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Mathematical analysis of the stationary Oldroyd model with diffusive stress

Laurent Chupin* and Sébastien Martin†

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Abstract

We present the mathematical analysis of the stationary Oldroyd model with diffusive stress: existence and uniqueness of weak solutions is shown if the source terms are small enough or if the Reynolds and Weissenberg numbers are small enough. Besides, in the corotational model, this condition on the data can be relaxed for the existence result. Finally, strong solutions are obtained with additional regularity on the data.

Keywords. viscoelastic fluid, Oldroyd model, diffusive stress

MSC. 76D03, 76A10, 35Q35

1 Introduction

The Oldroyd model describes the behaviour of a viscoelastic fluid. Its principle is built upon a description of the shear stress that interpolates between a purely viscous contribution and a purely elastic contribution. Let Ω be

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a bounded open set in \mathbb{R}^3 and \mathbf{f} be a given vector function on Ω . We are looking for a vector function $\mathbf{u} : \Omega \mapsto \mathbb{R}^3$, a scalar function $p : \Omega \mapsto \mathbb{R}$ and a symmetric tensor function $\boldsymbol{\sigma} : \Omega \mapsto \mathbb{R}^{3 \times 3}$, representing the velocity, the pressure and the elastic extra-stress of the fluid satisfying the following set of equations:

$$\begin{cases} \operatorname{Re}(\mathbf{u} \cdot \nabla \mathbf{u}) - (1 - r)\Delta \mathbf{u} + \nabla p = \operatorname{div} \boldsymbol{\sigma} + \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \Omega, \\ \operatorname{We}(\mathbf{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\nabla \mathbf{u}, \boldsymbol{\sigma})) + \boldsymbol{\sigma} - D \Delta \boldsymbol{\sigma} = 2r\mathbb{D}(\mathbf{u}), & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega, \\ D \partial_{\mathbf{n}} \boldsymbol{\sigma} = \mathbf{0}, & \text{on } \partial\Omega, \end{cases} \quad (1)$$

Here, Re denotes the Reynolds number that quantifies the inertial effects in the fluid flow; We is the Weissenberg number related to a relaxation time that characterizes the elasticity of the fluid; D is a diffusive parameter of the elastic stress; $r \in [0, 1]$ is an interpolation parameter : cases $r \in (0, 1)$ are often referred as Jeffreys models whereas case $r = 1$ is referred as the Maxwell model. Besides, $\mathbb{D}(\mathbf{u})$ (resp. $\mathbb{W}(\mathbf{u})$) denotes the symmetric (resp. skew-symmetric) part of the velocity gradient. The function g_a , $-1 \leq a \leq 1$, is a bilinear mapping related to the total derivative, in which the parameter a interpolates between the so-called upper-convected model ($a = 1$) and lower-convected model ($a = -1$). Note that the case $a = 0$ is known as the so-called corotational model. The function g_a is defined as

$$g_a(\nabla \mathbf{u}, \boldsymbol{\sigma}) = \mathbb{W}(\mathbf{u}) \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \mathbb{W}(\mathbf{u}) + a(\mathbb{D}(\mathbf{u}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbb{D}(\mathbf{u})).$$

Remark 1 *In standard derivations of Oldroyd model from kinetic models for dilute polymers, the diffusive term $D\Delta\boldsymbol{\sigma}$ is routinely omitted, on the grounds that it is several orders of magnitude smaller than the other terms in the equation. It physically corresponds to a centre-of-mass diffusion term in the dumbbell models and it is in the range of about 10^{-9} to 10^{-7} when the macroscopic length-scale of the domain is of order 1, see [1].*

Let us first discuss the transient case. The standard Oldroyd model (without diffusion: $D = 0$) has been the subject of intensive studies [4, 5, 6]: P.-L. Lions & N. Masmoudi [5] proved a global existence result of weak solutions for any data in the corotational case only. In the general case, C. Guillopé & J.-C. Saut [4] proved the existence and uniqueness of local strong solutions;

besides, if the fluid is not too elastic and if the data are sufficiently small, then solutions are global. Then L. Molinet & R. Talhouk [6] proved that the smallness assumption on the elasticity of the fluid could be relaxed. The diffusive Oldroyd model (with diffusion: $D > 0$) has been studied by a few authors: recently P. Constantin & M. Kliegl [3] proved the existence of global strong solutions in 2D for the Cauchy problem and uniqueness of the solution among a class of strong solutions. Notice also that other regularizations of the standard Oldroyd model have been studied, see in particular [1].

Let us now discuss the stationary case. To our knowledge, the only available result is due to M. Renardy [7] and focuses on the standard model (without diffusion) only: existence and uniqueness of strong solutions is proved under the assumption of small regular data. The method used by M. Renardy is based on a reformulation of the Oldroyd model as a “Newtonian generalized” fluid: the contribution of the stress $\text{div } \boldsymbol{\sigma}$ is expressed as an implicit function of the velocity field \mathbf{u} and, then, an iterative scheme is built upon this fully nonlinear system.

Now let us discuss the diffusive Oldroyd model in the stationary case. The mathematical analysis of the model can be approached with a completely different framework, as the diffusive contribution in the stress equation drastically changes the mathematical properties of the system. So far, the method that we present in the present paper is based on a classical weak formulation and then on energy estimates.

Let us underline the main differences with the standard Oldroyd model studied by M. Renardy:

- the diffusive model makes it possible to handle with irregular data;
- in the corotational case, the smallness of the data is not needed anymore.

The present paper is composed of five sections. In Section 2 we give the weak formulation of the problem, the mathematical framework and the main results. The last three sections are devoted to the proof of the main result (Theorem 1): in Section 3 the existence result, in Section 4, the uniqueness result and, in Section 5, the regularity result.

2 Weak formulation and main results

The variational formulation of Problem (1) is written

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{u}, \boldsymbol{\sigma}) \in V \times W \text{ such that, for all } (\mathbf{v}, \boldsymbol{\tau}) \in V \times W, \\ \text{Re} \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} + (1-r) \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} + \int_{\Omega} \boldsymbol{\sigma} : \mathbb{D}(\mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle, \\ \text{We} \int_{\Omega} (\mathbf{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\nabla \mathbf{u}, \boldsymbol{\sigma})) : \boldsymbol{\tau} + \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\tau} + \text{D} \int_{\Omega} \nabla \boldsymbol{\sigma} : \nabla \boldsymbol{\tau} \\ \hspace{15em} = 2r \int_{\Omega} \mathbb{D}(\mathbf{u}) : \boldsymbol{\tau}, \end{array} \right. \quad (2)$$

where the spaces V and W are defined by

$$\begin{aligned} V &= \{\mathbf{u} \in H_0^1(\Omega)^3; \text{div } \mathbf{u} = 0\}, \\ W &= \{\boldsymbol{\sigma} \in H^1(\Omega)^{3 \times 3}; \boldsymbol{\sigma} = {}^T \boldsymbol{\sigma}\}, \end{aligned}$$

and where $\langle \cdot, \cdot \rangle$ denotes the duality bracket between H^{-1} and H_0^1 . We will use the following norms:

$$\|\mathbf{u}\|_V^2 = \int_{\Omega} \|\nabla \mathbf{u}\|^2, \quad \|\boldsymbol{\sigma}\|_W^2 = \int_{\Omega} \|\boldsymbol{\sigma}\|^2 + \|\nabla \boldsymbol{\sigma}\|^2.$$

The main theorem which is proved in this article concerns an existence result for Problem (2). In its general form, it requires some assumptions on the data. For this, we introduce the following constants:

$$\begin{aligned} C_{(\text{I})} &:= \frac{8|a|C_{\Omega}^2 \text{We} \|\mathbf{f}\|_{H^{-1}}}{\min(1-r, \text{D})^2}, \\ C_{(\text{II})} &:= \frac{\sqrt{2r} \min(1-r, \text{D})}{4|a|C_{\Omega}^2 \text{We}} (1 - \sqrt{1 - C_{(\text{I})}}), \end{aligned}$$

where C_{Ω} is a constant which only depends on the domain Ω .

Remark 2

1. Note that $C_{(\text{II})}$ is defined provided $C_{(\text{I})} \leq 1$.
2. For $a = 0$, we have by continuity $C_{(\text{I})} = 0$ and $C_{(\text{II})} = \frac{\sqrt{2r} \|\mathbf{f}\|_{H^{-1}}}{\min(1-r, \text{D})}$.

3. Constant C_Ω is related to the Sobolev injection $W \subset L^4(\Omega)^{3 \times 3}$:

$$\forall \boldsymbol{\sigma} \in W, \|\boldsymbol{\sigma}\|_{L^4} \leq C_\Omega \|\boldsymbol{\sigma}\|_W.$$

4. The only physical parameter that is not involved in the definition of constant $C_{(I)}$ is the Reynolds number Re . In other words, the existence result that is further described does not depend on the value of the Reynolds number.

Theorem 1 *Let Ω be a Lipschitz bounded open set in \mathbb{R}^3 and $\mathbf{f} \in H^{-1}(\Omega)^3$. Let $\text{Re} \geq 0$, $\text{We} \geq 0$, $0 < r < 1$, $-1 \leq a \leq 1$ and $\text{D} > 0$.*

- *Existence. If $C_{(I)} \leq 1$ then, there exists a solution $(\mathbf{u}, \boldsymbol{\sigma})$ of problem (2) which satisfies*

$$2r\|\mathbf{u}\|_V^2 + \|\boldsymbol{\sigma}\|_W^2 \leq C_{(II)}^2. \quad (3)$$

Moreover there exists $p \in L^2(\Omega)$ such that (\mathbf{u}, p) satisfies $(P_i)_{i=1,2,3}$ in the sense of distributions.

- *Uniqueness. Problem (2) admits at most one solution if one of the following conditions is satisfied:*
 - a) $\|\mathbf{f}\|_{H^{-1}}$ is small enough;
 - b) Re and We are small enough.
- *Regularity. If Ω is of class C^∞ and if each component of \mathbf{f} belongs to $C^\infty(\overline{\Omega})$ then each component of any solution of (2) belongs to $C^\infty(\overline{\Omega})$ and the considered solution satisfies (1) in a classical sense.*

Let us mention two corollaries.

Corollary 2 *Problem (2) admits a unique solution if one of the following conditions is satisfied:*

- a) $\|\mathbf{f}\|_{H^{-1}}$ is small enough;
- b) Re and We are small enough.

Corollary 3 *If $a = 0$, there exists a solution for all data.*

The proof is decomposed into four parts. In section 3 we show the existence for the weak formulation (2) using a Galerkin approximations and compactness results to perform the limit. The existence of a pressure is obtained by De Rham theory. In section 4, we prove the uniqueness of the solution and, in section 5, we investigate the regularity of the weak solutions and prove that, if the data are regular, so is the solution.

3 Existence result

As a preliminary, $V \times W$ is endowed with the scalar product $(\cdot, \cdot)_{V \times W}$ defined by

$$((\mathbf{u}, \boldsymbol{\sigma}), (\mathbf{v}, \boldsymbol{\tau}))_{V \times W} = 2r(\mathbf{u}, \mathbf{v})_V + (\boldsymbol{\sigma}, \boldsymbol{\tau})_W.$$

As V and W are separable Hilbert spaces, we consider a countable orthonormal basis $(\mathbf{w}_k)_{k \in \mathbb{N}}$ in the space V , and a countable orthonormal basis $(\mathbf{s}_k)_{k \in \mathbb{N}}$ in the space W . We use the notation $V_k := \text{span}(\mathbf{w}_1, \dots, \mathbf{w}_k)$ and $W_k := \text{span}(\mathbf{s}_1, \dots, \mathbf{s}_k)$. For each fixed integer $k \in \mathbb{N}$, we would like to define an approximate solution $(\mathbf{u}_k, \boldsymbol{\sigma}_k)$ of (2) by

$$\mathbf{u}_k = \sum_{i=0}^k \alpha_{i,k} \mathbf{w}_i, \quad \boldsymbol{\sigma}_k = \sum_{i=0}^k \beta_{i,k} \mathbf{s}_i,$$

satisfying the variational problem

$$\begin{aligned} \text{Re} \int_{\Omega} (\mathbf{u}_k \cdot \nabla \mathbf{u}_k) \cdot \mathbf{v}_k + (1-r) \int_{\Omega} \nabla \mathbf{u}_k : \nabla \mathbf{v}_k + \int_{\Omega} \boldsymbol{\sigma}_k : \mathbb{D}(\mathbf{v}_k) &= \langle \mathbf{f}, \mathbf{v}_k \rangle, \\ \text{We} \int_{\Omega} (\mathbf{u}_k \cdot \nabla \boldsymbol{\sigma}_k + g_a(\nabla \mathbf{u}_k, \boldsymbol{\sigma}_k)) : \boldsymbol{\tau}_k + \int_{\Omega} \boldsymbol{\sigma}_k : \boldsymbol{\tau}_k + \text{D} \int_{\Omega} \nabla \boldsymbol{\sigma}_k : \nabla \boldsymbol{\tau}_k & \quad (4) \\ &= 2r \int_{\Omega} \boldsymbol{\tau}_k : \mathbb{D}(\mathbf{u}_k), \end{aligned}$$

for all $(\mathbf{v}_k, \boldsymbol{\tau}_k) \in V_k \times W_k$. Equations (4) form a system of nonlinear equations for $\alpha_{1,k}, \dots, \alpha_{k,k}, \beta_{1,k}, \dots, \beta_{k,k}$, and the existence of a solution of this system is not obvious. We use the following lemma:

Lemma 1 *Let X be a finite dimensional Hilbert space with scalar product (\cdot, \cdot) and norm $\|\cdot\|$, and let P be a continuous mapping from X into itself such that*

$$\exists R > 0; \forall \xi \in X \quad (\|\xi\| = R \implies (P(\xi), \xi) \geq 0). \quad (5)$$

Then there exists $\xi \in X$, $\|\xi\| \leq R$, such that $P(\xi) = 0$.

The related proof, based on the Brouwer fixed point theorem, can be found in [8, p.166]. We only note that the result which is proved in [8] corresponds to the case where the inequality $(P(\xi), \xi) \geq 0$ in the assertion (5) is a strict inequality. The case of a large inequality holds too, with the same proof.

We apply this lemma to prove the existence of $(\mathbf{u}_k, \boldsymbol{\sigma}_k)$ as follows: let X be the space defined as $X := V_k \times W_k$, endowed with the scalar product inherited from $V \times W$. Let P_k the mapping from X into itself defined by, for all $((\mathbf{u}, \boldsymbol{\sigma}), (\mathbf{v}, \boldsymbol{\tau})) \in X^2$,

$$\begin{aligned} & (P_k(\mathbf{u}, \boldsymbol{\sigma}), (\mathbf{v}, \boldsymbol{\tau}))_X \\ &= 2r \left[\operatorname{Re} \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} + (1-r) \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} + \int_{\Omega} \boldsymbol{\sigma} : \mathbb{D}(\mathbf{v}) - \langle \mathbf{f}, \mathbf{v} \rangle \right] \\ &+ \left[\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\tau} + \operatorname{D} \int_{\Omega} \nabla \boldsymbol{\sigma} : \nabla \boldsymbol{\tau} + \operatorname{We} \int_{\Omega} (\mathbf{u} \cdot \nabla \boldsymbol{\sigma} + g_a(\nabla \mathbf{u}, \boldsymbol{\sigma})) : \boldsymbol{\tau} \right. \\ &\quad \left. - 2r \int_{\Omega} \mathbb{D}(\mathbf{u}) : \boldsymbol{\tau} \right]. \end{aligned}$$

The continuity of P_k is obvious. Let us show that condition (5) holds. Denoting $\xi = (\mathbf{u}, \boldsymbol{\sigma}) \in X$ we have

$$\begin{aligned} (P_k(\xi), \xi)_X &= 2r(1-r) \int_{\Omega} \|\nabla \mathbf{u}\|^2 + \int_{\Omega} \|\boldsymbol{\sigma}\|^2 + \operatorname{D} \int_{\Omega} \|\nabla \boldsymbol{\sigma}\|^2 \\ &\quad + \operatorname{We} \int_{\Omega} g_a(\nabla \mathbf{u}, \boldsymbol{\sigma}) : \boldsymbol{\sigma} - 2r \langle \mathbf{f}, \mathbf{u} \rangle. \end{aligned}$$

From the Hölder inequality and the Sobolev injection $W \subset L^4(\Omega)^9$, there exists a positive constant C_{Ω} which only depends on the domain, such that $\|\boldsymbol{\sigma}\|_{L^4} \leq C_{\Omega} \|\boldsymbol{\sigma}\|_W$. Thus, the contribution g_a satisfied

$$\int_{\Omega} g_a(\nabla \mathbf{u}, \boldsymbol{\sigma}) : \boldsymbol{\sigma} \leq 2|a| \int_{\Omega} \|\nabla \mathbf{u}\| \|\boldsymbol{\sigma}\|^2 \leq 2|a| \|\nabla \mathbf{u}\|_{L^2} \|\boldsymbol{\sigma}\|_{L^4}^2,$$

which yields

$$\int_{\Omega} g_a(\nabla \mathbf{u}, \boldsymbol{\sigma}) : \boldsymbol{\sigma} \leq 2|a| C_{\Omega}^2 \|\mathbf{u}\|_V \|\boldsymbol{\sigma}\|_W^2. \quad (6)$$

We deduce that

$$\begin{aligned} (P_k(\xi), \xi)_X &\geq 2r(1-r) \|\mathbf{u}\|_V^2 + \min(1, \operatorname{D}) \|\boldsymbol{\sigma}\|_W^2 \\ &\quad - 2|a| C_{\Omega}^2 \operatorname{We} \|\mathbf{u}\|_V \|\boldsymbol{\sigma}\|_W^2 - 2r \|\mathbf{f}\|_{H^{-1}} \|\mathbf{u}\|_V. \end{aligned}$$

Using the definition of the norm $\|\xi\|_X^2 = 2r \|\mathbf{u}\|_V^2 + \|\boldsymbol{\sigma}\|_W^2$, we obtain

$$(P_k(\xi), \xi)_X \geq \min(1-r, \operatorname{D}) \|\xi\|_X^2 - \frac{\sqrt{2}|a| C_{\Omega}^2 \operatorname{We}}{\sqrt{r}} \|\xi\|_X^3 - \sqrt{2r} \|\mathbf{f}\|_{H^{-1}} \|\xi\|_X.$$

We write this inequality as $(P_k(\xi), \xi)_X \geq \|\xi\|_X(-\alpha\|\xi\|_X^2 + \beta\|\xi\|_X - \gamma)$.

• If $a \neq 0$ (*i.e.* $\alpha > 0$), we deduce that $(P_k(\xi), \xi)_X$ may be positive for some value of ξ if the discriminant $\beta^2 - 4\alpha\gamma$ is nonnegative; in this case we have

$$(P_k(\xi), \xi)_X \geq 0 \iff \frac{\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \leq \|\xi\|_X \leq \frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}.$$

In particular, by denoting

$$C_{(\text{II})} = \frac{\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha},$$

we obtain

$$\|\xi\|_X = C_{(\text{II})} \implies (P_k(\xi), \xi)_X \geq 0.$$

We also note that $C_{(\text{II})} > 0$ (except for the case $\mathbf{f} = \mathbf{0}$ (*i.e.* $\gamma = 0$) where we can use instead of $C_{(\text{II})}$ any constant C such that $0 < C < \frac{\beta}{\alpha}$).

• If $a = 0$ (*i.e.* $\alpha = 0$), we easily have

$$(P_k(\xi), \xi)_X \geq 0 \iff \|\xi\|_X \geq \frac{\gamma}{\beta}.$$

We then use

$$C_{(\text{II})} = \frac{\gamma}{\beta}.$$

As for the case $a \neq 0$ we have $C_{(\text{II})} > 0$ (except for $\mathbf{f} = \mathbf{0}$ (*i.e.* $\gamma = 0$) where we can use instead of $C_{(\text{II})}$ any positive constant).

Finally, assumptions given by Lemma 1 are satisfied taking $R = C_{(\text{II})}$ as soon as $\beta^2 - 4\alpha\gamma \geq 0$, which is equivalent to the condition $C_{(\text{I})} \leq 1$ introduced in Theorem 1. By Lemma 1, we deduce that, for any $k \in \mathbb{N}$, Equations (4) admit a solution $(\mathbf{u}_k, \boldsymbol{\sigma}_k)$ which satisfies

$$2r\|\mathbf{u}_k\|_V^2 + \|\boldsymbol{\sigma}_k\|_W^2 \leq C_{(\text{II})}^2. \quad (7)$$

This estimate (7) implies that the sequence $(\mathbf{u}_k, \boldsymbol{\sigma}_k)$ remains bounded in $V \times W$. Thus there exists some $(\mathbf{u}, \boldsymbol{\sigma}) \in V \times W$ and a subsequence (still denoted by k) such that $(\mathbf{u}_k, \boldsymbol{\sigma}_k) \rightharpoonup (\mathbf{u}, \boldsymbol{\sigma})$ for the weak topology of $V \times W$, as $k \rightarrow +\infty$. As $H^1 \subset L^2$ with compact injection, $(\mathbf{u}_k, \boldsymbol{\sigma}_k) \rightarrow (\mathbf{u}, \boldsymbol{\sigma})$ for the strong topology of $L^2(\Omega)^3 \times L^2(\Omega)^9$. The consequences are twofold:

- we can pass to the limit in all the terms of Equations (4), and deduce that $(\mathbf{u}, \boldsymbol{\sigma})$ is a solution of (2) ;
- due to the usual property of the weak limit, we have

$$2r\|\mathbf{u}\|_V^2 + \|\boldsymbol{\sigma}\|_W^2 \leq \liminf_{k \rightarrow +\infty} 2r\|\mathbf{u}_k\|_V^2 + \|\boldsymbol{\sigma}_k\|_W^2 \leq C_{(\text{II})}^2.$$

This concludes the proof of existence of a solution $(\mathbf{u}, \boldsymbol{\sigma})$ of problem (2) satisfying Eq. (3).

Now let us prove the existence of a pressure field associated to the incompressibility condition. For a solution $(\mathbf{u}, \boldsymbol{\sigma})$ of problem (2), we have, for all $\mathbf{v} \in \mathcal{D}(\Omega)$ such that $\text{div } \mathbf{v} = 0$,

$$\langle \text{Re}(\mathbf{u} \cdot \nabla \mathbf{u}) - (1-r)\Delta \mathbf{u} - \text{div } \boldsymbol{\sigma} - \mathbf{f}, \mathbf{v} \rangle = 0.$$

By De Rham theorem, there exists a pressure $p \in \mathcal{D}'(\Omega)$ such that Eq. (1) holds in $\mathcal{D}'(\Omega)$. The regularity of $\Delta \mathbf{u}$, $\mathbf{u} \cdot \nabla \mathbf{u}$, $\text{div } \boldsymbol{\sigma}$ and \mathbf{f} implies that the pressure p is more regular. For instance, by Sobolev embeddings, we have

$$\mathbf{u} \in V \subset L^6(\Omega)^3$$

and since $\nabla u \in L^2(\Omega)^{3 \times 3}$, then the convective term $\mathbf{u} \cdot \nabla \mathbf{u}$ belongs to $L^{3/2}(\Omega)$. Thus, we have

$$\nabla p = \Delta \mathbf{u} + \text{div } \boldsymbol{\sigma} + \mathbf{f} - \text{Re}(\mathbf{u} \cdot \nabla \mathbf{u}) \in H^{-1}(\Omega).$$

This implies that $p \in L^2(\Omega)$, see [8, p.14].

4 Uniqueness of the solution

Let $(\mathbf{u}_1, \boldsymbol{\sigma}_1)$ and $(\mathbf{u}_2, \boldsymbol{\sigma}_2)$ be two solutions of (2) and introduce the difference $(\mathbf{u}, \boldsymbol{\sigma}) = (\mathbf{u}_2, \boldsymbol{\sigma}_2) - (\mathbf{u}_1, \boldsymbol{\sigma}_1)$. By subtraction we obtain, for all $(\mathbf{v}, \boldsymbol{\tau}) \in V \times W$,

$$\begin{aligned} \text{Re} \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}_2 + \mathbf{u}_1 \cdot \nabla \mathbf{u}) \cdot \mathbf{v} + (1-r) \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} + \int_{\Omega} \boldsymbol{\sigma} : \mathbb{D}(\mathbf{v}) &= 0, \\ \text{We} \int_{\Omega} (\mathbf{u} \cdot \nabla \boldsymbol{\sigma}_2 + \mathbf{u}_1 \cdot \nabla \boldsymbol{\sigma} + g_a(\nabla \mathbf{u}, \boldsymbol{\sigma}_2) + g_a(\nabla \mathbf{u}_1, \boldsymbol{\sigma})) : \boldsymbol{\tau} + \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\tau} \\ + \text{D} \int_{\Omega} \nabla \boldsymbol{\sigma} : \nabla \boldsymbol{\tau} &= 2r \int_{\Omega} \boldsymbol{\tau} : \mathbb{D}(\mathbf{u}). \end{aligned}$$

Taking $(\mathbf{v}, \boldsymbol{\tau}) = (2r \mathbf{u}, \boldsymbol{\sigma})$ and adding the two equations, we obtain

$$2r(1-r)\|\mathbf{u}\|_V^2 + \min(1, D)\|\boldsymbol{\sigma}\|_W^2 \leq -2r\text{Re} \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u}_2) \cdot \mathbf{u} \\ - \text{We} \int_{\Omega} (\mathbf{u} \cdot \nabla \boldsymbol{\sigma}_2 + g_a(\nabla \mathbf{u}, \boldsymbol{\sigma}_2) + g_a(\nabla \mathbf{u}_1, \boldsymbol{\sigma})) : \boldsymbol{\sigma}.$$

By definition of the bilinear function g_a , we have

$$\int_{\Omega} g_a(\nabla \mathbf{u}, \boldsymbol{\sigma}_2) : \boldsymbol{\sigma} = (a+1) \int_{\Omega} (\nabla \mathbf{u} \cdot \boldsymbol{\sigma}_2) : \boldsymbol{\sigma} + (a-1) \int_{\Omega} (\boldsymbol{\sigma}_2 \cdot \nabla \mathbf{u}) : \boldsymbol{\sigma}.$$

Using the Hölder inequality and the Sobolev injection $W \subset L^4(\Omega)^9$ we deduce that

$$\left| \int_{\Omega} g_a(\nabla \mathbf{u}, \boldsymbol{\sigma}_2) : \boldsymbol{\sigma} \right| \leq 2C_{\Omega}^2 \|\mathbf{u}\|_V \|\boldsymbol{\sigma}_2\|_W \|\boldsymbol{\sigma}\|_W.$$

Estimating the other terms as in Equation (6), we obtain

$$2r(1-r)\|\mathbf{u}\|_V^2 + \min(1, D)\|\boldsymbol{\sigma}\|_W^2 \leq 2rC_{\Omega}^2 \text{Re} \|\mathbf{u}_2\|_V \|\mathbf{u}\|_V^2 \\ + 3C_{\Omega}^2 \text{We} \|\boldsymbol{\sigma}_2\|_W \|\mathbf{u}\|_V \|\boldsymbol{\sigma}\|_W \\ + 2|a|C_{\Omega}^2 \text{We} \|\mathbf{u}_1\|_V \|\boldsymbol{\sigma}\|_W^2.$$

As the solutions $(\mathbf{u}_1, \boldsymbol{\sigma}_1)$ and $(\mathbf{u}_2, \boldsymbol{\sigma}_2)$ satisfy Equation (3), we deduce that

$$2r(1-r)\|\mathbf{u}\|_V^2 + \min(1, D)\|\boldsymbol{\sigma}\|_W^2 \leq \frac{C_{\Omega}^2 C_{(\text{II})}}{\sqrt{2r}} \left(2r\text{Re} \|\mathbf{u}\|_V^2 \\ + 3\text{We}\sqrt{2r} \|\mathbf{u}\|_V \|\boldsymbol{\sigma}\|_W \quad (8) \\ + 2|a|\text{We} \|\boldsymbol{\sigma}\|_W^2 \right).$$

From Young inequality we deduce

$$3\text{We}\sqrt{2r} \|\mathbf{u}\|_V \|\boldsymbol{\sigma}\|_W \leq 2r\text{Re} \|\mathbf{u}\|_V^2 + \frac{9\text{We}^2}{4\text{Re}} \|\boldsymbol{\sigma}\|_W^2.$$

Consequently, Estimate (8) reads $2rA\|\mathbf{u}\|_V^2 + B\|\boldsymbol{\sigma}\|_W^2 \leq 0$ where A and B are given by

$$A = (1-r) - \frac{4\text{Re}C_{\Omega}^2 C_{(\text{II})}}{\sqrt{2r}}, \\ B = \min(1, D) - \frac{C_{\Omega}^2 C_{(\text{II})}}{\sqrt{2r}} \left(\frac{9\text{We}^2}{4\text{Re}} + 2|a|\text{We} \right).$$

Using the fact that $C_{(\text{II})}$ tends to 0 as $\|\mathbf{f}\|_{H^{-1}}$ tends to 0, we deduce that for $\|\mathbf{f}\|_{H^{-1}}$ small enough, the coefficients A and B are positive. In the same way, if Re is small enough then $A > 0$ and, if We is small, then $B > 0$. These inequalities imply $\|\mathbf{u}\|_V = \|\boldsymbol{\sigma}\|_W = 0$, which means $(\mathbf{u}_1, \boldsymbol{\sigma}_1) = (\mathbf{u}_2, \boldsymbol{\sigma}_2)$.

5 Strong solutions

As previously, we notice that $\mathbf{u} \in L^6(\Omega)^3$, so that $\mathbf{u} \cdot \nabla \mathbf{u} \in L^{3/2}(\Omega)^3$. We also have $\text{div } \boldsymbol{\sigma} \in L^2(\Omega)^3$ and if \mathbf{f} is regular then the regularity of the Stokes problem, see [2], implies that

$$\mathbf{u} \in W^{2,3/2}(\Omega)^3, \quad p \in W^{1,3/2}(\Omega).$$

In the same way, $\mathbf{u} \cdot \nabla \boldsymbol{\sigma}$ and $g_a(\nabla \mathbf{u}, \boldsymbol{\sigma})$ belong to $L^{3/2}(\Omega)^{3 \times 3}$. The regularity of the Laplace problem implies that

$$\boldsymbol{\sigma} \in W^{2,3/2}(\Omega)^{3 \times 3}.$$

Repeating such a process, we find that $\mathbf{u} \in W^{s,3/2}(\Omega)^3$, $p \in W^{s-1,3/2}(\Omega)$ and $\boldsymbol{\sigma} \in W^{s,3/2}(\Omega)^{3 \times 3}$ for any $s \in \mathbb{N}$. The proof is concluded.

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