About stratification by orbit types

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Abstract

When we have a proper action of a Lie group on a manifold, it is well known that we get a stratification by orbit types and it is known that this stratification satifies the Whitney (b) condition. In this article we will see that this stratification satisfies the strong Verdier condition.

This article is based on an idea of David Trotman.

1 Stratification by orbit types

In this section we will recall the definitions and principal results about stratifications by orbit type. The classical references underlying what will come are [8], [14]. We will follow mostly the notations of M. Pflaum in [15] which synthesizes works by [1], [3], [5], [7], [10], [16].
Let $\mathcal{M}$ be a manifold and G a Lie group.

Definition 1. *A (left) action of G is a smooth mapping (i.e. $\mathcal{C}^\infty$)*
$\Phi : G \times \mathcal{M} \to \mathcal{M}$, $(g, x) \mapsto \Phi(g, x) = \Phi_g(x) = gx$ such that:
$\forall g, h \in G, \forall x \in \mathcal{M}, \Phi_g(\Phi_h(x)) = \Phi_{gh}(x), \Phi_e(x) = x$, where e is the unit element of G.

Definition 2. *A G-action $\Phi : G \times \mathcal{M} \to \mathcal{M}$ is called* proper *if the mapping $\Phi_{ext} : G \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$, $(g, x) \mapsto (gx, x)$ is proper.*

With such proper actions several results are known, in particular $\mathcal{M}$ admits a G-invariant Riemannian metric. The most important result is the so called slice theorem ([8], [14]). Here it is as stated in [15]:

Theorem 1. *Let $\Phi : G \times \mathcal{M} \to \mathcal{M}$ be a proper group action, x a point of $\mathcal{M}$ and $\mathcal{V}_x = T_x\mathcal{M}/T_xGx$ the normal space to the orbit of x. Then there exists a G-equivariant diffeomorphism from a G-invariant neighborhood of the zero section of $G \times_{G_x} \mathcal{V}_x$ onto a G-invariant neighborhood of Gx such that the zero section is mapped onto Gx in a canonical way (where G_x is the isotropy group of x).*

If we denote $\mathcal{M}_{(H)}$ the set $\{x \in \mathcal{M} | G_x \sim H\}$ where G_x is the isotropy group of x and $\sim$ means "conjugate to", we get in particular that for a compact subgroup H of G each connected component of $\mathcal{M}_{(H)}$ is a submanifold of $\mathcal{M}$. The isotropy subgroups G_x are compact in the case of a proper group action. Assigning to each point $x \in \mathcal{M}$ the germ $\mathcal{S}_x$ of the set $\mathcal{M}_{(G_x)}$ we get a stratification of $\mathcal{M}$ in the sense of Mather ([13]), called stratification by orbit type.
This stratification has been studied a lot and has been also recently described in [6], [4]. This stratification is known to be Whitney (b) regular.

2 Verdier's condition

About Verdier's condition the reader may look for [19], [9], [17], [2]:

Definition 3. *Let X be a $\mathcal{C}^1$ submanifold of $\mathbb{R}^n$, and a subanalytic set. Let Y be an analytic submanifold of $\mathbb{R}^n$ such that $0 \in Y \subset \overline{X} \setminus X$. Verdier ([19]) defines X to be (w)-regular over Y at 0 if there is a constant $C > 0$ and a neighborhood U of 0 in $\mathbb{R}^n$ such that if $x \in U \cap X$ and $y \in U \cap Y$, then $d(T_yY, T_xX) \leqslant C|x - y|$.*

(with $d(A,B) = sup\{dist(x,B) | x \in A, |x| = 1\}$).
This condition is known to be stronger than the Whitney (b) condition for subanalytic sets (Kuo has shown that condition (w) implies condition (b) in [9] and Trotman in [18] has shown that the converse is false (in the real case)). But we also have a stronger version of Verdier's condition:

Definition 4. *Let X be a $\mathcal{C}^1$ submanifold of $\mathbb{R}^n$, and a subanalytic set. Let Y be an analytic submanifold of $\mathbb{R}^n$ such that $0 \in Y \subset \overline{X} \setminus X$. In [11] (see also [12]) Li, Kuo, Trotman and Wilson define X to be strongly Verdier regular over Y (or differentiably regular) at 0 if for all $\epsilon > 0$ and a neighborhood U of 0 in $\mathbb{R}^n$ such that if $x \in U \cap X$ and $y \in U \cap Y$, then $d(T_yY, T_xX) \leqslant \epsilon|x - y|$.*

The next theorem is an enhancement of the theorem 4.3.7 that can be found page 160 in [15], most of the notations will be conserved.

Theorem 2. *The stratification by orbit types of a G-manifold $\mathcal{M}$ with a proper action is a strong Verdier stratification.*

Proof. Suppose that $K \subsetneq H \subset G$ are two isotropy groups of $\mathcal{M}$, we have $\mathcal{M}_{(H)} < \mathcal{M}_{(K)}$. Let $y \in \mathcal{M}_{(H)}$. With the slice theorem, we can suppose that: $\mathcal{M} = G \times_H \mathcal{V} = (G \times_H \mathcal{W}) \times \mathcal{V}^H$ et $y = [(e, 0)]$ where $\mathcal{V}$ is an H-slice, $\mathcal{V}^H$ is the subspace of the H-invariant vectors, and $\mathcal{W} = (\mathcal{V}^H)^{\perp}$ is the orthogonal space relative to the H-invariant inner product on $\mathcal{V}$. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{h}$ that of H, and $\mathfrak{m}$ the orthogonal space of $\mathfrak{h} \subset \mathfrak{g}$ related to the H-invariant inner product on $\mathfrak{g}$. By the exponential map on G we have a natural chart on an open set $\mathcal{U}$ of $\mathcal{M}$ containing y:
$\phi : \mathcal{U} \to \mathfrak{m} \times \mathcal{V}, \phi([(exp(\xi), \nu]) = (\xi, \nu), \xi \in \mathfrak{m}, \nu \in \mathcal{V}$. We have ([15] page 159):$\mathcal{M}_{(K)} = (G \times_H \mathcal{W}_{(K)}) \times \mathcal{V}^H$ et $\mathcal{M}_{(H)} = G/H \times \{0\} \times \mathcal{V}^H$ and through this chart on $\mathcal{U}$ they become parts of $\mathfrak{m} \times \mathcal{W}_{(K)} \times \mathcal{V}^H$ and $\mathfrak{m} \times \{0\} \times \mathcal{V}^H$. This chart is smooth and so we can check Verdier's condition (which is $\mathcal{C}^2$-invariant) at $\tilde{y} = (0, 0, 0)$ in $\phi(\mathcal{M}_{(H)} \cap \mathcal{U})$ (open set of $\mathfrak{m} \times \{0\} \times \mathcal{V}^H$), let $\tilde{x} \in \phi(\mathcal{U} \cap \mathcal{M}_{(K)})$ (open set of $\mathfrak{m} \times \mathcal{W}_{(K)} \times \mathcal{V}^H$) we have,
$T_{\tilde{y}}(\phi(\mathcal{M}_{(H)} \cap \mathcal{U})) = \mathfrak{m} \times \{0\} \times \mathcal{V}^H \subset T_{\tilde{x}}(\phi(\mathcal{U} \cap \mathcal{M}_{(K)})$, (we also have a strict inclusion because $\mathcal{W}_{(K)}$ is invariant by multiplication by a non-vanishing scalar), so we have $d(T_{\tilde{y}}(\phi(\mathcal{M}_{(H)} \cap \mathcal{U})), T_{\tilde{x}}(\phi(\mathcal{U} \cap \mathcal{M}_{(K)})) = 0$ and so strong Verdier condition holds at $\tilde{y}$ (in fact we have something even stronger). □

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