



**HAL**  
open science

# Large deviation exponential inequalities for supermartingales

Xiequan Fan, Ion Grama, Quansheng Liu

► **To cite this version:**

Xiequan Fan, Ion Grama, Quansheng Liu. Large deviation exponential inequalities for supermartingales. *Electronic Communications in Probability*, 2012, 17 (59), pp.1-8. hal-00905554

**HAL Id: hal-00905554**

**<https://hal.science/hal-00905554>**

Submitted on 18 Nov 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Large deviation exponential inequalities for supermartingales\*

Xiequan Fan<sup>†</sup>      Ion Grama<sup>‡</sup>      Quansheng Liu<sup>§</sup>

### Abstract

Let  $(X_i, \mathcal{F}_i)_{i \geq 1}$  be a sequence of supermartingale differences and let  $S_k = \sum_{i=1}^k X_i$ . We give an exponential moment condition under which  $\mathbb{P}(\max_{1 \leq k \leq n} S_k \geq n) = O(\exp\{-C_1 n^\alpha\})$ ,  $n \rightarrow \infty$ , where  $\alpha \in (0, 1)$  is given and  $C_1 > 0$  is a constant. We also show that the power  $\alpha$  is optimal under the given moment condition.

**Keywords:** Large deviation; martingales; exponential inequality; Bernstein type inequality.

**AMS 2010 Subject Classification:** 60F10; 60G42; 60E15.

Submitted to ECP on September 17, 2012, final version accepted on December 9, 2012.

## 1 Introduction

Let  $(X_i, \mathcal{F}_i)_{i \geq 1}$  be a sequence of martingale differences and let  $S_k = \sum_{i=1}^k X_i$ ,  $k \geq 1$ . Under the Cramér condition  $\sup_i \mathbb{E}e^{|X_i|} < \infty$ , Lesigne and Volný [9] proved that

$$\mathbb{P}(S_n \geq n) = O(\exp\{-C_1 n^{\frac{1}{3}}\}), \quad n \rightarrow \infty, \quad (1.1)$$

for some constant  $C_1 > 0$ . Here and throughout the paper, for two functions  $f$  and  $g$ , we write  $f(n) = O(g(n))$  if there exists a constant  $C > 0$  such that  $|f(n)| \leq C|g(n)|$  for all  $n \geq 1$ . Lesigne and Volný [9] also showed that the power  $\frac{1}{3}$  in (1.1) is optimal even for stationary and ergodic sequence of martingale differences, in the sense that there exists a stationary and ergodic sequence of martingale differences  $(X_i, \mathcal{F}_i)_{i \geq 1}$  such that  $\mathbb{E}e^{|X_1|} < \infty$  and  $\mathbb{P}(S_n \geq n) \geq \exp\{-C_2 n^{\frac{1}{3}}\}$  for some constant  $C_2 > 0$  and infinitely many  $n$ 's. Liu and Watbled [10] proved that the power  $\frac{1}{3}$  in (1.1) can be improved to 1 under the conditional Cramér condition  $\sup_i \mathbb{E}(e^{|X_i|} | \mathcal{F}_{i-1}) \leq C_3$ , for some constant  $C_3$ . It is natural to ask under what condition

$$\mathbb{P}(S_n \geq n) = O(\exp\{-C_1 n^\alpha\}), \quad n \rightarrow \infty, \quad (1.2)$$

where  $\alpha \in (0, 1)$  is given and  $C_1 > 0$  is a constant. In this paper, we give some sufficient conditions in order that (1.2) holds for supermartingales  $(S_k, \mathcal{F}_k)_{k \geq 1}$ .

The paper is organized as follows. In Section 2, we present the main results. In Sections 3-5, we give the proofs of the main results.

---

\*Université de Bretagne-Sud, LMBA, UMR CNRS 6205, Campus de Tohannic, 56017 Vannes, France.

<sup>†</sup>E-mail: fanxiequan@hotmail.com

<sup>‡</sup>E-mail: ion.grama@univ-ubs.fr

<sup>§</sup>E-mail: quansheng.liu@univ-ubs.fr

## 2 Main Results

Our first result is an extension of the bound (1.1) of Lesigne and Volný [9].

**Theorem 2.1.** *Let  $\alpha \in (0, 1)$ . Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  is a sequence of supermartingale differences satisfying  $\sup_i \mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1$  for some constant  $C_1 \in (0, \infty)$ . Then, for all  $x > 0$ ,*

$$\mathbb{P} \left( \max_{1 \leq k \leq n} S_k \geq nx \right) \leq C(\alpha, x) \exp \left\{ - \left( \frac{x}{4} \right)^{2\alpha} n^\alpha \right\}, \quad (2.1)$$

where

$$C(\alpha, x) = 2 + 35C_1 \left( \frac{1}{x^{2\alpha} 16^{1-\alpha}} + \frac{1}{x^2} \left( \frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1-\alpha}{\alpha}} \right)$$

does not depend on  $n$ . In particular, with  $x = 1$ , it holds

$$\mathbb{P} \left( \max_{1 \leq k \leq n} S_k \geq n \right) = O \left( \exp \left\{ - \frac{1}{16} n^\alpha \right\} \right), \quad n \rightarrow \infty. \quad (2.2)$$

Moreover, the power  $\alpha$  in (2.2) is optimal in the class of martingale differences: for each  $\alpha \in (0, 1)$ , there exists a sequence of martingale differences  $(X_i, \mathcal{F}_i)_{i \geq 1}$  satisfying  $\sup_i \mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} < \infty$  and

$$\mathbb{P} \left( \max_{1 \leq k \leq n} S_k \geq n \right) \geq \exp\{-3n^\alpha\}, \quad (2.3)$$

for all  $n$  large enough.

In fact, we shall prove that the power  $\alpha$  in (2.2) is optimal even for stationary martingale difference sequences.

It is clear that when  $\alpha = \frac{1}{3}$ , the bound (2.2) implies the bound (1.1) of Lesigne and Volný.

Our second result shows that the moment condition  $\sup_i \mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} < \infty$  in Theorem 2.1 can be relaxed to  $\sup_i \mathbb{E} \exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} < \infty$ , where  $X_i^+ = \max\{X_i, 0\}$ , if we add a constraint on the sum of conditional variances

$$\langle S \rangle_k = \sum_{i=1}^k \mathbb{E}(X_i^2 | \mathcal{F}_{i-1}).$$

**Theorem 2.2.** *Let  $\alpha \in (0, 1)$ . Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  is a sequence of supermartingale differences satisfying  $\sup_i \mathbb{E} \exp\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\} \leq C_1$  for some constant  $C_1 \in (0, \infty)$ . Then, for all  $x, v > 0$ ,*

$$\begin{aligned} \mathbb{P} (S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ \leq \exp \left\{ - \frac{x^2}{2(v^2 + \frac{1}{3}x^{2-\alpha})} \right\} + nC_1 \exp\{-x^\alpha\}. \end{aligned} \quad (2.4)$$

For bounded random variables, some inequalities closely related to (2.4) can be found in Freedman [5], Dedecker [1], Dzharparidze and van Zanten [3], Merlevède, Peligrad and Rio [11] and Delyon [2].

Adding a hypothesis on  $\langle S \rangle_n$  to Theorem 2.2, we can easily obtain the following Bernstein type inequality which is similar to an inequality of Merlevède, Peligrad and Rio [12] for weakly dependent sequences.

**Corollary 2.3.** *Let  $\alpha \in (0, 1)$ . Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  is a sequence of supermartingale differences satisfying  $\sup_i \mathbb{E} \exp\{(X_i^+)^{1-\alpha}\} \leq C_1$  and  $\mathbb{E} \exp\{(\frac{\langle S \rangle_n}{n})^{1-\alpha}\} \leq C_2$  for some constants  $C_1, C_2 \in (0, \infty)$ . Then, for all  $x > 0$ ,*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx\right) \leq \exp\left\{-\frac{x^{1+\alpha}}{2(1+\frac{1}{3}x)}n^\alpha\right\} + (nC_1 + C_2)\exp\{-x^\alpha n^\alpha\}. \tag{2.5}$$

In particular, with  $x = 1$ , it holds

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq n\right) = O(\exp\{-Cn^\alpha\}), \quad n \rightarrow \infty, \tag{2.6}$$

where  $C > 0$  is an absolute constant. Moreover, the power  $\alpha$  in (2.6) is optimal for the class of martingale differences: for each  $\alpha \in (0, 1)$ , there exists a sequence of martingale differences  $(X_i, \mathcal{F}_i)_{i \geq 1}$  satisfying  $\sup_i \mathbb{E} \exp\{(X_i^+)^{1-\alpha}\} < \infty$ ,  $\sup_n \mathbb{E} \exp\{(\frac{\langle S \rangle_n}{n})^{1-\alpha}\} < \infty$  and

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq n\right) \geq \exp\{-3n^\alpha\} \tag{2.7}$$

for all  $n$  large enough.

Actually, just as (2.2), the power  $\alpha$  in (2.6) is optimal even for stationary martingale difference sequences.

In the i.i.d. case, the conditions of Corollary 2.3 can be weakened considerably, see Lanzinger and Stadtmüller [8] where it is shown that if  $\mathbb{E} \exp\{(X_1^+)^{\alpha}\} < \infty$  with  $\alpha \in (0, 1)$ , then

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq n\right) = O(\exp\{-C_\alpha n^\alpha\}), \quad n \rightarrow \infty. \tag{2.8}$$

### 3 Proof of Theorem 2.1

We shall need the following refined version of the Azuma-Hoeffding inequality.

**Lemma 3.1.** *Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  is a sequence of martingale differences satisfying  $|X_i| \leq 1$  for all  $i \geq 1$ . Then, for all  $x \geq 0$ ,*

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq \exp\left\{-\frac{x^2}{2n}\right\}. \tag{3.1}$$

A proof can be found in Laib [7].

For the proof of Theorem 2.1, we use a truncating argument as in Lesigne and Volný [9]. Let  $(X_i, \mathcal{F}_i)_{i \geq 1}$  be a sequence of supermartingale differences. Given  $u > 0$ , define

$$\begin{aligned} X'_i &= X_i \mathbf{1}_{\{|X_i| \leq u\}} - \mathbb{E}(X_i \mathbf{1}_{\{|X_i| \leq u\}} | \mathcal{F}_{i-1}), \\ X''_i &= X_i \mathbf{1}_{\{|X_i| > u\}} - \mathbb{E}(X_i \mathbf{1}_{\{|X_i| > u\}} | \mathcal{F}_{i-1}), \\ S'_k &= \sum_{i=1}^k X'_i, \quad S''_k = \sum_{i=1}^k X''_i, \quad S'''_k = \sum_{i=1}^k \mathbb{E}(X_i | \mathcal{F}_{i-1}). \end{aligned}$$

Then  $(X'_i, \mathcal{F}_i)_{i \geq 1}$  and  $(X''_i, \mathcal{F}_i)_{i \geq 1}$  are two martingale difference sequences and  $S_k = S'_k + S''_k + S'''_k$ . Let  $t \in (0, 1)$ . Since  $S'''_k \leq 0$ , for any  $x > 0$ ,

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x\right) &\leq \mathbb{P}\left(\max_{1 \leq k \leq n} S'_k + S'''_k \geq xt\right) + \mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right) \\ &\leq \mathbb{P}\left(\max_{1 \leq k \leq n} S'_k \geq xt\right) + \mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right). \end{aligned} \tag{3.2}$$

Using Lemma 3.1 and the fact that  $|X'_i| \leq 2u$ , we have

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S'_k \geq xt\right) \leq \exp\left\{-\frac{x^2 t^2}{8nu^2}\right\}. \quad (3.3)$$

Let  $F_i(x) = \mathbb{P}(|X_i| \geq x), x \geq 0$ . Since  $\mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1$ , we obtain, for all  $x \geq 0$ ,

$$F_i(x) \leq \exp\{-x^{\frac{2\alpha}{1-\alpha}}\} \mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} \leq C_1 \exp\{-x^{\frac{2\alpha}{1-\alpha}}\}.$$

Using the martingale maximal inequality (cf. e.g. p. 14 in [6]), we get

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right) \leq \frac{1}{x^2(1-t)^2} \sum_{i=1}^n \mathbb{E} X_i''^2. \quad (3.4)$$

It is easy to see that

$$\begin{aligned} \mathbb{E} X_i''^2 &= -\int_u^\infty t^2 dF_i(t) \\ &= u^2 F_i(u) + \int_u^\infty 2t F_i(t) dt \\ &\leq C_1 u^2 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\} + 2C_1 \int_u^\infty t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt. \end{aligned} \quad (3.5)$$

Notice that the function  $g(t) = t^3 \exp\{-t^{\frac{2\alpha}{1-\alpha}}\}$  is decreasing in  $[\beta, +\infty)$  and is increasing in  $[0, \beta]$ , where  $\beta = \left(\frac{3(1-\alpha)}{2\alpha}\right)^{\frac{1-\alpha}{2\alpha}}$ . If  $0 < u < \beta$ , we have

$$\begin{aligned} \int_u^\infty t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt &\leq \int_u^\beta t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt + \int_\beta^\infty t^{-2} t^3 \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt \\ &\leq \int_u^\beta t \exp\{-u^{\frac{2\alpha}{1-\alpha}}\} dt + \int_\beta^\infty t^{-2} \beta^3 \exp\{-\beta^{\frac{2\alpha}{1-\alpha}}\} dt \\ &\leq \frac{3}{2} \beta^2 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \end{aligned} \quad (3.6)$$

If  $\beta \leq u$ , we have

$$\begin{aligned} \int_u^\infty t \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt &= \int_u^\infty t^{-2} t^3 \exp\{-t^{\frac{2\alpha}{1-\alpha}}\} dt \\ &\leq \int_u^\infty t^{-2} u^3 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\} dt \\ &= u^2 \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \end{aligned} \quad (3.7)$$

By (3.5), (3.6) and (3.7), we get

$$\mathbb{E} X_i''^2 \leq 3C_1(u^2 + \beta^2) \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \quad (3.8)$$

From (3.4), it follows that

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right) \leq \frac{3nC_1}{x^2(1-t)^2} (u^2 + \beta^2) \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}. \quad (3.9)$$

Combining (3.2), (3.3) and (3.9), we obtain

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq 2 \exp\left\{-\frac{x^2 t^2}{8nu^2}\right\} + \frac{3nC_1}{(1-t)^2} \left(\frac{u^2}{x^2} + \frac{\beta^2}{x^2}\right) \exp\{-u^{\frac{2\alpha}{1-\alpha}}\}.$$

Taking  $t = \frac{1}{\sqrt{2}}$  and  $u = \left(\frac{x}{4\sqrt{n}}\right)^{1-\alpha}$ , we get, for all  $x > 0$ ,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq C_n(\alpha, x) \exp\left\{-\left(\frac{x^2}{16n}\right)^\alpha\right\},$$

where

$$C_n(\alpha, x) = 2 + 35nC_1 \left(\frac{1}{x^{2\alpha}(16n)^{1-\alpha}} + \frac{\beta^2}{x^2}\right).$$

Hence, for all  $x > 0$ ,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx\right) \leq C(\alpha, x) \exp\left\{-\left(\frac{x}{4}\right)^{2\alpha} n^\alpha\right\},$$

where

$$C(\alpha, x) = 2 + 35C_1 \left(\frac{1}{x^{2\alpha}16^{1-\alpha}} + \frac{1}{x^2} \left(\frac{3(1-\alpha)}{2\alpha}\right)^{\frac{1-\alpha}{\alpha}}\right).$$

This completes the proof of the first assertion of Theorem 2.1.

Next, we prove that the power  $\alpha$  in (2.2) is optimal by giving a stationary sequence of martingale differences satisfying (2.3). We proceed as in Lesigne and Volný ([9], p. 150). Take a positive random variable  $X$  such that

$$\mathbb{P}(X > x) = \frac{2e}{1 + x^{\frac{1+\alpha}{1-\alpha}}} \exp\left\{-x^{\frac{2\alpha}{1-\alpha}}\right\} \tag{3.10}$$

for all  $x > 1$ . Using the formula  $\mathbb{E}f(X) = f(1) + \int_1^\infty f'(t)\mathbb{P}(X > t)dt$  for  $f(t) = \exp\{t^{\frac{2\alpha}{1-\alpha}}\}$ ,  $t \geq 1$ , we obtain

$$\mathbb{E} \exp\{X^{\frac{2\alpha}{1-\alpha}}\} = e + \frac{4e\alpha}{1-\alpha} \int_1^\infty \frac{t^{\frac{3\alpha-1}{1-\alpha}}}{1+t^{\frac{1+\alpha}{1-\alpha}}} dt < \infty.$$

Assume that  $(\xi_i)_{i \geq 1}$  are Rademacher random variables independent of  $X$ , i.e.  $\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = \frac{1}{2}$ . Set  $X_i = X\xi_i$ ,  $\mathcal{F}_0 = \sigma(X)$  and  $\mathcal{F}_i = \sigma(X, (\xi_k)_{k=1, \dots, i})$ . Then,  $(X_i, \mathcal{F}_i)_{i \geq 1}$  is a stationary sequence of martingale differences satisfying

$$\sup_i \mathbb{E} \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} = \mathbb{E} \exp\{X^{\frac{2\alpha}{1-\alpha}}\} < \infty.$$

For  $\beta \in (0, 1)$ , it is easy to see that

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_i \geq n\right) \geq \mathbb{P}(S_n \geq n) \geq \mathbb{P}\left(\sum_{i=1}^n \xi_i \geq n^\beta\right) \mathbb{P}(X \geq n^{1-\beta}).$$

Since, for  $n$  large enough,

$$\mathbb{P}\left(\sum_{i=1}^n \xi_i \geq n^\beta\right) \geq \exp\{-n^{2\beta-1}\},$$

(cf. Corollary 3.5 in Lesigne and Volný [9]), we get, for  $n$  large enough,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_i \geq n\right) \geq \frac{2e}{1 + (n^{1-\beta})^{\frac{1+\alpha}{1-\alpha}}} \exp\left\{-n^{2\beta-1} - (n^{1-\beta})^{\frac{2\alpha}{1-\alpha}}\right\}. \tag{3.11}$$

Setting  $2\beta - 1 = \alpha$ , we obtain, for  $n$  large enough,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_i \geq n\right) \geq \frac{2e}{1 + n^{\frac{1+\alpha}{2}}} \exp\{-2n^\alpha\} \geq \exp\{-3n^\alpha\},$$

which proves (2.3). This ends the proof of Theorem 2.1.

#### 4 Proof of Theorem 2.2

To prove Theorem 2.2, we need the following inequality.

**Lemma 4.1** ([4], Remark 2.1). *Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  are supermartingale differences satisfying  $X_i \leq 1$  for all  $i \geq 1$ . Then, for all  $x, v > 0$ ,*

$$\mathbb{P}(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}x)} \right\}. \quad (4.1)$$

Assume that  $(X_i, \mathcal{F}_i)_{i \geq 1}$  are supermartingale differences. Given  $u > 0$ , set

$$X'_i = X_i \mathbf{1}_{\{X_i \leq u\}}, \quad X''_i = X_i \mathbf{1}_{\{X_i > u\}}, \quad S'_k = \sum_{i=1}^k X'_i \quad \text{and} \quad S''_k = \sum_{i=1}^k X''_i.$$

Then,  $(X'_i, \mathcal{F}_i)_{i \geq 1}$  is also a sequence of supermartingale differences and  $S_k = S'_k + S''_k$ . Since  $\langle S' \rangle_k \leq \langle S \rangle_k$ , we deduce, for all  $x, u, v > 0$ ,

$$\begin{aligned} & \mathbb{P}(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq \mathbb{P}(S'_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \quad + \mathbb{P}(S''_k \geq 0 \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq \mathbb{P}(S'_k \geq x \text{ and } \langle S' \rangle_k \leq v^2 \text{ for some } k \in [1, n]) + \mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq 0\right). \end{aligned} \quad (4.2)$$

Applying Lemma 4.1 to the supermartingale differences  $(X'_i/u, \mathcal{F}_i)_{i \geq 1}$ , we have, for all  $x, u, v > 0$ ,

$$\mathbb{P}(S'_k \geq x \text{ and } \langle S' \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xu)} \right\}. \quad (4.3)$$

Using the exponential Markov's inequality and the condition  $\mathbb{E} \exp\{(X_i^+)^{\frac{1}{1-\alpha}}\} \leq C_1$ , we get

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq 0\right) & \leq \sum_{i=1}^n \mathbb{P}(X_i > u) \\ & \leq \sum_{i=1}^n \mathbb{E} \exp\{(X_i^+)^{\frac{1}{1-\alpha}} - u^{\frac{1}{1-\alpha}}\} \\ & \leq nC_1 \exp\{-u^{\frac{\alpha}{1-\alpha}}\}. \end{aligned} \quad (4.4)$$

Combining the inequalities (4.2), (4.3) and (4.4) together, we obtain, for all  $x, u, v > 0$ ,

$$\begin{aligned} & \mathbb{P}(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xu)} \right\} + nC_1 \exp\{-u^{\frac{\alpha}{1-\alpha}}\}. \end{aligned} \quad (4.5)$$

Taking  $u = x^{1-\alpha}$ , we get, for all  $x, v > 0$ ,

$$\begin{aligned} & \mathbb{P}(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\ & \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}x^{2-\alpha})} \right\} + nC_1 \exp\{-x^\alpha\}. \end{aligned} \quad (4.6)$$

This completes the proof of Theorem 2.2.

## 5 Proof of Corollary 2.3.

To prove Corollary 2.3 we make use of Theorem 2.2. It is easy to see that

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx\right) &\leq \mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx, \langle S \rangle_n \leq nv^2\right) \\ &\quad + \mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx, \langle S \rangle_n > nv^2\right) \\ &\leq \mathbb{P}(S_k \geq nx \text{ and } \langle S \rangle_k \leq nv^2 \text{ for some } k \in [1, n]) \\ &\quad + \mathbb{P}(\langle S \rangle_n > nv^2). \end{aligned} \quad (5.1)$$

By Theorem 2.2, it follows that, for all  $x, v > 0$ ,

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq nx\right) &\leq \exp\left\{-\frac{x^2}{2(n^{\alpha-1}v^2 + \frac{1}{3}x^{2-\alpha})}n^\alpha\right\} \\ &\quad + nC_1 \exp\{-x^\alpha n^\alpha\} + \mathbb{P}(\langle S \rangle_n > nv^2), \end{aligned}$$

Using the exponential Markov's inequality and the condition  $\mathbb{E} \exp\left\{\left(\frac{\langle S \rangle_n}{n}\right)^{\frac{\alpha}{1-\alpha}}\right\} \leq C_2$ , we get, for all  $v > 0$ ,

$$\mathbb{P}(\langle S \rangle_n > nv^2) \leq \mathbb{E} \exp\left\{\left(\left(\frac{\langle S \rangle_n}{n}\right)^{\frac{\alpha}{1-\alpha}} - v^{\frac{2}{1-\alpha}}\right)\right\} \leq C_2 \exp\{-v^{\frac{2}{1-\alpha}}\}.$$

Taking  $v = (nx)^{\frac{1-\alpha}{2}}$ , we obtain, for all  $x > 0$ ,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} X_k \geq nx\right) \leq \exp\left\{-\frac{x^{1+\alpha}}{2\left(1 + \frac{1}{3}x\right)}n^\alpha\right\} + (nC_1 + C_2) \exp\{-x^\alpha n^\alpha\},$$

which gives inequality (2.5).

Next, we prove that the power  $\alpha$  in (2.6) is optimal. Let  $(X_i, \mathcal{F}_i)_{i \geq 1}$  be the sequence of martingale differences constructed in the proof of the second assertion of Theorem 2.1. Then  $\frac{\langle S \rangle_n}{n} = X^2$ ,

$$\sup_i \mathbb{E} \exp\left\{(X_i^+)^{\frac{\alpha}{1-\alpha}}\right\} = \frac{1}{2} \mathbb{E} \exp\{X^{\frac{\alpha}{1-\alpha}}\} < \infty$$

and

$$\sup_n \mathbb{E} \exp\left\{\left(\frac{\langle S \rangle_n}{n}\right)^{\frac{\alpha}{1-\alpha}}\right\} = \mathbb{E} \exp\{X^{\frac{2\alpha}{1-\alpha}}\} < \infty.$$

Using the same argument as in the proof of Theorem 2.1, we obtain, for  $n$  large enough,

$$\mathbb{P}\left(\max_{1 \leq k \leq n} S_k \geq n\right) \geq \exp\{-3n^\alpha\}.$$

This ends the proof of Corollary 2.3.

## References

- [1] J., Dedecker. Exponential inequalities and functional central limit theorems for random fields. *ESAIM Probab. Statist.* **5** (2001), 77–104.
- [2] B., Delyon. Exponential inequalities for sums of weakly dependent variables. *Electron. J. Probab.* **14** (2009), 752–779. MR-2495559
- [3] K., Dzhaparidze and J. H., van Zanten. On Bernstein-type inequalities for martingales. *Stochastic Process. Appl.* **93** (2001), 109–117. MR-1819486

- [4] X., Fan, I., Grama and Q., Liu. Hoeffding's inequality for supermartingales. *Stochastic Process. Appl.* **122** (2012), 3545–3559. MR-2956116
- [5] D. A., Freedman. On tail probabilities for martingales. *Ann. Probab.* **3** (1975), 100–118. MR-0380971
- [6] P., Hall and C. C., Heyde. *Martingale Limit Theory and Its Application*, Academic Press, 1980, 81–96.
- [7] N., Laib. Exponential-type inequalities for martingale difference sequences. Application to nonparametric regression estimation. *Commun. Statist.-Theory. Methods*, **28** (1999), 1565–1576. MR-1707103
- [8] H., Lanzinger and U., Stadtmüller. Maxima of increments of partial sums for certain subexponential distributions. *Stochastic Process. Appl.*, **86** (2000), 307–322. MR-1741810
- [9] E., Lesigne and D., Volný. Large deviations for martingales. *Stochastic Process. Appl.* **96** (2001), 143–159. MR-1856684
- [10] Q., Liu and F., Watbled. Exponential inequalities for martingales and asymptotic properties of the free energy of directed polymers in a random environment. *Stochastic Process. Appl.* **119** (2009), 3101–3132. MR-2568267
- [11] F., Merlevède, M., Peligrad and E., Rio. Bernstein inequality and moderate deviations under strong mixing conditions. *IMS Collections. High Dimensional Probability* **5** (2009), 273–292. MR-2797953
- [12] F., Merlevède, M., Peligrad and E., Rio. A Bernstein type inequality and moderate deviations for weakly dependent sequences. *Probab. Theory Relat. Fields* **151** (2011), 435–474. MR-2851689

**Acknowledgments.** We would like to thank the two referees for their helpful remarks and suggestions.