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Large deviation exponential inequalities for supermartingales*

Xiequan Fan[†] Ion Grama[‡] Quansheng Liu[§]

Abstract

Let $(X_i, \mathcal{F}_i)_{i \geq 1}$ be a sequence of supermartingale differences and let $S_k = \sum_{i=1}^k X_i$. We give an exponential moment condition under which $\mathbb{P}(\max_{1 \leq k \leq n} S_k \geq n) = O(\exp\{-C_1 n^\alpha\})$, $n \rightarrow \infty$, where $\alpha \in (0, 1)$ is given and $C_1 > 0$ is a constant. We also show that the power α is optimal under the given moment condition.

Keywords: Large deviation; martingales; exponential inequality; Bernstein type inequality.

AMS 2010 Subject Classification: 60F10; 60G42; 60E15.

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1 Introduction

Let $(X_i, \mathcal{F}_i)_{i \geq 1}$ be a sequence of martingale differences and let $S_k = \sum_{i=1}^k X_i$, $k \geq 1$. Under the Cramér condition $\sup_i \mathbb{E}e^{|X_i|} < \infty$, Lesigne and Volný [9] proved that

$$\mathbb{P}(S_n \geq n) = O(\exp\{-C_1 n^{\frac{1}{3}}\}), \quad n \rightarrow \infty, \quad (1.1)$$

for some constant $C_1 > 0$. Here and throughout the paper, for two functions f and g , we write $f(n) = O(g(n))$ if there exists a constant $C > 0$ such that $|f(n)| \leq C|g(n)|$ for all $n \geq 1$. Lesigne and Volný [9] also showed that the power $\frac{1}{3}$ in (1.1) is optimal even for stationary and ergodic sequence of martingale differences, in the sense that there exists a stationary and ergodic sequence of martingale differences $(X_i, \mathcal{F}_i)_{i \geq 1}$ such that $\mathbb{E}e^{|X_1|} < \infty$ and $\mathbb{P}(S_n \geq n) \geq \exp\{-C_2 n^{\frac{1}{3}}\}$ for some constant $C_2 > 0$ and infinitely many n 's. Liu and Watbled [10] proved that the power $\frac{1}{3}$ in (1.1) can be improved to 1 under the conditional Cramér condition $\sup_i \mathbb{E}(e^{|X_i|} | \mathcal{F}_{i-1}) \leq C_3$, for some constant C_3 . It is natural to ask under what condition

$$\mathbb{P}(S_n \geq n) = O(\exp\{-C_1 n^\alpha\}), \quad n \rightarrow \infty, \quad (1.2)$$

where $\alpha \in (0, 1)$ is given and $C_1 > 0$ is a constant. In this paper, we give some sufficient conditions in order that (1.2) holds for supermartingales $(S_k, \mathcal{F}_k)_{k \geq 1}$.

The paper is organized as follows. In Section 2, we present the main results. In Sections 3-5, we give the proofs of the main results.

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2 Main Results

Our first result is an extension of the bound (1.1) of Lesigne and Volný [9].

Theorem 2.1. *Let $\alpha \in (0, 1)$. Assume that $(X_i, \mathcal{F}_i)_{i \geq 1}$ is a sequence of supermartingale differences satisfying $\sup_i \mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1$ for some constant $C_1 \in (0, \infty)$. Then, for all $x > 0$,*

$$\mathbb{P} \left(\max_{1 \leq k \leq n} S_k \geq nx \right) \leq C(\alpha, x) \exp \left\{ - \left(\frac{x}{4} \right)^{2\alpha} n^\alpha \right\}, \quad (2.1)$$

where

$$C(\alpha, x) = 2 + 35C_1 \left(\frac{1}{x^{2\alpha} 16^{1-\alpha}} + \frac{1}{x^2} \left(\frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1-\alpha}{\alpha}} \right)$$

does not depend on n . In particular, with $x = 1$, it holds

$$\mathbb{P} \left(\max_{1 \leq k \leq n} S_k \geq n \right) = O \left(\exp \left\{ - \frac{1}{16} n^\alpha \right\} \right), \quad n \rightarrow \infty. \quad (2.2)$$

Moreover, the power α in (2.2) is optimal in the class of martingale differences: for each $\alpha \in (0, 1)$, there exists a sequence of martingale differences $(X_i, \mathcal{F}_i)_{i \geq 1}$ satisfying $\sup_i \mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} < \infty$ and

$$\mathbb{P} \left(\max_{1 \leq k \leq n} S_k \geq n \right) \geq \exp\{-3n^\alpha\}, \quad (2.3)$$

for all n large enough.

In fact, we shall prove that the power α in (2.2) is optimal even for stationary martingale difference sequences.

It is clear that when $\alpha = \frac{1}{3}$, the bound (2.2) implies the bound (1.1) of Lesigne and Volný.

Our second result shows that the moment condition $\sup_i \mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} < \infty$ in Theorem 2.1 can be relaxed to $\sup_i \mathbb{E} \exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} < \infty$, where $X_i^+ = \max\{X_i, 0\}$, if we add a constraint on the sum of conditional variances

$$\langle S \rangle_k = \sum_{i=1}^k \mathbb{E}(X_i^2 | \mathcal{F}_{i-1}).$$

Theorem 2.2. *Let $\alpha \in (0, 1)$. Assume that $(X_i, \mathcal{F}_i)_{i \geq 1}$ is a sequence of supermartingale differences satisfying $\sup_i \mathbb{E} \exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} \leq C_1$ for some constant $C_1 \in (0, \infty)$. Then, for all $x, v > 0$,*

$$\begin{aligned} \mathbb{P} (S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ \leq \exp \left\{ - \frac{x^2}{2(v^2 + \frac{1}{3}x^{2-\alpha})} \right\} + nC_1 \exp\{-x^\alpha\}. \end{aligned} \quad (2.4)$$

For bounded random variables, some inequalities closely related to (2.4) can be found in Freedman [5], Dedecker [1], Dzharparidze and van Zanten [3], Merlevède, Peligrad and Rio [11] and Delyon [2].

Adding a hypothesis on $\langle S \rangle_n$ to Theorem 2.2, we can easily obtain the following Bernstein type inequality which is similar to an inequality of Merlevède, Peligrad and Rio [12] for weakly dependent sequences.

Corollary 2.3. *Let $\alpha \in (0, 1)$. Assume that $(X_i, \mathcal{F}_i)_{i \geq 1}$ is a sequence of supermartingale differences satisfying $\sup_i \mathbb{E} \exp\{(X_i^+)^{1-\alpha}\} \leq C_1$ and $\mathbb{E} \exp\{(\frac{\langle S \rangle_n}{n})^{1-\alpha}\} \leq C_2$ for some constants $C_1, C_2 \in (0, \infty)$. Then, for all $x > 0$,*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx\right) \leq \exp\left\{-\frac{x^{1+\alpha}}{2(1+\frac{1}{3}x)}n^\alpha\right\} + (nC_1 + C_2)\exp\{-x^\alpha n^\alpha\}. \quad (2.5)$$

In particular, with $x = 1$, it holds

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq n\right) = O(\exp\{-Cn^\alpha\}), \quad n \rightarrow \infty, \quad (2.6)$$

where $C > 0$ is an absolute constant. Moreover, the power α in (2.6) is optimal for the class of martingale differences: for each $\alpha \in (0, 1)$, there exists a sequence of martingale differences $(X_i, \mathcal{F}_i)_{i \geq 1}$ satisfying $\sup_i \mathbb{E} \exp\{(X_i^+)^{1-\alpha}\} < \infty$, $\sup_n \mathbb{E} \exp\{(\frac{\langle S \rangle_n}{n})^{1-\alpha}\} < \infty$ and

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq n\right) \geq \exp\{-3n^\alpha\} \quad (2.7)$$

for all n large enough.

Actually, just as (2.2), the power α in (2.6) is optimal even for stationary martingale difference sequences.

In the i.i.d. case, the conditions of Corollary 2.3 can be weakened considerably, see Lanzinger and Stadtmüller [8] where it is shown that if $\mathbb{E} \exp\{(X_1^+)^{\alpha}\} < \infty$ with $\alpha \in (0, 1)$, then

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq n\right) = O(\exp\{-C_\alpha n^\alpha\}), \quad n \rightarrow \infty. \quad (2.8)$$

3 Proof of Theorem 2.1

We shall need the following refined version of the Azuma-Hoeffding inequality.

Lemma 3.1. *Assume that $(X_i, \mathcal{F}_i)_{i \geq 1}$ is a sequence of martingale differences satisfying $|X_i| \leq 1$ for all $i \geq 1$. Then, for all $x \geq 0$,*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq \exp\left\{-\frac{x^2}{2n}\right\}. \quad (3.1)$$

A proof can be found in Laib [7].

For the proof of Theorem 2.1, we use a truncating argument as in Lesigne and Volný [9]. Let $(X_i, \mathcal{F}_i)_{i \geq 1}$ be a sequence of supermartingale differences. Given $u > 0$, define

$$\begin{aligned} X'_i &= X_i \mathbf{1}_{\{|X_i| \leq u\}} - \mathbb{E}(X_i \mathbf{1}_{\{|X_i| \leq u\}} | \mathcal{F}_{i-1}), \\ X''_i &= X_i \mathbf{1}_{\{|X_i| > u\}} - \mathbb{E}(X_i \mathbf{1}_{\{|X_i| > u\}} | \mathcal{F}_{i-1}), \\ S'_k &= \sum_{i=1}^k X'_i, \quad S''_k = \sum_{i=1}^k X''_i, \quad S'''_k = \sum_{i=1}^k \mathbb{E}(X_i | \mathcal{F}_{i-1}). \end{aligned}$$

Then $(X'_i, \mathcal{F}_i)_{i \geq 1}$ and $(X''_i, \mathcal{F}_i)_{i \geq 1}$ are two martingale difference sequences and $S_k = S'_k + S''_k + S'''_k$. Let $t \in (0, 1)$. Since $S'''_k \leq 0$, for any $x > 0$,

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x\right) &\leq \mathbb{P}\left(\max_{1 \leq k \leq n} S'_k + S'''_k \geq xt\right) + \mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right) \\ &\leq \mathbb{P}\left(\max_{1 \leq k \leq n} S'_k \geq xt\right) + \mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right). \end{aligned} \quad (3.2)$$

Using Lemma 3.1 and the fact that $|X'_i| \leq 2u$, we have

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S'_k \geq xt\right) \leq \exp\left\{-\frac{x^2 t^2}{8nu^2}\right\}. \quad (3.3)$$

Let $F_i(x) = \mathbb{P}(|X_i| \geq x), x \geq 0$. Since $\mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1$, we obtain, for all $x \geq 0$,

$$F_i(x) \leq \exp\{-x^{\frac{2\alpha}{1-\alpha}}\} \mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1 \exp\{-x^{\frac{2\alpha}{1-\alpha}}\}.$$

Using the martingale maximal inequality (cf. e.g. p. 14 in [6]), we get

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right) \leq \frac{1}{x^2(1-t)^2} \sum_{i=1}^n \mathbb{E} X_i''^2. \quad (3.4)$$

It is easy to see that

$$\begin{aligned} \mathbb{E} X_i''^2 &= -\int_u^\infty t^2 dF_i(t) \\ &= u^2 F_i(u) + \int_u^\infty 2t F_i(t) dt \\ &\leq C_1 u^2 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\} + 2C_1 \int_u^\infty t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt. \end{aligned} \quad (3.5)$$

Notice that the function $g(t) = t^3 \exp\{-t^{\frac{2\alpha}{1-\alpha}}\}$ is decreasing in $[\beta, +\infty)$ and is increasing in $[0, \beta]$, where $\beta = \left(\frac{3(1-\alpha)}{2\alpha}\right)^{\frac{1-\alpha}{2\alpha}}$. If $0 < u < \beta$, we have

$$\begin{aligned} \int_u^\infty t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt &\leq \int_u^\beta t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt + \int_\beta^\infty t^{-2} t^3 \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt \\ &\leq \int_u^\beta t \exp\{-u^{\frac{2\alpha}{1-\alpha}}\} dt + \int_\beta^\infty t^{-2} \beta^3 \exp\{-\beta^{\frac{2\alpha}{1-\alpha}}\} dt \\ &\leq \frac{3}{2} \beta^2 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \end{aligned} \quad (3.6)$$

If $\beta \leq u$, we have

$$\begin{aligned} \int_u^\infty t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt &= \int_u^\infty t^{-2} t^3 \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt \\ &\leq \int_u^\infty t^{-2} u^3 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\} dt \\ &= u^2 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \end{aligned} \quad (3.7)$$

By (3.5), (3.6) and (3.7), we get

$$\mathbb{E} X_i''^2 \leq 3C_1(u^2 + \beta^2) \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \quad (3.8)$$

From (3.4), it follows that

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right) \leq \frac{3nC_1}{x^2(1-t)^2} (u^2 + \beta^2) \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \quad (3.9)$$

Combining (3.2), (3.3) and (3.9), we obtain

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq 2 \exp\left\{-\frac{x^2 t^2}{8nu^2}\right\} + \frac{3nC_1}{(1-t)^2} \left(\frac{u^2}{x^2} + \frac{\beta^2}{x^2}\right) \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}.$$

Taking $t = \frac{1}{\sqrt{2}}$ and $u = \left(\frac{x}{4\sqrt{n}}\right)^{1-\alpha}$, we get, for all $x > 0$,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq C_n(\alpha, x) \exp\left\{-\left(\frac{x^2}{16n}\right)^\alpha\right\},$$

where

$$C_n(\alpha, x) = 2 + 35nC_1 \left(\frac{1}{x^{2\alpha}(16n)^{1-\alpha}} + \frac{\beta^2}{x^2}\right).$$

Hence, for all $x > 0$,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx\right) \leq C(\alpha, x) \exp\left\{-\left(\frac{x}{4}\right)^{2\alpha} n^\alpha\right\},$$

where

$$C(\alpha, x) = 2 + 35C_1 \left(\frac{1}{x^{2\alpha}16^{1-\alpha}} + \frac{1}{x^2} \left(\frac{3(1-\alpha)}{2\alpha}\right)^{\frac{1-\alpha}{\alpha}}\right).$$

This completes the proof of the first assertion of Theorem 2.1.

Next, we prove that the power α in (2.2) is optimal by giving a stationary sequence of martingale differences satisfying (2.3). We proceed as in Lesigne and Volný ([9], p. 150). Take a positive random variable X such that

$$\mathbb{P}(X > x) = \frac{2e}{1 + x^{\frac{1+\alpha}{1-\alpha}}} \exp\left\{-x^{\frac{2\alpha}{1-\alpha}}\right\} \tag{3.10}$$

for all $x > 1$. Using the formula $\mathbb{E}f(X) = f(1) + \int_1^\infty f'(t)\mathbb{P}(X > t)dt$ for $f(t) = \exp\{t^{\frac{2\alpha}{1-\alpha}}\}$, $t \geq 1$, we obtain

$$\mathbb{E} \exp\{X^{\frac{2\alpha}{1-\alpha}}\} = e + \frac{4e\alpha}{1-\alpha} \int_1^\infty \frac{t^{\frac{3\alpha-1}{1-\alpha}}}{1+t^{\frac{1+\alpha}{1-\alpha}}} dt < \infty.$$

Assume that $(\xi_i)_{i \geq 1}$ are Rademacher random variables independent of X , i.e. $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2}$. Set $X_i = X\xi_i$, $\mathcal{F}_0 = \sigma(X)$ and $\mathcal{F}_i = \sigma(X, (\xi_k)_{k=1, \dots, i})$. Then, $(X_i, \mathcal{F}_i)_{i \geq 1}$ is a stationary sequence of martingale differences satisfying

$$\sup_i \mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} = \mathbb{E} \exp\{X^{\frac{2\alpha}{1-\alpha}}\} < \infty.$$

For $\beta \in (0, 1)$, it is easy to see that

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_i \geq n\right) \geq \mathbb{P}(S_n \geq n) \geq \mathbb{P}\left(\sum_{i=1}^n \xi_i \geq n^\beta\right) \mathbb{P}(X \geq n^{1-\beta}).$$

Since, for n large enough,

$$\mathbb{P}\left(\sum_{i=1}^n \xi_i \geq n^\beta\right) \geq \exp\{-n^{2\beta-1}\},$$

(cf. Corollary 3.5 in Lesigne and Volný [9]), we get, for n large enough,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_i \geq n\right) \geq \frac{2e}{1 + (n^{1-\beta})^{\frac{1+\alpha}{1-\alpha}}} \exp\left\{-n^{2\beta-1} - (n^{1-\beta})^{\frac{2\alpha}{1-\alpha}}\right\}. \tag{3.11}$$

Setting $2\beta - 1 = \alpha$, we obtain, for n large enough,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_i \geq n\right) \geq \frac{2e}{1 + n^{\frac{1+\alpha}{2}}} \exp\{-2n^\alpha\} \geq \exp\{-3n^\alpha\},$$

which proves (2.3). This ends the proof of Theorem 2.1.

4 Proof of Theorem 2.2

To prove Theorem 2.2, we need the following inequality.

Lemma 4.1 ([4], Remark 2.1). *Assume that $(X_i, \mathcal{F}_i)_{i \geq 1}$ are supermartingale differences satisfying $X_i \leq 1$ for all $i \geq 1$. Then, for all $x, v > 0$,*

$$\mathbb{P}(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}x)} \right\}. \quad (4.1)$$

Assume that $(X_i, \mathcal{F}_i)_{i \geq 1}$ are supermartingale differences. Given $u > 0$, set

$$X'_i = X_i \mathbf{1}_{\{X_i \leq u\}}, \quad X''_i = X_i \mathbf{1}_{\{X_i > u\}}, \quad S'_k = \sum_{i=1}^k X'_i \quad \text{and} \quad S''_k = \sum_{i=1}^k X''_i.$$

Then, $(X'_i, \mathcal{F}_i)_{i \geq 1}$ is also a sequence of supermartingale differences and $S_k = S'_k + S''_k$. Since $\langle S' \rangle_k \leq \langle S \rangle_k$, we deduce, for all $x, u, v > 0$,

$$\begin{aligned} & \mathbb{P}(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq \mathbb{P}(S'_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \quad + \mathbb{P}(S''_k \geq 0 \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq \mathbb{P}(S'_k \geq x \text{ and } \langle S' \rangle_k \leq v^2 \text{ for some } k \in [1, n]) + \mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq 0\right). \end{aligned} \quad (4.2)$$

Applying Lemma 4.1 to the supermartingale differences $(X'_i/u, \mathcal{F}_i)_{i \geq 1}$, we have, for all $x, u, v > 0$,

$$\mathbb{P}(S'_k \geq x \text{ and } \langle S' \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xu)} \right\}. \quad (4.3)$$

Using the exponential Markov's inequality and the condition $\mathbb{E} \exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} \leq C_1$, we get

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq 0\right) & \leq \sum_{i=1}^n \mathbb{P}(X_i > u) \\ & \leq \sum_{i=1}^n \mathbb{E} \exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}} - u^{\frac{\alpha}{1-\alpha}}\} \\ & \leq nC_1 \exp\{-u^{\frac{\alpha}{1-\alpha}}\}. \end{aligned} \quad (4.4)$$

Combining the inequalities (4.2), (4.3) and (4.4) together, we obtain, for all $x, u, v > 0$,

$$\begin{aligned} & \mathbb{P}(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xu)} \right\} + nC_1 \exp\{-u^{\frac{\alpha}{1-\alpha}}\}. \end{aligned} \quad (4.5)$$

Taking $u = x^{1-\alpha}$, we get, for all $x, v > 0$,

$$\begin{aligned} & \mathbb{P}(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}x^{2-\alpha})} \right\} + nC_1 \exp\{-x^\alpha\}. \end{aligned} \quad (4.6)$$

This completes the proof of Theorem 2.2.

5 Proof of Corollary 2.3.

To prove Corollary 2.3 we make use of Theorem 2.2. It is easy to see that

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx\right) &\leq \mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx, \langle S \rangle_n \leq nv^2\right) \\ &\quad + \mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx, \langle S \rangle_n > nv^2\right) \\ &\leq \mathbb{P}(S_k \geq nx \text{ and } \langle S \rangle_k \leq nv^2 \text{ for some } k \in [1, n]) \\ &\quad + \mathbb{P}(\langle S \rangle_n > nv^2). \end{aligned} \tag{5.1}$$

By Theorem 2.2, it follows that, for all $x, v > 0$,

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx\right) &\leq \exp\left\{-\frac{x^2}{2(n^{\alpha-1}v^2 + \frac{1}{3}x^{2-\alpha})}n^\alpha\right\} \\ &\quad + nC_1 \exp\{-x^\alpha n^\alpha\} + \mathbb{P}(\langle S \rangle_n > nv^2), \end{aligned}$$

Using the exponential Markov's inequality and the condition $\mathbb{E} \exp\left\{\left(\frac{\langle S \rangle_n}{n}\right)^{\frac{\alpha}{1-\alpha}}\right\} \leq C_2$, we get, for all $v > 0$,

$$\mathbb{P}(\langle S \rangle_n > nv^2) \leq \mathbb{E} \exp\left\{\left(\left(\frac{\langle S \rangle_n}{n}\right)^{\frac{\alpha}{1-\alpha}} - v^{\frac{2}{1-\alpha}}\right)\right\} \leq C_2 \exp\{-v^{\frac{2}{1-\alpha}}\}.$$

Taking $v = (nx)^{\frac{1-\alpha}{2}}$, we obtain, for all $x > 0$,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} X_k \geq nx\right) \leq \exp\left\{-\frac{x^{1+\alpha}}{2\left(1 + \frac{1}{3}x\right)}n^\alpha\right\} + (nC_1 + C_2) \exp\{-x^\alpha n^\alpha\},$$

which gives inequality (2.5).

Next, we prove that the power α in (2.6) is optimal. Let $(X_i, \mathcal{F}_i)_{i \geq 1}$ be the sequence of martingale differences constructed in the proof of the second assertion of Theorem 2.1. Then $\frac{\langle S \rangle_n}{n} = X^2$,

$$\sup_i \mathbb{E} \exp\left\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\right\} = \frac{1}{2} \mathbb{E} \exp\{X^{\frac{\alpha}{1-\alpha}}\} < \infty$$

and

$$\sup_n \mathbb{E} \exp\left\{\left(\frac{\langle S \rangle_n}{n}\right)^{\frac{\alpha}{1-\alpha}}\right\} = \mathbb{E} \exp\{X^{\frac{2\alpha}{1-\alpha}}\} < \infty.$$

Using the same argument as in the proof of Theorem 2.1, we obtain, for n large enough,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq n\right) \geq \exp\{-3n^\alpha\}.$$

This ends the proof of Corollary 2.3.

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