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# Finite speed of propagation for mixed problems in the $WR$ class.

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## Abstract

In this article we are interested in the propagation speed for solution of hyperbolic boundary value problem in the  $WR$  class. Using the Holmgren principle, we show that this speed is finite and we are able to give an explicit expression for the maximal speed. Due to propagation phenomenon along the boundary specific to the  $WR$  class, the maximal speed can be larger than the propagation speed for the Cauchy problem. This is consistent with examples of the litterature.

AMS subject classification : 35L50

## 1 Introduction.

The aim of this paper is to show a result of finite speed of propagation for mixed hyperbolic problems in the so-called *WR* class (see [1]). This class contains weakly well-posed mixed problems, more precisely for those problems the solution loses a derivative in the interior and a derivative on the boundary of the domain relative to the data of the problem (see [5]).

The property that the information propagation speed remains finite is one of the main feature of hyperbolic partial differential equations. Indeed it is easy to show using an integration by parts argument that for the Cauchy problem with symmetric coefficients, the maximal speed of propagation is the modulus of the largest eigenvalue of the spatial symbol (see for example [11]).

The generalization of this result to constantly hyperbolic Cauchy problems [2] and to well-posed mixed problems [[4] p.408-412-[13]] uses the analysis of variable coefficients problems in such a way that, thanks to the Holgrem principle, one can construct a foliation of the supposed cone of propagation. The main part of this process is that the straightened mixed problems with initial data prescribed on a sheet of the foliation inherits the properties of constant hyperbolicity and of well-posedness of the mixed problem.

It is this method that we will adapt here to mixed problems in the *WR* class. So we will have to show that the straightened mixed problem inherits weak well-posedness. In the proof of theorem 3.2 we will see that this property need that the speed of propagation is larger than the speed of propagation in the well-posed case.

More precisely due to propagation phenomenon along the boundary specific to the weakly well posed case, we will ask that the speed of propagation is also larger than the maximal speed of propagation along the boundary. This new requirement is not surprising, indeed the literature contains many examples of mixed problems in the *WR* class for which the propagation speed is larger than the propagation speed of the Cauchy problem (see for example [2]-[3]-[6]-[7]) and can even be chosen arbitrarily large.

Moreover using the lower bound of the propagation speed in [6], we will be able to show that the maximal speed of propagation found in this paper is sharp.

## 2 Notations.

In this article we will consider initial boundary value problems (ibvps in short) in the half-space  $\mathbb{R}_+^d = \{(x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid x_d \geq 0\}$ , for  $T > 0$  we will also note  $\Omega_T = [0, T] \times \mathbb{R}_+^d$  and  $\omega_T = \Omega_T \cap \{x_d = 0\}$ .

To simplify the notations we will denote by  $(t, x') := (t, x', 0)$  the elements of  $\omega_T$ .

$\mathcal{C}_b^\infty(\Omega_T, \mathcal{M}_{n \times m})$  (resp.  $\mathcal{C}_b^\infty(\omega_T, \mathcal{M}_{n \times m})$ ) will denote the set of matrices of size  $n \times m$  which are smooth bounded with bounded derivatives on  $\Omega_T$  (resp.  $\omega_T$ ) and which admit limits for  $t$  and  $x$  large.

Since it we be useful for energy estimates we also introduce the weighted Sobolev

spaces  $H^s(\Omega_T)$  defined by the norm :

$$\|\cdot\|_{H_\gamma^s(\Omega_T)} = \|e^{-\gamma t} \cdot\|_{H^s(\Omega_T)}.$$

Spaces  $H^s(\omega_T)$  are defined in a similar way.

Our ibvp of study reads :

$$\begin{cases} L(t, x, \partial)u = \partial_t u + \sum_{j=1}^d A_j(t, x) \partial_j u = f, & \text{on } \Omega_T \\ B(t, x')u = g, & \text{on } \omega_T \\ u(0, x) = u_0(x), & \text{on } \mathbb{R}_+^d \end{cases} \quad (1)$$

where  $A_j \in \mathcal{C}_b^\infty(\Omega_T, \mathcal{M}_{N \times N})$ , and  $B \in \mathcal{C}_b^\infty(\omega_T, \mathcal{M}_{p \times N})$ . The integer  $p$  is the number of positive eigenvalues of  $A_d$  (we stress that thanks to assumptions 2.1 and 2.2 below,  $p$  does not depend of  $(t, x)$ ).

In order to simplify the notations we will denote by  $A(t, x, \xi)$  (resp.  $A'(t, x, \xi')$ ) the spatial (resp. spatial tangential) symbol of  $L(t, x, \partial)$  that is to say  $A(t, x, \xi) = \sum_{j=1}^d \xi_j A_j(t, x)$  (resp.  $A'(t, x, \xi') = \sum_{j=1}^{d-1} \xi_j A_j(t, x)$ ).

From now on we will suppose that the ibvp (1) is constantly hyperbolic, with non-characteristic boundary that is to say that the following assumptions are satisfied :

**Assumption 2.1** *There exist an integer  $q \geq 1$ , smooth functions  $\lambda_1, \dots, \lambda_q$  on  $\Omega_T \times \mathbb{R}^d \setminus \{0\}$  and positive integers  $\nu_1, \dots, \nu_q$  such that :*

$$\forall \xi \in \mathbb{S}^{d-1}, \det(\tau + A(t, x, \xi)) = \prod_{k=1}^q (\tau + \lambda_k(t, x, \xi))^{\nu_k},$$

with  $\lambda_1 < \dots < \lambda_q$  and the eigenvalues  $\lambda_k(t, x, \xi)$  of  $A(t, x, \xi)$  are semi-simple.

**Assumption 2.2** *For all  $(t, x) \in \Omega_T$ ,  $\det A_d(t, x) \neq 0$ .*

Let  $u$  a solution of (1) in view to include  $\|u(t)\|_{L^2(\mathbb{R}_+^d)}$  in the energy estimate of  $u$  (see [5]) we need the following assumption.

**Assumption 2.3** *The mixed problem (1) is Friedrichs symmetrizable that is to say there is a symmetric positive definite matrix regular on  $\Omega_T$ ,  $S(t, x)$  such that for all  $j$  and for all  $(t, x) \in \Omega_T$ ,  $S(t, x)A_j(t, x)$  is a symmetric matrix.*

We introduce the frequency spaces :

$$\begin{aligned} \Xi &:= \{\zeta = (\sigma = \gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus \{\gamma \geq 0\}\}, \\ \Xi_0 &:= \Xi \cap \{\gamma = 0\}. \end{aligned}$$

For  $(t, x, \zeta) \in \Omega_T \times \Xi$ , the resolvent matrix  $\mathcal{A}(t, x, \zeta)$  associated to the ibvp (1) is defined by :

$$\mathcal{A}(t, x, \zeta) = -A_d(t, x)^{-1} (\sigma + iA'(t, x, \eta)),$$

we denote by  $E_-(t, x, \zeta)$  the stable subspace of  $\mathcal{A}(t, x, \zeta)$ . Thanks to Hersh lemma (see [2] p.103) we know that for all  $\zeta \in \Xi \setminus \Xi_0$ ,  $\mathcal{A}(t, x, \zeta)$  does not have

purely imaginary eigenvalues and that  $\dim(E_-(t, x, \zeta))$  is constant equal to  $p$  for all  $(t, x, \zeta) \in \Omega_T \times (\Xi \setminus \Xi_0)$ .

Moreover according to [8]-[9],  $E_-(t, x, \zeta)$  admits a continuous extension up to  $\Xi_0$ . Since it will be useful later on, we introduce a Lopatinskii determinant  $\Delta$  defined by

$$\Delta(t, x, \zeta) = \det(BE_-(t, x, \zeta)),$$

which is a holomorphic function in  $\sigma$  and smooth in  $\eta$  away from glancing set  $\mathcal{G}$  (see [2] p239 for a definition).

**Definition 2.1** *The hyperbolic region  $\mathcal{H}$  of  $L(t, x, \partial)$  is the set of  $(t, x', \zeta) \in \omega_T \times \Xi_0$  such that  $\mathcal{A}(t, x', i\tau, \eta)$  is diagonalizable with purely imaginary eigenvalues.*

*The glancing region  $\mathcal{G}$  is the set of  $(t, x', \zeta) \in \omega_T \times \Xi_0$  such that  $\mathcal{A}(t, x, i\tau, \eta)$  has at least one We denote by  $\Upsilon$  the set where the ibvp (1) does not satisfy the uniform Kreiss Lopatinskii condition ie*

$$\Upsilon = \{(t, x', \zeta) \in \omega_T \times \Xi \setminus \Delta(t, x', \zeta) = 0\}.$$

We give here an equivalent definition of the  $WR$  class introduced in [1] due to [12] (see [6] for a proof of equivalence).

**Assumption 2.4** *The ibvp (1) is said to be in the class  $WR$  if the following conditions are satisfied :*

*i) The ibvp (1) satisfies the weak Kreiss-Lopatinskii condition ie*

$$\forall (t, x', \zeta) \in \omega_T \times \Xi \setminus \Xi_0, \Delta(t, x', \zeta) \neq 0.$$

*ii)  $\Upsilon \neq \emptyset$  and  $\Upsilon \subset \overset{\circ}{\mathcal{H}}$ .*

*iii) For all  $(\underline{t}, \underline{x}', \underline{\zeta}) \in \Upsilon$ , there is a neighborhood  $\mathcal{V}$  of  $(\underline{t}, \underline{x}', \underline{\zeta})$  in  $\omega_T \times \Xi$ , a regular basis  $(E_1, \dots, E_p)(t, x', \zeta)$  of  $E_-(\zeta)$  on  $\mathcal{V}$ , an invertible matrix of size  $p$ ,  $P(t, x', \zeta)$  regular on  $\mathcal{V}$  and a smooth real valued function  $\Theta$  such that*

$$\forall (t, x', \zeta) \in \mathcal{V}, B[E_1, \dots, E_p](t, x', \zeta) = P(t, x', \zeta) \text{diag}(\gamma + i\Theta(t, x', \zeta), 1 \dots 1).$$

*In particular, on can find a Lopatinskii's determinant under the form :*

$$\forall (t, x', \zeta) \in \mathcal{V}, \Delta(t, x', \zeta) = (\gamma + i\Theta(t, x', \zeta)) \det P(t, x', \zeta).$$

### 3 Main result.

**Theorem 3.1** *If the ibvp (1) satisfies assumptions 2.1-2.2-2.3-2.4 then there exists a real positive  $V_0$  such that for all  $(t_0, x_0)$  in  $\Omega_T$ , if we denote by  $\mathcal{C}$  the cone define by :*

$$\mathcal{C} = \{(t, x) \in \mathbb{R}^{d+1} \setminus |x - x_0| \leq V_0(t_0 - t)\}. \quad (2)$$

Then the following property is true :  
 If  $u \in C([0, T], L^2(\mathbb{R}_+^d))$  is a solution of the ibvp :

$$\begin{cases} L(t, x, \partial)u = 0, & \text{on } \mathcal{C} \cap \Omega_T, \\ B(t, x')u = 0, & \text{on } \mathcal{C} \cap \omega_T, \\ u = 0, & \text{on } \mathcal{C} \cap \Omega_T \cap \{t = 0\}, \end{cases} \quad (3)$$

then  $u|_{\mathcal{C}} = 0$ .

Moreover the same propertie is true for all  $V > V_0$ .

The smallest real  $V$  such that theorem 3.1 is true is the sought maximal speed of propagation. We stress on the fact that this speed of propagation  $V$  only depends of the coefficients of (1). The value of  $V$  will be made precise in (6).

As mentionned in the introduction the proof of theorem 3.1 is based on the Holmgren principle.

Let  $u$  be a smooth solution of (3). We will construct a foliation of the cone  $\mathcal{C}$  and our goal will be to show that  $u$  is zero on any sheet of the foliation that is to say that  $u|_{\mathcal{C}} = 0$ . Then we will conclude the proof of theorem 3.1 by a mollification of the weak solution  $u$ .

To prove that  $u$  is zero on any sheet of the foliation we will show that the ibvp which takes the sheet as a space-like variety (see (4) ) remains in the  $WR$  class if we choose  $V$  large enough. Then using classical results on ibvp in the  $WR$  class, more precisely that the adjoint problem of an ibvp in the  $WR$  class is in the  $WR$  class (see [2] p.137) and a weakly well-posed result of [5] we will be able to conclude using Green's formula on  $\Omega_T$ .

The foliation  $(\mathcal{E}_\theta)_{\theta \in [0,1]}$  of  $\mathcal{C}$  used for this proof will be exactly the same as the foliation given in ([2] p.76) (of course restricted to  $\{x_d > 0\}$ ) that's why we will not give it explicitey in this paper. The only point to keep in mind is that for all  $\theta$  the sheet  $\mathcal{E}_\theta$  is a regular graph so one can find a smooth diffeomorphism

$$\psi_\theta : (t, x) \rightarrow (\tilde{t}, x)$$

such that  $\mathcal{E}_\theta = \psi_\theta(0, \mathbb{R}_+^d)$ . The straightened ibvp on the sheet  $\mathcal{E}_\theta$  reads

$$\begin{cases} \tilde{L}(\tilde{t}, x, \partial)u = 0, & \mathcal{C}_\varepsilon \cap \Omega_T \\ \tilde{B}(\tilde{t}, x')u = 0, & \mathcal{C}_\varepsilon \cap \omega_T \\ u = 0, & \mathcal{C}_\varepsilon \cap \Omega_T \cap \mathcal{E}_\theta \end{cases} \quad (4)$$

with  $\tilde{L}(\tilde{t}, x, \partial) = (I + A(t, x, \nabla_x \tilde{t})) \partial_{\tilde{t}} + \sum_{j=1}^d A_j(t, x) \partial_j$ ,  $\tilde{B}(\tilde{t}, x') = B(t, x')$ , and where  $\mathcal{C}_\varepsilon$  is defined as in (2) with  $t_0 - \varepsilon$  instead of  $t_0$  ( this restriction is needed for the final mollification argument).

We conclude this section by the definition of the resolvent matrix  $\tilde{\mathcal{A}}(\tilde{t}, x, \zeta)$  of our new ibvp (4) which is given by

$$\tilde{\mathcal{A}}(\tilde{t}, x, \zeta) = -(A_d(t, x))^{-1} (\sigma(I + A(t, x, \nabla_x \tilde{t})) + iA'(t, x, \eta)). \quad (5)$$

We denote by  $\tilde{E}_-(\tilde{t}, x, \zeta)$  the stable subspace of  $\tilde{\mathcal{A}}(\tilde{t}, x, \zeta)$ ,  $\tilde{\mathcal{H}}$  the new hyperbolic area and  $\tilde{\Upsilon}$  the new area in which Kreiss-Lopatinskii condition breaks down

(those objects are defined exactly as in definition 2.1).

Let

$$\begin{aligned} V_C &:= \sup_{\xi \in \mathbb{S}^{d-1}} \sup_{(t,x)} \max_i |\lambda_i(t, x, \xi)| \\ V_B &:= \sup_{(t,x',\zeta) \in \Upsilon} |\nabla_\eta \Theta(t, x', \zeta)| \end{aligned}$$

where  $\lambda_i(t, x, \xi)$  are defined in the assumption 2.1 and  $\Theta$  is defined in the assumption 2.4.

Then the maximal speed of propagation is  $V^{-1}$ , where  $V$  is given by

$$V = \min \left( \frac{1}{V_B}; \frac{1}{V_C} \right), \quad (6)$$

we denote by  $\mathcal{C}_{res} = \{(r, v) \in \mathbb{R} \times \mathbb{R}^d \setminus |v| < V|r|\}$ . As mentioned before the main point in the proof of theorem 3.1 is the following :

**Theorem 3.2** *For all  $(1, \nabla_x \tilde{t}) \in \mathcal{C}_{res}$  the ibvp (4) associated to the change of variable  $(1, \nabla_x \tilde{t})$  satisfies assumptions 2.1-2.2-2.3-2.4.*

**Remark** Lax lemma (see [2] p.29) shows that the ibvp (4) satisfies assumption 2.1 (we use the fact that  $V \leq \frac{1}{V_C}$ ), and (4) also satisfies assumption 2.2 because the change of variable does not change  $A_d$ . Moreover one can easily see that  $\mathcal{S}(t, x)$  symmetrizes (4) as well as (1).

So we will only prove that (4) satisfies assumption 2.4.

**Proof :** Let

$$\Omega = \left\{ (1, \nabla_x \tilde{t}) \in \mathcal{C}_{res} \setminus \left( \tilde{L}(\tilde{t}, x, \partial), \tilde{B}(t, x') \right) \text{ is in the WR class} \right\}$$

we will show by connectedness that  $\Omega = \mathcal{C}_{res}$ .

- Is it clear that  $(1, 0) \in \Omega$ .
- According to [1],  $\Omega$  is an open set in  $\mathcal{C}_{res}$ .
- So we just have to show that  $\Omega$  is a closed set in  $\mathcal{C}_{res}$ .

Let  $(1, v_n)$  be a sequence in  $\Omega$  which tends to  $(1, v)$  in  $\mathcal{C}_{res}$ . We denote by  $\tilde{E}_-^n(t, x, \zeta)$  (resp.  $\tilde{E}_-^\infty(t, x, \zeta)$ ) the stable subspace associated with the resolvent matrix after the change of variable  $(1, v_n)$  (resp.  $(1, v)$ ) and by  $\tilde{\Delta}_-^n(t, x, \zeta)$  and  $\tilde{\Delta}_-^\infty(t, x, \zeta)$  the corresponding Lopatinskii determinants.

We have to show that for all  $(t, x') \in \omega_T$ , conditions *i) – iii)* in the assumption 2.4 are satisfied.

◇ Proof of *i)* : We argue by contradiction. Let  $(\tilde{t}, x', \underline{\zeta}) \in \omega_T \times (\Xi \setminus \Xi_0)$  be a zero of  $\tilde{\Delta}_-^\infty$ , if  $\tilde{\Delta}_-^\infty$  is not identically zero, then  $(\tilde{t}, x', \underline{\zeta})$  is an isolated zero and thanks to Rouché's theorem we know that for  $n$  large enough  $\tilde{\Delta}_-^n$  admits a zero close to  $(\tilde{t}, x', \underline{\zeta})$  which is a contradiction because  $(1, v_n)$  is in  $\Omega$ .

Let us show that  $\tilde{\Delta}_-^\infty(\tilde{t}, x', 1, 0)$  is not zero.

A simple computation shows that,

$$\tilde{\mathcal{A}}(\tilde{t}, x', 1, 0) = \mathcal{A}(t, x', i, \nabla_{x'} \tilde{t}) - i \partial_{\tilde{t}} \tilde{t}.$$

That is to say that  $\tilde{\Delta}^\infty(\tilde{t}, x', 1, 0)$  is zero if and only if  $(\tilde{t}, x', 1, \nabla_{x'}\tilde{t}) \in \Upsilon$ .  
 But  $(1, \nabla_{x'}\tilde{t}) \in \mathcal{C}_{res}$  so we have :

$$|\nabla_{x'}\tilde{t}| \leq |\nabla_x\tilde{t}| < \frac{1}{V_B} \leq \inf_{(\tilde{t}, x', 1, \eta) \in \Upsilon} (|\eta|).$$

where we used Euler formula for the homogenous function  $\Theta$  to state the last inequality, which contradicts the fact that  $(\tilde{t}, x', 1, \nabla_{x'}\tilde{t})$  is in  $\Upsilon$ .

Since the following points are true for all  $n$  will feel free to drop out  $n$  in our notations.

◇ Proof of *ii*) :

The proof of *ii*) is also based on the explicit computation,

$$\tilde{\mathcal{A}}(\tilde{t}, x, \zeta) = \mathcal{A}(t, x, i\tau, \eta + \tau\nabla_{x'}\tilde{t}) - i\tau\partial_d\tilde{t} \quad (7)$$

which shows that  $\tilde{\mathcal{H}}$  (resp.  $\tilde{\Upsilon}$ ) is the translation of  $\mathcal{H}$  (resp.  $\Upsilon$ ) by the vector  $\tau\nabla_{x'}\tilde{t}$ .

So, if  $\Upsilon \subset \mathring{\mathcal{H}}$  it is also the case for  $\tilde{\Upsilon}$  and  $\tilde{\mathcal{H}}$ .

◇ Proof of *iii*) :

We will prove an equivalent condition *iii')* (see [1]) :

*iii')* Let  $(t, x', \underline{\zeta}) \in \tilde{\Upsilon}$  then  $\partial_\tau\tilde{\Delta}(t, x', \underline{\zeta})$  is not zero.

Using (7) we have the following relation between the Lopatinskii's determinant  $\tilde{\Delta}(t, x', i\tau, \eta) = \Delta(t, x', i\tau, \eta + \tau\nabla_{x'}\tilde{t})$ , in particular for  $(t, x', \underline{\zeta}) \in \tilde{\Upsilon}$ ,

$$0 = \partial_\tau\tilde{\Delta}(t, x', \underline{\zeta}) = [i\partial_\tau\Delta + \nabla_{x'}\tilde{t} \cdot \nabla_\eta\Delta](t, x', i\underline{\tau}, \underline{\eta} + \underline{\tau}\nabla_{x'}\tilde{t}). \quad (8)$$

Following [[6], proposition 3.5], one can suppose that

$$\nabla_\eta\Delta(t, x', i\underline{\tau}, \underline{\eta} + \underline{\tau}\nabla_{x'}\tilde{t}) = i[\partial_\tau\Delta\nabla_\eta\Theta](t, x', i\underline{\tau}, \underline{\eta} + \underline{\tau}\nabla_{x'}\tilde{t}).$$

So because  $\partial_\tau\Delta(t, x', i\underline{\tau}, \underline{\eta} + \underline{\tau}\nabla_{x'}\tilde{t})$  is not zero (8) becomes :

$$\nabla_{x'}\tilde{t} \cdot \nabla_\eta\Theta(t, x', i\underline{\tau}, \underline{\eta} + \underline{\tau}\nabla_{x'}\tilde{t}) = -1, \quad (9)$$

but the restriction  $|\nabla_{x'}\tilde{t}| < \frac{1}{V_B}$  makes (9) impossible.

□

We will now work on the adjoint ibvp of (4) :

$$\begin{cases} \tilde{L}^*(\tilde{t}, x, \partial)v = 0, \mathcal{C}_\varepsilon \cap \Omega_T \\ \tilde{C}(\tilde{t}, x')v = 0, \mathcal{C}_\varepsilon \cap \Omega_T \cap \{x_d = 0\} \\ v = h, \mathcal{C}_\varepsilon \cap \Omega_T \cap \mathcal{E}_\theta \end{cases} \quad (10)$$

where  $\tilde{L}^*(\tilde{t}, x, \partial) = -\partial_{\tilde{t}} - \sum_{j=1}^d A_j(t, x)^t \partial_j - \sum_{j=1}^d \partial_j A_j(t, x)^t$ , and the normal matrix  $A_d$  is decomposed in the following way

$$A_d(t, x') = C(t, x')^t M(t, x') + N(t, x')^t B(t, x') \quad (11)$$

with  $M$  and  $N$  in  $\mathcal{C}_b^\infty(\omega_T)$ .

Theorem 1.5 and 4.4 in [[2] p.29 and p137] show that since (4) satisfies assumptions 2.1-2.2-2.3-2.4, the same is true for its adjoint ibvp (10). We now give a result of weak well posedness for ibvps in the class  $WR$ .



**Theorem 3.3** [5] *If the ibvp (1) with homogeneous initial data satisfies assumptions 2.1-2.2-2.3-2.4 and moreover if for all  $j$ ,  $A_j \in W^{2,\infty}(\Omega_T, \mathcal{M}_N)$ ,  $B \in W^{2,\infty}(\omega_T, \mathcal{M}_{p \times N})$ , then for all source terms  $f$  and  $g$  such that  $f \in L^2(\Omega_T)$ ,  $\nabla_{t,x'} f \in L^2(\Omega_T)$ ,  $g \in H^1(\omega_T)$  and moreover satisfies  $f|_{t=0} = g|_{t=0} = 0$ , there exists a unique solution of the ibvp  $u \in L^2(\Omega_T)$ ,  $u|_{x_d=0} \in L^2(\omega_T)$  which admits the following energy estimate :*

$$\begin{aligned} e^{-2\gamma t} & \|u(t)\|_{L^2(\mathbb{R}_+^d)}^2 + \gamma \|u\|_{L_\gamma^2(\Omega_T)}^2 + \|u|_{x_d=0}\|_{L_\gamma^2(\omega_T)}^2 \\ & \lesssim \frac{1}{\gamma} \|f\|_{L_\gamma^2(\Omega_T)}^2 + \frac{1}{\gamma^3} \|\nabla_{t,x'} f\|_{L_\gamma^2(\Omega_T)}^2 + \|g\|_{L_\gamma^2(\omega_T)}^2 + \frac{1}{\gamma^2} \|\nabla g\|_{L_\gamma^2(\omega_T)}^2 \end{aligned} \quad (12)$$

Moreover the same result for inhomogeneous initial data is true if we ask that for all  $j$ ,  $A_j \in C_b^\infty(\Omega_T, \mathcal{M}_N)$ ,  $B \in C_b^\infty(\omega_T, \mathcal{M}_{p \times N})$  and that the initial data  $h \in H^2(\mathbb{R}_+^d)$  satisfies the compatibility condition  $B(0, x')h|_{x_d=0} = g|_{t=0}$ .

The end of the proof of theorem 3.1 is very similar to the proof of the finite speed of propagation for the Cauchy problem given in ([2] p.73-79) and ([4] p.320-324), which is the reason why we will not reproduce here all the details. Let  $u$  be a smooth solution of the ibvp (3) and  $v$  the solution of (10) given by theorem 3.3 for ibvp with non homogeneous initial data. Let  $\mathcal{L}_\theta$  be the lens between  $\mathcal{E}_\theta$  and  $\{t=0\}$ , then thanks to Green formula it follows :

$$\begin{aligned} 0 &= - \int_{\mathcal{L}_\theta \cap \{t=0\}} \langle u, v \rangle dx - \int_{\mathcal{L}_\theta \cap \{x_d=0\}} \langle A_d(t, x', 0)u, v \rangle dt dx' \\ &+ \int_{\mathcal{E}_\theta} \langle (\nu_0 + A(t, x, \vec{n}))u, v \rangle dt dx \end{aligned} \quad (13)$$

where  $\nu = (\nu_0, \vec{n})$  is the outgoing unitary normal of  $\mathcal{E}_\theta$ .

Since  $u$  is solution of (3),  $u|_{t=0}$  is zero so the first integral in (13) is also zero. Moreover using the decomposition (11), the fact that  $u$  is solution of (3) and  $v$  is solution of (10) then the second integral in (13) is also zero.

So (13) reads

$$\int_{\mathcal{E}_\theta} \langle (\nu_0 + A(t, x, \vec{n}))u, v \rangle dt dx = 0,$$

and we can conclude as in ([2] p.73-79) using the fact that  $v|_{\mathcal{E}_\theta} = h$  and the invertibility of  $(\nu_0 + A(t, x, \vec{n}))$  to show that  $u|_{\mathcal{E}_\theta} = 0$ , and that  $u = 0$  on  $\mathcal{C}$  because  $u$  is smooth enough.

To complete the proof of theorem 3.1 we have now to deal with the case where the solution  $u$  of (3) is in  $\mathcal{C}([0, T]; L^2(\mathbb{R}_+^d))$ . But, using a smoothing procedure by a mollifier and theorem 1.2 of [10], we can easily construct a sequence of continuous solutions  $(u_n)_{n \in \mathbb{N}}$  such that for  $n$  large enough  $u_n$  is a solution of (3) that is to say that  $u_n$  is zero on  $\mathcal{C}$ . Moreover the energy estimate (12) shows that  $u_n$  tends to  $u$  in  $\mathcal{C}([0, T]; L^2(\mathbb{R}_+^d))$ .

□

Thanks to theorem 4.5 in [6] which shows that  $V^{-1}$  is a lower bound of the speed of propagation in the case where the  $A_j$ 's and  $B$  do not depend on  $(t, x)$ , we can conclude that the speed  $V^{-1}$  is sharp.

We refer to [6] for an example of ibvp for which the maximal speed of propagation equals either  $V_C$  or  $V_B$  according to the boundary matrix  $B$ .

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