcal{E}_0} + \int_0^t |||Q_{e_\lambda}(h_\tau^N, h_\tau^N)|||_{\mathcal{E}_0}\,d\tau \\ &\leq |||h_{in}|||_{\mathcal{E}_0} + C\left(\sup_{\tau\geq 0} |||h_\tau^N|||_{\mathcal{E}_0}\right)\int_0^t |||h_\tau^N|||_{\mathcal{E}_0^1}\,d\tau \\ &\leq \frac{\varepsilon}{2} + \frac{C}{K}\varepsilon^2 \\ &\leq \varepsilon,\end{aligned}$$

as soon as $\varepsilon < K/(2C)$.

Step 2. Convergence of the scheme. Let us now denote $d^n := h^{n+1} - h^n$ and $s^n := h^{n+1} + h^n$ for $n \geq 0$. They satisfy

$$\forall\, n \geq 0, \quad \partial_t d^{n+1} = \mathcal{L}_\lambda d^{n+1} + Q_{e_\lambda}(d^n, s^n) + Q_{e_\lambda}(s^n, d^n)$$

and

$$\partial_t d^0 = \mathcal{L}_\lambda d^0 + Q_{e_\lambda}(h^0, h^0).$$

Let us denote

$$A^n(t) := \sup_{0\leq r\leq t}\left(|||d_r^n|||_{\mathcal{E}_0} + K\int_0^r \|d_\tau^n\|_{\mathcal{E}_0^1}\,d\tau\right).$$

We can prove by induction that

$$\forall\, t \geq 0, \quad \forall\, n \geq 0, \quad A^n(t) \leq (\overline{C}\varepsilon)^{n+2}$$

for some constant $\overline{C} > 0$.

Hence for ε small enough, the series $\sum_{n\geq 0} A^n(t)$ is summable for any $t \geq 0$ and the sequence h^n has the Cauchy property in $L^\infty_t(\mathcal{E}_0)$, which proves the convergence of the iterative scheme. The limit h as n goes to infinity satisfies the equation in the strong sense in $\mathcal{E}_0$.

Step 3. Rate of decay. We now consider the solution h constructed so far. From the first step, we first deduce by letting n go to infinity in the stability estimate that

$$\sup_{t\geq 0}\left(|||h_t|||_{\mathcal{E}_0} + K\int_0^t |||h_\tau|||_{\mathcal{E}_0^1}\,d\tau\right) \leq \varepsilon.$$

Second, we can apply the a priori estimate from Lemma 3.4 to this solution h which implies that

$$|||h_t|||_{\mathcal{E}_0} \leq e^{-\frac{K}{2}t}\|h_{in}\|_{\mathcal{E}_0}$$

under the appropriate smallness condition on ε. Using the fact that $|||h_t|||_{\mathcal{E}_0}$ converges to zero as $t \to +\infty$, we obtain

$$\int_t^\infty \|h_t\|_{\mathcal{E}_0^1}\,d\tau \leq \frac{2}{K\eta}\|h_t\|_{\mathcal{E}_0} \leq Ce^{-\frac{K}{2}t}\|h_{in}\|_{\mathcal{E}_0}.$$

We shall now perform a bootstrap argument in order to ensure that the solution h_t enjoys the same decay rate $O(e^{-\alpha' t})$ as the linearized semigroup (Theorem 2.14). Assuming that the solution is known to decay as

$$\|h_t\|_{\mathcal{E}_0} \leq Ce^{-\alpha_0 t}$$

for some constant $C > 0$, we can prove that it indeed decays

$$\|h_t\|_{\mathcal{E}_0} \leq C'e^{-\alpha_1 t}$$

with $\alpha_1 = \min(\alpha_0 + K/4, \alpha)$. It can be proved using Theorem 2.14 and Lemma 3.1. Hence, in a finite number of steps, it proves the desired decay rate $O(e^{-\alpha' t})$.

3.3.2. *Proof of Theorem 3.3.* We split the proof into three steps. We will only deal with the case $L^1_x L^1_v(\langle v\rangle m)$.

Step 1. The spatially homogeneous evolution. We consider the spatially homogeneous initial data g_{in}. From [24, Corollary 6.3], we know that it gives rise to a spatially homogeneous solution $g_t \in L^1_v(\langle v\rangle m)$ which satisfies

$$\|g_t - G_\lambda\|_{L^1_v(\langle v\rangle m)} \to 0$$

with explicit exponential rate and $g_t \in L^\infty_t(L^1_v(\langle v\rangle m)) \cap L^1_t(L^1_v(\langle v\rangle^2 m))$.

Step 2. Local in time stability estimate. The goal is to construct a solution f_t close to some spatially homogeneous solution g_t which is uniformly bounded in $L^1_x L^1_v(\langle v\rangle m)$. We consider the difference $d_t := f_t - g_t$ and we write its evolution equation:

$$\begin{aligned}\partial_t d + v\cdot\nabla_x d &= Q_{e_\lambda}(d,d) + Q^+_{e_\lambda}(g,d) + Q^+_{e_\lambda}(d,g) - Q^-_{e_\lambda}(g,d) - Q^-_{e_\lambda}(d,g) + \lambda^\gamma \Delta_v d \\ &= \mathcal{P}(d) + \lambda^\gamma\Delta_v d,\end{aligned}$$

where $\mathcal{P}(d) := Q_{e_\lambda}(d,d) + Q^+_{e_\lambda}(g,d) + Q^+_{e_\lambda}(d,g) - Q^-_{e_\lambda}(g,d) - Q^-_{e_\lambda}(d,g)$. We then estimate the time evolution of the $L^1_x L^1_v(\langle v\rangle m)$ norm:

$$\begin{aligned}&\frac{d}{dt}\|d_t\|_{L^1_xL^1_v(\langle v\rangle m)} = \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} (\mathcal{P}(d_t) + \lambda^\gamma\Delta_v d_t)\,\mathrm{sign}\, d_t\, dx\,\langle v\rangle m\, dv \\ &\le C\,\|Q_{e_\lambda}(d_t,d_t)\|_{L^1_xL^1_v(\langle v\rangle m)} + C\,\|Q^+_{e_\lambda}(g_t,d_t)\|_{L^1_xL^1_v(\langle v\rangle m)} + C\,\|Q^+_{e_\lambda}(d_t,g_t)\|_{L^1_xL^1_v(\langle v\rangle m)} \\ &\quad + C\,\|Q^-_{e_\lambda}(d_t,g_t)\|_{L^1_xL^1_v(\langle v\rangle m)} - \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} Q^-_{e_\lambda}(g_t,d_t)\,\mathrm{sign}\, d_t\, dx\,\langle v\rangle m\, dv \\ &\quad + \lambda^\gamma \int_{\mathbb{R}^3}\int_{\mathbb{T}^3} \Delta_v |d_t|\, dx\,\langle v\rangle m\, dv.\end{aligned}$$

First, using the bilinear estimates of Lemma 3.1, we have

$$\|Q_{e_\lambda}(d,d)\|_{L^1_xL^1_v(\langle v\rangle m)} \le C\|d\|_{L^1_xL^1_v(\langle v\rangle m)}\|d\|_{L^1_xL^1_v(\langle v\rangle^2 m)}$$

and

$$\begin{aligned}\|Q^+_{e_\lambda}(d,g)\|_{L^1_xL^1_v(\langle v\rangle m)} + \|Q^+_{e_\lambda}(g,d)\|_{L^1_xL^1_v(\langle v\rangle m)} \le\, &\eta\,\|g\|_{L^1_xL^1_v(\langle v\rangle^2 m)}\|d\|_{L^1_xL^1_v(\langle v\rangle^2 m)} \\ &+ C_\eta\,\|g\|_{L^1_xL^1_v(\langle v\rangle m)}\|d\|_{L^1_xL^1_v(\langle v\rangle m)}\end{aligned}$$

for any $\eta > 0$ as small as wanted, and some corresponding η-dependent constant C_η. Second, by trivial explicit computations we have

$$\|Q^-_{e_\lambda}(d,g)\|_{L^1_xL^1_v(\langle v\rangle m)} \le C\,\|d\|_{L^1_xL^1_v(\langle v\rangle m)}\|g\|_{L^1_xL^1_v(\langle v\rangle^2 m)}.$$

Third, we have for some $K > 0$,

$$-\int_{\mathbb{R}^3}\int_{\mathbb{T}^3} Q^-_{e_\lambda}(g,d)\,\mathrm{sign}\, d_t\, dx\,\langle v\rangle m\, dv \le -K\|d\|_{L^1_xL^1_v(\langle v\rangle^2 m)}.$$

Fourth and last,

$$\lambda^\gamma\int_{\mathbb{R}^3}\int_{\mathbb{T}^3}\Delta_v|d|\,dx\,\langle v\rangle m\,dv \le C\,\|d\|_{L^1_xL^1_v(\langle v\rangle m)} \le C\,\|d\|_{L^1_xL^1_v(\langle v\rangle^2 m)}.$$

Gathering all these estimates, we finally obtain

$$\frac{d}{dt}\|d_t\|_{L^1_x L^1_v(\langle v\rangle m)} \leq (C\,\|d_t\|_{L^1_x L^1_v(\langle v\rangle m)} + \lambda^\gamma - K)\|d_t\|_{L^1_x L^1_v(\langle v\rangle^2 m)} + C\,\|g_t\|_{L^1_x L^1_v(\langle v\rangle^2 m)}\|d_t\|_{L^1_x L^1_v(\langle v\rangle m)}.$$

We then introduce an iterative scheme

$$\partial_t d^{n+1} = Q_{e_\lambda}(d^n, d^n) + Q_{e_\lambda}(g, d^n) + Q_{e_\lambda}(d^n, g), \quad n \geq 0,$$

and

$$\partial_t d^0 = Q_{e_\lambda}(g, d^0) + Q_{e_\lambda}(d^0, g)$$

with $d^n_{in} = d_{in} = f_{in} - g_{in}$ for all $n \geq 0$, just as the previous subsection. At each step, a global solution d_n is constructed in $L^1_x L^1_v(\langle v\rangle m)$ using the estimates above. We assume that $\|d_{in}\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \varepsilon/2$. By passing to the limit in the a priori estimates, we deduce that, as long as

$$C\,\|d_t\|_{L^1_x L^1_v(\langle v\rangle m)} \leq K - \lambda^\gamma \tag{3.3}$$

we have

$$\|d_t\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \frac{\varepsilon}{2}\exp\left(C\int_0^t \|g_\tau\|_{L^1_x L^1_v(\langle v\rangle^2 m)}\,d\tau\right).$$

We then choose ε small enough so that $C\varepsilon \leq K - \lambda^\gamma$, and then since $g_t \in L^1_t\left(L^1_x L^1_v(\langle v\rangle^2 m)\right)$, we can choose $T_1 = T_1(\varepsilon) > 0$ so that the smallness condition (3.3) is satisfied and

$$\forall\, t \in [0, T_1], \quad \|d_t\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \varepsilon.$$

Observe that $T_1(\varepsilon) \xrightarrow[\varepsilon\to 0]{} +\infty$. This completes the proof of stability.

Step 3. The trapping mechanism. Consider δ the smallness constant of the stability neighborhood in Theorem 3.2 in $L^1_x L^1_v(\langle v\rangle m)$. Then from [24], we deduce that there is some time $T_2 = T_2(M) > 0$ such that

$$\forall\, t \geq T_2, \quad \|g_t - G_{\lambda,g}\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \frac{\delta}{3}$$

where $G_{\lambda,g}$ is the equilibrium associated to g_{in}. We then choose ε small enough such that

$$\|f_{in} - g_{in}\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \varepsilon \Rightarrow \|G_{\lambda,f} - G_{\lambda,g}\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \frac{\delta}{3}$$

where $G_{\lambda,f}$ is the equilibrium associated to f_{in}, $T_1(\varepsilon) \geq T_2(M)$ and

$$\|f_{T_2} - g_{T_2}\|_{L^1_x L^1_v(\langle v\rangle m)} \leq \frac{\delta}{3},$$

from the stability result.

We deduce that

$$\begin{aligned}\|f_{T_2} - G_{\lambda,f}\|_{L^1_x L^1_v(\langle v\rangle m)} &\leq \|f_{T_2} - g_{T_2}\|_{L^1_x L^1_v(\langle v\rangle m)} + \|g_{T_2} - G_{\lambda,g}\|_{L^1_x L^1_v(\langle v\rangle m)} \\ &\quad + \|G_{\lambda,f} - G_{\lambda,g}\|_{L^1_x L^1_v(\langle v\rangle m)} \\ &\leq \delta\end{aligned}$$

and we can therefore use the perturbative Theorem 3.2 for $t \geq T_2$ which concludes the proof.

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