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Large deviation exponential inequalities for supermartingales

Xiequan Fan†  Ion Grama‡  Quansheng Liu§

Abstract
Let \((X_i, F_i)_{i \geq 1}\) be a sequence of supermartingale differences and let \(S_k = \sum_{i=1}^{k} X_i\).
We give an exponential moment condition under which \(P(\max_{1 \leq k \leq n} S_k \geq n) = O(\exp\{-C_1 n^{\alpha}\})\), \(n \to \infty\), where \(\alpha \in (0, 1)\) is given and \(C_1 > 0\) is a constant. We also show that the power \(\alpha\) is optimal under the given moment condition.

Keywords: Large deviation; martingales; exponential inequality; Bernstein type inequality.
AMS 2010 Subject Classification: 60F10; 60G42; 60E15.

1 Introduction
Let \((X_i, F_i)_{i \geq 1}\) be a sequence of martingale differences and let \(S_k = \sum_{i=1}^{k} X_i, k \geq 1\).
Under the Cramér condition \(\sup_i E(e^{\|X_i\|}) < \infty\), Lesigne and Volný [9] proved that
\[
P(S_n \geq n) = O(\exp\{-C_1 n^{\frac{1}{3}}\}), \quad n \to \infty,
\]
for some constant \(C_1 > 0\). Here and throughout the paper, for two functions \(f\) and \(g\), we write \(f(n) = O(g(n))\) if there exists a constant \(C > 0\) such that \(|f(n)| \leq C|g(n)|\) for all \(n \geq 1\). Lesigne and Volný [9] also showed that the power \(\frac{1}{3}\) in (1.1) is optimal even for stationary and ergodic sequence of martingale differences, in the sense that there exists a stationary and ergodic sequence of martingale differences \((X_i, F_i)_{i \geq 1}\) such that \(E(e^{\|X_i\|}) < \infty\) and \(P(S_n \geq n) \geq \exp\{-C_2 n^{\frac{1}{3}}\}\) for some constant \(C_2 > 0\) and infinitely many \(n's\). Liu and Watbled [10] proved that the power \(\frac{1}{3}\) in (1.1) can be improved to 1 under the conditional Cramér condition \(\sup_i E(e^{\|X_i\|}|F_{i-1}) \leq C_3\), for some constant \(C_3\). It is natural to ask under what condition
\[
P(S_n \geq n) = O(\exp\{-C_1 n^{\alpha}\}), \quad n \to \infty,
\]
where \(\alpha \in (0, 1)\) is given and \(C_1 > 0\) is a constant. In this paper, we give some sufficient conditions in order that (1.2) holds for supermartingales \((S_k, F_k)_{k \geq 1}\).

The paper is organized as follows. In Section 2 we present the main results. In Sections 3-5 we give the proofs of the main results.

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2 Main Results

Our first result is an extension of the bound (1.1) of Lesigne and Volný [9].

**Theorem 2.1.** Let \( \alpha \in (0, 1) \). Assume that \((X_i, F_i)_{i \geq 1}\) is a sequence of supermartingale differences satisfying 
\[
\sup_i \mathbb{E} \exp \left\{ |X_i|^2 \alpha \right\} \leq C_1
\]
for some constant \( C_1 \in (0, \infty) \). Then, for all \( x > 0 \),
\[
P \left( \max_{1 \leq k \leq n} S_k \geq nx \right) \leq C(\alpha, x) \exp \left\{ - \frac{x^2}{4} n^\alpha \right\},
\]
where
\[
C(\alpha, x) = 2 + 35C_1 \left( \frac{1}{x^{2\alpha} 16^{1-\alpha}} + \frac{1}{x^2} \left( \frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1-\alpha}{\alpha}} \right)
\]
does not depend on \( n \). In particular, with \( x = 1 \), it holds
\[
P \left( \max_{1 \leq k \leq n} S_k \geq n \right) = O \left( \exp \left\{ -\frac{1}{16} n^\alpha \right\} \right), \quad n \to \infty.
\]

Moreover, the power \( \alpha \) in (2.2) is optimal in the class of martingale differences: for each \( \alpha \in (0, 1) \), there exists a sequence of martingale differences \((X_i, F_i)_{i \geq 1}\) satisfying
\[
\sup_i \mathbb{E} \exp \left\{ |X_i|^2 \alpha \right\} < \infty \text{ and }
\]
\[
P \left( \max_{1 \leq k \leq n} S_k \geq n \right) \geq \exp\{-3n^\alpha\},
\]
for all \( n \) large enough.

In fact, we shall prove that the power \( \alpha \) in (2.2) is optimal even for stationary martingale difference sequences.

It is clear that when \( \alpha = \frac{1}{3} \), the bound (2.2) implies the bound (1.1) of Lesigne and Volný.

Our second result shows that the moment condition \( \sup_i \mathbb{E} \exp \left\{ |X_i|^2 \alpha \right\} < \infty \) in Theorem 2.1 can be relaxed to \( \sup_i \mathbb{E} \exp \left\{ (X_i^+)^{2 \alpha} \right\} < \infty \), where \( X_i^+ = \max\{X_i, 0\} \), if we add a constraint on the sum of conditional variances
\[
\langle S \rangle_k = \sum_{i=1}^k \mathbb{E}(X_i^2|F_{i-1}).
\]

**Theorem 2.2.** Let \( \alpha \in (0, 1) \). Assume that \((X_i, F_i)_{i \geq 1}\) is a sequence of supermartingale differences satisfying 
\[
\sup_i \mathbb{E} \exp \left\{ (X_i^+)^{2 \alpha} \right\} \leq C_1
\]
for some constant \( C_1 \in (0, \infty) \). Then, for all \( x, v > 0 \),
\[
P \left( S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) \leq \exp \left\{ - \frac{x^2}{2(v^2 + \frac{1}{3}x^2 \alpha)} \right\} + nC_1 \exp\{-x^\alpha\}.
\]

For bounded random variables, some inequalities closely related to (2.4) can be found in Freedman [5], Dedecker [1], Dzhaparidze and van Zanten [3], Merlevède, Peligrad and Rio [11] and Delyon [2].

Adding a hypothesis on \( \langle S \rangle_n \) to Theorem 2.2, we can easily obtain the following Bernstein type inequality which is similar to an inequality of Merlevède, Peligrad and Rio [12] for weakly dependent sequences.
Corollary 2.3. Let \( \alpha \in (0, 1) \). Assume that \( (X_i, \mathcal{F}_i)_{i \geq 1} \) is a sequence of supermartingale differences satisfying \( \sup_i \mathbb{E}\exp\left(\frac{(X_i^+)^{\alpha}}{\mathbb{1}_{\{X_i > 0\}}}ight) \leq C_1 \) and \( \mathbb{E}\exp\left(\frac{(\langle S_i \rangle_n)^{\alpha}}{n}\right) \leq C_2 \) for some constants \( C_1, C_2 \in (0, \infty) \). Then, for all \( x > 0 \),

\[
P\left(\max_{1 \leq k \leq n} S_k \geq nx\right) \leq \exp \left\{ -\frac{x^{1+\alpha}}{2(1+\frac{1}{3}x)} n^\alpha \right\} + (nC_1 + C_2) \exp\{-x^\alpha n^\alpha\}. \tag{2.5}\]

In particular, with \( x = 1 \), it holds

\[
P\left(\max_{1 \leq k \leq n} S_k \geq n\right) = O(\exp\{-C n^\alpha\}), \quad n \to \infty, \tag{2.6}\]

where \( C > 0 \) is an absolute constant. Moreover, the power \( \alpha \) in (2.6) is optimal for the class of supermartingale differences: for each \( \alpha \in (0, 1) \), there exists a sequence of martingale differences \( (X_i, \mathcal{F}_i)_{i \geq 1} \) satisfying \( \sup_i \mathbb{E}\exp\left(\frac{(X_i^+)^{\alpha}}{\mathbb{1}_{\{X_i > 0\}}}ight) < \infty \), \( \sup_n \mathbb{E}\exp\left(\frac{(\langle S_i \rangle_n)^{\alpha}}{n}\right) < \infty \) and

\[
P\left(\max_{1 \leq k \leq n} S_k \geq n\right) \geq \exp\{-3n^\alpha\}. \tag{2.7}\]

for all \( n \) large enough.

Actually, just as (2.2), the power \( \alpha \) in (2.6) is optimal even for stationary martingale difference sequences.

In the i.i.d. case, the conditions of Corollary 2.3 can be weakened considerably, see Lanzinger and Stadtmüller [8] where it is shown that if \( \mathbb{E}\exp\left(\frac{(X_1^+)^{\alpha}}{\mathbb{1}_{\{X_1 > 0\}}}ight) < \infty \) with \( \alpha \in (0, 1) \), then

\[
P\left(\max_{1 \leq k \leq n} S_k \geq n\right) = O(\exp\{-C_n n^\alpha\}), \quad n \to \infty. \tag{2.8}\]

3 Proof of Theorem 2.1

We shall need the following refined version of the Azuma-Hoeffding inequality.

**Lemma 3.1.** Assume that \( (X_i, \mathcal{F}_i)_{i \geq 1} \) is a sequence of martingale differences satisfying \( |X_i| \leq 1 \) for all \( i \geq 1 \). Then, for all \( x \geq 0 \),

\[
P\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq \exp\left\{-\frac{x^2}{2n}\right\}. \tag{3.1}\]

A proof can be found in Laib [7].

For the proof of Theorem 2.1 we use a truncating argument as in Lesigne and Volný [9]. Let \( (X_i, \mathcal{F}_i)_{i \geq 1} \) be a sequence of supermartingale differences. Given \( u > 0 \), define

\[
X'_i = X_i \mathbb{1}_{\{|X_i| \leq u\}} - \mathbb{E}(X_i \mathbb{1}_{\{|X_i| \leq u\}} | \mathcal{F}_{i-1}),
\]

\[
X''_i = X_i \mathbb{1}_{\{|X_i| > u\}} - \mathbb{E}(X_i \mathbb{1}_{\{|X_i| > u\}} | \mathcal{F}_{i-1}),
\]

\[
S'_k = \sum_{i=1}^{k} X'_i, \quad S''_k = \sum_{i=1}^{k} X''_i, \quad S''''_k = \sum_{i=1}^{k} \mathbb{E}(X_i | \mathcal{F}_{i-1}).
\]

Then \( (X'_i, \mathcal{F}_i)_{i \geq 1} \) and \( (X''_i, \mathcal{F}_i)_{i \geq 1} \) are two martingale difference sequences and \( S_k = S'_k + S''_k + S''''_k \). Let \( t \in (0, 1) \). Since \( S''''_k \leq 0 \), for any \( x > 0 \),

\[
P\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq P\left(\max_{1 \leq k \leq n} S'_k + S''''_k \geq x(1-t)\right) + P\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right).
\]

\[
\leq P\left(\max_{1 \leq k \leq n} S'_k \geq x\right) + P\left(\max_{1 \leq k \leq n} S''_k \geq x(1-t)\right). \tag{3.2}\]
Using Lemma 3.1 and the fact that \(|X_i| \leq 2u\), we have

\[
P \left( \max_{1 \leq k \leq n} S_k' \geq xt \right) \leq \exp \left\{ -\frac{x^2t^2}{8nu^2} \right\}. \tag{3.3}\]

\[\text{Let } F_i(x) = P(|X_i| \geq x), x \geq 0. \text{ Since } E \exp\{|X_i|^{\frac{2\alpha}{\beta}}\} \leq C_1, \text{ we obtain, for all } x \geq 0,\]

\[
F_i(x) \leq \exp\{-x^{\frac{2\alpha}{\beta}}\} E \exp\{|X_i|^{\frac{2\alpha}{\beta}}\} \leq C_1 \exp\{-x^{\frac{2\alpha}{\beta}}\}.\]

Using the martingale maximal inequality (cf. e.g. p. 14 in [6]), we get

\[
P \left( \max_{1 \leq k \leq n} S_k'' \geq x(1-t) \right) \leq \frac{1}{x^2(1-t)^2} \sum_{i=1}^{n} EX_i''^2. \tag{3.4}\]

It is easy to see that

\[
EX_i''^2 = -\int_{u}^{\infty} t^2 dF_i(t) = u^2 F_i(u) + \int_{u}^{\infty} 2t F_i(t) dt \leq C_1 u^2 \exp\{-u^{\frac{2\alpha}{\beta}}\} + 2C_1 \int_{u}^{\infty} t \exp\{-t^{\frac{2\alpha}{\beta}}\} dt. \tag{3.5}\]

Notice that the function \(g(t) = t^3 \exp\{-t^{\frac{2\alpha}{\beta}}\}\) is decreasing in \([\beta, +\infty)\) and is increasing in \([0, \beta]\), where \(\beta = \left( \frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1}{\alpha}}\). If \(0 < u < \beta\), we have

\[
\int_{u}^{\infty} t \exp\{-t^{\frac{2\alpha}{\beta}}\} dt \leq \int_{u}^{\beta} t \exp\{-t^{\frac{2\alpha}{\beta}}\} dt + \int_{\beta}^{\infty} t^{-2} t^3 \exp\{-t^{\frac{2\alpha}{\beta}}\} dt \leq \int_{u}^{\beta} t \exp\{-u^{\frac{2\alpha}{\beta}}\} dt + \int_{\beta}^{\infty} t^{-2} \beta^3 \exp\{-\beta^{\frac{2\alpha}{\beta}}\} dt \leq \frac{3}{2} \beta^2 \exp\{-u^{\frac{2\alpha}{\beta}}\}. \tag{3.6}\]

If \(\beta \leq u\), we have

\[
\int_{u}^{\infty} t \exp\{-t^{\frac{2\alpha}{\beta}}\} dt = \int_{u}^{\beta} t^{-2} t^3 \exp\{-t^{\frac{2\alpha}{\beta}}\} dt \leq \int_{u}^{\beta} t^{-2} u^3 \exp\{-u^{\frac{2\alpha}{\beta}}\} dt = u^2 \exp\{-u^{\frac{2\alpha}{\beta}}\}. \tag{3.7}\]

By (3.5), (3.6) and (3.7), we get

\[
EX_i''^2 \leq 3C_1 (u^2 + \beta^2) \exp\{-u^{\frac{2\alpha}{\beta}}\}. \tag{3.8}\]

From (3.4), it follows that

\[
P \left( \max_{1 \leq k \leq n} S_k'' \geq x(1-t) \right) \leq \frac{3nC_1}{x^2(1-t)^2} (u^2 + \beta^2) \exp\{-u^{\frac{2\alpha}{\beta}}\}. \tag{3.9}\]

Combining (3.2), (3.3) and (3.9), we obtain

\[
P \left( \max_{1 \leq k \leq n} S_k \geq x \right) \leq 2 \exp\left\{ -\frac{x^2t^2}{8nu^2} \right\} + \frac{3nC_1}{(1-t)^2} \left( \frac{u^2}{x^2} + \frac{\beta^2}{x^2} \right) \exp\{-u^{\frac{2\alpha}{\beta}}\}.\]
Assume that $x > 150$ for all $X$. Take a positive random variable of martingale differences satisfying (2.3). We proceed as in Lesigne and Volný ([9], p. 150). Take $\sqrt{\lim_{n \to \infty} 2n \beta}$ large enough, which proves (2.3). This ends the proof of Theorem 2.1.

Next, we prove that the power $\alpha$ in (2.2) is optimal by giving a stationary sequence of martingale differences satisfying (2.3). We proceed as in Lesigne and Volný ([9], p. 150). Take a positive random variable $X$ such that

$$
P(X > x) = \frac{2e}{1 + x^{1/\alpha}} \exp \left\{ -x^{2/\alpha} \right\}
$$

for all $x > 1$. Using the formula $E f(X) = f(1) + \int_1^\infty f(t) P(X > t) dt$ for $f(t) = \exp \left\{ t^{2/\alpha} \right\}, t \geq 1$, we obtain

$$
E \exp \left\{ X^{2/\alpha} \right\} = e + \frac{4e \alpha}{1 - \alpha} \int_1^\infty \frac{t^{n-1}}{1 + t^{1/\alpha}} dt < \infty.
$$

Assume that $(\xi_i)_{i \geq 1}$ are Rademacher random variables independent of $X$, i.e. $P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}$. Let $X_i = X \xi_i$, $F_0 = \sigma(X)$ and $F_i = \sigma(X, (\xi_k)_{k=1, \ldots, i})$. Then, $(X_i, F_i)_{i \geq 1}$ is a stationary sequence of martingale differences satisfying

$$
\sup_i E \exp \left\{ |X_i|^{2/\alpha} \right\} = E \exp \left\{ X^{2/\alpha} \right\} < \infty.
$$

For $\beta \in (0, 1)$, it is easy to see that

$$
P \left( \max_{1 \leq k \leq n} S_k \geq n \right) \geq P \left( S_n \geq n \right) \geq P \left( \sum_{i=1}^n \xi_i \geq n^\beta \right) \geq P \left( X \geq n^{1-\beta} \right).
$$

Since, for $n$ large enough,

$$
P \left( \sum_{i=1}^n \xi_i \geq n^\beta \right) \geq \exp \left\{ -n^{2\beta-1} \right\},
$$

(cf. Corollary 3.5 in Lesigne and Volný ([9])), we get, for $n$ large enough,

$$
P \left( \max_{1 \leq k \leq n} S_k \geq n \right) \geq \frac{2e}{1 + (n^{1-\beta})^{-2/\alpha}} \exp \left\{ -n^{2\beta-1} - (n^{1-\beta})^{2/\alpha} \right\}. \tag{3.11}
$$

Setting $2\beta - 1 = \alpha$, we obtain, for $n$ large enough,

$$
P \left( \max_{1 \leq k \leq n} S_k \geq n \right) \geq \frac{2e}{1 + n^{1-\beta}} \exp \left\{ -2n^\alpha \right\} \geq \exp \left\{ -3n^\alpha \right\},
$$

which proves (2.3). This ends the proof of Theorem 2.1.
4 Proof of Theorem 2.2

To prove Theorem 2.2, we need the following inequality.

**Lemma 4.1** ([20], Remark 2.1). Assume that \((X_i, \mathcal{F}_i)_{i \geq 1}\) are supermartingale differences satisfying \(X_i \leq 1\) for all \(i \geq 1\). Then, for all \(x, v > 0\),

\[
\mathbb{P}(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp\left\{-\frac{x^2}{2(v^2 + \frac{1}{3}x)}\right\}.
\]

(4.1)

Assume that \((X_i, \mathcal{F}_i)_{i \geq 1}\) are supermartingale differences. Given \(u > 0\), set

\[
X'_i = X_i 1_{\{X_i \leq u\}}, \quad X''_i = X_i 1_{\{X_i > u\}}, \quad S'_k = \sum_{i=1}^k X'_i \quad \text{and} \quad S''_k = \sum_{i=1}^k X''_i.
\]

Then, \((X'_i, \mathcal{F}_i)_{i \geq 1}\) and \((X''_i, \mathcal{F}_i)_{i \geq 1}\) are supermartingale differences. Given \(u > 0\), we have, for all \(x, u, v > 0\),

\[
\mathbb{P}(S'_k \geq x \text{ and } \langle S' \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \mathbb{P}(S''_k \geq x \text{ and } \langle S'' \rangle_k \leq v^2 \text{ for some } k \in [1, n]) + \mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq 0\right).
\]

(4.2)

Applying Lemma 4.1 to the supermartingale differences \((X'_i/u, \mathcal{F}_i)_{i \geq 1}\), we have, for all \(x, u, v > 0\),

\[
\mathbb{P}(S''_k \geq x \text{ and } \langle S'' \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp\left\{-\frac{x^2}{2(v^2 + \frac{1}{3}xu)}\right\}.
\]

(4.3)

Using the exponential Markov’s inequality and the condition \(\mathbb{E}\exp\{(X'_i)^{\alpha/n}\} \leq C_1\), we get

\[
\mathbb{P}\left(\max_{1 \leq k \leq n} S''_k \geq 0\right) \leq \sum_{i=1}^n \mathbb{P}(X_i > u) \leq \sum_{i=1}^n \mathbb{E}\exp\{(X'_i)^{\alpha/n} - u^{\alpha/n}\} \leq nC_1 \exp\{-u^{\alpha/n}\}.
\]

(4.4)

Combining the inequalities (4.2), (4.3) and (4.4) together, we obtain, for all \(x, u, v > 0\),

\[
\mathbb{P}(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp\left\{-\frac{x^2}{2(v^2 + \frac{1}{3}xu)}\right\} + nC_1 \exp\{-u^{\alpha/n}\}.
\]

(4.5)

Taking \(u = x^{1-\alpha}\), we get, for all \(x, v > 0\),

\[
\mathbb{P}(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp\left\{-\frac{x^2}{2(v^2 + \frac{1}{3}x^{2-\alpha})}\right\} + nC_1 \exp\{-x^{\alpha}\}.
\]

(4.6)

This completes the proof of Theorem 2.2.
5 Proof of Corollary 2.3.

To prove Corollary 2.3 we make use of Theorem 2.2. It is easy to see that
\[
\mathbb{P}
\left(\max_{1 \leq k \leq n} S_k \geq nx, \langle S \rangle_n \leq nx^2 \right)
\leq \mathbb{P}
\left(\max_{1 \leq k \leq n} S_k \geq nx, \langle S \rangle_n > nx^2 \right)
\leq \mathbb{P}(S_k \geq nx \text{ and } \langle S \rangle_k \leq nx^2 \text{ for some } k \in [1, n])
\leq \mathbb{P}(S_k \geq nx) + \mathbb{P}(\langle S \rangle_k > nx^2).
\]

By Theorem 2.2 it follows that, for all \( x, v > 0 \),
\[
\mathbb{P}
\left(\max_{1 \leq k \leq n} S_k \geq nx \right)
\leq \exp \left\{ -\frac{x^2}{2 (n^\alpha - \frac{1}{3} v^2 + \frac{1}{3} x^2)} n^\alpha \right\}
+ nC_1 \exp \left\{ -x^\alpha n^\alpha \right\} + \mathbb{P}(\langle S \rangle_n > nx^2),
\]
Using the exponential Markov’s inequality and the condition \( \mathbb{E} \exp\left\{ \left(\frac{\langle S \rangle_n}{n}\right)^{\frac{1}{1-\alpha}} \right\} \leq C_2 \), we get, for all \( v > 0 \),
\[
\mathbb{P}(\langle S \rangle_n > nx^2) \leq \mathbb{E} \exp\left\{ \left(\frac{\langle S \rangle_n}{n}\right)^{\frac{1}{1-\alpha}} - v^{\frac{2}{1-\alpha}} \right\} \leq C_2 \exp\{-v^{\frac{2}{1-\alpha}}\}.
\]

Taking \( v = (nx)^{\frac{1-\alpha}{2}} \), we obtain, for all \( x > 0 \),
\[
\mathbb{P}
\left(\max_{1 \leq k \leq n} X_k \geq nx \right)
\leq \exp \left\{ -\frac{x^{1+\alpha}}{2 (1 + \frac{1}{3} x^2)} n^\alpha \right\}
+ (nC_1 + C_2) \exp \{-x^\alpha n^\alpha\},
\]
which gives inequality (2.5).

Next, we prove that the power \( \alpha \) in (2.6) is optimal. Let \( (X_i, \mathcal{F}_i)_{i \geq 1} \) be the sequence of martingale differences constructed in the proof of the second assertion of Theorem 2.1. Then \( \frac{\langle S \rangle_n}{n} = X^2 \),
\[
\sup_i \mathbb{E} \exp\left\{ (X_{i}^+)^{\frac{\alpha}{1-\alpha}} \right\} = \frac{1}{2} \mathbb{E} \exp\{X^{\frac{\alpha}{1-\alpha}}\} < \infty
\]
and
\[
\sup_n \mathbb{E} \exp\left\{ \left(\frac{\langle S \rangle_n}{n}\right)^{\frac{\alpha}{1-\alpha}} \right\} = \mathbb{E} \exp\{X^{\frac{\alpha}{1-\alpha}}\} < \infty.
\]
Using the same argument as in the proof of Theorem 2.1 we obtain, for \( n \) large enough,
\[
\mathbb{P}
\left(\max_{1 \leq k \leq n} S_k \geq n \right)
\geq \exp\{-3n^\alpha\}.
\]
This ends the proof of Corollary 2.3.

References


Exponential inequalities for supermartingales


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