L^∞-estimates for the Vlasov-Poisson-Fokker-Planck equation.

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Abstract

We consider the Vlasov-Poisson-Fokker-Planck equation, in three dimensions, as the backward Kolmogorov equation associated with a nonlinear diffusion process. In this way, we derive new L^∞-estimates on the spatial density, which are uniform in the diffusion parameters.

1 Introduction

The Vlasov-Poisson-Fokker-Planck (VPFP) equation is an evolution equation which describes the dynamics of a plasma of Coulomb particles in a thermal reservoir (see Eq.s (1)-(4) in the following section). When the action of the reservoir is neglected (i.e. $\beta = \sigma = 0$), we have the well-known Vlasov-Poisson (VP) equation.

The specific difficulty encountered in solving the Cauchy problem associated with the VP equation relies on the singularity of the potential term; such a difficulty is increasing with the dimension of the physical space.
In dimension three, the problem was relatively recently solved in Ref.s [6], [11], [12] and [14] by performing a careful analysis of the behaviour of the characteristics and in Ref.[7] by means of a control on the velocity moments of the distribution function. As a matter of fact, the method of the first group of references produces a control on the spatial density ρ in L^∞-norm, while the second approach yields an L^p, $1 < p < +\infty$ control. Both kinds of control ensure sufficient regularity properties of the potential field E, to construct uniquely the solution.

Coming back to the Cauchy problem for the VPFP equation, one expects that this last problem should be easier to treat. Indeed, a diffusion term (even though it is degenerate, because it involves only the velocity variables) should regularize the behaviour of the solutions. Nevertheless, if one tries to get estimates on the spatial density, which are uniform in the diffusion parameters β and σ, one is faced at least with the same difficulties as for the VP equation.
Bouchut (see Ref.s [1] and [2]) used the "propagation of moments" techniques

from Ref.[7] to construct the solution to the VPFP initial value problem. Various contributions to the VPFP problem have also appeared to deal with different situations (see e.g. [3] and its references).

The purpose of the present paper is to approach the VPFP equation in the same spirit as in Ref.s [6], [11], [12] and [14], namely by controlling the behaviour of the characteristics which are the solutions of the ordinary differential equation underlying the VP equation. In our case, however, due to the presence of a diffusion term, we are quite naturally lead to consider a stochastic differential equation rather than an ordinary differential problem. This idea in not completely new: it goes back to McKean (see Ref.[9]) and it has been successfully applied to the VPFP equation in the much easier two-dimensional case (see Ref.[10]).
We must comment that, due to the fact that the diffusion coefficient in Eq.(1) is constant, the stochastic content of this work is poor (the only property of diffusion processes we need is a standard and rather elementary feature of the Brownian motion).
In conclusion, the interest of our approach is two-fold: from one side, we present an alternative method to solve the VPFP initial value problem; from the other one, we obtain L^∞-estimates (uniform in β and σ) on the spatial density in Eq.(3) which are, as far as we know, new.

2 Preliminaries

We consider the VPFP equation in $\mathbb{R}^3$, that is

$$\frac{\partial}{\partial t} f + v \cdot \nabla_x f + \nabla_v \cdot (E - \beta\, v) f = \frac{\sigma^2}{2}\, \Delta_v f, \tag{1}$$

where

$$f = f(t, x, v) \tag{2}$$

denotes the probability density on the phase space at time $t \in \mathbb{R}$, x and v represent position and velocity respectively, then β and σ are non-negative parameters. Moreover,

$$\rho(t, x) = \int f(t, x, v)\, dv \tag{3}$$

denotes the spatial density and

$$E(t, x) = \gamma \int \frac{x - y}{|x - y|^3}\, \rho(t, y)\, dy, \quad \gamma = \pm\, \frac{1}{4\,\pi}, \tag{4}$$

is the field acting on the test particles, which gives rise to a gravitational or an electric field for negative or positive γ respectively.

Let $f(t, x, v)$ be any classical solution to Eq.(1), associated with a sufficiently regular initial condition $f_0 = f_0(x, v)$. The following estimates are well-known and consequence of the control on the energy

$$\mathcal{E}(t) = \frac{1}{2} \int |v|^2\, f(t, x, v)\, dx\, dv + \frac{\gamma}{2} \int \int \frac{\rho(t, x)\, \rho(t, y)}{|x - y|}\, dx\, dy, \tag{5}$$

for both positive and negative γ values.
Denoting by T an arbitrary but fixed $T>0$, it holds (see Ref.s [12] and [14]) that

$$\sup_{t\in[0,T]} \|\rho(t)\|_{L^{5/3}} \leq C_1, \tag{6}$$

$$\|E(t)\|_{L^\infty} \leq C_2 \, \|\rho(t)\|_{L^\infty}^{4/9}. \tag{7}$$

Here and after C_i, $i=1,2,...$, denote positive constants depending on f_0 and T only.

Let $b \in \mathcal{C}_0^*([0,T])$, where we define

$$\mathcal{C}_0^*([0,T]) = \left\{\xi : [0,T] \longrightarrow \mathbb{R}^3 \ / \ \ \xi \in \mathcal{C}([0,T]),\ \xi(0)=0\right\} \tag{8}$$

and we consider the pair $(X(t,s,x,v;b), V(t,s,x,v;b))$, for $t>s$, which is a solution of the integral system

$$\begin{cases} X(t,s,x,v;b) = x + \int_s^t V(\tau,s,x,v;b)\, d\tau \\ V(t,s,x,v;b) = v\, e^{-\beta(t-s)} + \int_s^t E(\tau, X(\tau,s,x,v;b))\, e^{-\beta(t-\tau)}\, d\tau + \sigma B(t,s,b), \end{cases} \tag{9}$$

with

$$B(t,s,b) = b(t) - b(s)\, e^{-\beta(t-s)} - \beta \int_s^t b(\tau)\, e^{-\beta(t-\tau)}\, d\tau. \tag{10}$$

We note that a unique solution to Eq.(9) can be found under appropriate regularity hypotheses on the field E, to be discussed later on.
On $\mathcal{C}_0^*([0,T])$, endowed with the *sup*-norm topology, we introduce (see e.g. Ref.[13]) the *Wiener* measure $\mu(db)$, which is concentrated on the Hölder continuous functions of exponent $\alpha<\frac{1}{2}$. Therefore, the following identity holds,

$$\int f(t,x,v)\, \varphi(x,v)\, dx\, dv = \mathbb{E}\left[\int f_0(x,v)\, \varphi(X(t,0,x,v;b), V(t,0,x,v;b))\, dx\, dv\right], \tag{11}$$

where $\mathbb{E}$ denotes "expectation" with respect to μ and φ is any regular test function.

The formulations we have introduced above are motivated by the fact that the VPFP equation can be interpreted as the evolution equation for the probability density associated with a diffusion process $(X(t), V(t)) \in \mathbb{R}^6$, initially distributed according to f_0 and solution of the stochastic differential system

$$\begin{cases} dX(t) = V(t)\, dt \\ dV(t) = (E(t,X(t)) - \beta V(t))\, dt + \sigma\, db(t). \end{cases} \tag{12}$$

Indeed, as the diffusion coefficient σ is constant, we can integrate Eq.(12) for each sample b of the Brownian motion and the resulting trajectory $(X(t;b), V(t;b))$ is a sample of the diffusion process in Eq.s (9)-(10), whose statistical weight is that assigned to b by the *Wiener* measure. We remark that Eq.(10) follows from the stochastic integral

$$\int_s^t e^{-\beta(t-\tau)}\, db(\tau) \tag{13}$$

and a "formal" integration by parts (see Ref.[9]). The explicit form of Eq.(10) thus avoids the use of the stochastic calculus.
We also note that, for all $b \in \mathcal{C}_0^*([0,T])$, the application

$$(x,v) \longrightarrow (X(t,s,x,v;b), V(t,s,x,v;b)) \tag{14}$$

is an invertible continuous mapping on $\mathbb{R}^6$ and, for the *Liouville*'s theorem, we have that the Jacobian is $e^{-\beta(t-s)}$. As a matter of fact, Eq.(11) can be replaced by the more convenient one

$$f(t,x,v) = e^{\beta t}\, \mathbb{E}\left[\, f_0(X(0,t,x,v;b), V(0,t,x,v;b))\,\right], \tag{15}$$

which will be the starting point of our analysis.

The main result of the present paper is summarized by the following

Theorem 1 *Let $f_0(x,v) \in L^1 \cap L^\infty(\mathbb{R}^6)$ be a probability density such that*

$$f_0(x,v) = 0 \quad if \quad |v| > Q_0. \tag{16}$$

Let $f(t,x,v)$ be given by Eq.(15), with E computed in a selfconsistent way by Eq.(4) and with $\beta \in (0,\beta_0]$ and $\sigma \in (0,\sigma_0]$. Then, there exists a constant C, depending only on f_0, β_0, σ_0 and T, for which

$$\sup_{t\in[0,T]} \|\rho(t)\|_{L^\infty} \le C. \tag{17}$$

Remark 1 In Theorem 1, we assume the existence of f which satisfies Eq.(15). In fact, the result above allows us to construct such f, which turns out to be also a weak solution of Eq.(1). This can be done by a standard procedure, for instance by regularizing the singular kernel $\frac{x}{|x|^3}$ by means of a suitable cutoff and then removing it by using Eq.(17), which is independent of the cutoff's parameters. The details are given in Ref.[10], where the two-dimensional case is handled. Classical solutions can be easily obtained by assuming appropriate regularity on f_0.

Remark 2 The uniformity of the estimate in Eq.(17) with respect to β and σ close to zero allows us to show the weak convergence of the solution, in the limit $\beta \to 0$ and $\sigma \to 0$, to the corresponding one associated to the VP equation. This is consequence of the convergence of the stochastic trajectories in Eq.(9) to the usual characteristics of the VP equation. This procedure is similar to that exploited for the vanishing viscosity limit, applied to Incompressible Fluid Dynamics (see Ref.[8]). We do not give any detail here.

Remark 3 We shall not use in the sequel the fully structure of Eq.(1), but only its representation through Eq.(11) or Eq.(15), together with the energy estimates given in Eq.s (6)-(7).

3 The proof of Theorem 1

Let $\hat{b}$ be a sample of Brownian motion and $(\hat{X}(t,0,x,v;\hat{b}), \hat{V}(t,0,x,v;\hat{b}))$, $t \in [0,T]$, the corresponding solution to Eq.s (9)-(10). We define

$$Q(t) = \sup_{\substack{(x,v)\in\mathbb{R}^6 \\ \hat{b}\in\mathcal{C}_0^*([0,T])}} \left\{ |\hat{V}(t,0,x,v;\hat{b})| \;/\; |v| \le Q_0 \text{ and } \sup_{\tau\in[0,t]} |\hat{b}(\tau)| \le \frac{|\hat{V}(t,0,x,v;\hat{b})|^\alpha}{40\,\sigma_0} \right\},$$

with $0<\alpha<1$ to be chosen later. This quantity, even thought it is implicitly defined, is meaningful because of the property of the mapping in Eq.(14) to be invertible. Finally, let be

$$Q = 1 + \sup_{t\in[0,T]} Q(t). \tag{18}$$

It follows from Eq.(15) that

$$\rho(t,x) \le M\, e^{\beta T}\, \mathbb{E}\left[\int \mathcal{X}(|V(0,t,x,v;b)| \le Q_0)\, dv\right], \tag{19}$$

where $M = \|f_0\|_{L^\infty}$ and $\mathcal{X}$ denotes the characteristic function of generic events. Then, we have

$$\int \mathcal{X}(|V(0,t,x,v;b)| \le Q_0)\, dv = \int \mathcal{X}(|V(0,t,x,v;b)| \le Q_0,\ |v| \le Q)\, dv + \int \mathcal{X}(|V(0,t,x,v,b)| \le Q_0,\ |v| > Q)\, dv,$$

that induces a similar decomposition on Eq.(19). The first term is easily bounded by

$$\mathbb{E}\left[\int \mathcal{X}(|v| \le Q)\, dv\right] = \frac{4}{3}\,\pi\, Q^3.$$

To deal with the second one, we need the *Lévy*'s inequality (see e.g. Ref.[13])

$$\mathbb{E}\left[\max_{\tau\in[0,t]} |b(\tau)| \ge \lambda\right] \le 2\,\mathbb{E}\left[|b(t)| \ge \lambda\right], \quad \lambda \ge 0. \tag{20}$$

Therefore, by the definition of Q in Eq.(18), we can write

$$\mathbb{E}\left[\int \mathcal{X}(|V(0,t,x,v;b)| \le Q_0,\ |v| > Q)\right]$$

$$= \int \mathbb{E}\left[\sup_{\tau\in[0,t]} |b(\tau)| > \frac{|v|^\alpha}{40\,\sigma_0}\right] dv \le 2 \int \mathbb{E}\left[|b(t)| > \frac{|v|^\alpha}{40\,\sigma_0}\right] dv.$$

As $\mathbb{E}\left[|b(t)| \ge \lambda\right]$ is exponentially decreasing in λ^2, this last integral is uniformly bounded by a constant. In this sense we state that the high velocities, due to

the "degenerate" samples of the Brownian motion, are not very probable (this remark will be useful later on). In conclusion, we obtain that

$$\rho(t,x) \leq C\, Q^3,$$

where C depends only on f_0 and T.

To achieve the proof, it is sufficient to show that Q is bounded as function of T. From Eq.s (9)-(10), we deduce that

$$|\hat{V}(t,0,x,v;\hat{b})| \leq |v| + \int_0^t |E(\tau, \hat{X}(\tau,0,x,v;\hat{b}))|\, d\tau + \sigma\, |\hat{b}(t)| + \sigma\, \beta \int_0^t |\hat{b}(\tau)|\, d\tau$$

and, according to the condition on the sample $\hat{b}$ in the definition of $Q(t)$, we have

$$|\hat{V}(t,0,x,v;\hat{b})| \leq Q_0 + \int_0^t |E(\tau, \hat{X}(\tau,0,x,v;\hat{b}))|\, d\tau + \sigma\, \frac{|\hat{V}(t,0,x,v;\hat{b})|^\alpha}{40\,\sigma_0}\, (1+\beta\, t), \tag{21}$$

with

$$\sigma\, \frac{|\hat{V}(t,0,x,v;\hat{b})|^\alpha}{40\,\sigma_0} \leq \frac{Q^\alpha}{40}$$

for $\sigma \in (0, \sigma_0]$. Our task in the sequel is to estimate the term

$$\int_0^t |E(\tau, \hat{X}(\tau,0,x,v;\hat{b}))|\, d\tau.$$

We note that it follows from an immediate application of Eq.(7) that

$$\sup_{t\in[0,T]} \|E(t)\|_{L^\infty} \leq C_3\, Q^{\frac{4}{3}},$$

but this is not enough to conclude the desired estimate.

To improve this result and obtain a better power for Q, we develop arguments similar to those formulated by Schaeffer (see Ref.[12]) and Wollman (see Ref.[14]) for solving the VP equation.
Let $P = Q^\alpha$, $0<\alpha<1$, as introduced above, so that $0 < P < Q$. We set

$$\Delta T = \min\left\{T,\ \frac{P}{10\, C_3\, Q^{\frac{4}{3}}}\right\},$$

where C_3 is the same constant as in Eq.(7). If $\frac{T}{\Delta T}$ is an integer, let be $N = \frac{T}{\Delta T} - 1$, otherwise let be $N = [\frac{T}{\Delta T}]$; then, for $t_i = i\Delta T$, $i = 0, ..., N$ and $t_{N+1} = T$, we have

$$[0,T] = \bigcup_{i=0}^{N} [t_i, t_{i+1}], \quad |t_{i+1} - t_i| \leq \Delta T.$$

It follows that, for any $t \in [0,T]$,

$$\int_0^t |E(\tau, \hat{X}(\tau,0,x,v;\hat{b}))|\, d\tau \leq \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} |E(\tau, \hat{X}(\tau,0,x,v;\hat{b}))|\, d\tau. \tag{22}$$

For simplicity, we will denote by $(\hat{X}(t), \hat{V}(t))$ the sample of trajectory in the definition of $Q(t)$. From Eq.(4) and Eq.(11), thanks to a change of variables, we deduce

$$\begin{aligned}\int_{t_i}^{t_{i+1}} |E(\tau, \hat{X}(\tau))|\, d\tau &\leq \int_{t_i}^{t_{i+1}} d\tau \int\int f(\tau, y, w)\, |y - \hat{X}(\tau)|^{-2}\, dy\, dw \\ &= \int_{t_i}^{t_{i+1}} \mathbb{E}\Big[\int\int f(t_i, x, v)\, |X(\tau, t_i, x, v; b) - \hat{X}(\tau)|^{-2}\, dx\, dv\Big]\, d\tau \qquad (23) \\ &= \int_{t_i}^{t_{i+1}} d\tau \int\int\int f(t_i, x, v)\, |X(\tau, t_i, x, v; b) - \hat{X}(\tau)|^{-2}\, dx\, dv\, \mu(db).\end{aligned}$$

The domain of integration of the last integral, that is

$$S = [t_i, t_{i+1}] \times \mathbb{R}^6 \times \mathcal{C}_0^*([0, T]),$$

can be partitioned as follows,

$$S_1 = \left\{(\tau, x, v; b) \in S \;/\; |v| \leq P \text{ or } |v - \hat{V}(t_i)| \leq P, \sup_{\tau \in [t_i, t_{i+1}]} |b(\tau)| \leq \frac{P}{40\, \sigma_0}\right\},$$

$$S_2 = \left\{(\tau, x, v; b) \in S \;/\; |v| \leq P \text{ or } |v - \hat{V}(t_i)| \leq P, \sup_{\tau \in [t_i, t_{i+1}]} |b(\tau)| > \frac{P}{40\, \sigma_0}\right\},$$

$$S_3 = \Big\{(\tau, x, v; b) \in S \;/\; |v| > P \text{ and } |v - \hat{V}(t_i)| > P,\ |X(\tau, t_i, x, v; b) - \hat{X}(\tau)| \leq d\Big\},$$

$$S_4 = \Big\{(\tau, x, v; b) \in S \;/\; |v| > P \text{ and } |v - \hat{V}(t_i)| > P,$$

$$|X(\tau, t_i, x, v; b) - \hat{X}(\tau)| > d, \sup_{\tau \in [t_i, t_{i+1}]} |b(\tau)| \leq \frac{P}{40\, \sigma_0}\right\},$$

$$S_5 = \Big\{(\tau, x, v; b) \in S \;/\; |v| > P \text{ and } |v - \hat{V}(t_i)| > P,$$

$$|X(\tau, t_i, x, v; b) - \hat{X}(\tau)| > d, \sup_{\tau \in [t_i, t_{i+1}]} |b(\tau)| > \frac{P}{40\, \sigma_0}\right\},$$

where $d > 0$ is to be chosen later on, as a function of Q, P and ΔT.
Therefore, we derive that

$$\int_{t_i}^{t_{i+1}} |E(\tau, \hat{X}(\tau))|\, d\tau \leq I_1 + I_2 + I_3 + I_4 + I_5, \qquad (24)$$

with

$$I_j = \int_{S_j} f(t_i, x, v)\, |X(\tau, t_i, x, v; b) - \hat{X}(\tau)|^{-2}\, d\tau\, dx\, dv\, \mu(db), \quad j = 1, ..., 5.$$

We are going to estimate the contribution of each integral I_j, $j = 1, ..., 5$, separately. If $(\tau, x, v; b) \in S_1$ and $|v| \leq P$, then

$$|V(\tau, t_i, x, v; b)| \leq |v| + \int_{t_i}^{\tau} |E(s, X(s, t_i, x, v; b))|\, ds + \sigma\, |B(\tau, t_i, b)|$$

$$\leq P + C_3\, Q^{\frac{4}{3}}\, \Delta T + \sigma\, \frac{P}{40\, \sigma_0}\, (2 + \beta\, T) < 2\, P,$$

for $\sigma \in (0, \sigma_0]$ and $\beta \in (0, \beta_0]$, as σ_0 and β_0 have been usefully fixed.
Similarly, if $|v - \hat{V}(t_i)| \leq P$, then

$$|V(\tau, t_i, x, v; b) - \hat{V}(\tau)| \leq |v - \hat{V}(t_i)| + \int_{t_i}^{\tau} \left| E(s, X(s, t_i, x, v; b)) - E(s, \hat{X}(s)) \right| ds$$

$$+ \sigma\, |B(\tau, t_i, b)| + \sigma\, |\hat{B}(\tau, t_i, \hat{b})|$$

$$\leq P + 2\, C_3\, Q^{\frac{4}{3}}\, \Delta T + 2\, \sigma \left(\frac{P}{40\, \sigma_0} + \frac{Q^{\alpha}}{40\, \sigma_0} \right) (2 + \beta\, T) < 2\, P.$$

Thus, coming back to the initial variables, we have

$$I_1 \leq \int_{t_i}^{t_{i+1}} d\tau \int\int \frac{f(\tau, y, w)}{|y - \hat{X}(\tau)|^2} \left[\mathcal{X}(|w| < 2P) + \mathcal{X}(|w - \hat{V}(\tau)| < 2P) \right] dy\, dw$$

$$= 2 \int_{t_i}^{t_{i+1}} d\tau \int\int f(\tau, y, w)\, |y - \hat{X}(\tau)|^{-2}\, \mathcal{X}(|w| < 2P)\, dy\, dw.$$

Taking $l > 0$, the inner integral can also be decomposed into two terms, according to $|y - \hat{X}(\tau)| > l$ or $|y - \hat{X}(\tau)| \leq l$. For the first one, we use the *Hölder*'s inequality (for $p = \frac{5}{3}$) and the estimate in Eq.(6); for the second one, we apply directly the *maximum principle* as $\|f(t)\|_{L^\infty} \leq e^{\beta\, T}\, \|f_0\|_{L^\infty}$ and we write

$$I_1 \leq 2\, C \int_{t_i}^{t_{i+1}} d\tau \left[\left(\int_{|y - \hat{X}(\tau)| > l} \frac{1}{|y - \hat{X}(\tau)|^5}\, dy \right)^{\frac{2}{5}} + P^3 \int_{|y - \hat{X}(\tau)| \leq l} \frac{1}{|y - \hat{X}(\tau)|^2}\, dy \right],$$

to obtain

$$I_1 \leq C\, (P^3\, l + l^{-\frac{4}{5}})\, \Delta T.$$

Finally, we minimize over l (by choosing $l = P^{-5/3}$) and we conclude that

$$I_1 \leq C\, P^{\frac{4}{3}}\, \Delta T. \tag{25}$$

We deal with the integral I_2 in a similar way, that is

$$I_2 \leq \int_{t_i}^{t_{i+1}} d\tau \int\int\int \mathcal{X}\left(\sup_{\tau \in [t_i, t_{i+1}]} |b(\tau)| > \frac{P}{40\, \sigma_0} \right) \frac{f(t_i, x, v)}{|X(\tau, t_i, x, v; b) - \hat{X}(\tau)|^2}\, dx\, dv\, \mu(db),$$

so that we have

$$I_2^{\wedge} = \int_{t_i}^{t_{i+1}} d\tau \int_{|y-\hat{X}(\tau)|>l} dy \int f(\tau, y, w)\, |y - \hat{X}(\tau)|^{-2}\, dw \leq C_1\, l^{-\frac{4}{5}}\, \Delta T$$

and

$$\begin{aligned} I_2^{\vee} &= \int_{t_i}^{t_{i+1}} d\tau \int_{|y-\hat{X}(\tau)|\leq l} dy \int f(\tau, y, w)\, |y - \hat{X}(\tau)|^{-2}\, dw \\ &\leq \int_{t_i}^{t_{i+1}} d\tau \left(\int\int\int f(t_i, x, v)\, \frac{\mathcal{X}\Big(|X(\tau, t_i, x, v; b) - \hat{X}(\tau)| \leq l\Big)}{|X(\tau, t_i, x, v; b) - \hat{X}(\tau)|^{\frac{5}{2}}}\, dx\, dv\, \mu(db) \right)^{\frac{4}{5}} \\ &\qquad \times \left(\int\int\int f(t_i, x, v)\, \mathcal{X}\left(\sup_{\tau\in[t_i, t_{i+1}]} |b(\tau)| > \frac{P}{40\,\sigma_0} \right) dx\, dv\, \mu(db) \right)^{\frac{1}{5}}. \end{aligned}$$

The second part of the last integral can be rewritten as

$$(\|f(t_i)\|_{L^1})^{\frac{1}{5}} \left(\int \mathcal{X}\left(\sup_{\tau\in[t_i, t_{i+1}]} |b(\tau)| > \frac{P}{40\,\sigma_0} \right) \mu(db) \right)^{\frac{1}{5}}$$

and it is exponentially decreasing in P^2, while the first part becomes

$$\left(\int\int f(\tau, y, w)\, \frac{\mathcal{X}\Big(|y - \hat{X}(\tau)| \leq l\Big)}{|y - \hat{X}(\tau)|^{\frac{5}{2}}}\, dy\, dw \right)^{\frac{4}{5}} \leq C\, Q^{\frac{12}{5}}\, l^{\frac{4}{10}}.$$

Then, choosing $l = P^{-5/3}$, we conclude that

$$I_2^{\vee} \leq C\, P^{\frac{16+5\alpha}{15\alpha}}\, \exp\left\{-C\, P^2\right\}$$

and it can be neglected with respect to $I_2^{\wedge}$. In conclusion, we have

$$I_2 \leq C\, P^{\frac{4}{3}}\, \Delta T. \tag{26}$$

The bounds of I_3 and I_5 are almost immediate,

$$I_3 \leq \int_{t_i}^{t_{i+1}} d\tau \int_{|y-\hat{X}(\tau)|\leq d} dy \int f(s, y, w)\, \frac{1}{|y - \hat{X}(\tau)|^2}\, dw \leq C\, Q^3\, d\, \Delta T \tag{27}$$

and

$$I_5 \leq \int_{t_i}^{t_{i+1}} d\tau \int\int \frac{f(t_i, x, v)}{d^2}\, \mathbb{E}\left[\sup_{\tau\in[t_i, t_{i+1}]} |b(\tau)| > \frac{P}{40\,\sigma_0} \right] dx\, dv \leq \frac{\Delta T}{d^2}\, \exp\left\{-C\, P^2\right\}. \tag{28}$$

It remains I_4 to estimate, that we will treat as in Ref.s [12] and [14].
If $(s,x,v;b) \in S_4$, then

$$|V(\tau,t_i,x,v;b)-\hat{V}(\tau)| \geq |v-\hat{V}(t_i)|-\int_{t_i}^{\tau}\Big|E(s,X(s,t_i,x,v;b))-E(s,\hat{X}(s))\Big|\,ds$$

$$-\sigma\,|B(\tau,t_i,b)|-\sigma\,|\hat{B}(\tau,t_i,\hat{b})| \tag{29}$$

$$\geq P-2\,C_3\,Q^{\frac{4}{3}}\,\Delta T-2\,\sigma\left(\frac{P}{40\,\sigma_0}+\frac{Q^{\alpha}}{40\,\sigma_0}\right)(2+\beta\,T) \geq \frac{1}{2}\,P.$$

To expand on the time integral, we consider $(x,v;b)$ fixed in S_4 and we define

$$Z(\tau) = X(\tau,t_i,x,v;b)-\hat{X}(\tau),$$

so there exists $\tau_0=\tau_0(t_i,x,v;b)\in[t_i,t_{i+1}]$ such that, for all $\tau\in[t_i,t_{i+1}]$,

$$|Z(\tau_0)| \leq |Z(\tau)|.$$

Moreover, it follows that

$$\dot{Z}(\tau)=\dot{Z}(\tau_0)+\int_{\tau_0}^{\tau}\Big[E(s,X(s,t_i,x,v;b))-E(s,\hat{X}(s))\Big]\,ds+\sigma\left[B(\tau,\tau_0,b)-\hat{B}(\tau,\tau_0,\hat{b})\right].$$

Let be

$$\bar{Z}(\tau)=Z(\tau_0)+\dot{Z}(\tau_0)(\tau-\tau_0),$$

then $\bar{Z}(\tau_0)=Z(\tau_0)$ and $\dot{\bar{Z}}(\tau_0)=\dot{Z}(\tau_0)$, so that

$$Z(\tau)-\bar{Z}(\tau)=\int_{\tau_0}^{\tau}\Big[\dot{Z}(s)-\dot{\bar{Z}}(s)\Big]\,ds$$

$$=\int_{\tau_0}^{\tau}ds\int_{\tau_0}^{s}\Big[E(s',X(s',t_i,x,v;b))-E(s',\hat{X}(s'))\Big]\,ds'$$

$$+\sigma\int_{\tau_0}^{\tau}B(s,\tau_0,b)\,ds+\sigma\int_{\tau_0}^{\tau}\hat{B}(s,\tau_0,\hat{b})\,ds$$

$$\leq \frac{1}{2}\,(\tau-\tau_0)^2\,C_3\,Q^{\frac{4}{3}}+2\,\sigma\,\frac{P}{40\,\sigma_0}\,(\tau-\tau_0)$$

$$\leq \frac{C_3}{2}\,(\tau-\tau_0)\,Q^{\frac{4}{3}}\,\Delta T+\frac{1}{6}\,P(\tau-\tau_0)\leq\frac{1}{4}\,P\,(\tau-\tau_0),$$

and hence

$$|Z(\tau)| \geq |\bar{Z}(\tau)|-\frac{1}{4}\,P\,|\tau-\tau_0|. \tag{30}$$

To get a lower bound for $|\bar{Z}(\tau)|$, we note that if $\tau_0=t_i$ then $Z(\tau_0)\cdot\dot{Z}(\tau_0)\geq 0$, if $\tau_0=t_{i+1}$ then $Z(\tau_0)\cdot\dot{Z}(\tau_0)\leq 0$ and if $\tau_0\in(t_i,t_{i+1})$ then $Z(\tau_0)\cdot\dot{Z}(\tau_0)=0$; in all three cases, $(\tau-\tau_0)\,Z(\tau_0)\cdot\dot{Z}(\tau_0)\geq 0$ for $\tau\in[t_i,t_{i+1}]$, so that by Eq.(29) we deduce

$$|\bar{Z}(\tau)|^2=|Z(\tau_0)|^2+|\dot{Z}(\tau_0)\,(\tau-\tau_0)|^2+2\,(\tau-\tau_0)\,Z(\tau_0)\cdot\dot{Z}(\tau_0)$$

$$\geq |\dot{Z}(\tau_0)|^2 \, |\tau - \tau_0|^2 \geq \left(\frac{1}{2}\, P\right)^2 |\tau - \tau_0|^2.$$

Using this inequality in Eq.(30) yields

$$|Z(\tau)| \geq \frac{P}{2}\, |\tau - \tau_0|, \text{ for all } \tau \in [t_i, t_{i+1}]. \tag{31}$$

We shall use the last bound to estimate

$$\int_{t_i}^{t_{i+1}} |Z(\tau)|^{-2}\, \mathcal{X}_{S_4}(\tau, x, v; b)\, d\tau.$$

By construction of S_4, it is $\mathcal{X}_{S_4}(\tau, x, v; b) = 0$ if $|Z(\tau)| \leq d$. We define an application $\Sigma : [0, +\infty) \longrightarrow \mathbb{R}$ by

$$\Sigma(r) = \begin{cases} d^{-2} & \text{if } \ 0 \leq r \leq d \\ r^{-2} & \text{if } \ r > d \end{cases}$$

and we note, by Eq.(31) and since Σ is non-increasing, that

$$|Z(\tau)|^{-2}\, \mathcal{X}_{S_4}(\tau, x, v; b) \leq \Sigma(|Z(\tau)|) \leq \Sigma\left(\frac{P}{2}\, (\tau - \tau_0)\right).$$

Thus, by means of a short computation, we obtain

$$\int_{t_i}^{t_{i+1}} |Z(\tau)|^{-2}\, \mathcal{X}_{S_4}(\tau, x, v; b)\, d\tau \leq 2 \int_0^{+\infty} \Sigma\left(\frac{P}{2}\, \tau\right) d\tau = \frac{2}{Pd}.$$

Finally, by substituting in integral I_4, we have

$$\begin{aligned} I_4 &\leq \frac{2}{Pd} \int\int\int f(t_i, x, v)\, dx\, dv\, \mu(db) \\ &\leq \frac{2}{d}\, P^{-3} \int\int\int |v|^2\, f(t_i, x, v)\, dx\, dv\, \mu(db) \leq 2\, K\, P^{-3}\, d^{-1}, \end{aligned} \tag{32}$$

where K is a constant, only depending on f_0 and T, which bounds the *kinetic energy* (see e.g. Ref.s [4] and [5]).
Returning to Eq.(24), by combining the estimates above in Eq.s (25), (26), (27), (28) and (32), we get

$$\int_{t_i}^{t_{i+1}} |E(\tau, \hat{X}(\tau))|\, d\tau \leq C\left(P^{\frac{4}{3}}\, \Delta T + Q^3\, d\, \Delta T + d^{-2}\, \Delta T\, \exp\left\{-C\, P^2\right\} + P^{-3}\, d^{-1}\right).$$

By choosing $d = P^{-\frac{3}{2}}\, Q^{-\frac{3}{2}}\, \Delta T^{-\frac{1}{2}}$, for the definition of ΔT, it yields

$$\begin{aligned} \int_{t_i}^{t_{i+1}} |E(\tau, \hat{X}(\tau))|\, ds &\leq C\left(P^{\frac{4}{3}}\, \Delta T + Q^{\frac{3}{2}}\, P^{-\frac{3}{2}}\, \Delta T^{\frac{1}{2}} + Q^3\, \Delta T^2\, P^3\, \exp\left\{-C\, P^2\right\}\right) \\ &\leq C\left(P^{\frac{7}{3}}\, Q^{-\frac{4}{3}} + P^{-1}\, Q^{-\frac{5}{6}} + Q^{\frac{1}{3}}\, P^5\, \exp\left\{-C\, P^2\right\}\right). \end{aligned}$$

We note that, as $Q = P^{\frac{1}{\alpha}}$, the third term of the last line can be neglected with respect to the other ones, because it is exponentially decreasing.
So, we conclude that

$$\int_{t_i}^{t_{i+1}} |E(\tau, \hat{X}(\tau))|\, d\tau \leq C\, Q^{-\frac{4}{3}+\frac{7}{3}\,\alpha}.$$

After a short computation from Eq.(22), it yields

$$\int_0^t |E(\tau, \hat{X}(\tau, 0, x, v; \hat{b}))|\, d\tau \leq (N+1)\, C\, Q^{-\frac{4}{3}+\frac{7}{3}\,\alpha}. \tag{33}$$

We now observe that, if $T \leq P/10\, C_1\, Q^{\frac{4}{3}}$ then $\Delta T = T$ and $N = 0$, if $T > P/10\, C_1\, Q^{\frac{4}{3}}$ then $\Delta T = P/10\, C_1\, Q^{\frac{4}{3}} = Q^{\alpha-\frac{4}{3}}/10\, C_1$ and $N < \frac{T}{\Delta T} = 10\, C_1\, T\, Q^{\frac{4}{3}-\alpha}$.
In both cases, we have

$$N + 1 \leq 1 + C\, T\, Q^{\frac{4}{3}-\alpha} \leq C\, (T+1)\, Q^{\frac{4}{3}-\alpha},$$

and hence

$$\int_0^t |E(\tau, \hat{X}(\tau, 0, x, v; \hat{b}))|\, d\tau \leq C\, (T+1)\, Q^{\frac{4}{3}\,\alpha}. \tag{34}$$

Let $\gamma = \frac{4}{3}\,\alpha$ (if we choose $0 < \alpha < \frac{3}{4}$ then $\gamma < 1$). It follows from Eq.(21) that

$$|\hat{V}(t, 0, x, v; \hat{b})| \leq Q_0 + C\, (T+1)\, Q^{\gamma} + \frac{1}{40}\, Q^{\alpha} \leq Q_0 + C\, (T+1)\, Q^{\beta}, \tag{35}$$

where $\beta = \max\{\alpha, \gamma\}$ and $0 < \beta < 1$.
The upper bound in Eq.(35) is independent of the choice of the sample $(\hat{X}, \hat{V})$ and so, by taking the supremum, we conclude

$$Q \leq Q_0 + C\, (T+1)\, Q^{\beta} \leq [Q_0 + C\, (T+1)]^{\frac{1}{1-\beta}}\,.$$

This completes the proof.

References

[1] Bouchut, F., Existence and uniqueness of a global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions, *J. Funct. Anal.*, **111** (1993), no. 1, 239-258

[2] Bouchut, F., Smoothing effects for the non-linear Vlasov-Poisson-Fokker-Planck system, *J. Differential Equations*, **122** (1995), 225-238

[3] Castella, F., Effets dispersifs pour les équations de Vlasov et de Schrödinger, *Thése de doctorat*, Université Paris VI, Décembre 1997

[4] Horst, E., On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation I, *Math. Methods Appl. Sci.*, **3** (1981), 229-248

[5] Horst, E., On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation II, *Math. Methods Appl. Sci.*, **4** (1982), 19-32

[6] Horst, E., On the asymptotic growth of the solutions of the Vlasov-Poisson system, *Math. Methods Appl. Sci.*, **16** (1993), no. 2, 75-8

[7] Lions, P.L., Perthame, B., Propagation of moments and regularity for the three-dimensional Vlasov-Poisson system, *Invent. Math.*, **105** (1991), no. 2, 415-430

[8] Marchioro, C., Pulvirenti, M., *Hydrodynamics in two dimensions and Vortex Theory*, Commun. Math. Phys., **84** (1982), 483-503

[9] Mc Kean Jr H.P., Propagation of chaos for a class of non-linear parabolic equations, *Lectures in Differential Equations* (ed. A.K. Aziz), vol II. (1969), 177-194

[10] Neunzert, H., Pulvirenti, M., Triolo, L., On the Vlasov-Fokker-Planck equation, *Math. Meth. Appl. Sci.*, **6** (1984), 527-538

[11] Pfaffelmoser, K., Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data, *J. Differential Equations*, **95** (1992), no. 2, 281-303

[12] Schaeffer, J., Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions, *Comm. Partial Differential Equations*, **16** (1991), 1313-1335

[13] Simon, B., *Functional Integration and Quantum Physics*, Pure and Applied Mathematics, Academic Press, New York, 1979

[14] Wollman, S., Global-in-time solutions to the three-dimensional Vlasov-Poisson system, *J. Math. Anal. Appl.*, **176** (1993), no. 1, 76-91